Journal of Fourier Analysis and Applications (2019) 25:2141–2149
https://doi.org/10.1007/s00041-018-09660-y

Oscillatory Singular Integral Operators with Hölder Class Kernels

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Received: 6 July 2018 / Revised: 11 November 2018 / Published online: 2 January 2019

Abstract
We establish the boundedness on $L^p(\mathbb{R}^n)$ of oscillatory singular integral operators whose kernels are the products of an oscillatory factor with bilinear phase and a Calderón–Zygmund kernel $K(x, y)$ satisfying a Hölder condition. Our results also hold on weighted L^p spaces with A_p weights.

Keywords L^p spaces · Oscillatory integrals · Singular integrals

Mathematics Subject Classification Primary 42B20 · Secondary 42B35

1 Introduction

In [6], Phong and Stein studied oscillatory singular integrals with bilinear phases as a hybrid between the Fourier transforms and singular integral operators.

Communicated by Arieh Iserles.

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Let $n \in \mathbb{N}$, $B = (b_{jk})$ be an $n \times n$ matrix with real entries. For $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, let

$$B(x, y) = xBy^T = \sum_{j=1}^{n} \sum_{k=1}^{n} b_{jk} x_j y_k.$$

For a singular kernel $K(x, y)$, the oscillatory singular integral operator T_B, acting initially on test functions, is given by

$$T_B f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iB(x,y)} K(x, y) f(y) dy. \tag{1}$$

Let's begin by recalling the following:

Theorem 1.1 *Let $A > 0$ and $\Delta = \{(x, x) : x \in \mathbb{R}^n\}$. Suppose that*

(i) *For all $(x, y) \in (\mathbb{R}^n \times \mathbb{R}^n) \backslash \Delta$,*

$$|K(x, y)| \le \frac{A}{|x - y|^n}; \tag{2}$$

(ii) *$K(x, y) \in C^1((\mathbb{R}^n \times \mathbb{R}^n) \backslash \Delta)$, and for $(x, y) \in (\mathbb{R}^n \times \mathbb{R}^n) \backslash \Delta$*

$$|\nabla_x K(x, y)| + |\nabla_y K(x, y)| \le \frac{A}{|x - y|^{n+1}}; \tag{3}$$

(iii)

$$\|T_o\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \le A, \tag{4}$$

where

$$T_o f(x) = p.v. \int_{\mathbb{R}^n} K(x, y) f(y) dy. \tag{5}$$

Then, for $1 < p < \infty$, there exists a positive C_p which may depend on p, n and A, but is independent of the matrix B, such that

$$\|T_B f\|_{L^p(\mathbb{R}^n)} \le C_p \|f\|_{L^p(\mathbb{R}^n)} \tag{6}$$

for all $f \in L^p(\mathbb{R}^n)$.

The above result first appeared in [6, p. 130] for smooth convolutional kernels K, while the fact it also holds for C^1 nonconvolutional kernels was pointed out in [7, p. 192].

Since the publication of [6], many papers have been written regarding oscillatory integrals with singular kernels, successfully extending the results of Phong and Stein to more general phase functions (see, for example, [1,3,5,7,8]). The focus of the current paper is to consider the L^p boundedness of T_B without assuming that $K(x, y)$ is C^1 away from the diagonal [or that $K(x, y)$ has a special form such as $K(x, y) = |x - y|^{-n}\Omega((x - y)/|x - y|)$]. The C^1 condition on $K(x, y)$ has been generally viewed as a key assumption due to the historically important role played by van der Corput type arguments. In our main result presented below, the C^1 assumption will be replaced by a well-known weaker condition of Hölder type on $K(x, y)$.

Theorem 1.2 *Let $A, \delta > 0$. Suppose that*

(i) *For all* $(x, y) \in (\mathbb{R}^n \times \mathbb{R}^n)\backslash\Delta$,

$$|K(x, y)| \leq \frac{A}{|x - y|^n}; \tag{7}$$

(ii)

$$|K(x, y) - K(x', y)| \leq \frac{A|x - x'|^\delta}{(|x - y| + |x' - y|)^{n+\delta}} \tag{8}$$

whenever $|x - x'| < (1/2)\max\{|x - y|, |x' - y|\}$, *and*

$$|K(x, y) - K(x, y')| \leq \frac{A|y - y'|^\delta}{(|x - y| + |x - y'|)^{n+\delta}} \tag{9}$$

whenever $|y - y'| < (1/2)\max\{|x - y|, |x - y'|\}$;

(iii)

$$\|T_o\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq A. \tag{10}$$

Then, for $1 < p < \infty$, there exists a positive C_p which may depend on p, n, δ and A, but is independent of the matrix B, such that

$$\|T_B f\|_{L^p(\mathbb{R}^n)} \leq C_p\|f\|_{L^p(\mathbb{R}^n)} \tag{11}$$

for all $f \in L^p(\mathbb{R}^n)$.

