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Divergence Behavior of Sequences of Linear Operators with Applications

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Abstract In this paper we study the spaceability of divergence sets of sequences of bounded linear operators on Banach spaces. For Banach spaces with the s-property, we can give a sufficient condition that guarantees the unbounded divergence on a set that contains an infinite dimensional closed subspace after the zero element has been added. This generalizes the classical Banach–Steinhaus theorem which implies that the divergence set is a residual set. We further prove that many important spaces, e.g., ℓ^p, $1 \leq p < \infty$, $C[0, 1]$, L^p, $1 < p < \infty$, as well as Paley–Wiener and Bernstein spaces, have the s-property. Finally, consequences for the convergence behavior of sampling series and system approximation processes are shown.

Keywords System and signal approximation · Sampling series · Paley–Wiener spaces · Spaceability · Banach–Steinhaus theorem

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1 Introduction

In [11] we gave a sufficient condition for the spaceability of the divergence set of a sequence of linear operators on an arbitrary Banach space. This result can be seen as an extension of the classical theory of Banach and Steinhaus [3,5]. The Banach–Steinhaus theory is particularly valuable for the analysis of the approximation behavior in Fourier analysis and approximation theory. Typical examples are the analysis of norm convergence and pointwise convergence of Fourier series, summability of Fourier series, convergence behavior of sampling series, signal reconstructions, and system approximations.

However, so far it was unclear what happens if the sufficient condition that was given in [11] is not satisfied. Our first contribution is to show that the condition is not necessary. To this end, we choose a different approach for the analysis in this paper. We will show for a large class of Banach spaces that we have spaceability of the divergence set of a sequence of linear operators under similar conditions as in the Banach–Steinhaus theorem. Hence, for these spaces we can considerably strengthen the Banach–Steinhaus theorem. The class of Banach spaces for which we can show this behavior comprises important spaces that are used in Fourier analysis and approximation theory, e.g., all separable Hilbert spaces, $C[0,1]$, $C^{\mathrm{e}}[-\pi,\pi]$, L^p, $1 \le p < \infty$, c_0, ℓ^p, $1 < p < \infty$, Paley–Wiener spaces $\mathcal{PW}_\pi^p$, $1 \le p < \infty$, and Bernstein spaces $\mathcal{B}_\pi^p$, $1 \le p < \infty$. As an application of the results we show the spaceability of the divergence set for sampling series, system approximations, and convolution sums.

2 Notation

In this work we will use several Banach spaces, which we briefly introduce next. By c_0 we denote the space of all sequences that converge to zero. ℓ^p, $1 < p < \infty$, are the usual spaces of p-th power summable sequences $x = \{x_n\}_{n\in\mathbb{N}}$ with the norm $\|x\|_{\ell^p} = (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}$. The space of all continuous functions on $[0,1]$ is denoted by $C[0,1]$, and the space of all continuous functions f on $[-\pi,\pi]$, satisfying $f(\pi) = f(-\pi)$ by $C^{\mathrm{e}}[-\pi,\pi]$. Let Ω be a Lebesgue measurable subset of $\mathbb{R}$. By $L^p(\Omega)$, $1 \le p \le \infty$, we denote the usual L^p-spaces on Ω, equipped with the norm $\|\cdot\|_p$. The Bernstein space $\mathcal{B}_\sigma^p$, $\sigma > 0$, $1 \le p \le \infty$, consists of all functions of exponential type at most σ, whose restriction to the real line is in $L^p(\mathbb{R})$ [20, p. 49]. $\mathcal{B}_{\sigma,0}^\infty$ denotes the set of all signals in $\mathcal{B}_\sigma^\infty$ that vanish on the real axis at infinity.

Let $\hat{f}$ denote the Fourier transform of a function f, where $\hat{f}$ is to be understood in the distributional sense. For $\sigma > 0$ and $1 \le p \le \infty$, we denote by $\mathcal{PW}_\sigma^p$ the Paley–Wiener space of signals f with a representation $f(z) = 1/(2\pi) \int_{-\sigma}^{\sigma} g(\omega)\, \mathrm{e}^{iz\omega}\ \mathrm{d}\omega$, $z \in \mathbb{C}$, for some $g \in L^p[-\sigma,\sigma]$. The norm for $\mathcal{PW}_\sigma^p$, $1 \le p < \infty$, is given by $\|f\|_{\mathcal{PW}_\sigma^p} = (1/(2\pi) \int_{-\sigma}^{\sigma} |\hat{f}(\omega)|^p\ \mathrm{d}\omega)^{1/p}$. Note that $\mathcal{PW}_\sigma^2 \subset \mathcal{PW}_\sigma^1$.

A subset $\mathcal{M}$ of a metric space X is said to be nowhere dense in X if the closure $[\mathcal{M}]$ does not contain a non-empty open subset of X. $\mathcal{M}$ is said to be meager (or of the first category) if $\mathcal{M}$ is the countable union of sets each of which is nowhere dense in X. $\mathcal{M}$ is said to be nonmeager (or of the second category) if it is not meager. The complement of a meager set is called a residual set. Meager sets may be considered as "small". According to Baire's theorem [26], in a complete metric space any residual set is dense and nonmeager. One property that shows the richness of residual sets is the following: the countable intersection of residual sets is always a residual set. Further, any subset of a meager set is a meager set and any superset of a residual set is a residual set.

3 General Questions on Spaceability

Before we start with our program that was outlined in the introduction, we introduce the concept of spaceability. Spaceability, which has recently been used for example in [1, 2,6,16,18], is a concept that describes the structure of some given subset of an ambient normed space or, more generally, topological space. A set S in a linear topological space X is said to be spaceable if $S \cup \{0\}$ contains a closed infinite dimensional subspace of X.

Next, we review the Banach–Steinhaus theorem [25, p. 98].

Theorem 1 (Banach–Steinhaus). *Let B_1 be a Banach space and B_2 a normed space. Assume that $\{T_N\}_{N\in\mathbb{N}}$ is a sequence of bounded linear operators from B_1 into B_2, satisfying* $\limsup_{N\to\infty} \|T_N\|_{B_1\to B_2} = \infty$. *Then the set*

$$\left\{ f \in B_1 : \limsup_{N\to\infty} \|T_N f\|_{B_2} = \infty \right\} \tag{3.1}$$

is a residual set.

The goal of this work is to strengthen this result. In particular we are interested in extending it towards showing a linear structure in the set (3.1), or, more specifically spaceability. Such a linear structure is important in applications, because it implies that any linear combination of vectors, which is not the zero vector, leads to divergence as well.

Note that it is significantly more difficult to show a linear structure in the set of vectors for which we have divergence compared to showing a linear structure in the set of vectors for which we have convergence. If we have two vectors f_1 and f_2, for which $T_N f_1$ and $T_N f_2$ converge, it is clear that for their sum $f_1 + f_2$ we have convergence as well. However, for divergence this is not true. Given two vectors g_1 and g_2 for which $T_N g_1$ and $T_N g_2$ diverge, we cannot conclude that $T_N(g_1 + g_2)$ diverges: indeed, choose $g_2 = f_1 - g_1$, where f_1 is any vector for which we have convergence and g_1 any vector for which we have divergence.

The above example shows that in general we cannot expect that the set of vectors with divergent approximation process is a linear space. However, we can ask if this set contains an infinite dimensional subspace with linear structure.

We want to analyze the following question.

Question 1 *Let B_1 and B_2 be two separable Banach spaces. Assume that $\{T_N\}_{N\in\mathbb{N}}$ is a sequence of bounded linear operators from B_1 into B_2, satisfying:*

(A1) $\limsup_{N\to\infty} \|T_N\|_{B_1\to B_2} = \infty$, *and*

(A2) There exists a bounded linear operator $T: B_1 \to B_2$ *as well as a dense subset* $\mathcal{M} \subset B_1$ *such that* $\lim_{N\to\infty}\|Tf - T_N f\|_{B_2} = 0$ *for all* $f \in \mathcal{M}$.

Is the set

$$\left\{ f \in B_1 : \limsup_{N\to\infty} \|T_N f\|_{B_2} = \infty \right\} \tag{3.2}$$

spaceable?

We do not know whether the answer to Question 1 is positive for arbitrary separable Banach spaces B_1, B_2. In [11], we were able to answer the question positively under an additional assumption on the sequence $\{T_N\}_{N\in\mathbb{N}}$. If it is additionally assumed that

(A3) There exists an infinite dimensional closed subspace $\underline{B}_1$ of B_1 such that $\sup_{N\in\mathbb{N}}\|T_N f\|_{B_2} \le C\|f\|_{\underline{B}_1}$ for all $f \in \underline{B}_1$,

then it can be concluded that the set (3.2) is spaceable. Thus, the assumptions (A1)–(A3) are sufficient for spaceability. In Sect. 5 we will give an example where we have spaceability, but where the assumptions from [11] are not satisfied. Hence, the assumptions in [11] are sufficient but not necessary for spaceability.

Remark 1 Condition (A2) in Question 1 entails that the set of functions with convergence is dense lineable.

It is possible to reduce Question 1 to a simpler question about sequences of functionals on $C[0, 1]$.

Question 2 *Let* B_1 *be a closed subspace of* $C[0, 1]$. *Assume that* $\{\psi_N\}_{N\in\mathbb{N}}$ *is a sequence of continuous linear functionals on* B_1, *satisfying:*

(A1') $\limsup_{N\to\infty}\|\psi_N\|_{B_1\to\mathbb{C}} = \infty$, *and*
(A2') there exists a dense subset $\mathcal{M} \subset B_1$ *such that* $\lim_{N\to\infty}\psi_N f = 0$ *for all* $f \in \mathcal{M}$.

Is the set

$$\left\{ f \in B_1 : \limsup_{N\to\infty} |\psi_N f| = \infty \right\} \tag{3.3}$$

spaceable?

The answer to Question 1 is positive if and only if Question 2 can be answered positively. The details of this reduction are provided in Appendix 1.

Hence, it is possible to significantly reduce the problem of answering Question 1. However, we still do not know the answer to Question 2, or, equivalently, the full answer to Question 1, i.e., if for all separable Banach spaces B_1, assumptions (A1) and (A2) are sufficient for the spaceability of the divergence set (3.2).

Although we are not able to prove the sufficiency of assumptions (A1) and (A2) for the spaceability of the divergence set (3.2), we will identify important Banach spaces, for which (A1) and (A2) are sufficient for spaceability. We call those Banach spaces, Banach spaces that have the s-property, where “s” stands for “spaceability”.

Definition 1 We say that an infinite dimensional separable Banach space B_1 has the s-property if for every separable Banach space B_2 and every sequence of bounded linear operators $\{T_N\}_{N\in\mathbb{N}}$ from B_1 into B_2, satisfying:

(A1) $\limsup_{N\to\infty}\|T_N\|_{B_1\to B_2} = \infty$, and
(A2) there exists a bounded linear operator $T\colon B_1 \to B_2$ as well as a dense subset $\mathcal{M} \subset B_1$ such that $\lim_{N\to\infty}\|Tf - T_N f\|_{B_2} = 0$ for all $f \in \mathcal{M}$,

the set

$$\left\{ f \in B_1 \colon \limsup_{N\to\infty}\|T_N f\|_{B_2} = \infty \right\}$$

is spaceable.

Remark 2 If T and T_N have further structural properties, e.g., shift invariance, one may expect the divergence subspace to have additional properties as well. We will resume this discussion later in Remark 7, where we discuss shift invariance.

According to the considerations regarding Questions 1 and 2, we can define the s-property of a Banach space also in a different but equivalent way.

Definition 2 We say that an infinite dimensional separable Banach space B_1 has the s-property if for every sequence of linear and continuous functionals $\{\psi_N\}_{N\in\mathbb{N}}$ on B_1, satisfying:

(A1') $\limsup_{N\to\infty}\|\psi_N\|_{B_1\to\mathbb{C}} = \infty$, and
(A2') there exists a dense subset $\mathcal{M} \subset B_1$ such that $\lim_{N\to\infty}\psi_N f = 0$ for all $f \in \mathcal{M}$,

the set

$$\left\{ f \in B_1 \colon \limsup_{N\to\infty}|\psi_N f| = \infty \right\}$$

is spaceable.

