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Pointwise Estimates for Block-Radial Functions of Sobolev Classes

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Abstract The paper gives sharp pointwise estimates for functions belonging to $\dot{H}^{s,p}(\mathbb{R}^N)$ with radial symmetry in m blocks of variables, for $m < sp < N$. The estimates are formulated in terms of multiradial monomials. The form of the monomials depends on the structure of the group of block-radial symmetries and the distances of the given point to the hyperplanes in $\mathbb{R}^N$ that contain the singular orbits of the group. For some exceptional set of parameters the logarithmic factor is needed. Weak continuity related to the estimates is also considered.

Keywords Multiradial functions · Sobolev spaces · Strauss inequality · Pointwise estimates

Mathematics Subject Classification Primary 46E35 · Secondary 46B50 · 46N20 · 42C99

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1 Introduction

This paper studies embeddings of multiradial subspaces of homogeneous Sobolev spaces $\dot{H}^{s,p}$ into weighted L^∞-spaces. The main result, Theorem 4.1, is the multiradial analog of the Straus estimate for radial functions. W. Strauss proved in [19] that every radial function belonging to the inhomogeneous Sobolev spaces $H^{1,2}(\mathbb{R}^N)$, $N > 2$, is almost everywhere equal to a continuous function and there is a positive constant C such that

$$|f(x)| \leq C|x|^{-\frac{1-N}{2}} \|f\|_{H^{1,2}}.$$

We refer to [3] for the proof of this inequality and further historical references. To the best of our knowledge a block-radial symmetry was first considered by L.P.Lions in [10]. Here the author is interested in the compactness of embeddings of subspaces of radial and block-radial functions of inhomogeneous Sobolev space. To prove the compactness one needs some decay of the function at infinity and an inequality similar to above one involving the inhomogeneous Sobolev norm is proved there. In contrast to this paper we work with homogeneous spaces and we are looking for the block-radial counterpart of the following estimate

$$|f(x)| \leq C|x|^{s-\frac{N}{2}} \|f\|_{\dot{H}^{s,2}}$$

that was proved for radial function by Cho and Ozawa in [2], but some preliminary version can be found in [11]. We refer also to [14] and [13] for comparison of the behaviour at infinity of the radial function belonging to homogeneous and inhomogeneous Sobolev-Besov type spaces respectively.

A preliminary and more coarse pointwise estimate for multiradial functions in $\dot{H}^{1,p}(\mathbb{R}^N)$ with the right hand side as in (2.9) below has been previously given by the authors in Corollary 1, [17], for a subspace of $\dot{H}^{1,p}(\mathbb{R}^N)$. For a large range of parameters this subspace is strictly smaller than $\dot{H}^{1,p}(\mathbb{R}^N)$, and perhaps more significantly, for a large range of parameters inequality (2.9) is not an optimal estimate. In this paper we give a pointwise estimate for multiradial functions in $\dot{H}^{1,p}(\mathbb{R}^N)$ by a monomial of block radii $r_i(x)$, $i = 1, \ldots, m$, with exponents that differ in different cones $\{x \in \mathbb{R}^N : r_{i_1}(x) \geq \cdots \geq r_{i_m}(x)\}$. These monomials are dominated by the right hand side of (2.9) (where the exponents in the monomial are the same for the whole space). For exceptional values of parameters, the estimate has to be amended by an additional logarithmic factor. In addition to the pointwise estimate (which we also generalize to the spaces $\dot{H}^{s,p}$), and to its optimality, we prove cocompactness of a related (non-compact) embedding relative to the group of dilations at the origin.

Let $m \in 1, \ldots, N$ and let $\gamma \in \mathbb{N}^m$ be an m-tuple $\gamma = (\gamma_1, \ldots, \gamma_m)$, $\gamma_1 + \cdots + \gamma_m = |\gamma| = N$. The m-tuple γ describes decomposition of $\mathbb{R}^{|\gamma|} = \mathbb{R}^{\gamma_1} \times \cdots \times \mathbb{R}^{\gamma_m}$ into m subspaces of dimensions $\gamma_1, \ldots, \gamma_m$ respectively. Let

$$SO(\gamma) = SO(\gamma_1) \times \ldots SO(\gamma_m) \subset SO(N)$$

be a group of isometries on $\mathbb{R}^{|\gamma|}$. An element $g = (g_1, \ldots, g_m)$, $g_i \in SO(\gamma_i)$ acts on $x = (\tilde{x}_1, \ldots, \tilde{x}_m)$, $\tilde{x}_i \in \mathbb{R}^{\gamma_i}$ by $x \mapsto g(x) = (g_1(\tilde{x}_1), \ldots, g_m(\tilde{x}_m))$. If $m = 1$ then $SO(\gamma) = SO(N)$ is a special orthogonal group acting on $\mathbb{R}^N$. If $m = N$ then the group is trivial since then $\gamma_1 = \cdots = \gamma_m = 1$ and $SO(1) = \{\mathrm{id}\}$.

We will denote a subspace of any space X of functions on $\mathbb{R}^{|\gamma|}$ consisting of functions invariant with respect to the action the group $SO(\gamma)$ as X_γ. If $SO(\gamma) = SO(N)$, then we will write X_{rad} since in that case the subspace consists of radial functions.

The spaces of our concern here are homogeneous Sobolev spaces of invariant functions $\dot{H}^{s,p}_\gamma(\mathbb{R}^{|\gamma|})$, $s > 0$, $p > 1$, defined as the completion of $C^\infty_{0,\gamma}(\mathbb{R}^{|\gamma|})$ in the norm $\|u\|_{s,p} = \|(-\Delta)^s u\|_p = \|\mathcal{F}^{-1}(|\xi|^s \mathcal{F} u)\|_p$, which generalizes the norm $\|\nabla u\|_p$ in the case $s = 1$. The space $\dot{H}^{s,p}_\gamma(\mathbb{R}^{|\gamma|})$ can be identified as subspace of homogeneous Sobolev space $\dot{H}^{s,p}(\mathbb{R}^{|\gamma|})$ defined as the completion of $C^\infty_0(\mathbb{R}^{|\gamma|})$. The space $\dot{H}^{s,p}(\mathbb{R}^{|\gamma|})$ is a spaces of functions if $sp < N$, cf. Theorem 3.11 and Proposition 3.41 in [20].

We will work also with the inhomogeneous Sobolev spaces $H^{s,p}(\mathbb{R}^N)$ and their invariant subspaces $H^{s,p}_\gamma(\mathbb{R}^N)$. Both spaces are equipped with the norm

$$\|f\|^*_{s,p} = \|f\|_p + \|f\|_{s,p}.$$

For technical reasons it will be convenient to work also with Besov spaces $B^{s,p}_1(\mathbb{R}^N)$ and $B^{s,p}_\infty(\mathbb{R}^N)$. The spaces can be defined by the real method of interpolation

$$B^{s,p}_q(\mathbb{R}^N) = \left(H^{2s,p}(\mathbb{R}^N), L^p(\mathbb{R}^N)\right)_{1/2,q}, \quad q = 1, \infty.$$

Many different characterizations of the Besov spaces can be found in [20], in particular we have

$$B^{s,p}_1(\mathbb{R}^N) \hookrightarrow H^{s,p}(\mathbb{R}^N) \hookrightarrow B^{s,p}_\infty(\mathbb{R}^N).$$

The norm of $B^{s,p}_q(\mathbb{R}^N)$ will be denoted as $\|\cdot\|_{s,p,q}$.

The main technical tool used in the paper is the method of atomic decompositions. Strauss type inequalities for Sobolev spaces of integer smoothness can be also proved by more elementary methods, and, as mentioned above, such inequalities were obtained by authors in [17] $s = 1$. However, the inequality that could be obtained by a simpler argument, (2.9), is less sharp than (2.8), and is verified for, generally, a more narrow class of functions. The main objective of using the approach of this paper was to refine the estimates of [17] for the classical Sobolev spaces, but as it happened, generalization to Sobolev spaces of the fractional did not complicate the proofs.

Precise pointwise estimates of radial functions belonging to different inhomogeneous spaces with fractional smoothness can be found in [13]. The similar estimates in the case of homogeneous spaces can be found in [2] and [14]. The applications of block-radial functions to semi-linear elliptic equations can be found for example in [4,8,12] and [9].

2 Main Results

In this section we state the pointwise estimate in the generic case, when the function in $\dot{H}^{s,p}_\gamma$ is dominated by $R_m(x)^{-\frac{1}{p}} r_{\min}(x)^{s-m/p}$ at the point x belonging to principal orbits. (cf. (2.8)), where the following notations are used:

$$
\begin{aligned}
r_j &= r_j(x) = \left(x^2_{\gamma_1+\cdots+\gamma_{j-1}+1} + \cdots + x^2_{\gamma_1+\cdots+\gamma_{j-1}+\gamma_j}\right)^{1/2};\\
r_{\min}(x) &= \min\{r_1(x), \ldots, r_m(x)\}, \quad x \in \mathbb{R}^N;\\
R_m(x) &= \prod_{i=1}^m r_i(x)^{\gamma_i - 1}, \quad x \in \mathbb{R}^N.
\end{aligned} \tag{2.1}
$$

It is needed, however, to define the exceptional set of parameters for which this simple estimate does not hold, therefore for any subset J of $\{1, \ldots, m\}$ we define the associated effective dimension

$$
d_J = \sum_{i \in J} \gamma_i + \#\{i \notin J\} \tag{2.2}
$$

For $J \subsetneqq \{1, \ldots, m\}$ we also put

$$
R_J(x) = \prod_{i \notin J} r_i(x)^{\gamma_i - 1} \quad \text{and} \quad r_J(x) = \min_{i \notin J} r_i(x). \tag{2.3}
$$

Let $m < sp < N$, $1 < p$. The following expression takes a decisive part in our estimates.

$$
R_J(x)^{-1/p} r_J(x)^{s-d_J/p} \tag{2.4}
$$

One can easily check that the expression is a homogeneous function of x of order $s - \frac{N}{p}$.