An extension of the above result to the weighted L^p spaces with A_p weights will be given in Sect. 3.

2 Proof of Theorem 1.2

For $B = (b_{jk})_{n\times n}$, let

$$b = \max\{|b_{jk}| : 1 \leq j, k \leq n\}. \tag{12}$$

If $b = 0$, then $T_B = T_o$. It is well-known that, under the conditions (7)–(10), T_o is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Thus, from this point on, we may assume that $b > 0$.

For $m \in \mathbb{N}$, $u \in \mathbb{R}^m$ and $r > 0$, let $D_m(u, r) = \{v \in \mathbb{R}^m : |v - u| < r\}$. Let ϕ be a real-valued C^∞ function on $(0, \infty)$ such that $0 \le \phi \le 1$,

$$\text{supp}(\phi) \subset \left(\frac{1}{2}, 4\right), \tag{13}$$

and

$$\sum_{\nu \in \mathbb{Z}} \phi(2^{-\nu} t) = 1 \tag{14}$$

for all $t > 0$.

For $\nu \ge 0$, define the operator S_ν by

$$S_\nu f(x) = \int_{\mathbb{R}^n} e^{iB(x,y)} K(x, y)\phi(2^{-\nu}\sqrt{b}|x - y|) f(y) dy, \tag{15}$$

and let

$$K_\nu(x, y) = \left(\frac{2^\nu}{\sqrt{b}}\right)^n K\left(\frac{2^\nu x}{\sqrt{b}}, \frac{2^\nu y}{\sqrt{b}}\right). \tag{16}$$

It is easy to see that (7)–(10) remain valid with the same constants A and δ if $K(x, y)$ is substituted by $K_\nu(x, y)$. Clearly one may also assume that $\delta < 1$.

For $f \in L^2(\mathbb{R}^n)$,

$$\begin{aligned} \|S_\nu f\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} S_\nu f(z) \overline{S_\nu f(z)} dz \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} L_\nu(x, y) f_\nu(x) \overline{f_\nu(y)} dx dy, \end{aligned} \tag{17}$$

where

$$f_\nu(x) = \left(\frac{2^\nu}{\sqrt{b}}\right)^{n/2} f\left(\frac{2^\nu x}{\sqrt{b}}\right), \tag{18}$$

$$L_\nu(x, y) = \int_{\mathbb{R}^n} e^{i(b^{-1}2^{2\nu})B(z,x-y)} K_\nu(z, x)\overline{K_\nu(z, y)}\phi(|z - x|)\phi(|z - y|)dz. \tag{19}$$

Without loss of generality, we may assume that $b = \pm b_{1k_0}$ holds for some $k_0 \in \{1, 2, \ldots, n\}$. For $x \in \mathbb{R}^n$, let $\tilde{x} = (x_2, \ldots, x_n)$, $P(x) = \sum_{k=1}^n b_{1k} x_k$, and $G_\nu(x, y, z) = K_\nu(z, x)\overline{K_\nu(z, y)}\phi(|z - x|)\phi(|z - y|)$. Then,

$$|L_\nu(x,y)| \le \chi_{D_n(0,8)}(x-y) \times \int_{D_{n-1}(\tilde{x},4)\cap D_{n-1}(\tilde{y},4)} \left| \int_{\mathbb{R}} e^{i(b^{-1}2^{2\nu})z_1 P(x-y)} G_\nu(x,y,z)\,dz_1 \right| d\tilde{z}. \tag{20}$$

Let $s \in \mathbb{R}$. For $z = (z_1, \tilde{z}) \in \mathbb{R}^n$, let $z' = (z_1 + s, \tilde{z})$. We will need the following inequality:

$$|G_\nu(x,y,z) - G_\nu(x,y,z')| \le C|s|^\delta, \tag{21}$$

where C is independent of s, ν, x, y and z. We will first verify that

$$|K_\nu(z,x)\phi(|z-x|) - K_\nu(z',x)\phi(|z'-x|)| \le C|s|^\delta \tag{22}$$

uniformly in s, ν, x and z.

When $|s| \ge 1/4$, (22) follows trivially from (7) and (13). Thus, we may now assume that $|s| < 1/4$.