In the next section we will show that the corresponding question for sequences of bounded linear functionals on separable Hilbert spaces can be answered positively. Later in Sect. 6 we will extend this results to other relevant Banach spaces.

Remark 3 There is probably no universal approach for showing lineability. This conjecture is supported by an observation made in [13], where it was discussed that for a certain set the question of dense lineability is equivalent to the Riemann hypothesis on the zeros of the Riemann zeta function.

4 Separable Hilbert Spaces

Next we show that Question 1 can be answered positively for all separable Hilbert spaces. Thus, all separable Hilbert spaces have the s-property.

Theorem 2 *Let H_1 be an infinite dimensional separable Hilbert space. Then H_1 has the s-property.*

Before we prove Theorem 2, we present a useful lemma, which we will also employ in the proof, and in several corollaries.

Lemma 1 *Let B_1 and B_2 be two separable isomorphic Banach spaces. Then B_1 has the s-property if and only if B_2 has the s-property.*

Proof Since B_1 and B_2 are isomorphic, there exists a linear and bounded bijection $U: B_1 \to B_2$. Without loss of generality we assume that B_1 has the s-property. Let $\{\psi_N\}_{N\in\mathbb{N}}$ be a sequence of functionals on B_2, satisfying the properties (A1') and (A2'), i.e., $\lim_{N\to\infty}\|\psi_N\|_{B_2\to\mathbb{C}} = \infty$, and $\lim_{N\to\infty}\psi_N f = 0$ for all f in some dense subset $\mathcal{M}$ of B_2. Then $U^{-1}(\mathcal{M})$ is dense in B_1. Let $\phi_N = \psi_N U$, $N \in \mathbb{N}$. $\{\phi_N\}_{N\in\mathbb{N}}$ is a sequence of functionals on B_1 that satisfies $\lim_{N\to\infty}\phi_N f = 0$ for all f in the dense subset $U^{-1}(\mathcal{M})$ of B_1. Further, we have $\psi_N = \phi_N U^{-1}$, $N \in \mathbb{N}$, and consequently

$$\|\psi_N\|_{B_2\to\mathbb{C}} \leq \|\phi_N\|_{B_1\to\mathbb{C}}\|U^{-1}\|_{B_2\to B_1},$$

which shows that

$$\limsup_{N\to\infty}\|\phi_N\|_{B_1\to\mathbb{C}} = \infty.$$

Thus, $\{\phi_n\}_{n\in\mathbb{N}}$ is a sequence of functionals on B_1 that satisfies the conditions (A1') and (A2'). Since B_1 has the s-property, there exists an infinite dimensional closed subspace $\underline{B}_1 \subset B_1$ such that

$$\limsup_{N\to\infty}|\phi_N f| = \infty$$

for all $f \in \underline{B}_1 \setminus \{0\}$. It follows that $\underline{B}_2 := U(\underline{B}_1)$ is an infinite dimensional closed subspace, and we have

$$\limsup_{N\to\infty}|\psi_N f| = \limsup_{N\to\infty}|\phi_N U^{-1} f| = \infty$$

for all $f \in \underline{B}_2 \setminus \{0\}$. This shows that B_2 has the s-property because the sequence of functionals $\{\psi_N\}_{N\in\mathbb{N}}$ was chosen arbitrary. □

Theorem 2 together with Lemma 1 immediately gives the following corollary.

Corollary 1 *In $L^2[0,1]$ every infinite dimensional closed subspace has the s-property.*

Remark 4 If we could prove the statement of Corollary 1 for $C[0,1]$ instead of $L^2[0,1]$, i.e., if we could prove that every infinite dimensional closed subspace of $C[0,1]$ has the s-property, then we would know that every separable Banach space has the s-property.

Theorem 2 has further interesting consequences. An infinite dimensional Banach space B_1 is called homogeneous if every infinite dimensional closed subspace $\underline{B}_2$ of B_1 is isomorphic to B_1. This definition leads us to the following corollary.

Corollary 2 *Every homogeneous separable Banach space B_1 has the s-property, and every infinite dimensional closed subspace $\underline{B}_1$ of B_1 has the s-property.*

Proof Gowers has shown in [17] that every homogeneous separable Banach space is isomorphic to some separable Hilbert space and therefore to $L^2[0,1]$. Application of Lemma 1 and Theorem 2 gives the result. □

Now we come to the proof of Theorem 2. As we have discussed in the previous section, instead of studying sequences of linear operators, it is sufficient to study sequences of linear functionals. Further, since all infinite dimensional separable Hilbert spaces are isomorphic to ℓ^2, it suffices, thanks to Lemma 1, to consider sequences of linear functionals on ℓ^2.

Theorem 3 *Let $\{\psi_N\}_{N\in\mathbb{N}}$ be a sequence of continuous linear functionals on ℓ^2, satisfying:*

- *(A1') $\limsup_{N\to\infty}\|\psi_N\| = \infty$, and*
- *(A2') there exists a dense subset $\mathcal{M} \subset \ell^2$ such that $\lim_{N\to\infty}\psi_N f = 0$ for all $f \in \mathcal{M}$.*

Then the set

$$\left\{f \in \ell^2 \colon \limsup_{N\to\infty}|\psi_N f| = \infty\right\}$$

is spaceable.

Remark 5 Theorem 3 shows that ℓ^2 has the s-property.

For the proof of Theorem 3 we use the following lemma, the proof of which we postpone until the end of this section.

Lemma 2 *Let $\{g_N\}_{N\in\mathbb{N}}$ be a sequence in ℓ^2, satisfying*

1. *$\limsup_{N\to\infty}\|g_N\|_{\ell^2} = \infty$ and*
2. *$\lim_{N\to\infty}|g_N(n)| = 0$ for all $n \in \mathbb{N}$.*

For $N \in \mathbb{N}$, we define

$$\psi_N f = \sum_{k=1}^{\infty} f(k)\overline{g_N(k)}.$$

Then the set

$$\left\{f \in \ell^2 \colon \limsup_{N\to\infty}|\psi_N f| = \infty\right\}$$

is spaceable.

Equipped with Lemma 2, we are in the position to prove Theorem 3.

Proof of Theorem 3 Let $\{\psi_N\}_{N\in\mathbb{N}}$ be a sequence of continuous linear functionals defined on ℓ^2, satisfying (A1') and (A2'). Let $L = \operatorname{span}(\mathcal{M})$. Since ℓ^2 is separable, L is separable and therefore contains a countable linearly independent subset $\{u_n\}_{n\in\mathbb{N}} \subset L$ such that $\operatorname{span}(\{u_n\}_{n\in\mathbb{N}})$, i.e., the finite linear span, is dense in L. Next, we can apply the Gram–Schmidt orthonormalization process on $\{u_n\}_{n\in\mathbb{N}}$, which gives us a complete orthonormal system $\{\phi_n\}_{n\in\mathbb{N}}$ in ℓ^2, where each ϕ_n, $n \in \mathbb{N}$, is a linear combination of $\{u_1, \dots, u_n\}$, and hence an element of L. Since the functionals ψ_N are linear, it follows that

$$\lim_{N\to\infty} \psi_N\phi_n = 0 \tag{4.1}$$

for all $n \in \mathbb{N}$. Using the isomorphic isomorphism

$$U : \ell^2 \to \ell^2,\ f \mapsto \left\{ \sum_{k=1}^{\infty} f(k)\overline{\phi_n(k)} \right\}_{n\in\mathbb{N}},$$

we can work directly in the coefficient space of the complete orthonormal system $\{\phi_n\}_{n\in\mathbb{N}}$. Let us consider $\psi_N^* f = \psi_N U^{-1} f$, $f \in \ell^2$. Then $\{\psi_N^*\}_{N\in\mathbb{N}}$ is a sequence of continuous linear functionals defined on ℓ^2, for which we have

$$\limsup_{N\to\infty} \|\psi_N^*\|_{\ell^2\to\mathbb{C}} = \infty \tag{4.2}$$

and

$$\lim_{N\to\infty} \psi_N^* U\phi_n = 0 \tag{4.3}$$

for all $n \in \mathbb{N}$ according to (4.1). Due to the Riesz representation theorem, there exists $g_N \in \ell^2$ such that

$$\psi_N^* f = \sum_{k=1}^{\infty} f(k)\overline{g_N(k)}$$

for all $f \in \ell^2$. For $n \in \mathbb{N}$ we have

$$\psi_N^* U\phi_n = \sum_{k=1}^{\infty} (U\phi_n)(k)\overline{g_N(k)} = \sum_{k=1}^{\infty} \left(\sum_{m=1}^{\infty} \phi_n(m)\overline{\phi_k(m)} \right) \overline{g_N(k)} = \overline{g_N(n)}.$$

Hence, we see from (4.3) that $\lim_{N\to\infty} g_N(n) = 0$ for all $n \in \mathbb{N}$. Further, since $\|\psi_N^*\|_{\ell^2\to\mathbb{C}} = \|g_N\|_{\ell^2}$, it follows from (4.2) that $\limsup_{N\to\infty} \|g_N\|_{\ell^2} = \infty$. Thus, we can apply Lemma 2 which gives that the set

$$\left\{ f \in \ell^2 :\ \limsup_{N\to\infty} |\psi_N^* f| = \infty \right\}$$

is spaceable, i.e., that there exists an infinite dimensional closed subspace $\underline{B}_1 \subset \ell^2$ such that $\limsup_{N\to\infty} |\psi_N^* f| = \infty$ for all $f \in \underline{B}_1 \backslash \{0\}$. It follows that $\underline{B}_2 := U^{-1}(\underline{B}_1)$ is an infinite dimensional closed subspace, and we have

$$\limsup_{N\to\infty} |\psi_N f| = \limsup_{N\to\infty} |\psi_N^* U f| = \infty$$

for all $f \in \underline{B}_2$. In other words,

$$\left\{ f \in \ell^2 :\ \limsup_{N\to\infty} |\psi_N f| = \infty \right\}$$

is spaceable. □

Having proved Theorem 3, we are in the position to give the short proof of Theorem 2.

Proof of Theorem 2 From Theorem 3 we know that ℓ^2 has the s-property. Since every infinite dimensional separable Hilbert space is isomorphic to ℓ^2, the statement of Theorem 2 follows directly from Lemma 1. □

Finally, we come to the remaining proof of Lemma 2. Given a sequence $G = \{g_N\}_{N\in\mathbb{N}}$ in ℓ^2 we define

$$\mathrm{Div}(G) = \left\{ f \in \ell^2 \colon \limsup_{N\to\infty} |\langle f, g_N\rangle| = \infty \right\}.$$

Further, to keep the notation compact we introduce the following abbreviation. For $S \subseteq \mathbb{N}$ we set

$$\mathbf{1}_S(k) = \begin{cases} 1, & k \in S, \\ 0, & k \in \mathbb{N} \setminus S, \end{cases}$$

and for the case $S = \{1, \ldots n\}$ we simply write $\mathbf{1}_n$.

Lemma 3 *Assume that $G = \{g_n\}_{n\in\mathbb{N}}$ and $H = \{h_n\}_{n\in\mathbb{N}}$ are two sequences in ℓ^2 with*

$$\lim_{N\to\infty} \|h_N - g_N\|_{\ell^2} = 0.$$

Then

1. $\limsup_{N\to\infty} \|g_N\|_{\ell^2} = \infty$ *if and only if* $\limsup_{N\to\infty} \|h_N\|_{\ell^2} = \infty$,
2. $\lim_{N\to\infty} g_N(k) = 0$ *for all* $k \in \mathbb{N}$ *if and only if* $\lim_{N\to\infty} h_N(k) = 0$ *for all* $k \in \mathbb{N}$, *and*
3. $\mathrm{Div}(H) = \mathrm{Div}(G)$.