Lemma 2.1 *If $J \subset I \subsetneqq \{1, \ldots, m\}$, $d_I \le sp$ and $R_J(x) > 0$ then*

$$
R_J(x)^{-1/p} r_J(x)^{s-d_J/p} \le R_I(x)^{-1/p} r_I(x)^{s-d_I/p}. \tag{2.5}
$$

Proof It is sufficient to consider the case $I = J \cup \{i_o\}$, $i_o \in \{1, \ldots m\} \setminus J$. The rest follows by iteration. One can easily see that $d_J \le d_I$ and

$$
R_I(x)^{-1/p} r_I(x)^{s-d_I/p} = R_J(x)^{-1/p} r_I(x)^{s-d_J/p} r_{i_o}(x)^{\frac{\gamma_{i_o}-1}{p}} r_I(x)^{-\frac{\gamma_{i_o}-1}{p}}.
$$

If $r_I(x) = r_J(x)$ then $r_I(x) \le r_{i_o}(x)$ and the inequality (2.5) holds.
If $r_I(x) > r_J(x)$ then $r_J(x) = r_{i_o}(x)$. Then

$$
R_I(x)^{-1/p} r_I(x)^{s-d_I/p} > R_J(x)^{-1/p} r_J(x)^{s-d_J/p} \left(\frac{r_I(x)}{r_{i_o}(x)}\right)^{s-\frac{d_I}{p}}.
$$

This proves (2.5). □

Remark 2.2 The notation above brings up the matter of regularity of the functions in $\dot{H}^{s,p}_\gamma(\mathbb{R}^N)$. Let $f \in \dot{H}^{s,p}_\gamma(\mathbb{R}^N)$, $p > 1$, $m < sp < N$. If $R_m(x)$ is bounded away from zero, then f is locally a $H^{s,p}$-function of m variables $r_1, \dots, r_m$ and is therefore continuous in such region since $m < sp$.

Without loss of generality let us assume that $r_1(x) \geq \cdots \geq r_m(x)$. Whenever $r_j(x) = \cdots = r_k(x)$, also without loss of generality, we assume that $\gamma_j \leq \cdots \leq \gamma_k$. Consider a region where $R_n(x) = R_J(x)$, $J = \{n+1, \dots, m\}$, is bounded away from zero. In such region, f can be considered locally as a $\dot{H}^{s,p}$-function of $d_n = d_J$ variables $r_1, \dots, r_n, x_{1+\sum_{i=1}^n \gamma_i}, \dots, x_N$, and therefore f is continuous whenever $R_n(x) \neq 0$ and $d_n < sp$. Such region contains all orbits of the form $\Gamma = \{x : r_{n+1}(x) = \cdots = r_m(x) = 0, r_i(x) = \rho_i > 0, i = 1, \dots n\}$, and therefore the number d_n can be called *effective dimension* of such orbits.

Since d_n is a monotone decreasing function of n, $d_m = m$, and $d_0 = N$, there exists $n^* \in \{1, \dots, m\}$, which is the smallest n such that $d_n < sp$. The function f is continuous whenever $d_{n*} < sp$ and $R_{n^*}(x) \neq 0$, but may be discontinuous at the orbits where R_{n^*} equals 0. The similar statement holds for any permutation of indices.

To any subset $J \subsetneqq \{1, \dots, m\}$ we assign a cone domain in $\mathbb{R}^N$ defined by

$$\mathcal{C}_J = \left\{ x \in \mathbb{R}^N : \; R_J(x) > 0 \quad \text{and} \quad r_J(x) \geq \max_{j \in J} r_j(x) \right\}. \tag{2.6}$$

Here $\max_{j \in J} r_j(x) = 0$ if $J = \emptyset$. A given point x can belong to several sets $\mathcal{C}_J$ if $r_J(x)$ coincides with several values of $r_j(x)$, $j \notin J$. Note however that the expression (2.4) is independent of the ordering by the values of $r_i(x)$ that coincide.

If $x \in \mathcal{C}_J$ then there exists $i_x \in \{1, \dots, m\} \setminus J$ such that $r_{i_x}(x) = r_J(x)$. Let $s > 0$, $1 < p$, $m < sp < N$ and $\gamma_i \geq 2$. For a given J, s and p we define a subdomain $\mathcal{C}(J, s, p)$ of $\mathcal{C}_J$ by

$$\mathcal{C}(J, s, p) = \left\{ x \in \mathcal{C}_J : \; sp \leq d_{J \cup \{i_x\}} \right\}. \tag{2.7}$$

By definition $d_J < d_{J \cup \{i_x\}}$. So if $d_J < sp$ then $\mathcal{C}(J, s, p)$ consists of these points of $\mathcal{C}_J$, for which sp separates consecutive effective dimensions.

We first give a pointwise estimate for points x that belong to principal $SO(\gamma)$-orbits, i.e. if $R_m(x) \neq 0$, and when sp does not take any of the values d_I, $I \subsetneqq \{1, \dots, m\}$, in particular when $sp \notin \mathbb{N}$.

Theorem 2.3 *Let $s > 0$, $m \in \mathbb{N}$, $p > 1$, $m < sp < N$ and assume that $\gamma_i \geq 2$, $i = 1, \dots, m$. Assume also that $sp \neq d_I$ for any $I \subset \{1, \dots, m\}$.*
(i) If $R_m(x) \neq 0$ then there exists $J \subsetneqq \{1, \dots, m\}$ such that $d_J < sp$ and $x \in \mathcal{C}(J, s, p)$.
(ii) There exists $C > 0$, $C = C(\gamma, s, p)$ such that for any $J \subsetneqq \{1, \dots, m\}$ with $d_J < sp$, and for every $f \in \dot{H}^{s,p}_\gamma(\mathbb{R}^N)$, the inequality

$$|f(x)| \leq C R_J(x)^{-1/p} r_J(x)^{s - d_J/p} \, \|f\|_{s,p}, \tag{2.8}$$

holds for any $x \in \mathcal{C}(J, s, p)$.

The above theorem is the simplified version of Theorem 4.1 in Sect. 4. We formulate here this version for the convenience of the reader.

We also state a simpler, but more coarse, estimate.

Theorem 2.4 *Let $s > 0$, $m \in \mathbb{N}$, $p > 1$, $m < sp < N$ and assume that $\gamma_i \geq 2$, $i = 1, \dots, m$. Then there exists $C > 0$, $C = C(\gamma, s, p)$ such that for every $f \in \dot{H}^{s,p}_\gamma(\mathbb{R}^N)$, the inequality*

$$|f(x)| \leq C R_m(x)^{\frac{s-\frac{N}{p}}{N-m}} \|f\|_{s,p}, \tag{2.9}$$

holds for all $x \in \mathbb{R}^N$ such that $R_m(x) \neq 0$.

Remark 2.5 (1) Note that the right-hand side of (2.8) defines on the set $\cup_{J \subsetneq \{1,\dots,m\}, d_J < sp} \mathcal{C}(\gamma, s, p) = \{x \in \mathbb{R}^N : R_m(x) > 0\}$ a continuous positive-homogeneous function of degree $s - \frac{N}{p}$. This is the same homogeneity as $|x|^{s-N/p}$ that appears in the right hand side of the Strauss inequality, which is in fact the case $m = 1$ of (2.8) and (2.9).

(2) The above mentioned homogeneity implies, with $\hat{x} = \frac{x}{\|x\|}$, that

$$|f(x)| \leq C R_J(\hat{x})^{-1/p} r_J(\hat{x})^{s-d_J/p} \|x\|^{s-\frac{N}{p}} \|f\|_{s,p}.$$

for $x \in \mathcal{C}(J, s, p)$.

(3) For the case $s = 1$ the estimate (2.9) for a subspace of $\dot{H}^{1,p}_\gamma(\mathbb{R}^N)$ was given previously as Corollary 1 in [17]. Part (ii) of the corollary extends it to the whole $\dot{H}^{1,p}_\gamma(\mathbb{R}^N)$ under an additional assumption $\min_i \gamma_i \geq p$. The authors would also like to bring it to the attention of the reader that Proposition 1 of [17] that alleges that the subspace of the functions considered in [17] cannot be generally enlarged to $\dot{H}^{1,p}_\gamma(\mathbb{R}^N)$, contains a computational error (a request for publication an erratum has been made) and should be ignored.

For certain ranges of parameters inequality (2.8) takes a simpler form. In particular, if sp is sufficiently close to N (in particular, if $m = 1$), the estimate is the same as for radial functions.