The first case to be examined is when $\phi(|z-x|)$ and $\phi(|z'-x|)$ are both nonzero. Then, we have $|z-x| \ge 1/2$, $|z'-x| \ge 1/2$, $|z-z'| = |s| < (1/2)|z-x|$, and

$$|K_\nu(z',x)| \le C.$$

Therefore, it follows from (8) that

$$\begin{aligned} &|K_\nu(z,x)\phi(|z-x|) - K_\nu(z',x)\phi(|z'-x|)| \\ &\quad \le |K_\nu(z,x) - K_\nu(z',x)||\phi(|z-x|)| + |K_\nu(z',x)||\phi(|z-x|) - \phi(|z'-x|)| \\ &\quad \le C(|z-z'|^\delta + \|\phi'\|_\infty |z-z'|) \le C|s|^\delta. \end{aligned}$$

Next, if $\phi(|z-x|) \ne 0$ and $\phi(|z'-x|) = 0$, then $|z-x| \ge 1/2$ and

$$\begin{aligned} |K_\nu(z,x)\phi(|z-x|) - K_\nu(z',x)\phi(|z'-x|)| &= |K_\nu(z,x)||\phi(|z-x|) - \phi(|z'-x|)| \\ &\le C\|\phi'\|_\infty |z-z'| \le C|s|^\delta. \end{aligned}$$

Finally, the case of $\phi(|z-x|) = 0$ and $\phi(|z'-x|) \ne 0$ can be treated in the same manner as above, which completes the proof of (22).

From (22), one gets

$$|\overline{K_\nu(z,y)}\phi(|z-y|) - \overline{K_\nu(z',y)}\phi(|z'-y|)| \le C|s|^\delta \tag{23}$$

uniformly in s, ν, y and z.

By (7), (13), (22) and (23), we have

$$\begin{aligned} &|G_\nu(x,y,z) - G_\nu(x,y,z')| \\ &\quad \le |K_\nu(z,x)\phi(|z-x|) - K_\nu(z',x)\phi(|z'-x|)||\overline{K_\nu(z,y)}\phi(|z-y|)| \end{aligned}$$

$$+ |\overline{K_\nu(z,y)}\phi(|z-y|) - \overline{K_\nu(z',y)}\phi(|z'-y|)||K_\nu(z',x)\phi(|z'-x|)|$$
$$\le C|s|^\delta(|\overline{K_\nu(z,y)}\phi(|z-y|)| + |K_\nu(z',x)\phi(|z'-x|)|) \le C|s|^\delta,$$

which proves (21).

By letting $s = \pi b[2^{2\nu}P(x-y)]^{-1}$ and using (22), we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}} e^{i(b^{-1}2^{2\nu})z_1 P(x-y)} G_\nu(x,y,z)dz_1\right| \\
&\quad = (1/2)\left|\int_{\mathbb{R}} e^{i(b^{-1}2^{2\nu})z_1 P(x-y)}(G_\nu(x,y,z) - G_\nu(x,y,z'))dz_1\right| \\
&\quad \le C|s|^\delta |D_1(x_1,4)\cup D_1(x_1-s,4)| \\
&\quad \le C2^{-2\nu\delta}b^\delta \left|\sum_{k=1}^n b_{1k}(x_k-y_k)\right|^{-\delta}.
\end{aligned} \tag{24}$$

It follows from (19), (20), (24) and the proposition on p. 182 of [7] that

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}|L_\nu(x,y)|dy = \sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}|L_\nu(x,y)|dx \\
&\quad \le C2^{-2\nu\delta}b^\delta \int_{D_n(0,8)}\left|\sum_{k=1}^n b_{1k}x_k\right|^{-\delta}dx \\
&\quad \le C2^{-2\nu\delta}\left(\frac{b}{\sum_{k=1}^n |b_{1k}|}\right)^\delta \le C2^{-2\nu\delta}.
\end{aligned} \tag{25}$$

Thus, by (17), (18) and (25),

$$\|S_\nu f\|_{L^2(\mathbb{R}^n)} \le (C2^{-2\nu\delta}\|f_\nu\|^2_{L^2(\mathbb{R}^n)})^{1/2} = C2^{-\nu\delta}\|f\|_{L^2(\mathbb{R}^n)}.$$

By (7) and (13), we have

$$\|S_\nu f\|_{L^1(\mathbb{R}^n)} \le C\|f\|_{L^1(\mathbb{R}^n)}.$$

By first interpolating between L^1 and L^2 and then using a duality argument, we obtain that, for $1 < p < \infty$ and $\nu \in \mathbb{N}$,

$$\|S_\nu f\|_{L^p(\mathbb{R}^n)} \le C2^{-\nu\delta_p}\|f\|_{L^p(\mathbb{R}^n)}, \tag{26}$$

where δ_p is the positive number given by

$$\delta_p = \begin{cases} \frac{2\delta(p-1)}{p} & \text{if} \quad 1 < p \le 2, \\ \frac{2\delta}{p} & \text{if} \quad 2 < p < \infty. \end{cases}$$