As a consequence of Lemma 3, the simple proof of which is omitted, we can assume, without loss of generality, in the proof of Lemma 2 that, for each $N \in \mathbb{N}$, the support of g_N is finite. This is easily justified: For each N there exists a $n(N) \in \mathbb{N}$ such that $\|g_N - g_N \mathbf{1}_{n(N)}\|_{\ell^2} \le 1/N$. Defining $H = \{h_N\}_{N\in\mathbb{N}} = \{g_N \mathbf{1}_{n(N)}\}_{N\in\mathbb{N}}$ the situation of Lemma 3 is given.

Proof of Lemma 2 Let $G = \{g_N\}_{N\in\mathbb{N}}$ be a sequence in ℓ^2, satisfying the assumptions of the lemma. As justified above, we work with the finitely supported functions h_N, $N \in \mathbb{N}$ in the following. Since $\lim_{N\to\infty} h_N(k) = 0$ for all $k \in \mathbb{N}$, we even have $\lim_{N\to\infty} \|h_N \mathbf{1}_n\|_{\ell^2} = 0$ for each fixed $n \in \mathbb{N}$.

Next, we will construct by induction:

1. A strictly increasing index sequence $\{N(j)\}_{j\in\mathbb{N}}$ of natural numbers such that $\|h_{N(j)}\|_{\ell^2} \ge j + 1$,
2. A sequence of disjoint intervals $I_j = \{l_j, \ldots, r_j\} \subset \mathbb{N}$ with $l_j < r_j < l_{j+1}$ for all $j \in \mathbb{N}$,
3. An ONS $\{\phi_j\}_{j\in\mathbb{N}}$ with $\mathrm{supp}\, \phi_j \subseteq I_j$ for all $j \in \mathbb{N}$,

4. The index sequence $\{N(j)\}_{j\in\mathbb{N}}$ and the intervals $I_j = \{l_j, \dots, r_j\}$, $j \in \mathbb{N}$, are such that $\|h_N \mathbf{1}_{r_{j-1}}\|_{\ell^2} < 1/j$ for all $j \ge 2$ and all $N \ge N(j)$.

Let $N(1)$ be the smallest natural number such that $\|h_{N(1)}\|_{\ell^2} \ge 2$. By r_1 we denote the smallest natural number such that $\operatorname{supp} h_{N(1)} \subseteq \{1, \dots, r_1\}$. We set $I_1 = \{1, \dots, r_1\}$ and $\phi_1 = h_{N(1)}/\|h_{N(1)}\|_{\ell^2}$.

Given $N(j-1)$, $I_{j-1} = \{l_{j-1}, \dots, r_{j-1}\}$, and ϕ_{j-1}, we choose $N(j) > N(j-1)$ such that

$$\|h_{N(j)}\|_{\ell^2} \ge j + 1 \tag{4.4}$$

and

$$\|h_N \mathbf{1}_{r_{j-1}}\|_{\ell^2} < 1/j \tag{4.5}$$

for all $N \ge N(j)$. We set $l_j = r_{j-1} + 1$, $r_j = \max \operatorname{supp} h_{N(j)}$, and $I_j = \{l_j, \dots, r_j\}$. Then $h_{N(j)} \mathbf{1}_{I_j}$ satisfies

$$\|h_{N(j)} \mathbf{1}_{I_j}\|_{\ell^2} \ge \|h_{N(j)}\|_{\ell^2} - \|h_{N(j)} \mathbf{1}_{r_{j-1}}\|_{\ell^2} \ge j, \tag{4.6}$$

and we see that $h_{N(j)} \mathbf{1}_{I_j}$ is non-zero. We set

$$\phi_j = h_{N(j)} \mathbf{1}_{I_j} / \|h_{N(j)} \mathbf{1}_{I_j}\|_{\ell^2}.$$

Having constructed all above quantities for all j by induction, we set

$$\eta_n = \phi_{2^n+n-1} + \sum_{k=n+1}^{\infty} \frac{1}{k} \phi_{2^k+n-1}, \quad n \in \mathbb{N}.$$

Since the supports of the ϕ_k, $k \in \mathbb{N}$, are all pairwise disjoint, and we have $\|\phi_k\|_{\ell^2} = 1$, $k \in \mathbb{N}$, it follows that

$$\|\eta_n\|_{\ell^2}^2 = 1 + \sum_{k=n+1}^{\infty} \frac{1}{k^2}$$

and consequently that

$$1 < \|\eta_n\|_{\ell^2}^2 \le \pi^2/6 \tag{4.7}$$

for all $n \in \mathbb{N}$. We set

$$\kappa_n = \eta_n / \|\eta_n\|_{\ell^2}, \quad n \in \mathbb{N}.$$

Thus, $\|\kappa_n\|_{\ell^2} = 1$, $n \in \mathbb{N}$, and all κ_n, $n \in \mathbb{N}$, have pairwise disjoint supports and hence are orthogonal.

For $n \in \mathbb{N}$ and $s > n$ we have

$$\begin{aligned} \psi_{N(2^s+n-1)} \kappa_n &= \langle \kappa_n, h_{N(2^s+n-1)} \rangle \\ &= \langle \kappa_n \mathbf{1}_{r_{2^s+n-2}}, h_{N(2^s+n-1)} \rangle + \langle \kappa_n \mathbf{1}_{I_{2^s+n-1}}, h_{N(2^s+n-1)} \rangle \\ &\quad + \langle \kappa_n (1 - \mathbf{1}_{r_{2^s+n-1}}), h_{N(2^s+n-1)} \rangle. \end{aligned} \tag{4.8}$$

Next, we analyze the three summands on the right hand side of (4.8). For the first term we have

$$\begin{aligned}|\langle \kappa_n \mathbf{1}_{r_{2^s+n-2}}, h_{N(2^s+n-1)}\rangle| &\leq \|\kappa_n\|_{\ell^2} \|h_{N(2^s+n-1)} \mathbf{1}_{r_{2^s+n-2}}\|_{\ell^2} \\ &\leq \frac{1}{2^s+n-1},\end{aligned}$$

where we used (4.5) in the last inequality. For the second term we have

$$\begin{aligned}\langle \kappa_n \mathbf{1}_{I_{2^s+n-1}}, h_{N(2^s+n-1)}\rangle &= \frac{1}{s\|\eta_n\|_{\ell^2}} \langle \phi_{2^s+n-1}, h_{N(2^s+n-1)}\rangle \\ &= \frac{1}{s\|\eta_n\|_{\ell^2}} \left\| h_{N(2^s+n-1)} \mathbf{1}_{I_{2^s+n-1}} \right\|_{\ell^2} \\ &\geq \frac{(2^s+n-1)\sqrt{6}}{s\pi},\end{aligned}$$

where we used (4.6) and (4.7) in the last inequality. The third term gives

$$\langle \kappa_n (1-\mathbf{1}_{r_{2^s+n-1}}), h_{N(2^s+n-1)}\rangle = \langle \kappa_n, h_{N(2^s+n-1)}(1-\mathbf{1}_{r_{2^s+n-1}})\rangle = 0,$$

because, according to the definition of r_{2^s+n-1}, $h_{N(2^s+n-1)}(k) = 0$ for all $k > r_{2^s+n-1}$. Combining all partial results it follows that

$$|\psi_{N(2^s+n-1)}\kappa_n| \geq \frac{(2^s+n-1)\sqrt{6}}{s\pi} - \frac{1}{2^s+n-1},$$

which shows that for each fixed $n \in \mathbb{N}$

$$\lim_{s\to\infty} |\psi_{N(2^s+n-1)}\kappa_n| = \infty. \tag{4.9}$$

We define now the space $\mathcal{D} = \overline{\operatorname{span}(\{\kappa_n\}_{n\in\mathbb{N}})}^{\ell^2}$. Since the set $\{\kappa_n\}_{n\in\mathbb{N}}$ is a complete orthonormal system in $\mathcal{D}$ we have for the coefficients $\alpha_n = \langle f, \kappa_n\rangle$

$$f = \sum_{n=1}^{\infty} \alpha_n \kappa_n, \quad \text{and} \quad \|f\|_{\ell^2}^2 = \sum_{n=1}^{\infty} |\alpha_n|^2, \;\; f \in \mathcal{D}. \tag{4.10}$$

Let $f \in \mathcal{D}$, $f \not\equiv 0$, be arbitrary but fixed. Further let n_0 be the smallest natural number such that $\alpha_{n_0} \neq 0$. Then we have for $s \in \mathbb{N}$ that

$$\psi_{N(2^s+n_0-1)} f = \alpha_{n_0} \psi_{N(2^s+n_0-1)} \kappa_{n_0} + \sum_{n=n_0+1}^{\infty} \alpha_n \psi_{N(2^s+n_0-1)} \kappa_n. \tag{4.11}$$

We study the expression $\psi_{N(2^s+n_0-1)}\kappa_n$ for $n > n_0$ next. We have to distinguish two cases: $n_0 < n < s$ and $n_0 < s \le n$. For $n_0 < n < s$, we have

$$\begin{aligned}\psi_{N(2^s+n_0-1)}\kappa_n &= \langle \kappa_n, h_{N(2^s+n_0-1)}\rangle \\ &= \langle \kappa_n \mathbf{1}_{r_{2^s+n_0-2}}, h_{N(2^s+n_0-1)}\rangle + \langle \kappa_n \mathbf{1}_{I_{2^s+n_0-1}}, h_{N(2^s+n_0-1)}\rangle \\ &\quad + \langle \kappa_n(1-\mathbf{1}_{r_{2^s+n_0-1}}), h_{N(2^s+n_0-1)}\rangle.\end{aligned}$$

For the first summand we have

$$|\langle \kappa_n \mathbf{1}_{r_{2^s+n_0-2}}, h_{N(2^s+n_0-1)}\rangle| \le \|\kappa_n\|_{\ell^2}\|h_{N(2^s+n_0-1)}\mathbf{1}_{r_{2^s+n_0-2}}\|_{\ell^2} \le \frac{1}{2^s+n_0-1},$$

where we used (4.5) in the last inequality. For the second summand we have

$$\langle \kappa_n \mathbf{1}_{I_{2^s+n_0-1}}, h_{N(2^s+n_0-1)}\rangle = 0,$$

because $\kappa_n(k) = 0$ for all $k \in I_{2^s+n_0-1}$. For the third summand we have

$$\langle \kappa_n(1-\mathbf{1}_{r_{2^s+n_0-1}}), h_{N(2^s+n_0-1)}\rangle = \langle \kappa_n, h_{N(2^s+n_0-1)}(1-\mathbf{1}_{r_{2^s+n_0-1}})\rangle = 0$$

because, according to the definition of $r_{2^s+n_0-1}$, $h_{N(2^s+n_0-1)}(k) = 0$ for all $k > r_{2^s+n_0-1}$. Combining all partial results, we see that for $n_0 < n < s$

$$|\psi_{N(2^s+n_0-1)}\kappa_n| \le \frac{1}{2^s+n_0-1}. \tag{4.12}$$

For $n_0 < s \le n$ we have

$$\begin{aligned}\psi_{N(2^s+n_0-1)}\kappa_n &= \langle \kappa_n, h_{N(2^s+n_0-1)}\rangle \\ &= \langle \kappa_n(1-\mathbf{1}_{r_{2^n+n-2}}), h_{N(2^s+n_0-1)}\mathbf{1}_{r_{2^s+n_0-1}}\rangle \\ &= \langle \kappa_n, h_{N(2^s+n_0-1)}\mathbf{1}_{r_{2^s+n_0-1}}(1-\mathbf{1}_{r_{2^n+n-2}})\rangle = 0,\end{aligned} \tag{4.13}$$

where we used that $\kappa_n(k) = 0$ for all $k \le r_{2^n+n-2}$, $h_{N(2^s+n_0-1)}(k) = 0$ for all $k > r_{2^s+n_0-1}$, and the fact that $2^s + n_0 - 1 \le 2^n + n - 2$.