Corollary 2.6 *Assume the conditions of Theorem 2.3.*

(i) If, additionally, $sp > N - \gamma_i + 1$ for all $i = 1, \dots, m$, then inequality (2.8) becomes

$$|f(x)| \leq C|x|^{s-N/p} \|f\|_{s,p}\,. \tag{2.10}$$

(ii) If, on the other hand, $sp < m + \gamma_i - 1$ for all $i = 1, \dots, m$, then inequality (2.8) becomes

$$|f(x)| \leq C R_m(x)^{-1/p} r_{\min}(x)^{s-m/p} \|f\|_{s,p}\,. \tag{2.11}$$

Proof We have $N - \gamma_i + 1 = d_{J_i}$, $J_i = \{1, \dots, m\} \setminus \{i\}$, $i = 1, \dots, m\}$. Thus the assumption (i) implies

$$d_{J_i} < sp < N = d_{\{1,\dots,m\}}.$$

If $R_m(x) \neq 0$ then there exists i such that $x \in \mathcal{C}(J_i, s, p)$. But in that case $r_{J_i}(x) = R_{J_i}(x) \sim \|x\|$. This proofs (i).

Analogously the assumption (ii) implies that $x \in \mathcal{C}(\emptyset, s, p)$ if $R_m(x) \neq 0$ since $d_\emptyset = m < sp < d_{\{i\}}$ for any i. But $R_\emptyset(x) = R_m(x)$ and $r_\emptyset(x) = r_{\min}(x)$. This proofs (ii). □

The second assertion of Theorem 2.3 is restated in Sect. 4 as Theorem 4.1, and for the exceptional region characterized by $sp = d_J$ a similar estimate, but with a logarithmic term, is provided by Theorem 4.2. Corollaries 4.3 and 4.4 simplify the respective statements of Theorems 4.1 and 4.2 in the bi-radial case ($m = 2$).

Further results of the paper are dealing with optimality of the estimate in Theorem 4.1 and with weak continuity properties of the estimate, in Sects. 5 and 6 respectively.

3 Preparations: Atomic Decomposition

We assume that $N \geq 2$, $\gamma_i \geq 2$ for any $i = 1, \ldots, m$.

To prove our statements we use the approach of atomic decomposition. In the context of Sobolev and Besov spaces the method goes back to the seminal papers by M. Frazier and B. Jawerth, cf. [6,7]. The atomic decomposition adapted to the radial case was constructed by J. Epperson and M. Frazier in [5]. Here we use slightly different approach described in [16]. It should be pointed out that we use the atomic decomposition of inhomogeneous spaces and afterwards we use the homogeneity argument to transfer the result to homogeneous spaces. This can be done since estimating functions are homogeneous. In contrast to [5] our atoms are supported on balls not on annuli centred at zero. Here we recall the main idea of the method, with the needed modifications, and we refer the reader to [16] for more details.

We start with notions of separation and discretization in $\mathbb{R}^N$ since they are needed for description of the atomic decomposition. Let $B(x, r)$ denote the ball of radius r in $\mathbb{R}^N$.

Definition 3.1 Let $\varepsilon > 0$ be a positive number, $\alpha = 1, 2, \ldots$ be a positive integer and X a nonempty subset of $\mathbb{R}^N$.

(a) A subset $\mathcal{H}$ of X is said to be ε–separation of X, if the distance between any two distinct points of $\mathcal{H}$ is greater than or equal to ε.

(b) A subset $\mathcal{H}$ of X is called an (ε, α)–discretization of X if it is an ε–separation of X and

$$X \subset \bigcup_{x \in \mathcal{H}} B(x, \alpha\varepsilon).$$

Remark 3.1 1. Both notations are well known and important in geometry eg. cf. [1, Chapter 4]. Please note, that our notion of discretization is a bit different to that one in Chavel's book.

2. Let m be a positive integer. If $\mathcal{H}$ is an (ε, α)–discretization of $\mathbb{R}^N$ and $m \geq \alpha$, then the family $\{B(x, m\varepsilon)\}_{x \in \mathcal{H}}$ is a uniformly locally finite covering of $\mathbb{R}^N$ whose multi-

plicity is bounded from above by a constant depending on N and m, but independent of ε.

We describe the needed discretizations related to the group $SO(\gamma)$. In this case we can proceed in the following way. Let $\{x^{(j,i)}_{k,\ell}\}$, $\ell = 1, \ldots, \max\{1, k\}^{\gamma_i - 1}$ and $k \in \mathbb{N}_0$, be a $(2^{-j}, \alpha_i)$-discretization in $\mathbb{R}^{\gamma_i}$ constructed in [15, Sect. 3.2], with values of $x^{(j,i)}_{0,0}$ set instead of zero, as in [15], to $|x^{(j,i)}_{0,0}| = 2^{-2j}$.

We put

$$\mathcal{H}_j = \left\{ x^{(j)}_{\mathbf{k},\mathbf{l}} = \left(x^{(j,1)}_{k_1,l_1}, \ldots, x^{(j,m)}_{k_m,l_m} \right) : \quad k_i \in \mathbb{N}_0\,, \quad l_i = 1, \ldots, \max\{1, k_i\}^{\gamma_i - 1} \right\},$$

and

$$\alpha = \sqrt{n} \max_i \{\alpha_i\}.$$

The set $\mathcal{H}_j$ is a $(2^{-j}, \alpha)$-discretization of $\mathbb{R}^N$ Let $x^{(j)}_{\mathbf{k},\mathbf{l}} \in \mathcal{H}_j$. Then

$$SO(\gamma)\big(x^{(j)}_{\mathbf{k},\mathbf{l}}\big) = \prod_{i=1}^m SO(\gamma_i)\big(x^{(j)}_{k_i,l_i}\big)$$

and

$$\operatorname{card}\Big(\mathcal{H}_j \cap SO(\gamma)\big(x^{(j)}_{\mathbf{k},\mathbf{l}}\big)\Big) = \prod_{i=1}^m \max\{1, k_i\}^{\gamma_i - 1} = C(j, \mathbf{k}) \tag{3.1}$$

Any point $x^{(j)}_{\mathbf{k},\mathbf{l}}$ defined above belongs to an orbit of $SO(\gamma)$ of the maximal dimension $N - m$.

The function $a_{j,\mathbf{k},\mathbf{l}}$ is called $(1, p)$-atom centered at the point $x^{(j)}_{\mathbf{k},\mathbf{l}} \in \mathcal{H}_j$ if:

$$\operatorname{supp} a_{j,\mathbf{k},\mathbf{l}} \subset B\big(x^{(j)}_{\mathbf{k},\mathbf{l}}, \alpha 2^{-j}\big), \tag{3.2}$$

$$\sup_{y \in \mathbb{R}^N} \left|\partial^\beta a_{j,\mathbf{k},\mathbf{l}}(y)\right| \le 2^{-j\left(s - |\beta| - \frac{N}{p}\right)} \quad |\beta| \le [s + 1]. \tag{3.3}$$

Proposition 3.2 *Let $s > 0$ and $1 \le p, q \le \infty$. Let $\mathcal{H}_j$ be a sequence of discretizations described above. Then*
(i) any function $f \in B^{s,p}_{q,\gamma}(\mathbb{R}^N)$ can be decomposed in the following way

$$f = \sum_{j=0}^\infty \sum_{\mathbf{k} \in \mathbb{N}_0^m} \sum_{\mathbf{l}} s_{j,\mathbf{k}}\, a_{j,\mathbf{k},\mathbf{l}}, \quad (\textit{convergence in} \quad \mathcal{S}') \tag{3.4}$$

with

$$\left(\sum_{j=0}^{\infty}\left(\sum_{\mathbf{k}\in\mathbb{N}_0^m} C(j,\mathbf{k})\,|s_{j,\mathbf{k}}|^p\right)^{q/p}\right)^{1/q} < \infty, \quad s_{j,\mathbf{k}} \in \mathbb{C} \tag{3.5}$$

(usual change if $q = \infty$).
(ii) Conversely, any distribution represented by (3.4) *with* (3.5) *belongs to* $B^{s,p}_{q,\gamma}(\mathbb{R}^N)$.
Moreover the infimum over all possible representations of the expression (3.5) *gives an equivalent norm in* $B^{s,p}_{q,\gamma}(\mathbb{R}^N)$.

$$\|f\|^*_{s,p,q} = \inf\left(\sum_{j=0}^{\infty}\left(\sum_{\mathbf{k}} C(j,\mathbf{k})\,|s_{j,\mathbf{k}}|^p\right)^{q/p}\right)^{1/q} \sim \|f\|_{s,p,q}. \tag{3.6}$$

For the proof we refer to [16], cf. Theorem 1, Lemma 2, Remark 7 and Step 2 of the proof of Theorem 2 ibidem. A similar proposition holds for the case of non-positive smoothness index ($s \leq 0$), but in this case additional assumptions concerning the atoms, so call moment conditions, are needed. We refer once more to [16].

4 Pointwise Estimate for Multiradial Functions

In this section we will state and prove the main result in full generality, that is, including the exceptional cases when a logarithmic term appears. Let $s > 0$ and $1 < p < \infty$ be such that $m < sp < N$.

First we consider the case when the value of sp is distinct from the effective dimension of any orbit.

Theorem 4.1 *Let $s > 0$, $m \in \mathbb{N}$, $p > 1$, $m < sp < N$. Assume that $\gamma_i \geq 2$, $i = 1,\ldots,m$. Let $J \subsetneq \{1,\ldots,m\}$ and $d_J < sp$. We assume moreover that $sp \neq d_I$ for any set I such that $J \subset I \subset \{1,\ldots,m\}$. Then there exists $C > 0$, $C = C(\gamma,s,p)$, such that for every $f \in \dot{H}^{s,p}_{\gamma}(\mathbb{R}^N)$, and every $x \in \mathcal{C}(J,s,p)$ the estimates* (2.8) *hold.*

We now formulate the result for the logarithmic pointwise estimate that occurs in the regions whose effective dimension coincides with the value of sp.