Let

$$\psi(t) = \sum_{\nu=-\infty}^{0} \phi(2^{-\nu} t),$$

and

$$\tilde{S} f(x) = \int_{\mathbb{R}^n} e^{iB(x,y)} K(x, y) \psi(\sqrt{b}|x - y|) f(y) dy. \tag{27}$$

Since

$$\operatorname{supp}(\psi) \subseteq [0, 4],$$

the localization technique described on pp. 118–119 of [6] can be used to get

$$\|\tilde{S} f\|_{L^p(\mathbb{R}^n)} \le C_p \|f\|_{L^p(\mathbb{R}^n)} \tag{28}$$

for $1 < p < \infty$. Since the argument, after proper scaling, uses the size condition (2) but not the smoothness condition (3), we will omit the details of the proof of (28).

It follows from (14), (26) and (28) that

$$\begin{aligned} \|T_B f\|_{L^p(\mathbb{R}^n)} &\le \|\tilde{S} f\|_{L^p(\mathbb{R}^n)} + \sum_{\nu=1}^{\infty} \|S_\nu f\|_{L^p(\mathbb{R}^n)} \\ &\le C_p \left(1 + \sum_{\nu=1}^{\infty} 2^{-\nu\delta_p}\right) \|f\|_{L^p(\mathbb{R}^n)} = C_p \|f\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Theorem 1.2 is proved.

3 Weighted L^p Spaces

Theorem 1.2 can be extended to the weighted L^p spaces with the Muckenhoupt A_p weights. First, let us recall the definition of A_p weights for $1 < p < \infty$. Let $w(\cdot)$ be a nonnegative, locally integrable function on $\mathbb{R}^n$.

Definition 3.1 For $1 < p < \infty$, w is said to be in the Muckenhoupt weight class $A_p(\mathbb{R}^n)$ if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|Q|} \int_Q w(y) dy\right) \left(\frac{1}{|Q|} \int_Q w(y)^{-1/(p-1)} dy\right)^{p-1} \le C \tag{29}$$

holds for all cubes Q in $\mathbb{R}^n$. The smallest such constant C in (29) is the corresponding A_p constant of w.

We recall that $A_{p_1}(\mathbb{R}^n) \subset A_{p_2}(\mathbb{R}^n)$ when $p_1 < p_2$ and

Lemma 3.1 [2] *Let $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$. Then there exists a $\theta \in (0, 1)$ such that $w^{1+\theta} \in A_p(\mathbb{R}^n)$. Both θ and the A_p constant of $w^{1+\theta}$ depend on n, p and the A_p constant of w only.*

Let $L^p_w(\mathbb{R}^n) = \left\{ f : \int_{\mathbb{R}^n} |f(x)|^p w(x)dx < \infty \right\}$ and

$$\|f\|_{L^p_w(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx \right)^{1/p}.$$

Then, we have the following:

Theorem 3.1 *Let the operator T_B be given as in Theorem 1.2, $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$. Then there exists a positive $C_{p,w}$ which may depend on p, n, δ, A and the A_p constant of w, but is independent of the matrix B, such that*

$$\|T_B f\|_{L^p_w(\mathbb{R}^n)} \leq C_{p,w} \|f\|_{L^p_w(\mathbb{R}^n)} \tag{30}$$

for all $f \in L^p_w(\mathbb{R}^n)$.

We will end the paper with a brief description of the proof of Theorem 3.1.

First, a weighted version of (28) follows from the L^p_w boundedness of T_o (see [4, p. 712]) and an application of the localization technique mentioned earlier. Essentially all one needs now is to find a weighted analogue of (26), which can be done as follows.

By Lemma 3.1, there exists a $\theta > 0$ such that $w^{1+\theta} \in A_p(\mathbb{R}^n)$. By (7), (13) and (15), we have

$$|S_\nu f(x)| \leq C M_{HL} f(x), \tag{31}$$

where M_{HL} denotes the Hardy–Littlewood maximal operator. Thus,

$$\|S_\nu f\|_{L^p_{w^{1+\theta}}(\mathbb{R}^n)} \leq C_{p,w} \|f\|_{L^p_{w^{1+\theta}}(\mathbb{R}^n)}. \tag{32}$$

By using (26), (32) and an interpolation with change of measures (see [9]), one obtains that

$$\|S_\nu f\|_{L^p_w(\mathbb{R}^n)} \leq C_{p,w} 2^{-\nu\delta_p\theta/(1+\theta)} \|f\|_{L^p_w(\mathbb{R}^n)}$$

for all $\nu \in \mathbb{N}$. The rest of the details are omitted.

Acknowledgements We thank the referees for their helpful comments.

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