From (4.11) it follows that

$$\begin{aligned}\left|\psi_{N(2^s+n_0-1)}f - \alpha_{n_0}\psi_{N(2^s+n_0-1)}\kappa_{n_0}\right| &= \left|\sum_{n=n_0+1}^{\infty}\alpha_n\psi_{N(2^s+n_0-1)}\kappa_n\right| \\ &\le \sum_{n=n_0+1}^{s-1}|\alpha_n\psi_{N(2^s+n_0-1)}\kappa_n| + \sum_{n=s}^{\infty}|\alpha_n\psi_{N(2^s+n_0-1)}\kappa_n|.\end{aligned}$$

The second term equals 0 due to (4.13), and thanks to (4.12) and the Cauchy–Schwarz inequality we have

$$\sum_{n=n_0+1}^{s-1} |\alpha_n \psi_{N(2^s+n_0-1)} \kappa_n| \le \frac{1}{2^s+n_0-1} \sum_{n=n_0+1}^{s-1} |\alpha_n|$$
$$\le \frac{\sqrt{s}}{2^s+n_0-1} \sqrt{\sum_{n=n_0+1}^{s-1} |\alpha_n|^2} \le \frac{\sqrt{s}}{2^s} \|f\|_{\ell^2}.$$

Thus, we obtain altogether

$$\left|\psi_{N(2^s+n_0-1)} f - \alpha_{n_0} \psi_{N(2^s+n_0-1)} \kappa_{n_0}\right| \le \frac{\sqrt{s}}{2^s} \|f\|_{\ell^2} \tag{4.14}$$

for all $s > n_0$. Combining (4.9) and (4.14), we finally see that

$$\lim_{s\to\infty} |\psi_{N(2^s+n_0-1)} f| = \infty.$$

Since $f \in \mathcal{D}$, $f \not\equiv 0$, was chosen arbitrarily, we have proved the spaceability of the set $\left\{f \in \ell^2 \colon \limsup_{N\to\infty} |\psi_N f| = \infty\right\}$. □

5 Spaceability Without the Control of the Boundedness Behavior

In [11] a sufficient condition for the spaceability of the divergence set of a sequence of linear operators on an arbitrary Banach space was given. For completeness, we state this theorem next.

Theorem 4 *Let B_1 and B_2 be two separable Banach spaces. Assume that $\{T_N\}_{N\in\mathbb{N}}$ is a sequence of bounded linear operators from B_1 into B_2, satisfying:*

- *(A1)* $\limsup_{N\to\infty} \|T_N\|_{B_1\to B_2} = \infty$,
- *(A2) There exists a bounded linear operator $T\colon B_1 \to B_2$ as well as a dense subset $\mathcal{M}$ of B_1 such that* $\lim_{N\to\infty} \|Tf - T_N f\|_{B_2} = 0$ *for all $f \in \mathcal{M}$, and*
- *(A3) There exists an infinite dimensional closed subspace $\underline{B}_1$ of B_1 such that* $\sup_{N\in\mathbb{N}} \|T_N f\|_{B_2} \le C\|f\|_{\underline{B}_1}$ *for all $f \in \underline{B}_1$.*

Then, the set

$$\left\{f \in B_1 \colon \limsup_{N\to\infty} \|T_N f\|_{B_2} = \infty\right\}$$

is spaceable.

Theorem 4 shows that the conditions (A1)–(A3) together are sufficient for the spaceability of the divergence set. However, the next theorem shows that they are not necessary, because we can find a sequence of bounded linear operators $\{T_N\}_{N\in\mathbb{N}}$, satisfying (A1) and (A2), such that the divergence set is spaceable, but (A3) does not hold. Again we can reduce the problem to sequences of continuous linear functionals on ℓ^2.

Theorem 5 *There exists a sequence of continuous linear functionals* $\{\psi_N\}_{N\in\mathbb{N}}$ *on* ℓ^2*, satisfying:*

(A1”) $\lim_{N\to\infty}\|\psi_N\| = \infty$*, and*
(A2’) there exists a dense subspace $\mathcal{M} \subset \ell^2$ *such that* $\lim_{N\to\infty}\psi_N(f) = 0$ *for all* $f \in \mathcal{M}$*,*

such that the set

$$\left\{f \in \ell^2 \colon \limsup_{N\to\infty}|\psi_N f| = \infty\right\}$$

is spaceable, but

$$\left\{f \in \ell^2 \colon \limsup_{N\to\infty}|\psi_N f| < \infty\right\} \tag{5.1}$$

is not spaceable.

Note that the divergence set

$$\left\{f \in \ell^2 \colon \limsup_{N\to\infty}|\psi_N f| = \infty\right\}$$

in Theorem 5 is spaceable according to Theorem 3. However, since (5.1) is not spaceable, we see that (A3) is not satisfied.

Proof For $\gamma > 0$, we consider the functionals $\psi_n \colon \ell^2 \to \mathbb{C}$, $n \in \mathbb{N}$, defined by

$$\psi_n f = f(n)n^\gamma.$$

Let

$$e_n(k) = \begin{cases} 1, & k = n, \\ 0, & k \neq n, \end{cases}$$

denote the standard basis of ℓ^2. Since we have $\psi_n e_n = n^\gamma$, it follows that $\lim_{n\to\infty}\|\psi_n\| = \infty$. Further, for all finite sequences $f \in \ell^2$ we have $\lim_{n\to\infty}\psi_n f = 0$. It remains to show that the set

$$K = \left\{f \in \ell^2 \colon \limsup_{n\to\infty}|\psi_n f| < \infty\right\}$$

is not spaceable, or, in other words, that K contains no infinite dimensional closed subspace. We use an indirect proof and assume that there exists an infinite dimensional closed subspace $H_1 \subset K$. We have $f \in K$ if and only if

$$\sup_{n\in\mathbb{N}}|f(n)|n^\gamma < \infty.$$

For $N \in \mathbb{N}$, let

$$P_N(f,\gamma) = \max_{1\leq n\leq N}|f(n)|n^\gamma.$$

$\{P_N\}_{N\in\mathbb{N}}$ is a sequence of convex homogeneous continuous functionals. For $f \in H_1$ we have

$$\lim_{N\to\infty} P_N(f,\gamma) = \sup_{n\in\mathbb{N}} |f(n)|n^\gamma < \infty.$$

According to the generalized uniform boundedness principle [21] there exists a constant C_1 such that

$$P_N(f,\gamma) \le C_1 \|f\|_{\ell^2}$$

for all $f \in H_1$, which in turn implies that

$$\sup_{n\in\mathbb{N}} |f(n)|n^\gamma \le C_1 \|f\|_{\ell^2} \tag{5.2}$$

for all $f \in H_1$.

We choose $\gamma = 2$. For $M \in \mathbb{N}$ and $f \in H_1$ we have

$$\begin{aligned} \left(\sum_{n=1}^{M} |f(n)|^2 n^2\right)^{\frac{1}{2}} &= \left(\sum_{n=1}^{M} |f(n)|^2 n^4 \frac{1}{n^2}\right)^{\frac{1}{2}} \\ &\le \left(\sup_{n\in\mathbb{N}} |f(n)|n^2\right) \left(\sum_{n=1}^{M} \frac{1}{n^2}\right)^{\frac{1}{2}} \\ &\le \frac{\pi}{\sqrt{6}} \sup_{n\in\mathbb{N}} |f(n)|n^2. \end{aligned} \tag{5.3}$$

It follows that

$$\left(\sum_{n=1}^{\infty} |f(n)|^2 n^2\right)^{\frac{1}{2}} < \infty$$

for all $f \in H_1$, and, by (5.3) and (5.2), there exists a constant C_2 such that

$$\left(\sum_{n=1}^{\infty} |f(n)|^2 n^2\right)^{\frac{1}{2}} \le C_2 \|f\|_{\ell^2} \tag{5.4}$$

for all $f \in H_1$.

Next, we show that H_1 needs to be finite dimensional, which is a contradiction to our assumption. Let $\{f^{(m)}\}_{m\in\mathbb{N}}$ be an arbitrary bounded sequence of elements in H_1. Without loss of generality, we assume that $\|f^{(m)}\|_{\ell^2} \le 1, m \in \mathbb{N}$. Since ℓ^2 is reflexive, $\{f^{(m)}\}_{m\in\mathbb{N}}$ has a weakly convergent subsequence. It follows that there exists a $f^* \in \ell^2$ and a subsequence $\{m_k\}_{k\in\mathbb{N}}$ of the natural numbers such that

$$\lim_{k\to\infty} f^{(m_k)}(n) = f^*(n) \tag{5.5}$$

for all $n \in \mathbb{N}$. We have $\|f^*\|_{\ell^2} \le 1$. It follows that

$$|f^*(n)|n \le C_2, \quad n \in \mathbb{N}, \tag{5.6}$$

and

$$|f^{(m_k)}(n)|n \leq C_2, \quad n \in \mathbb{N}, \tag{5.7}$$

where we used (5.4). We further have

$$\begin{aligned}
&\left(\sum_{n=1}^{\infty}|f^{(m_k)}(n) - f^*(n)|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{n=1}^{M}|f^{(m_k)}(n) - f^*(n)|^2 + \sum_{n=M+1}^{\infty}|f^{(m_k)}(n) - f^*(n)|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{n=1}^{M}|f^{(m_k)}(n) - f^*(n)|^2\right)^{\frac{1}{2}} + \left(\sum_{n=M+1}^{\infty}|f^{(m_k)}(n) - f^*(n)|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

For the second term on the right hand side we obtain

$$\begin{aligned}
&\left(\sum_{n=M+1}^{\infty}|f^{(m_k)}(n) - f^*(n)|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{n=M+1}^{\infty}|f^{(m_k)}(n)|^2\right)^{\frac{1}{2}} + \left(\sum_{n=M+1}^{\infty}|f^*(n)|^2\right)^{\frac{1}{2}} \\
&\leq C_2\left(\sum_{n=M+1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}} + C_2\left(\sum_{n=M+1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}} \\
&\leq \frac{2C_2}{M},
\end{aligned}$$

where we used (5.6) and (5.7) in the second to last line. Let $\epsilon > 0$ be arbitrary, and let M be the smallest natural number such that

$$\frac{2C_2}{M} < \frac{\epsilon}{2}.$$

Due to (5.5) there exists a $k_0 = k_0(\epsilon)$ such that

$$\max_{1\leq n\leq M}|f^{(m_k)}(n) - f^*(n)| < \frac{\epsilon}{2M}$$

for all $k \geq k_0$. It follows that

$$\left(\sum_{n=1}^{\infty}|f^{(m_k)}(n) - f^*(n)|^2\right)^{\frac{1}{2}} < \epsilon$$

for all $k \geq k_0$. That is, we have norm convergence of the subsequence $\{f^{(m_k)}\}_{k\in\mathbb{N}}$. Thus, every bounded sequence has a norm convergent subsequence, i.e., the unit ball in H_1 is compact. This implies that H_1 is finite dimensional. □

6 Extension to Other Banach Spaces

We want to use the results from Sect. 4 to prove the s-property for further Banach spaces. To this end, we start with a general result.