Theorem 4.2 *Let $s > 0$, $m \in \mathbb{N}$, $p > 1$, $m < sp < N$. Assume that $\gamma_i \geq 2$, $i = 1,\ldots,m$. Let $J \subsetneq \{1,\ldots,m\}$ and $d_J < sp$.*
Then there exists $C > 0$, $C = C(\gamma,s,p)$, such that for every $f \in \dot{H}^{s,p}_{\gamma}(\mathbb{R}^N)$ and every $x \in \mathcal{C}(J,s,p)$

$$|f(x)| \leq C\,\|f\|_{s,p} \times \begin{cases} r_J(x)^{s-d_J/p} R_J(x)^{-1/p} & \text{if } sp < d_{J\cup\{i_x\}}, \\ \left(1+\log\frac{\|x\|}{r_J(x)}\right) r_J(x)^{s-d_J/p} R_J(x)^{-1/p} & \text{if } sp = d_{J\cup\{i_x\}}. \end{cases} \tag{4.1}$$

As the statements of the theorems above are rather complicated, we would like to give some corollaries with simpler statements. One such corollary is the already stated Theorem 2.4. The following two corollaries are straightforward elaborations, respectively, of Theorem 4.1 and of Theorem 4.2 in the bi-radial case ($m = 2$).

Corollary 4.3 *Let $s > 0$, $m = 2 < sp < N$, and assume that $\gamma_i \geq 2$, $i = 1, 2$.*
(i) If $2 < sp < \gamma_1 + 1$ and $r_2(x) \geq r_1(x) > 0$ then

$$|f(x)| \leq C r_1(x)^{s - \frac{\gamma_1+1}{p}} r_2(x)^{\frac{1-\gamma_2}{p}} \|f\|_{s,p}\,. \tag{4.2}$$

(ii) If $2 < sp < \gamma_2 + 1$ and $r_1(x) \geq r_2(x) > 0$ then

$$|f(x)| \leq C r_2(x)^{s - \frac{\gamma_2+1}{p}} r_1(x)^{\frac{1-\gamma_1}{p}} \|f\|_{s,p}\,. \tag{4.3}$$

(iii) Let $\gamma_1 + 1 < sp < N$, $r_2(x) > 0$ and $r_2(x) \geq r_1(x)$ or let $\gamma_2 + 1 < sp < N$, $r_1(x) > 0$ and $r_1(x) \geq r_2(x)$. Then

$$|f(x)| \leq C|x|^{s - \frac{N}{p}} \|f\|_{s,p}. \tag{4.4}$$

Corollary 4.4 *Let $s > 0$, $m = 2 < sp < N$, and assume that $\gamma_i \geq 2$, $i = 1, 2$.*
(i) If $sp = \gamma_1 + 1 \neq \gamma_2 + 1$ then

$$|f(x)| \leq C\|f\|_{s,p} \begin{cases} \left(1 + \log \frac{r_2(x)}{r_1(x)}\right) r_2(x)^{\frac{1-\gamma_2}{p}} & \text{if } r_2(x) \geq r_1(x) > 0, \\ r_2(x)^{s - \frac{\gamma_2+1}{p}} r_1(x)^{\frac{1-\gamma_1}{p}} & \text{if } r_1(x) \geq r_2(x) > 0. \end{cases} \tag{4.5}$$

(ii) If $sp = \gamma_2 + 1 \neq \gamma_1 + 1$ then

$$|f(x)| \leq C\|f\|_{s,p} \begin{cases} \left(1 + \log \frac{r_1(x)}{r_2(x)}\right) r_1(x)^{\frac{1-\gamma_1}{p}} & \text{if } r_1(x) \geq r_2(x) > 0, \\ r_1(x)^{s - \frac{\gamma_1+1}{p}} r_2(x)^{\frac{1-\gamma_2}{p}} & \text{if } r_2(x) \geq r_1(x) > 0. \end{cases} \tag{4.6}$$

(iii) If $sp = \gamma_1 + 1 = \gamma_2 + 1$
then

$$|f(x)| \leq C\|f\|_{s,p} \begin{cases} \left(1 + \log \frac{r_2(x)}{r_1(x)}\right) r_2(x)^{\frac{1-\gamma_2}{p}} & \text{if } r_2(x) \geq r_1(x) > 0, \\ \left(1 + \log \frac{r_1(x)}{r_2(x)}\right) r_1(x)^{\frac{1-\gamma_1}{p}} & \text{if } r_1(x) \geq r_2(x) > 0. \end{cases} \tag{4.7}$$

In order to prove Theorems 4.1 and 4.2 we rewrite their assertions as the theorem below for the case when the values $r_i(x)$ are ordered in i, and prove the latter instead.

Theorem 4.5 *Let $s > 0$, $p > 1$, $m \in \mathbb{N}$, $m < sp < N$, and assume that $\gamma_i \geq 2$, $i = 1, \ldots, m$. Let n, $1 \leq n \leq m$, be the smallest integer such that $d_n = \sum_{i=n+1}^{m} \gamma_i + n < sp$. Let $x \in \mathbb{R}^N$ with $r_1(x) \geq r_2(x) \geq \ldots r_m(x)$ and $R_n(x) > 0$.*

If $d_{n-1} > sp$ *then there exists* $C = C(s, p, \gamma, n)$ *such that for every* $f \in \dot{H}^{s,p}_\gamma(\mathbb{R}^N)$

$$|f(x)| \le C\, r_n(x)^{s-d_n/p} R_n(x)^{-1/p} \|f\|_{s,p} \tag{4.8}$$

If $d_{n-1} = sp$ *then there exists* $C = C(s, p, \gamma, n)$ *such that for every* $f \in \dot{H}^{s,p}_\gamma(\mathbb{R}^N)$

$$|f(x)| \le C \left(1 + \log \frac{r_{n-1}(x)}{r_n(x)}\right) r_n(x)^{s-d_n/p} R_n(x)^{-1/p} \|f\|_{s,p}\,. \tag{4.9}$$

Proof We first prove the inequalities (4.8) and (4.9) for $f \in B^{s,p}_{\infty,\gamma}(\mathbb{R}^N)$ and $x \neq 0$, $\max\{r_i(x) : i = 1, \dots, n\} \le 1$, $n \le m$.

By the assumption $x \in \mathcal{C}_J$ where $J = \{n+1, \dots, m\}$. We assume in addition that $r_i(x) \le 1$, $i = 1, \dots, m$. If $r_i(x) > 0$, then one can find $j_i \in \mathbb{N}$ such that $2^{-j_i-1} \le r_i(x) \le 2^{-j_i+1}$. The inequality $r_i(x) > 0$ is satisfied for $i = 1, \dots, \nu$, with some ν, $n \le \nu \le m$. We assume that ν is the largest integer with this property. We may assume that $j_1 \le j_2 \le \dots \le j_\nu$.

We have the following atomic decomposition, cf (3.4):

$$f = \sum_{j=0}^{\infty} \sum_{\mathbf{k}\in\mathbb{N}_0^m} \sum_{\mathbf{l}} s_{j,\mathbf{k}}\, a_{j,\mathbf{k},\mathbf{l}}, \tag{4.10}$$

of f with

$$\sup_j \left(\sum_{\mathbf{k}} C(j,\mathbf{k})\, |s_{j,\mathbf{k}}|^p\right)^{1/p} \le C\|f\|_{s,p,\infty}\,. \tag{4.11}$$

If $\mathbf{k} = (k_1, \dots k_m) \in \mathbb{N}^m$ then

$$C(j,\mathbf{k}) = k_1^{\gamma_1-1} \dots k_m^{\gamma_m-1} \sim 2^{j(N-m)} r_1\left(x^{(j)}_{\mathbf{k},\mathbf{l}}\right)^{\gamma_1-1} \dots r_m\left(x^{(j)}_{\mathbf{k},\mathbf{l}}\right)^{\gamma_m-1}, \tag{4.12}$$

cf. (3.1). By our construction $r_i(x^{(j)}_{\mathbf{k},\mathbf{l}}) > 0$ for any $i = 1, \dots, m$ and for any point $x^{(j)}_{\mathbf{k},\mathbf{l}}$.

The point x belongs to supp $a_{j,\mathbf{k},\mathbf{l}}$ if

$$\begin{aligned} \max\{0, [2^{j-j_i}] - n_0\} \le k_i \le [2^{j-j_i}] + n_0 \quad &\text{for} \quad i = 1, \dots, \nu, \\ 0 \le k_i \le n_0 \quad &\text{for} \quad i = \nu+1, \dots, m, \end{aligned} \tag{4.13}$$

with some fixed $n_0 \in \mathbb{N}$.