Theorem 6 *Let B_1 be a separable Banach space. If B_1 contains an infinite dimensional closed subspace $\underline{B}_1$ that has a basis and the s-property, then B_1 has the s-property.*

Corollary 3 $C[0, 1]$ *has the s-property.*

Proof of Corollary 3 According to the Banach–Mazur theorem, there exists an infinite dimensional closed subspace $\underline{B}_1 \subset C[0, 1]$ that is isometrically isomorphic to ℓ^2. Since ℓ^2 has the s-property, according to Theorem 3, it follows from Lemma 1 that $\underline{B}_1$ has the s-property. Theorem 6 gives the assertion. □

Proof of Theorem 6 Let $\underline{B}_1$ be the infinite dimensional closed subspace of B_1 that has a basis and the s-property. Let $\{e_n\}_{n\in\mathbb{N}}$ be a basis of $\underline{B}_1$. Further, let $\{\psi_N\}_{N\in\mathbb{N}}$ a sequence of continuous linear functionals on B_1, satisfying properties (A1') and (A2'), i.e., $\limsup_{N\to\infty} \|\psi_N\|_{B_1\to\mathbb{C}} = \infty$ and $\lim_{N\to\infty} \psi_N f = 0$ for all $f \in \mathcal{M}$, where $\mathcal{M}$ is some dense subset of B_1.

We have to distinguish two cases: First, $\sup_{N\in\mathbb{N}} \|\psi_N\|_{\underline{B}_1\to\mathbb{C}} < \infty$, and second, $\sup_{N\in\mathbb{N}} \|\psi_N\|_{\underline{B}_1\to\mathbb{C}} = \infty$. In the first case we can use Theorem 4 from [11], which immediately gives that the set

$$\left\{ f \in B_1 \colon \limsup_{N\to\infty} |\psi_N f| = \infty \right\} \tag{6.1}$$

is spaceable and thus completes the proof. Hence, we can assume in the following that we have the second case, i.e., that $\sup_{N\in\mathbb{N}} \|\psi_N\|_{\underline{B}_1\to\mathbb{C}} = \infty$. Let $\{e_n^*\}_{n\in\mathbb{N}} \subset \underline{B}_1^*$ denote the coefficient functionals of the basis $\{e_n\}_{n\in\mathbb{N}}$. Since $\mathcal{M}$ is dense in B_1, we can find for every $n \in \mathbb{N}$ a function $f_n \in \mathcal{M}$ such that

$$\|e_n - f_n\|_{B_1} < \frac{1}{2^{n+1}\|e_n^*\|_{\underline{B}_1^*}}$$

It follows that

$$\sum_{n=1}^{\infty} \|e_n^*\|_{\underline{B}_1^*} \|e_n - f_n\|_{B_1} < \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2},$$

which shows that $\{f_n\}_{n\in\mathbb{N}}$ is a basic sequence in B_1 that is equivalent to $\{e_n\}_{n\in\mathbb{N}}$ [14, Theorem 9]. Let $\underline{B}_2 = \overline{\operatorname{span}(\{f_n\}_{n\in\mathbb{N}})}^{B_1}$. Then $\underline{B}_2$ is a infinite dimensional closed subspace of B_1, which is isomorphic to $\underline{B}_1$ [14, Theorem 5]. Hence, we can use

Lemma 1 to deduce that $\underline{B}_2$ has the s-property. For $f \in \text{span}(\{f_n\}_{n\in\mathbb{N}})$ we have the representation

$$f = \sum_{l=1}^{L} \alpha_{k_l} f_{k_l}$$

and it follows that

$$\lim_{N\to\infty} \psi_N f = \sum_{l=1}^{L} \alpha_{k_l} \lim_{N\to\infty} \psi_N f_{k_l} = 0,$$

because $f_n \in \mathcal{M}$ for all $n \in \mathbb{N}$. Hence, we have convergence for the dense subset $\text{span}(\{f_n\}_{n\in\mathbb{N}})$ of $\underline{B}_2$. Again, we distinguish the two cases: $\sup_{N\in\mathbb{N}} \|\psi_N\|_{\underline{B}_2} < \infty$ and $\sup_{N\in\mathbb{N}} \|\psi_N\|_{\underline{B}_2} = \infty$. In the first case we can apply Theorem 4 from [11] again, which immediately gives that the set (6.1) is spaceable and thus completes the proof. Thus, we can assume that we are in the second case, i.e., that $\limsup_{N\to\infty} \|\psi_N\|_{\underline{B}_2} = \infty$ Since $\underline{B}_2$ has the s-property, there exists an infinite dimensional closed subspace $\underline{\underline{B}}_2 \subset \underline{B}_2$ such that

$$\limsup_{N\to\infty} |\psi_N f| = \infty$$

for all $f \in \underline{\underline{B}}_2 \setminus \{0\}$. Clearly, $\underline{\underline{B}}_2$ is also an infinite dimensional closed subspace of B_1. Since the sequence of functionals $\{\psi_N\}_{N\in\mathbb{N}}$ was arbitrary, we have completed the proof. □

Next, we show that the ℓ^p spaces, $1 < p < \infty$, and c_0 have the s-property. Later, in Sect. 7 we will use this result together with Theorem 6 to prove the s-property for the Paley–Wiener spaces $\mathcal{PW}_\pi^p$, $1 \le p < \infty$, and the Bernstein spaces $\mathcal{B}_\pi^p$, $1 < p < \infty$.

Theorem 7 *Let $B_1 = \ell^p$ for some $p \in (1, \infty)$ or $B_1 = c_0$. Then B_1 has the s-property.*

Proof Let $\{\psi_N\}_{N\in\mathbb{N}}$ be an arbitrary sequence of continuous linear functionals on B_1, satisfying the properties (A1') and (A2'), i.e., $\limsup_{N\to\infty} \|\psi_N\|_{B_1\to\mathbb{C}} = \infty$, and $\lim_{N\to\infty} \psi_N f = 0$ for all f is some dense subset $\mathcal{M}$ of B_1. Let q be the conjugate index to p, i.e., satisfying $1/p + 1/q = 1$. Then we have $B_1^* = \ell^q$ if $B_1 = \ell^p$, $1 < p < \infty$, and $B_1^* = \ell^1$ if $B_1 = c_0$. Further, let

$$e_n(k) = \begin{cases} 1, & k = n, \\ 0, & k \neq n, \end{cases}$$

denote the standard basis of ℓ^p and c_0, and $\{e_n^*\}_{n\in\mathbb{N}} \subset B_1^*$ the coefficient functionals of the basis $\{e_n\}_{n\in\mathbb{N}}$. We have $\|e_n^*\|_{B_1^*} = 1$, $n \in \mathbb{N}$. Since $\mathcal{M}$ is dense in B_1 we can find for every $n \in \mathbb{N}$ an element $f_n \in \mathcal{M}$ such that

$$\|e_n - f_n\|_{B_1} < \frac{1}{2^{n+1}}.$$

Thus, we have

$$\sum_{n=1}^{\infty} \|e_n^*\|_{B_1^*} \|e_n - f_n\|_{B_1} = \frac{1}{2},$$

which shows that $\{f_n\}_{n\in\mathbb{N}}$ is a basic sequence in B_1 that is equivalent to $\{e_n\}_{n\in\mathbb{N}}$ [14, Theorem 9]. Let $\underline{B}_1 = \overline{\text{span}\,(\{f_n\}_{n\in\mathbb{N}})}^{B_1}$. Then $\underline{B}_1$ is an infinite dimensional closed subspace of B_1 that is isomorphic to B_1 [14, Theorem 5]. Further, every $f \in \underline{B}_1$ has the representation

$$f = \sum_{n=1}^{\infty} \alpha_n(f) f_n \tag{6.2}$$

with a unique sequence of coefficients $\{\alpha_n(f)\}_{n\in\mathbb{N}}$. Since the basis $\{f_n\}_{n\in\mathbb{N}}$ of $\underline{B}_1$ is equivalent to the standard basis $\{e_n\}_{n\in\mathbb{N}}$ of B_1, it follows that there exist two positive constants C_L and C_R such that

$$C_\text{L} \left(\sum_{n=1}^{\infty} |\alpha_n(f)|^p \right)^{\frac{1}{p}} \le \|f\|_{\underline{B}_1} \le C_\text{R} \left(\sum_{n=1}^{\infty} |\alpha_n(f)|^p \right)^{\frac{1}{p}} \tag{6.3}$$

for all $f \in \underline{B}_1$ if $B_1 = \ell^p$, $1 < p < \infty$. If $B_1 = c_0$ then the above norm has to be replaced by the maximum norm.

We have to distinguish two cases: First, $\sup_{N\in\mathbb{N}} \|\psi_N\|_{\underline{B}_1\to\mathbb{C}} < \infty$, and second $\sup_{N\in\mathbb{N}} \|\psi_N\|_{\underline{B}_1\to\mathbb{C}} = \infty$. In the first case we can use Theorem 4, which immediately gives that the set

$$\left\{ f \in B_1 : \limsup_{N\to\infty} |\psi_N f| = \infty \right\}$$

is spaceable. Hence, we can assume in the following that we have the second case, i.e., that $\sup_{N\in\mathbb{N}} \|\psi_N\|_{\underline{B}_1\to\mathbb{C}} = \infty$. For $f \in \text{span}(\{f_n\}_{n\in\mathbb{N}})$ we have the representation

$$f = \sum_{l=1}^{L} \beta_{k_l} f_{k_l}$$

and it follows that

$$\lim_{N\to\infty} \psi_N f = \sum_{l=1}^{L} \beta_{k_l} \lim_{N\to\infty} \psi_N f_{k_l} = 0,$$

because $f_n \in \mathcal{M}$ for all $n \in \mathbb{N}$. Hence, we have convergence for the dense subset $\text{span}(\{f_n\}_{n\in\mathbb{N}})$ of $\underline{B}_1$.

We consider the operator $T : \underline{B}_1 \to \ell^p$, $f \mapsto \{\alpha_n(f)\}_{n\in\mathbb{N}}$. From (6.3) we see that T is a linear bounded operator mapping $\underline{B}_1$ onto ℓ^p and that T^{-1} is also a linear and bounded. For $\alpha \in \ell^p$ we consider

$$\psi_N^* \alpha = \psi_N T^{-1} \alpha.$$

We have $\psi_N^* T f = \psi_N f$, and it follows that

$$\begin{aligned} |\psi_N f| &\le \|\psi_N^*\|_{\ell^p\to\mathbb{C}} \|Tf\|_{\ell^p} \\ &\le \|\psi_N^*\|_{\ell^p\to\mathbb{C}} \|T\|_{\underline{B}_1\to\ell^p} \|f\|_{\underline{B}_1} \end{aligned}$$

$$\leq C_{\mathrm{L}}^{-1}\|\psi_N^*\|_{\ell^p\to\mathbb{C}}\|f\|_{\underline{B}_1}.$$

Thus, we have $\|\psi_N\|_{\underline{B}_1\to\mathbb{C}} \leq C_{\mathrm{L}}^{-1}\|\psi_N^*\|_{\ell^p\to\mathbb{C}}$, which implies that

$$\limsup_{N\to\infty}\|\psi_N^*\|_{\ell^p\to\mathbb{C}} = \infty.$$

Further, we have $Tf_n = e_n$, i.e., $f_n = T^{-1}e_n$. Hence, we have

$$\psi_N^* e_n = \psi_N T^{-1} e_n = \psi_N f_n$$

and consequently

$$\lim_{N\to\infty} \psi_N^* e_n = 0.$$

According to the Riesz representation theorem there exists a $g_N \in \ell^q$ such that

$$\psi_N^*\alpha = \sum_{n=1}^{\infty}\alpha(n)g_N(n)$$

for all $\alpha \in \ell^p$. We have

$$\|\psi_N^*\|_{\ell^p\to\mathbb{C}} = \left(\sum_{n=1}^{\infty}|g_N(n)|^q\right)^{\frac{1}{q}},$$

and consequently

$$\limsup_{N\to\infty}\left(\sum_{n=1}^{\infty}|g_N(n)|^q\right)^{\frac{1}{q}} = \infty.$$

Further, we have

$$\psi_N^*(e_n) = g_N(n)$$

which implies that

$$\lim_{N\to\infty}|g_N(n)| = 0$$

for all $n \in \mathbb{N}$. Using Lemma 4, which follows below, we obtain an infinite dimensional closed subspace $S \subset \ell^p$ with

$$\limsup_{N\to\infty}|\psi_N^*\alpha| = \infty$$

for all $\alpha \in S \setminus \{0\}$. Then $\underline{\underline{B}}_1 = T^{-1}[S]$ is an infinite dimensional closed subspace of $\underline{B}_1$. For $f \in \underline{\underline{B}}_1$, $f \neq 0$, we have

$$\psi_N f = \psi_N^* T f,$$

and, since $Tf \in S$, it follows that

$$\limsup_{N\to\infty} |\psi_N f| = \infty.$$

Hence, B_1 has the s-property. □

Lemma 4 is a simple extension of Lemma 2.

Lemma 4 *Let* $1 \le q < \infty$ *and* $\{g_N\}_{N\in\mathbb{N}} \subset \ell^q$ *be a sequence, satisfying*

1. $\lim_{N\to\infty} \|g_N\|_{\ell^q} = \infty$ *and*
2. $\lim_{N\to\infty} g_N(k) = 0$ *for all* $k \in \mathbb{N}$.

Further, let $B_1 = c_0$ *if* $q = 1$ *and* $B_1 = \ell^p$, $1/p + 1/q = 1$, *if* $q > 1$. *For* $N \in \mathbb{N}$, *define*

$$\psi_N f = \sum_{k=1}^{\infty} f(k)\overline{g_N(k)}.$$

Then the set

$$\left\{ f \in B_1 : \limsup_{N\to\infty} |\psi_N f| = \infty \right\}$$

is spaceable.

Proof The proof is very similar to the proof of Lemma 2. For $1 < p < \infty$ we use duality and replace the ℓ^2 norm conditions with ℓ^q norm conditions. For c_0 we replace the ℓ^2 norm conditions with ℓ^1 norm conditions. □

7 Applications

In this section we prove the s-property for classes of Paley–Wiener and Bernstein spaces and present several applications.

7.1 Paley–Wiener Space $\mathcal{PW}_\sigma^1$

We start with the Paley–Wiener spaces. The Paley–Wiener space $\mathcal{PW}_\sigma^p$, $0 < \sigma < \infty$, $1 \le p < \infty$, consists of all functions f with a representation $f(z) = 1/(2\pi) \int_{-\sigma}^{\sigma} g(\omega)\, \mathrm{e}^{iz\omega}\, \mathrm{d}\omega$, $z \in \mathbb{C}$, for some $g \in L^p[-\sigma, \sigma]$. The norm for $\mathcal{PW}_\sigma^p$ is given by

$$\|f\|_{\mathcal{PW}_\sigma^p} = \left(\frac{1}{2\pi} \int_{-\sigma}^{\sigma} |g(\omega)|^p \, \mathrm{d}\omega \right)^{\frac{1}{p}}.$$

Observation 1 $\mathcal{PW}_\sigma^1$, $0 < \sigma < \infty$, *has the s-property.*

Proof Let $0 < \sigma < \infty$ be arbitrary but fixed. For $n \in \mathbb{N}$, let

$$q_n(t) = \frac{\sin(\pi(\frac{\sigma}{\pi}t - 2^n))}{\pi(\frac{\sigma}{\pi}t - 2^n)}, \quad t \in \mathbb{R}.$$

According to Paley's theorem [15, p. 104], $\{q_n\}_{n\in\mathbb{N}}$ is a basic sequence in $\mathcal{PW}_\sigma^1$. Let

$$\mathcal{D}_\sigma = \overline{\operatorname{span}(\{q_n\}_{n\in\mathbb{N}})}^{\mathcal{PW}_\sigma^1}. \tag{7.1}$$

Then $\mathcal{D}_\sigma$ is an infinite dimensional closed subspace of $\mathcal{PW}_\sigma^1$. Further, for all $f \in \mathcal{D}_\sigma$ we have

$$\frac{1}{C}\left(\sum_{k=-\infty}^{\infty}\left|f\left(\frac{\pi k}{\sigma}\right)\right|^2\right)^{\frac{1}{2}} \le \|f\|_{\mathcal{PW}_\sigma^1} \le \left(\sum_{k=-\infty}^{\infty}\left|f\left(\frac{\pi k}{\sigma}\right)\right|^2\right)^{\frac{1}{2}},$$

where C is a constant that does not depend on f. Hence, $\mathcal{D}_\sigma$ is isomorphic to ℓ^2. From Theorem 7 we know that ℓ^2 has the s-property, and, since $\mathcal{D}_\sigma$ and ℓ^2 are isomorphic, Lemma 1 implies that $\mathcal{D}_\sigma$ has the s-property. Thus, $\mathcal{PW}_\sigma^1$ contains an infinite dimensional closed subspace that has a basis and the s-property. It follows from Theorem 6 that $\mathcal{PW}_\sigma^1$ has the s-property. □

7.1.1 Approximation of the Identity

We study the approximation behavior of sampling series next, and use the fact that $\mathcal{PW}_\pi^1$ has the s-property to show that the set of divergence is spaceable. We consider sampling series of the type

$$\sum_{k=-\infty}^{\infty} f(t_k)\phi_k(t),$$

where ϕ_k, $k \in \mathbb{Z}$, are certain reconstruction functions and $\{t_k\}_{k\in\mathbb{Z}}$ is the sequence of real sampling points. We assume that the sequence of sampling points is ordered strictly increasingly, i.e.,

$$\cdots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \cdots,$$

and, without loss of generality, that $t_0 = 0$. Further, we assume that the set of sampling points $\{t_k\}_{k\in\mathbb{Z}}$ is the zero set of a function of sine type.

Definition 3 An entire function f of exponential type π is said to be of sine type if the zeros of f are separated and simple, and there exist positive constants A, B, and H such that $A\,\mathrm{e}^{\pi|y|} \le |f(x+iy)| \le B\,\mathrm{e}^{\pi|y|}$ whenever x and y are real and $|y| \ge H$.

Under the above assumptions on the sequence $\{t_k\}_{k\in\mathbb{Z}}$, the limit

$$\phi(z) = (z - t_0)\lim_{N\to\infty}\prod_{\substack{|t_k|\le N\\ k\neq 0}}\left(1 - \frac{z}{t_k}\right)$$

exists for all finite $z \in \mathbb{C}$ and represents an entire function of exponential type π, and the reconstruction functions ϕ_k, $k \in \mathbb{Z}$, are given by

$$\phi_k(t) = \frac{\phi(t)}{\phi'(t_k)(t - t_k)}. \tag{7.2}$$

There is an important connection between the set of zeros $\{t_k\}_{k\in\mathbb{Z}}$ of a function of sine type, the basis properties of the system of exponentials $\{\mathrm{e}^{i\omega t_k}\}_{k\in\mathbb{Z}}$, and complete interpolating sequences, which we will summarize in the following lemma. Please see [22] and [27] for details.

Lemma 5 *If $\{t_k\}_{k\in\mathbb{Z}}$ is the set of zeros of a function of sine type, then $\{t_k\}_{k\in\mathbb{Z}}$ is a complete interpolating sequence for $\mathcal{PW}_\pi^2$, the system $\{\mathrm{e}^{i\omega t_k}\}_{k\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\pi, \pi]$, and $\{\phi_k\}_{k\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{PW}_\pi^2$.*

For a subclass of the just introduced class of sampling patterns, we have an interesting divergence result. This subclass consists of all separated and strictly increasingly ordered sequences $\{t_k\}_{k\in\mathbb{Z}}$ that are the zero set of an entire function ϕ that has the Fourier-Stieltjes integral representation

$$\phi(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{i\omega t} \; \mathrm{d}\mu(\omega), \tag{7.3}$$

where $\mu(\omega)$ is a real function of bounded variation on the interval $[-\pi, \pi]$ and has a jump discontinuity at each endpoint. It can be shown that all functions ϕ with this property are sine-type functions [27, p. 143], and hence the class of functions ϕ that we consider here is a subclass of the functions of sine type.

For this subclass, we have the following result [7].

Theorem 8 *Let ϕ be a function of sine type that has the representation (7.3), and whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real and ordered increasingly. Let ϕ_k, $k \in \mathbb{Z}$, be the corresponding reconstruction functions as defined in (7.2). Then there exists a $f_1 \in \mathcal{PW}_\pi^1$ such that*

$$\limsup_{N\to\infty} \max_{t\in\mathbb{R}} \left| f_1(t) - \sum_{k=-N}^{N} f_1(t_k)\phi_k(t) \right| = \infty. \tag{7.4}$$

Remark 6 In particular $\sin(\pi t)$ is a sine-type function that has the representation (7.3). Hence, Theorem 8 implies that

$$\limsup_{N\to\infty} \max_{t\in\mathbb{R}} \left| f_1(t) - \sum_{k=-N}^{N} f_1(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = \infty, \tag{7.5}$$

i.e., the peak approximation error of the Shannon sampling series with equidistant sampling points diverges for some signal $f_1 \in \mathcal{PW}_\pi^1$.

The next corollary shows that the sets of divergence in (7.4) and (7.5) are spaceable.

Corollary 4 *Under the assumptions and notations of Theorem 8, the set of functions* $f_1 \in \mathcal{PW}_\pi^1$ *that satisfy* (7.4) *is spaceable.*

Proof We consider the operators $\psi_N : \mathcal{PW}_\pi^1 \to \mathcal{B}_{\pi,0}^\infty$, $N \in \mathbb{N}$, defined by

$$\psi_N f = \sum_{k=-N}^{N} f(t_k)\phi_k.$$

Clearly, $\{\psi_N\}_{N\in\mathbb{N}}$ is a sequence of bounded linear operators on $\mathcal{PW}_\pi^1$. For all f in the dense subset $\mathcal{D}_\pi \subset \mathcal{PW}_\pi^1$, which was defined in (7.1), we have $\lim_{N\to\infty}\|f - \psi_N f\|_{\mathcal{B}_{\pi,0}^\infty} = 0$, as the following quick argument shows. We have $\mathcal{D}_\pi \subset \mathcal{PW}_\pi^2$, and, since $\{\phi_k\}_{k\in\mathbb{N}}$ is a Riesz basis for $\mathcal{PW}_\pi^2$, it follows that

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| \le \lim_{N\to\infty} \left\| f - \sum_{k=-N}^{N} f(t_k)\phi_k \right\|_{\mathcal{PW}_\pi^2} = 0$$

for all $f \in \mathcal{D}_\pi$. Further, from Theorem 8 we see that

$$\limsup_{N\to\infty} \|\psi_N\|_{\mathcal{PW}_\pi^1 \to \mathcal{B}_{\pi,0}^\infty} = \infty.$$

Since $\mathcal{PW}_\pi^1$ has the s-property, it follows that there exists an infinite dimensional closed subspace $\underline{B}_1 \subset \mathcal{PW}_\pi^1$ such that $\limsup_{N\to\infty}\|\psi_N f\|_{\mathcal{B}_{\pi,0}^\infty} = \infty$ for all $f \in \underline{B}_1$. □

Remark 7 An interesting question about shift invariance is the following: If the operators of the sequence $\{T_N\}_{N\in\mathbb{N}}$ are shift invariant, is it then possible to construct a closed infinite dimensional subspaces with divergence that is additionally shift invariant? As it turns out, for the kind of operators that we are considering in this paper, this is never possible [13]. This is a deep result because it is equivalent to Carleson's theorem on almost everywhere convergence of Fourier series of L^2 functions [13].

7.1.2 Approximation of Stable LTI Systems

In addition to the reconstruction of signals from their samples as discussed in Sect. 7.1.1, the approximation of linear time-invariant (LTI) systems is of practical relevance. The canonical approximation process in this case is given by

$$\sum_{k=-\infty}^{\infty} f(t_k)(T\phi_k)(t), \tag{7.6}$$

where T denotes the stable LTI system.