By the normalization (3.3) of the atoms, condition (3.2) on their supports, and (4.13), we get

$$|f(x)| \le \sum_{j=0}^{\infty} \sum_{\mathbf{k},\mathbf{l}:\, x\in \operatorname{supp} a_{j,\mathbf{k},\mathbf{l}}} |s_{j,\mathbf{k}}|\, |a_{j,\mathbf{k},\mathbf{l}}(x)| \le$$

$$\le C \sum_{j=0}^{\infty} \sum_{\substack{\max\{0,[2^{j-j_i}]-n_0\}\le k_i \le [2^{j-j_i}]+n_0 \\ i=1,\dots,\nu}} \sum_{\substack{0\le k_i\le n_0 \\ i=\nu+1,\dots,m}} 2^{-j(s-N/p)} |s_{j,\mathbf{k}}| \,. \tag{4.14}$$

Now by (3.5) and (4.14), we get

$$|f(x)| \le C\, \|f\|_{s,p,\infty} \times \sum_{j=0}^{\infty} \sum_{\substack{\max\{0,[2^{j-j_i}]-n_0\}\le k_i \le [2^{j-j_i}]+n_0 \\ i=1,\dots,\nu}} \sum_{\substack{0\le k_i\le n_0 \\ i=\nu+1,\dots,m}} C_{j,\mathbf{k}}^{-1/p}\, 2^{-j(s-N/p)} \,. \tag{4.15}$$

In what follows we will omit in writing the last factor $\|f\|_{1,p,\infty}$ from the right hand side, remembering it as an implicit factor in a multiplicative constant. Taking into account our estimate (4.12) for $C_{j,\mathbf{k}}$, we can evaluate the sum above by splitting the summation in j into $\nu+1$ intervals $0 = j_0 \le j < j_1$, $j_\ell \le j < j_{\ell+1}$, $\ell = 0, \dots \nu-1$, and $j_\nu \le j < \infty$. Some of these intervals may be empty, and the corresponding sums are assigned value zero.

So, taking into account only the atoms whose support contains x, we have

$$|f(x)| \le C\left(\sum_{j=0}^{j_1-1} + \dots + \sum_{j=j_\ell}^{j_{\ell+1}-1} + \dots + \sum_{j=j_\nu}^{\infty}\right) 2^{-j(s-N/p)} \times \sum_{\substack{\max\{[2^{j-j_i}]-n_0,0\}\le k_i \le [2^{j-j_i}]+n_0 \\ i=1,\dots,\nu}} \sum_{\substack{0\le k_i\le n_0 \\ i=\nu+1,\dots,m}} C_{j,\mathbf{k}}^{-1/p}. \tag{4.16}$$

For $j_\ell \le j < j_{\ell+1}$, whenever this is a non-empty interval, we have the summation over k_i of a uniformly finite number of terms. Moreover, for the values of k_i involved in the sum, we have

$$C_{j,\mathbf{k}}^{-1/p} = k_1^{-\frac{\gamma_1-1}{p}} \dots k_m^{-\frac{\gamma_m-1}{p}} \le C\, k_1^{-\frac{\gamma_1-1}{p}} \dots k_\ell^{-\frac{\gamma_\ell-1}{p}}, \quad \text{whenever} \quad j_\ell \le j, \tag{4.17}$$

cf. (4.12), and the constant C depends only on the atomic decomposition and n, in particular it is independent of j, ℓ and j_ℓ. Thus for any $C_{j,\mathbf{k}}$ that appear in the right hand side of (4.16) we have, given our assumption that $r_i(x) \le 1$,

$$C_{j,\mathbf{k}}^{-1/p} \le C \prod_{i=1}^{\ell} 2^{(-j+j_i)(\gamma_i-1)/p} \quad \text{if} \quad j_\ell \le j, \tag{4.18}$$

and the constant C is independent of j, ℓ and j_ℓ.

In consequence,

$$
\begin{aligned}
|f(x)| \le C \sum_{j=0}^{j_1-1} 2^{-j(s-N/p)} + \cdots + \\
+ C \sum_{j=j_\ell}^{j_{\ell+1}-1} 2^{-j(s-N/p)} \prod_{i=1}^{\ell} 2^{(-j+j_i)(\gamma_i-1)/p} + \cdots + \\
+ C \sum_{j=j_\nu}^{\infty} 2^{-j(s-N/p)} \prod_{i=1}^{\nu} 2^{(-j+j_i)(\gamma_i-1)/p} .
\end{aligned} \tag{4.19}
$$

If $n \le \ell < \nu$ then $d_\nu \le d_\ell \le d_n < sp$ and

$$
\begin{aligned}
\sum_{j=j_\ell}^{j_{\ell+1}-1} 2^{-j(s-N/p)} \prod_{i=1}^{\ell} 2^{(-j+j_i)(\gamma_i-1)/p} &\le C r_\ell(x)^{\frac{sp-d_\ell}{p}} R_\ell(x)^{-1/p} \\
&\le C r_n(x)^{\frac{sp-d_n}{p}} R_n(x)^{-1/p},
\end{aligned} \tag{4.20}
$$

where the last inequality follows from (2.5). Here we use the already introduced notation $d_\ell = \sum_{i=\ell+1}^{m} \gamma_i + \ell$, $\ell = 0, \ldots, m$. The similar estimates hold for the sum $\sum_{j=j_\nu}^{\infty}$ since $d_\nu < sp$.

If $0 \le \ell < n$ then $d_n < d_\ell$. Tedious, but elementary computations, which we have confined to Lemma 4.6 below, show that if $sp \ne d_{n-1}$, then the sums $\sum_{j=j_\ell}^{j_{\ell+1}-1}$ are dominated by the sum $\sum_{j=j_n}^{j_{n+1}-1}$.

If, however, $sp = d_{n-1}$, then, again by Lemma 4.6, the terms in (4.19) are estimated by

$$
C\left(1 + \log \frac{r_{n-1}(x)}{r_n(x)}\right) r_n(x)^{s-d_n/p} R_n(x)^{-1/p}.
$$

Note that this function has the same homogeneity $\frac{sp-N}{p}$ as every other term in (4.19), once we take into account that $ps = d_{n-1}$.

Note that the term $r_1(x)^{\frac{p-N}{p}}$ is in fact $r_{\max}(x)^{\frac{p-N}{p}}$, and since ℓ^∞-norm in $\mathbb{R}^N$ is equivalent to the Euclidean norm, this term is equivalent to $|x|^{\frac{p-N}{p}}$.

Thus we have proved that

$$
|f(x)| \le C Q(x)^{1/p} \|f\|_{s,p,\infty}, \tag{4.21}
$$

with a function Q, positively homogeneous of degree $N - sp$ that appears in the right hand side of (4.8) and (4.9), whenever $|x| \le 1$.

Furthermore, by a well-known embedding, the norm in the right hand side is dominated by the $H^{s,p}$-norm. Setting $f(x) = g(tx)$ with $g \in C_0^\infty(\mathbb{R}^N)$, $|x| \le 1$, and $t > 0$

large, and using homogeneity properties of the norms we get

$$|g(tx)|^p \le CQ(tx)\left(\|g\|_{s,p}^p + t^{-N}\|g\|_p^p\right).$$

setting $x = z/t$ with arbitrary fixed z, and taking the limit at $t \to \infty$, we have

$$|g(z)|^p \le CQ(z)\|g\|_{s,p}^p.$$

Since $C_0^\infty(\mathbb{R}^N)$ is dense in $\dot{H}^{s,p}(\mathbb{R}^N)$, (2.8) follows. □

In order to complete the argument above we have to prove the following elementary technical statement. We recall that any sum over an empty set is understood as zero.

Lemma 4.6 *Let* $1 < n \le m$ *and* $d_n < sp \le d_{n-1}$. *Assume that* $j_1 \le j_2 \le \cdots \le j_n$, *and that for some* $q > 0$ *we have* $2^{-j_i - q} \le r_i(x) \le 2^{-j_i + q}$, $i = 1, \ldots, n$.
(i) Let $0 < \ell \le n$. *If* $sp \ne d_{n-1}$ *then*

$$\sum_{j=j_\ell}^{j_{\ell+1}-1} 2^{-j\frac{sp-d_\ell}{p}} R_\ell(x)^{-1/p} \le C\, r_n(x)^{s-d_n/p} R_n(x)^{-1/p} \tag{4.22}$$

(ii) Let $0 < \ell \le n$. *If* $sp = d_{n-1}$ *then*

$$\sum_{j=j_\ell}^{j_{\ell+1}-1} 2^{-j\frac{sp-d_\ell}{p}} R_\ell(x)^{-1/p} \le C\left(1 + \log\frac{r_{n-1}(x)}{r_n(x)}\right) r_n(x)^{s-\frac{n}{p}} R_n(x)^{-1/p}. \tag{4.23}$$

Proof The proof is based on the observation that the sums in (4.22) and (4.23) are geometric sums and thus are evaluated by the upper term or by the lower term and by the number of terms if the power is zero. Observe that the mapping $\ell \mapsto d_\ell$ is monotone decreasing on $\{0, \ldots, n\}$ from N to d_n.