We briefly review some definitions and facts. A linear system $T : \mathcal{PW}_\pi^p \to \mathcal{PW}_\pi^p$, $1 \leq p \leq \infty$, is called stable if the operator T is bounded, i.e., if $\|T\| := \sup_{\|f\|_{\mathcal{PW}_\pi^p} \leq 1} \|Tf\|_{\mathcal{PW}_\pi^p} < \infty$. Furthermore, it is called time-invariant if $(Tf(\cdot - a))(t) = (Tf)(t-a)$ for all $f \in \mathcal{PW}_\pi^p$ and $t, a \in \mathbb{R}$. For every stable LTI system $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ there exists exactly one function $\hat{h}_T \in L^\infty[-\pi, \pi]$ such that

$$(Tf)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \hat{h}_T(\omega) \, \mathrm{e}^{i\omega t} \, \mathrm{d}\omega, \quad t \in \mathbb{R}, \tag{7.7}$$

for all $f \in \mathcal{PW}_\pi^1$ [8]. Conversely, every function $\hat{h}_T \in L^\infty[-\pi, \pi]$ defines a stable LTI system $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$. The operator norm of a stable LTI system T is given by $\|T\| = \|\hat{h}\|_{L^\infty[-\pi,\pi]}$.

In [9] it was shown that for any sampling pattern that is a complete interpolating sequence and all $t \in \mathbb{R}$ there exists a stable LTI system $T_1 : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ and a signal $f_1 \in \mathcal{PW}_\pi^1$ such that the approximation process (7.6) diverges. This result even holds true in the case of oversampling.

Theorem 9 *Let $\{t_k\}_{k \in \mathbb{Z}} \subset \mathbb{R}$ be an ordered complete interpolating sequence for $\mathcal{PW}_\pi^2$, ϕ_k as defined in (7.2), and $t \in \mathbb{R}$. Then there exists a stable LTI system $T_1 : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ such that for every $0 < \sigma \leq \pi$ there exists a signal $f_1 \in \mathcal{PW}_\sigma^1$ such that*

$$\limsup_{N \to \infty} \left| (T_1 f_1)(t) - \sum_{k=-N}^{N} f_1(t_k)(T_1 \phi_k)(t) \right| = \infty. \tag{7.8}$$

Again we can use the results from this paper to derive that the set of signals creating divergence is spaceable.

Corollary 5 *Under the assumptions and notations of Theorem 9, for any $0 < \sigma \leq \pi$, the set of functions $f_1 \in \mathcal{PW}_\sigma^1$ that satisfies (7.8) is spaceable.*

Proof Let $\{t_k\}_{k \in \mathbb{Z}} \subset \mathbb{R}$ be an ordered complete interpolating sequence for $\mathcal{PW}_\pi^2$ and $t \in \mathbb{R}$, both arbitrary but fixed. From Theorem 9 we know that there exists a stable LTI system T_1 such that for every $0 < \sigma \leq \pi$ there exists a signal $f_1 \in \mathcal{PW}_\sigma^1$ such that (7.8) holds. Let $0 < \sigma \leq \pi$ be arbitrary but fixed. We consider the functionals $\psi_N : \mathcal{PW}_\sigma^1 \to \mathbb{C}$, $N \in \mathbb{N}$, defined by

$$\psi_N f = \sum_{k=-N}^{N} f(t_k)(T_1 \phi_k)(t).$$

Clearly, $\{\psi_N\}_{N \in \mathbb{N}}$ is a sequence of bounded linear functionals on $\mathcal{PW}_\sigma^1$. For $f \in \mathcal{D}_\sigma \subset \mathcal{PW}_\sigma^2$ we have

$$\left| (T_1 f)(t) - \sum_{k=-N}^{N} f(t_k)(T_1 \phi_k)(t) \right|$$

$$\leq \left\| T_1 f - \sum_{k=-N}^{N} f(t_k) T_1 \phi_k \right\|_{\mathcal{PW}_\pi^1}$$

$$\leq \|T_1\|_\infty \left\| f - \sum_{k=-N}^{N} f(t_k)\phi_k \right\|_{\mathcal{PW}_\pi^2},$$

and, since $\{\phi_k\}_{k\in\mathbb{N}}$ is a Riesz basis for $\mathcal{PW}_\pi^2$, we further have

$$\lim_{N\to\infty} \left\| f - \sum_{k=-N}^{N} f(t_k)\phi_k \right\|_{\mathcal{PW}_\pi^2} = 0.$$

Thus, it follows that $\lim_{N\to\infty}|(T_1 f)(t) - \psi_N f| = 0$ for all f in the dense subset $\mathcal{D}_\sigma \subset \mathcal{PW}_\sigma^1$. Further, from Theorem 9 we see that that

$$\limsup_{N\to\infty} \|\psi_N\|_{\mathcal{PW}_\sigma^1 \to \mathbb{C}} = \infty.$$

Since $\mathcal{PW}_\sigma^1$ has the s-property, it follows that there exists an infinite dimensional closed subspace $\underline{B}_1 \subset \mathcal{PW}_\sigma^1$ such that $\limsup_{N\to\infty}|\psi_N f| = \infty$ for all $f \in \underline{B}_1 \setminus \{0\}$. □

7.2 Paley–Wiener Space $\mathcal{PW}_\sigma^2$

Since $\mathcal{PW}_\sigma^2$, $0 < \sigma < \infty$, is isomorphic to ℓ^2, which has the s-property, it follows immediately from Lemma 1 that $\mathcal{PW}_\sigma^2$ has the s-property.

Observation 2 *$\mathcal{PW}_\sigma^2$, $0 < \sigma < \infty$, has the s-property.*

Let $\mathcal{T}_C$ denote the set of all energetically stable LTI systems $T\colon \mathcal{PW}_\pi^2 \to \mathcal{PW}_\pi^2$ with $\hat{h}_T \in C[-\pi,\pi]$, and $\mathcal{T}_{C^e}$ denote the set of all energetically stable LTI systems $T\colon \mathcal{PW}_\pi^2 \to \mathcal{PW}_\pi^2$ with $\hat{h}_T \in C^e[-\pi,\pi]$.

Next, we study the convolution sum

$$\sum_{k=-\infty}^{\infty} f(t-k) h_T(k), \tag{7.9}$$

where the time variable is in the argument of the function f. It is easy to see that (7.9) converges uniformly for all $f \in \mathcal{PW}_\pi^2$ and all stable LTI systems $T\colon \mathcal{PW}_\pi^2 \to \mathcal{PW}_\pi^2$. However, the L^2 norm of the convolution sum (7.9) diverges for certain functions and systems. In [10] it was shown that there exists a spaceable set $V_{\text{sig}} \subset \mathcal{PW}_\pi^2$ and a spaceable set $V_{\text{sys}} \subset \mathcal{T}_C$, such that

$$\limsup_{N\to\infty} \int_{-\infty}^{\infty} \left| \sum_{k=-N}^{N} f(t-k) h_T(k) \right|^2 \mathrm{d}t = \infty. \tag{7.10}$$

for all $f \in V_{\text{sig}}$, $T \in V_{\text{sys}}$.

This results about joint spaceability gives no information about individual spaceability, i.e., whether for a given system T the set of signals f with divergence is spaceable. The following corollary, which is a consequence of the previous results, gives an answer to this question.

Corollary 6 *Let $T : \mathcal{PW}_\pi^2 \to \mathcal{PW}_\pi^2$ be an energetically stable LTI system and $f \in \mathcal{PW}_\pi^2$ such that*

$$\limsup_{N\to\infty} \int_{-\infty}^{\infty} \left| \sum_{k=-N}^{N} f(t-k) h_T(k) \right|^2 \mathrm{d}t = \infty. \tag{7.11}$$

Then the set of all functions $f \in \mathcal{PW}_\pi^2$ for which (7.11) holds is spaceable.

Proof Let T_1 be an energetically stable LTI system and $f_1 \in \mathcal{PW}_\pi^2$ be such that

$$\limsup_{N\to\infty} \int_{-\infty}^{\infty} \left| \sum_{k=-N}^{N} f_1(t-k) h_{T_1}(k) \right|^2 \mathrm{d}t = \infty.$$

We consider the operators $\psi_N : \mathcal{PW}_\pi^2 \to \mathcal{PW}_\pi^2$, defined by

$$\psi_N f = \sum_{k=-N}^{N} f(\cdot - k) h_{T_1}(k).$$

Clearly $\{\psi_N\}_{N\in\mathbb{N}}$ is a sequence of bounded linear operators on $\mathcal{PW}_\pi^2$. Let

$$D = \left\{ \sum_{k=-N}^{N} c_k \frac{\sin(\pi(t-k))}{\pi(t-k)} : N \in \mathbb{N}, c_k \in \mathbb{C} \right\}.$$

Then D is a dense subset of $\mathcal{PW}_\pi^2$. For $f \in D$ we have

$$\begin{aligned}
&\|T_1 f - \psi_N f\|_{\mathcal{PW}_\pi^2}^2 \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \hat{f}(\omega)\hat{h}_{T_1}(\omega) - \hat{f}(\omega) \sum_{k=-N}^{N} \mathrm{e}^{-i\omega k} h_{T_1}(k) \right|^2 \mathrm{d}\omega \\
&\leq \frac{\|\hat{f}\|_{L^\infty[-\pi,\pi]}^2}{2\pi} \int_{-\pi}^{\pi} \left| \hat{h}_{T_1}(\omega) - \sum_{k=-N}^{N} \mathrm{e}^{-i\omega k} h_{T_1}(k) \right|^2 \mathrm{d}\omega \\
&= \|\hat{f}\|_{L^\infty[-\pi,\pi]}^2 \sum_{|k|>N} |h_{T_1}(k)|^2,
\end{aligned}$$

and it follows that $\lim_{N\to\infty}\|T_1 f - \psi_N f\|_{\mathcal{PW}_\pi^2} = 0$. Further, from (7.11) we know that $\limsup_{N\to\infty}\|\psi_N\|_{\mathcal{PW}_\pi^2\to\mathcal{PW}_\pi^2} = \infty$. Since $\mathcal{PW}_\pi^2$ has the s-property, it follows that there exists an infinite dimensional closed subspace $\underline{B}_1 \subset \mathcal{PW}_\pi^2$ such that $\limsup_{N\to\infty}\|\psi_N f\|_{\mathcal{PW}_\pi^2} = \infty$ for all $f \in \underline{B}_1 \setminus \{0\}$. □

The next corollary gives, for a fixed signal f, a statement about the set of systems, creating divergence.

Corollary 7 *Let $T \in \mathcal{T}_{C^e}$ be an energetically stable LTI system and $f \in \mathcal{PW}_\pi^2$ such that*

$$\limsup_{N\to\infty} \int_{-\infty}^{\infty} \left| \sum_{k=-N}^{N} f(t-k)h_T(k) \right|^2 \mathrm{d}t = \infty. \tag{7.12}$$

Then the set of all energetically stable LTI systems $T \in \mathcal{T}_{C^e}$ for which (7.12) holds is spaceable.

For the proof of Corollary 7 we need the following observation.

Observation 3 *$C^e[-\pi,\pi]$ has the s-property.*

Proof of Observation 3 For $n \in \mathbb{N}$, let $m_n = \pi\left(1 - 3/2^{n+1}\right)$ and define the functions $\phi_n\colon [-\pi,\pi] \to \mathbb{R}$ by

$$\phi_n(\omega) = \begin{cases} 1 - \frac{2^{n+1}}{\pi}|\omega - m_n|, & |\omega - m_n| \le \frac{\pi}{2^{n+1}}, \\ 0, & \text{otherwise.} \end{cases}$$

We have $\phi_n \in C^{\mathrm{e}}[-\pi,\pi]$, $n \in \mathbb{N}$. Next, consider

$$G = \overline{\operatorname{span}(\{\phi_n\}_{n\in\mathbb{N}})}^{C^{\mathrm{e}}[-\pi,\pi]}.$$

Then G is a closed subspace of $C^{\mathrm{e}}[-\pi,\pi]$ and $\{\phi_n\}_{n\in\mathbb{N}}$ is a basis for G [19, Theorem 5.17]. For $f \in G$ we have the representation

$$f = \sum_{n=1}^{\infty} \alpha_n \phi_n$$

with $\{\alpha_n\}_{n\in\mathbb{N}} \in c_0$. Further we have $\|f\|_\infty = \|\{\alpha_n\}_{n\in\mathbb{N}}\|_{c_0}$. Thus G is isometric isomorphic to c_0. Since c_0 has the s-property according to Theorem 7, it follows from Lemma 1 that G has the s-property. Finally, Theorem 6 shows that $C^{\mathrm{e}}[-\pi,\pi]$ has the s-property. □

Proof of Corollary 7 Let $T_1 \in \mathcal{T}_{C^e}$ be an energetically stable LTI system and $f_1 \in \mathcal{PW}_\pi^2$ such that

$$\limsup_{N\to\infty} \int_{-\infty}^{\infty} \left| \sum_{k=-N}^{N} f_1(t-k)h_{T_1}(k) \right|^2 \mathrm{d}t = \infty.$$

We consider the operators $\psi_N\colon C^{\mathrm{e}}[-\pi,\pi] \to \mathcal{PW}_\pi^2$, $N \in \mathbb{N}$, defined by

$$\psi_N h_T = \sum_{k=-N}^{N} f_1(\cdot - k) h_T(k).$$

Clearly $\{\psi_N\}_{N\in\mathbb{N}}$ is a sequence of bounded linear operators on $C^{\mathrm{e}}[-\pi,\pi]$. Let

$$D = \left\{ \sum_{k=-N}^{N} c_k \, \mathrm{e}^{i\omega k} \colon N \in \mathbb{N}, c_k \in \mathbb{C} \right\}.$$

Then D is a dense subset of $C^{\mathrm{e}}[-\pi,\pi]$. For $h_T \in D$ we have

$$\begin{aligned} &\|T f_1 - \psi_N h_T\|^2_{\mathcal{PW}_\pi^2} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \hat{f}_1(\omega)\hat{h}_T(\omega) - \hat{f}_1(\omega) \sum_{k=-N}^{N} \mathrm{e}^{-i\omega k} \, h_T(k) \right|^2 \mathrm{d}\omega \\ &\le \left\| \hat{h}_T(\omega) - \sum_{k=-N}^{N} \mathrm{e}^{-i\omega k} \, h_T(k) \right\|^2_{L^\infty[-\pi,\pi]} \|f\|^2_{\mathcal{PW}_\pi^2}, \end{aligned}$$

and further

$$\lim_{N\to\infty} \left\| \hat{h}_T(\omega) - \sum_{k=-N}^{N} \mathrm{e}^{-i\omega k} \, h_T(k) \right\|^2_{L^\infty[-\pi,\pi]} = 0.$$

Hence, it follows that $\lim_{N\to\infty} \|T f_1 - \psi_N h_T\|_{\mathcal{PW}_\pi^2} = 0$ for all $h_T \in D$. Further, from (7.12) we know that $\limsup_{N\to\infty} \|\psi_N\|_{C^{\mathrm{e}}[-\pi,\pi]\to\mathcal{PW}_\pi^2} = \infty$. Since $C^{\mathrm{e}}[-\pi,\pi]$ has the s-property, it follows that there exists an infinite dimensional closed subspace $\underline{B}_1 \subset C^{\mathrm{e}}[-\pi,\pi]$ such that $\limsup_{N\to\infty} \|\psi_N h_T\|_{\mathcal{PW}_\pi^2} = \infty$ for all $h_T \in \underline{B}_1 \setminus \{0\}$. □

7.3 Further Spaces

So far we have seen that the spaces c_0, ℓ^p, $1 < p < \infty$, $C[0,1]$, $C^{\mathrm{e}}[-\pi,\pi]$, $L^2[0,1]$, $\mathcal{PW}_\sigma^1$, $\mathcal{PW}_\sigma^2$, $0 < \sigma < \infty$, have the s-property. More Banach spaces that have the s-property are summarized in the following theorem.

Theorem 10 *The following Banach space have the s-property:*

1. *The Bernstein spaces* $\mathcal{B}_\pi^p$, $1 < p < \infty$,
2. $L^p[-\pi,\pi]$, $1 \le p < \infty$,
3. *The Paley–Wiener spaces* $\mathcal{PW}_\pi^p$, $1 \le p < \infty$.

Proof 1. For $1 < p < \infty$, $\mathcal{B}_\pi^p$ is isomorphic to ℓ^p [22, p. 152], and ℓ^p has the s-property according to Theorem 7. 2. Let $1 \le p < \infty$ and let r_n, $n \in \mathbb{N}$, denote the Rademacher functions scaled to $[-\pi,\pi]$. Further, let

$$D = \overline{\operatorname{span}(\{r_n\}_{n\in\mathbb{N}})}^{L^p[-\pi,\pi]}.$$

D is a closed subspace of $L^p[-\pi,\pi]$, and, by Khintchine's inequality [23, p. 66], D has a basis and is isomorphic to ℓ^2. Hence, by Theorem 6, Lemma 1, and the fact that ℓ^2 has the s property (Theorem 7), it follows that $L^p[-\pi,\pi]$ has the s property. 3. The s-property of the Paley–Wiener space $\mathcal{PW}_\pi^p$, $1 \leq p < \infty$, follows from the fact that $\mathcal{PW}_\pi^p$ is isomorphic to $L^p[-\pi,\pi]$ and Lemma 1. □

Appendix: Equivalence of Questions 1 and 2

In this section, we show the equivalence of Questions 1 and 2. We start with reducing Question 1 to a simpler question.

Theorem 11 *The answer to Question 1 is positive for arbitrary separable Banach spaces B_1, B_2 if and only if the answer to Question 1 is positive for arbitrary closed subspaces B_1, B_2 of $C[0,1]$.*

Proof ⇒: This direction is trivial, because every closed subspace of $C[0,1]$ is a Banach space.

⇐: Let B_1 and B_2 two arbitrary separable Banach spaces, and assume that the assumptions of Question 1 are fulfilled. According to the Banach–Mazur theorem [4,24] there exist closed subspaces H_1, H_2 of $C[0,1]$ and isometric isomorphisms U_1, U_2 such that $U_l(B_l) = H_l$, $l = 1, 2$.

$$\begin{array}{ccc} B_1 & \xrightarrow{T,T_N} & B_2 \\ {\scriptstyle U_1}\downarrow & & \downarrow{\scriptstyle U_2} \\ H_1 & \xrightarrow{T^c,T_N^c} & H_2 \end{array}$$

Let $T^c = U_2 \circ T \circ U_1^{-1}$ and $T_N^c = U_2 \circ T_N \circ U_1^{-1}$, $N \in \mathbb{N}$, both of which map from H_1 into H_2. We have $\|T^c\|_{H_1\to H_2} = \|T\|_{B_1\to B_2}$ and $\|T_N^c\|_{H_1\to H_2} = \|T_N\|_{B_1\to B_2}$, $N \in \mathbb{N}$. According to our assumption there exists a dense subset $\mathcal{M} \subset B_1$ such that

$$\lim_{N\to\infty} \|T_N f - T f\|_{B_2} = 0$$

for all $f \in \mathcal{M}$. It follows that $\mathcal{M}_c := U_1(\mathcal{M})$ is dense in H_1 and that for all $f \in \mathcal{M}_c$ we have

$$\lim_{N\to\infty} \|T_N^c f - T^c f\|_{H_2} = 0.$$

The assertion for arbitrary closed subspaces of $C[0,1]$, which we assume to be true, gives that there exists an infinite dimensional closed subspace $\underline{H_1} \subset H_1$ such that

$$\limsup_{N\to\infty} \|T_N^c f\|_{H_2} = \infty$$

for all $f \in \underline{H_1} \setminus \{0\}$. It follows that $\underline{B_1} = U_1^{-1}\underline{H_1}$ is an infinite dimensional closed subspace of B_1, and that we have

$$\limsup_{N\to\infty} \|T_N f\|_{B_2} = \infty$$

for all $f \in \underline{B_1} \setminus \{0\}$. □

We want to further reduce the question by showing that the structure of the space B_2 is not particularly significant; it suffices to consider $B_2 = \mathbb{C}$. This leads us to the following question.

Question 3 *Let B_1 be a closed subspace of $C[0,1]$ and $\{\psi_N\}_{N\in\mathbb{N}}$ a sequence of continuous linear functionals on B_1, satisfying*

(A1') $\limsup_{N\to\infty} \|\psi_N\|_{B_1\to\mathbb{C}} = \infty$, *and*
(A2') there exists a continuous linear functional $\psi : B_1 \to \mathbb{C}$ as well as a dense subset $\mathcal{M} \subset B_1$ such that $\lim_{N\to\infty} \psi_N f = \psi f$ for all $f \in \mathcal{M}$.

Is the set

$$\left\{ f \in B_1 : \limsup_{N\to\infty} |\psi_N f| = \infty \right\}$$

spaceable?

Theorem 12 *The answer to Question 1 is positive if and only if the answer to Question 3 is positive.*

Proof ⇒: If the answer to Question 1 is positive, then the assertion is true for arbitrary separable Banach spaces B_1 and B_2, and thus, in particular for B_1 being an arbitrary closed subspace of $C[0,1]$ and $B_2 = \mathbb{C}$.

⇐: Let B_1, B_2 be two arbitrary closed subspaces of $C[0,1]$. The set of all $f \in B_1$ satisfying $\limsup_{N\to\infty} \|T_N f\|_{C[0,1]} = \infty$ is a residual set. Hence, there exist a $f \in B_1$ and two sequences $\{N_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ and $\{t_k\}_{k\in\mathbb{N}} \subset [0,1]$, such that

$$\lim_{k\to\infty} |(T_{N_k} f)(t_k)| = \infty.$$

Further, there exists a $t_* \in [0,1]$ and a subsequence $\{k_l\}_{l\in\mathbb{N}}$ such that

$$\lim_{l\to\infty} |t_* - t_{k_l}| = 0.$$

We consider the functionals

$$\psi_l f = (T_{N_{k_l}} f)(t_{k_l}), \quad l \in \mathbb{N}.$$

For $f \in \mathcal{M}$ we have that for all $\epsilon > 0$ there exists a $l_0 = l_0(\epsilon)$ such that

$$\|T_{N_{k_l}} f - Tf\|_{C[0,1]} < \epsilon$$

for all $l \geq l_0$. Since Tf is continuous, there exists a $l_1 = l_1(\epsilon)$ such that

$$|(Tf)(t_*) - (Tf)(t_{k_l})| < \epsilon$$

for all $l \geq l_1$. Hence, for all $l \geq \max\{l_0, l_1\}$ we have

$$|(T_{N_{k_l}} f)(t_{k_l}) - (Tf)(t_*)| < 2\epsilon.$$

Therefore, we have for all $f \in \mathcal{M}$ that

$$\lim_{l \to \infty} \psi_l f = \psi f,$$

where $\psi f = (Tf)(t_*)$. Thus, the set

$$\left\{ f \in B_1 : \limsup_{l \to \infty} |\psi_l f| = \infty \right\}$$

is spaceable. Further, we have

$$\|T_{N_{k_l}} f\|_{C[0,1]} \geq |\psi_l f|, \quad l \in \mathbb{N}.$$

and therefore spaceability of the set

$$\left\{ f \in B_1 : \limsup_{N \to \infty} \|T_N f\|_{B_2} = \infty \right\}.$$

□

Next we show that it is sufficient to consider specific sequences of linear functionals, which is the the final simplification.

Corollary 8 *The answer to Question 1 is positive if and only if Question 3 can be answered positively for all sequences of functionals* $\{\psi_N\}_{N \in \mathbb{N}}$ *with* $\lim_{N \to \infty} \psi_N(f) = 0$ *for all* $f \in \mathcal{M}$.

Proof Choose $\psi_N^* = \psi_N - \psi$. □

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