First we prove (i). We have

$$s - \frac{N}{p} = s - \frac{d_0}{p} < s - \frac{d_1}{p} < \cdots < s - \frac{d_\ell}{p} < \cdots s - \frac{d_{n-1}}{p} < 0 < s - \frac{d_n}{p}$$

If $1 \le \ell < n$ then

$$\sum_{j=j_\ell}^{j_{\ell+1}-1} 2^{-j\frac{sp-d_\ell}{p}} R_\ell^{-1/p} \sim r_{\ell+1}^{s-d_\ell/p} R_\ell^{-1/p} =: T_\ell.$$

But $d_{\ell-1} = d_\ell + \gamma_\ell - 1$, $s - \frac{d_\ell}{p} < 0$ and $r_{\ell-1} \geq r_\ell$ therefore $T_{\ell-1} \leq T_\ell$. Thus

$$\sum_{j=j_\ell}^{j_{\ell+1}-1} 2^{-j\frac{sp-d_\ell}{p}} R_\ell(x)^{-1/p} \leq C r_n(x)^{s-\frac{d_{n-1}}{p}} R_{n-1}(x)^{-\frac{1}{p}} = C r_n(x)^{s-\frac{d_n}{p}} R_n(x)^{-\frac{1}{p}}.$$

Similarly we have

$$\sum_{j=0}^{j_1-1} 2^{-j\frac{sp-N}{p}} = C r_1(x)^{\frac{sp-d_0}{p}} \leq T_1.$$

Now we prove (ii). We have

$$s - \frac{m}{p} < s - \frac{d_1}{p} < \cdots < s - \frac{d_{n-1}}{p} = 0 < s - \frac{d_n}{p}.$$

If $\ell < n-1$ then the same calculations as above show us that

$$\sum_{j=j_\ell}^{j_{\ell+1}-1} 2^{-j\frac{sp-d_\ell}{p}} R_\ell^{-1/p} \leq C R_{n-1}(x)^{-\frac{1}{p}} = C r_n(x)^{s-\frac{d_n}{p}} R_n(x)^{-\frac{1}{p}}.$$

If $\ell = n$ then

$$\sum_{j=j_n}^{j_{n+1}-1} 2^{-j\frac{sp-d_n}{p}} R_\ell^{-1/p} \leq C r_n(x)^{s-d_n/p} R_n(x)^{-1/p},$$

since $sp > d_n$. The same estimate holds if we sum up from $j = j_n$ to infinity.

At the end if $\ell = n-1$ then

$$\begin{aligned}\sum_{j=j_{n-1}}^{j_n-1} 2^{-j\frac{sp-d_{n-1}}{p}} R_{n-1}(x)^{-1/p} &= (j_n - j_{n-1}) R_{n-1}(x)^{-1/p} \\ &\leq C\left(1 + \log_2 \frac{r_{n-1}(x)}{r_n(x)}\right) R_{n-1}(x)^{-1/p} \\ &= C\left(1 + \log_2 \frac{r_{n-1}(x)}{r_n(x)}\right) r_n(x)^{s-d_n/p} R_n(x)^{-1/p}.\end{aligned}$$

□

Proof of Theorem 4.1 and 4.2

Let $x \in \mathcal{C}(J, s, p)$. Then these is a permutation $j_1, \dots j_m$ of the set $\{1, \dots m\}$ such that $\{j_{k+1}, \dots j_m\} = J$ and

$$r_{j_1}(x) \geq \cdots \geq r_{j_{k-1}}(x) \geq r_{j_k}(x) \geq r_{j_m}(x).$$

and

$$sp \leq d_{J_1}, \quad \text{where } J_1 = J \cup \{x_{k-1}\}.$$

The arguments used in the proof of Theorem 4.5 show that

$$|f(x)| \leq \; C r_J(x)^{s-d_J/p} R_J(x)^{-1/p} \|f\|_{s,p}$$

if $sp \neq d_{J_1}$, and

$$|f(x)| \leq \; C \left(1 + \log \frac{r_{J_1}(x)}{r_J(x)}\right) r_J(x)^{s-d_{J_1}/p} R_J(x)^{-1/p} \|f\|_{s,p},$$

if $sp = d_{J_1}$. □

Proof of Theorem 2.4. The assertion of the theorem follows by replacement of the weight in right hand side of the estimates in Theorems 4.1 and 4.2 by a simpler expression $R_m(x)^{\frac{s-\frac{N}{p}}{N-m}}$ that dominates them in all cases. It suffices to carry out the comparison with the weights in (4.8) and (4.9) under the ordering assumption $r_1(x) \geq \cdots \geq r_m(x)$, which is an elementary computation. Indeed, in case of (4.8) we should prove the inequality

$$R_m(x)^{N-sp} \leq R_n(x)^{N-m}(x) r_n(x)^{(d_n - sp)(N-m)}, \quad 1 \leq n \leq m.$$

The last inequality is equivalent to

$$\prod_{i=1}^{n} r_i(x)^{(\gamma_i - 1)(m-sp)} \prod_{i=n+1}^{m} r_i(x)^{(\gamma_i - 1)(N-sp)} \leq r_n(x)^{(d_n - sp)(N-m)}. \tag{4.24}$$

But $r_j(x) \leq r_n(x) \leq r_i(x)$ if $i \leq n \leq j$, so

$$\begin{aligned} \prod_{i=1}^{n} r_i(x)^{(\gamma_i - 1)(m-sp)} &\leq r_n(x)^{(m-np)(\gamma_1 + \cdots + \gamma_n - n)}, \\ \prod_{i=n+1}^{m} r_i(x)^{(\gamma_i - 1)(N-sp)} &\leq r_n(x)^{(N-sp)(\gamma_{n+1} + \cdots + \gamma_m - m + n)}. \end{aligned}$$

The last inequalities imply (4.24). If $d_{n-1} = sp$ then

$$\prod_{i=1}^{n-1} r_i(x)^{(\gamma_i-1)(m-sp)} \leq r_{n-1}(x)^{(m-sp)(N-sp)},$$
$$\prod_{i=n}^{m} r_i(x)^{(\gamma_i-1)(N-sp)} \leq r_n(x)^{(sp-m))(N-sp)}.$$

Therefore

$$\begin{aligned} R_m(x)^{N-sp} &\leq \left(\frac{r_{n-1}(x)}{r_n(x)}\right)^{(N-sp)(m-sp)} R_{n-1}(x)^{N-m} \\ &\leq \left(1+\log\frac{r_{n-1}(x)}{r_n(x)}\right)^{p(m-N)} r_n(x)^{(d_n-sp)(N-m)} R_n(x)^{N-m}, \end{aligned}$$

where we use the elementary inequality $1 + \log t \leq C(a)t^a$, $t \geq 1$, $a > 0$. □

Lemma 4.7 *Let $s > 0$, $1 < p < \infty$ and $m < sp < N$. Let $w_J(x)$, $J \subsetneq \{1, \ldots, m\}$, denote the weight in the right hand side of* (2.8) *and* (4.1) *and let*

$$\mathcal{A}_{sp} = \bigcup_{J:\, d_J < sp} \mathcal{C}(J, s, p).$$

(i) The set $\mathcal{A}_{sp}$ is a dense conical subset of $\mathbb{R}^N$.
(ii) If $x \in \mathcal{C}(J, s, p) \cap \mathcal{C}(I, s, p)$, for some $J, I \subsetneq \{1, \ldots, m\}$, $d_J, d_I < sp$, then

$$sp \neq d_{J\cup\{j_x\}}, \quad sp \neq d_{I\cup\{i_x\}} \quad \text{and} \quad w_J(x) = w_I(x), \tag{4.25}$$

or

$$sp = d_{J\cup\{j_x\}} = d_{I\cup\{i_x\}} \quad \text{and} \quad w_J(x) = w_I(x), . \tag{4.26}$$

Proof If $x \in \mathbb{R}^N$ then for some permutation $(j_1, \ldots, j_m)$ of $(1, \ldots, m)$ we have

$$r_{j_1}(x) \geq r_{j_2}(x) \geq \cdots \geq r_{j_m}(x). \tag{4.27}$$

Let $J_k = \{j_{k+1}, \ldots, j_m\}$, $k = 0, \ldots, m-1$, and $J_m = \emptyset$. Then there is exactly one $k > 0$ such that $d_{J_k} < sp \leq d_{J_{k+1}}$. One can easily see that the set $\{x \in \mathbb{R}^N : R_m(x) > 0\}$ is dense in $\mathbb{R}^N$ and contained in $\mathcal{A}_{sp}$.

If in (4.27) all inequalities are strict, then the relations (4.27) are satisfied only by one permutation and in consequence x belongs only to one set $\mathcal{C}(J, s, p)$. However if some equalities occur in (4.27) then another permutation is possible. Let $(i_1, \ldots, i_m)$ be such permutation. We may assume that the last permutation is an inversion of $(j_1, \ldots, j_m)$ i.e. $i_m = j_{m+1}$ and $i_{m+1} = j_m$ for some m. If (4.27) holds for $(i_1, \ldots, i_m)$ then $r_{i_m}(x) = r_{i_{m+1}}(x) = r_{j_m}(x) = r_{j_{m+1}}(x)$. So for $m \neq k$ we have $R_J(x) = R_I(x)$,

$r_J(x) = r_I(x)$ and $d_J = d_I$, $d_{J\cup\{j_x\}} = d_{I\cup\{i_x\}}$. If $m = k$ then simple calculations prove (4.25) and (4.26). □

It follows from the last lemma that the function

$$W_{sp}(x) = w_J(x)^{-1}, \quad x \in \mathcal{C}(J, s, p),$$

is well defined and continuous on the dense set $\mathcal{A}_{sp}$. For any $x \in \mathbb{R}^N$ the inequalities (4.27) hold for some permutation. If J_k denotes the same set as above, then $x \notin \mathcal{A}_{sp}$ implies that $r_{j_k}(x) = r_{J_k}(x) = 0$. So if $x_n \to x$ then $W_{sp}(x_n) \to 0$ if $sp < d_{J_{k+1}}$ and $W_{sp}(x_n) \to (r_{j_1}(x)\ldots r_{j_{k-1}}(x))^{1/p}$ if $sp = d_{J_{k+1}}$. Thus W_{sp} can be extended to the continuous function on $\mathbb{R}^N$.

Proposition 4.8 *Let $s > 0$, $m \in \mathbb{N}$, $p > 1$, $m < sp < N$. Assume that $\gamma_i \geq 2$, $i = 1,\ldots,m$. Then for every $f \in \dot{H}^{s,p}_\gamma(\mathbb{R}^N)$ the function $W_{sp}f$ is continuous on $\mathbb{R}^N$.*

Proof Let $f \in \dot{H}^{s,p}_\gamma(\mathbb{R}^N)$. The inequalities (2.8) and (4.1) implies

$$\sup_x W_{sp}(x)|f(x)| \leq C\|f\|_{s,p}. \tag{4.28}$$

Smooth compactly supported $SO(\gamma)$-invariant functions are dense in $\dot{H}^{s,p}_\gamma(\mathbb{R}^N)$. So there exists a sequence f_n of test $SO(\gamma)$-invariant functions convergent to f in $\dot{H}^{s,p}_\gamma(\mathbb{R}^N)$. But then by (4.28) the sequence $W(x)f_n(x)$ is uniformly convergent to $W(x)f(x)$. □

5 Optimality of the Estimate

We prove that the estimates (2.8) are optimal.

Proposition 5.1 *Let $m < sp < N$ and $s \neq d_J$ for any $J \subset \{1,\ldots,m\}\}$. There exists a constant $c > 0$ such that for all $x \in \mathbb{R}^N$ such that $0 < r_i(x) < 1$, $i = 1,\ldots,m$, there exists a smooth $SO(\gamma)$-invariant function $f \in \dot{H}^{s,p}_\gamma(\mathbb{R}^N)$, $\|f\| = 1$, such that*

$$|f(x)| \geq c\,R_J(x)^{-1/p} r_J(x)^{s-\frac{d_J}{p}} \quad \textit{if } x \in \mathcal{C}(J, s, p). \tag{5.1}$$

Proof Without loss of generality, it suffices to prove the optimality of (4.8). This means that we assume that $r_1(x) \geq \cdots \geq r_m(x) > 0$. Let n be such that $d_n < sp < d_{n-1}$ $d_k = \sum_{i=k+1}^m \gamma_i + k$. We consider a point $z \in \mathbb{R}^N$ with $r_i(z) = 2^{-j_i}$ for some $j_i \in \mathbb{N}$, $i = 1,\ldots,m$ satisfying $j_1 \leq j_2 \leq \cdots \leq j_n = j_{n+1} = \cdots = j_m$.

Let $\psi \in C_0^\infty(\mathbb{R})$, supp $\psi = [-\frac{1}{2}, \frac{1}{2}]$ and $\psi(t) = 1$ if $t \in [-\frac{1}{4}, \frac{1}{4}]$. Consider a smooth $SO(\gamma)$-symmetric function $\psi_{j_1,\ldots,j_m}$ on $\mathbb{R}^N$ defined by

$$\psi_{j_1,\ldots,j_n}(x) = \prod_{i=1}^m \psi\bigl(2^{j_n} r_i(x) - 2^{j_n - j_i}\bigr).$$

The function $\psi_{j_1,\dots,j_n}$ is supported in

$$\Omega_{j_1,\dots,j_n} = \left\{x \in \mathbb{R}^N,\ 2^{-j_i} - 2^{-j_n-1} \le r_i(x) \le 2^{-j_i} + 2^{-j_n-1},\ i = 1, \dots, m\right\},$$

and $\psi_{j_1,\dots,j_n}(z) = 1$. Moreover,

$$\left|\partial^\beta \psi_{j_1,\dots,j_n}(x)\right| \le C(|\beta|)2^{j_n|\beta|}, \quad x \in \mathbb{R}^N, \quad \beta \in \mathbb{N}^N,$$

with some $C(|\beta|) > 0$.

Let $j_0 = j_n + q$ with some $q \in \mathbb{N}$ sufficiently large to be chosen later. We take the partition of unity $\varphi_{j_0,\mathbf{k},\ell}$ subordinated to the covering $B(x_{j_0,\mathbf{k},\ell}, \alpha 2^{-j_0})_{\mathbf{k},\ell}$ constructed in Sect. 3. We recall that $\alpha > 0$ is a fixed positive integer independent of j_0. The covering is uniformly locally finite with the multiplicity constant independent of j_0. Due to the properties of the covering the partition of unity can be chosen in such a way that

$$|\partial^\beta \varphi_{j_0,\mathbf{k},\ell}(x)| \le C 2^{j_0|\beta|}$$

and the constant C depends on multi-index β but it is independent of $j_0, \mathbf{k}, \ell$. Let us now choose q such that $1 > \alpha 2^{-q+2} + 2^{-q+1}$. If $x \in B(x_{j_0,\mathbf{k},\ell}, \alpha 2^{-j_0})$ and the ball $B(x_{j_0,\mathbf{k},\ell}, \alpha 2^{-j_0})$ has a nonempty intersection with $\Omega_{j_1,\dots,j_n}$ then $r_n(x) \ge 2^{-j_0}$.

Using the partition of unity we define the following atomic decomposition of $\psi_{j_1,\dots,j_n}$:

$$\psi_{j_1,\dots,j_n}(x) = \sum_{\mathbf{k}} \sum_{\ell} c_M 2^{j_0(s-N/p)}\, a_{j_0,\mathbf{k},\ell}(x), \tag{5.2}$$

with atoms

$$a_{j_0,\mathbf{k},\ell}(x) = c_M^{-1} 2^{-j_0(s-N/p)} \varphi_{j_0,\mathbf{k},\ell}(x) \psi_{j_1,\dots,j_n}(x).$$

We can take in the expansion only the nonzero atoms. Such atoms are supported only in the balls that intersect $\Omega_{j_1,\dots,j_n}$.

Applying the atomic decomposition estimates (cf. (3.4)–(3.6)) we get

$$\begin{aligned} \|\psi_{j_1,\dots,j_n}\|_{s,p} &\le C\|\psi_{j_1,\dots,j_n}\|_{s,p,1} \le C\, 2^{j_0(\frac{sp-N}{p})} \left(\sum_{\mathbf{k}} C(j_0, \mathbf{k})\right)^{1/p} \\ &\le C 2^{j_0(\frac{sp-N}{p})} 2^{j_0 \frac{N-m}{p}} \left(\sum_{\mathbf{k}} r_1(x_{j_0,\mathbf{k},\ell})^{\gamma_1 - 1} \dots r_m(x_{j_0,\mathbf{k},\ell})^{\gamma_n - 1}\right)^{1/p}, \end{aligned} \tag{5.3}$$

where the last inequality follows from (4.12). But

$$r_i(x_{j_0,\mathbf{k},\ell}) \sim r_i(z) \quad \text{and} \quad 2^{-j_0} \sim 2^{-j_n} \sim r_n(z) \tag{5.4}$$

with the constants independent of j_0, j_n and z.

The number of elements in the sum over **k** is uniformly bounded. Therefore (5.4) implies that

$$\left(\sum_{\mathbf{k}} r_1(x_{j_0,\mathbf{k},\ell})^{\gamma_1 - 1} \dots r_m(x_{j_0,\mathbf{k},\ell})^{\gamma_n - 1} \right)^{1/p} \leq C R_n(z)^{\frac{1}{p}} r_n(z)^{\frac{d_n}{p} - m}. \tag{5.5}$$

Now by (5.3) and (5.5) we get

$$\|\psi_{j_1,\dots,j_n}\|_{s,p} \leq C r_n(z)^{\frac{d_n - sp}{p}} R_n(z)^{\frac{1}{p}}$$

and the constant $C > 0$ is independent of z. Now if we put

$$f(x) = r_n(z)^{\frac{sp - d_n}{p}} R_n(z)^{-\frac{1}{p}} \psi_{j_1,\dots,j_m}(x), \tag{5.6}$$

then

$$\|f\|_{s,p} \leq C \qquad \text{and} \qquad f(z) = r_n(z)^{\frac{sp - d_n}{p}} R_n(z)^{-\frac{1}{p}}.$$

This proves (5.1). □

6 Cocompactness and Defect of Weak Convergence

Inequalities (4.28), define a continuous embedding

$$\dot{H}^{s,p}_{\gamma}(\mathbb{R}^N) \hookrightarrow L^{\infty}(W_{sp}, \mathbb{R}^N), \quad m < sp < N, \ \gamma_i \geq 2, \tag{6.1}$$

where

$$\|f\|_{\infty, W_{sp}} = \|f\|_{L^{\infty}(W_{sp}, \mathbb{R}^N)} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^N} |f(x)| W_{sp}(x),$$

We recall that W_{sp} is a positively homogeneous function of degree $\frac{N - sp}{p}$. Embedding (6.1) is not compact, which can be verified on a sequence

$$(u_j)_{j \in \mathbb{Z}}, \ u_j(x) = g_j u := 2^{j \frac{N - sp}{p}} u_0(2^j x), \tag{6.2}$$

which leaves both norms in (6.1) constant in j, but converges weakly to zero whenever $|j| \to \infty$. On the other hand, we have compactness of the following trace embedding.

Lemma 6.1 *Let* $A_1 = \{x \in \mathbb{R}^N : \ 1 < |x| < 2\}, p > 1, \ m < sp < N, \ \gamma_i \geq 2,$ $i = 1, \dots, m$, *then for every* $s' \in (s/p, s)$ *the embedding*

$$\dot{H}^{s,p}_{\gamma}(\mathbb{R}^N) \hookrightarrow L^{\infty}(W_{s'p}, A_1) \tag{6.3}$$

is compact.

Proof Let $A_1' = \{x \in \mathbb{R}^N : \ 1/2 < |x| < 4\}$ and let $\chi \in C_0^\infty(A_1')$ be a function that equals 1 on A_1. Then for every $s' \in (s/p, s)$ the multiplication operator $(Tu)(x) = \chi(x)u(x)$ is a compact operator $T : \dot{H}_\gamma^{s,p}(\mathbb{R}^N) \to \dot{H}_\gamma^{s',p}(\mathbb{R}^N)$. Since the embedding $\dot{H}_\gamma^{s',p}(\mathbb{R}^N) \hookrightarrow L^\infty(W_{s'p}, \mathbb{R}^N)$ is continuous, T is a compact operator from $\dot{H}_\gamma^{s,p}(\mathbb{R}^N)$ to $L^\infty(W_{s'p}, \mathbb{R}^N)$, which implies that the trace embedding $\dot{H}_\gamma^{s,p}(\mathbb{R}^N) \hookrightarrow L^\infty(W_{s'p}, A_1)$ is compact. □

The following counterexample shows that the assertion of Lemma 6.1 becomes false if we replace s' with s.

Proposition 6.2 *Let $m < sp < N$ and $s \neq d_J$ for any $J \subset \{1, \ldots, m\}\}$. Let $A = \{x \in \mathbb{R}^N : \ c_1 < |x| < c_2\}$, $0 < c_1 < c_2 < \infty$. There exists a sequence $(f_k)_k$ of $SO(\gamma)$-invariant functions such that:*
(a) $(f_k)_k$ is a bounded sequence in $\dot{H}_\gamma^{s,p}(\mathbb{R}^N)$,
(b) supp $f_k \subset A$ for any k,
(c) $(f_k)_k$ does not contain a subsequence convergent in $L^\infty(W_{sp}, A)$.

Proof Let $d_n < r_n(x) < d_{n-1}$. We take a sequence

$$(x^{(k)})_k \subset A \cap \{x : r_1(x) \geq \ldots \geq r_m(x)\}$$

such that:

$$r_n(x^{(k)}) = 2^{-i_k}, \quad i_k \in \mathbb{N}, \quad i_k < i_{k+1} \quad \text{and} \quad i_k \to \infty \quad \text{if} \quad k \to \infty. \tag{6.4}$$

Let f_k be a function (5.6) constructed in the proof of Proposition5.1 with $z = x^{(k)}$. Then the sequence is bounded in $\dot{H}_\gamma^{s,p}(\mathbb{R}^N)$. But

$$\|f_{k_1} - f_{k_2}\|_{\infty, W_{sp}} = \sup_x |\psi_{k_1}(x) - \psi_{k_2}(x)| > |\psi_{k_1}(x^{(k_1)}) - \psi_{k_2}(x^{(k_1)})| = 1\,,$$

if $k_1 \neq K - 2$. Here ψ_k is the function $\psi_{j_1,\ldots,j_n}$ used in the definition of the function f_k, cf. the proof of Proposition 5.1. □

We recall the notation of cocompact embeddings. Let G be a group of bijective linear isometries of a reflexive Banach space X. We say that the sequence (u_n) in X is G-weakly convergent to 0 if $g_n u_n \rightharpoonup 0$ for any choice of the sequence $(g_n) \subset G$. A continuous embedding of a reflexive Banach space X into a normed linear space Y is called cocompact relative to the group G, if any sequence (u_n) in X, that is G-weakly convergent to 0, converges to zero in the norm of Y. Embedding (6.1) is not cocompact relative to the group of dilations (6.2). Sequence $(f_k)_k$ provided by Proposition 6.2 has the property that for any sequence of dilations $(g_{j_k})_k$ of the form (6.2), the sequence $(g_{j_k} f_k)_k$ is weakly convergent to zero without vanishing in the $L^\infty(W_{sp}, \mathbb{R}^N)$-norm. We conjecture that there is a group of bijective linear isometries on $\dot{H}_\gamma^{s,p}$ (larger than the set of dilations (6.2)) relative to which the embedding (6.1) becomes cocompact.

On the other hand, replacing W_{sp} with another homogeneous weight of the same degree $\frac{N-sp}{p}$,

$$W_{s;s'p}(x) = W_{s'p}\left(x/|x|\right)|x|^{\frac{N-sp}{p}}, \quad s' \in (s/p, s),$$

as stated in the theorem below, yields a cocompact embedding. The new weight $W_{s;s'p}$ is bounded by a constant multiple of W_{sp}, and therefore the embedding $\dot{H}^{s,p}_{\gamma}(\mathbb{R}^N) \hookrightarrow L^{\infty}(W_{s;s'p}, \mathbb{R}^N)$ is continuous. On the other hand, $W_{s;s'p}$ is equivalent to W_{sp} and to $|x|^{\frac{N-sp}{p}}$ on the set $\{x \in \mathbb{R}^N : \min\{r_1(x), \ldots, r_m(x)\} \geq \epsilon \max\{r_1(x), \ldots, r_m(x)\}\}$ with any fixed $\epsilon > 0$.

Theorem 6.3 *Assume that $p > 1$, $m < sp < N$, $\gamma_i \geq 2$, $i = 1, \ldots, m$. Then the embedding $\dot{H}^{s,p}_{\gamma}(\mathbb{R}^N) \hookrightarrow L^{\infty}(W_{s;s'p}, \mathbb{R}^N)$ is cocompact relative to the group*

$$G = \left\{ g_j \in L\big(\dot{H}^{s,p}_{\gamma}(\mathbb{R}^N), \dot{H}^{s,p}_{\gamma}(\mathbb{R}^N)\big) \ : \ g_j(f)(x) = 2^{j\frac{N-sp}{p}} f(2^j x), \quad j \in \mathbb{Z} \right\}.$$

Proof Let $(u_n)_n \subset \dot{H}^{s,p}$ and assume that for any sequence (j_n) of integers, $v_n := 2^{j_n \frac{N-sp}{p}} u_n(2^{j_n}\cdot) \rightharpoonup 0$. Assume that $x_n \in \mathbb{R}^N$ is such that that

$$\left|u_n(x_n) W_{s;s'p}(x_n)\right| \geq \frac{1}{2}\|u_n\|_{\infty, W_{s;s'p}}.$$

Let us choose (j_n) so that $2^{j_n} \leq |x| \leq 2^{j_n+1}$. Then, taking into account homogeneity of $W_{s;s'p}$, we have

$$\begin{aligned}\|u_n\|_{\infty, W_{s;s'p}} &\leq 2\left|u_n(x_n) W_{s;s'p}(x_n)\right| \\ &= 2\left|v_n(2^{-j_n}x_n) W_{s;s'p}(2^{-j_n}x_n)\right| \leq 2 \sup_{1<|y|<2} \left|v_n(y) W_{s;s'p}(y)\right| \to 0,\end{aligned}$$

since $v_n \rightharpoonup 0$, the weights $W_{s;s'p}$ and $W_{s'p}$ are equivalent on A_1, and the embedding $H^{s,p}_{\gamma}(A_1) \hookrightarrow L^{\infty}(W_{s'p}, A_1)$ is compact by Lemma 6.1. □

As a consequence we have the following structural result for bounded multiradial sequences. The theorem is an immediate consequence of profile decomposition in [18](Definition 2.5 and Theorem 2.6), taking into account that the equivalent norm of $\dot{H}^{s,p}(\mathbb{R}^N)$ defined by means of the Littlewood-Paley decomposition satisfies Opial's condition.

Theorem 6.4 *Assume that $p > 1$, $m < sp < N$, $\gamma_i \geq 2$, $i = 1, \ldots, m$. Any bounded sequence $(u_k)_{k\in\mathbb{N}}$ in $\dot{H}^{s,p}_{\gamma}(\mathbb{R}^N)$ has a renamed subsequence such that there exist sequences $(j^{(n)}_k)_{k\in\mathbb{N}}$ and functions $w^{(n)} \in \dot{H}^{1,p}_{\gamma}(\mathbb{R}^N)$, $n \in \mathbb{N}$, such that the following holds:*

$$
\begin{aligned}
&2^{-j_k^{(n)}\frac{N-sp}{p}} u_k(2^{-j_k^{(n)}}\cdot) \rightharpoonup w^{(n)},\\
&\left|j_k^{(n)} - j_k^{(m)}\right| \to \infty \text{ whenever } m \neq n,\\
&\|w^{(n)}\|_{s,p} \to 0 \text{ as } n \to \infty,\\
&u_k - \sum_{n\in\mathbb{N}} 2^{j_k^{(n)}\frac{N-sp}{p}} w^{(n)}\big(2^{j_k^{(n)}}\cdot\big) \to 0 \text{ in } L^\infty\big(W_{s;s'p}, \mathbb{R}^N\big), m/p < s' < s,
\end{aligned}
$$

and the series in the last relation converges unconditionally and uniformly with respect to k.

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