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On Pointwise Convergence for Schrödinger Operator in a Convex Domain

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Abstract
In this paper, we prove that the maximal inequality

$$\Big\| \sup_{|t|<1} |e^{it\Delta_D} f(x,y)| \Big\|_{L^2_{\text{loc}}(\Omega)} \leq C\|f\|_{H^s_D(\Omega)}, \quad \forall\, f \in H^s_D(\Omega)$$

holds for any $s > \frac{1}{2}$ with $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x > 0\}$ and $\Delta_D = \partial_x^2 + (1+x)\partial_y^2$. As a direct application, we obtain the pointwise convergence for the free Schrödinger equation $i\partial_t u + \Delta_D u = 0$ with initial data $u(0) = f$ inside strictly convex domain.

Keywords Schrödinger operator · Pointwise convergence · Airy function

Mathematics Subject Classification 35Q55 · 33C10 · 42B25

1 Introduction

Let Ω be the upper right plane $\{(x,y) \in \mathbb{R}^2 \mid x > 0\}$. Define the Laplacian on Ω to be $\Delta_D = \partial_x^2 + (1+x)\partial_y^2$, together with Dirichlet boundary conditions on $\partial\Omega$: one may easily see that Ω, with the metric inherited from Δ_D, is a strictly convex domain, we refer the reader to [17,18] about the dispersive estimate and Strichartz estimates for wave equation in such convex domain. In this paper, we study the following local maximal inequality

$$\Big\| \sup_{|t|<1} |S(t)f(x,y)| \Big\|_{L^2_{\text{loc}}(\Omega)} \leq C\|f\|_{H^s_D(\Omega)}, \quad \forall\, f \in H^s_D(\Omega), \tag{1.1}$$

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where $u(t, x, y) := S(t)f(x, y)$ solves

$$\begin{cases} i\partial_t u + \Delta_D u = 0, & (t, x, y) \in \mathbb{R} \times \Omega \\ u(t, x, y) = 0, & (x, y) \in \partial\Omega, \\ u(0, x, y) = f(x, y), \end{cases} \tag{1.2}$$

and we define $H^s_D(\Omega)$ as the completion of $C^\infty_c(\Omega)$ equipped with norm

$$\|f\|^2_{H^s_D(\Omega)} := \|f\|^2_{L^2(\Omega)} + \left\|(-\Delta_D)^{\frac{s}{2}} f\right\|^2_{L^2(\Omega)}. \tag{1.3}$$

Using the standard process of approximation (see Corollary 1.2 below), we obtain the point-wise convergence by (1.1)

$$\lim_{t\to 0} S(t)f(x, y) = f(x, y), \text{ a.e. } (x, y) \in \Omega, \quad \forall f \in H^s_D(\Omega), \tag{1.4}$$

We easily check that (1.1) is valid for $s > 1$ by Sobolev embedding [1]: $H^s_D(\Omega) \hookrightarrow L^\infty(\Omega)$. In this paper, we want to look for the minimal s to ensure (1.1). The impetus to consider this problem stems from a series of recent works of the dispersive operators, including the Schrödinger operator and the wave operator in the flat space $\mathbb{R}^d$, since Carleson [8] on the Schrödinger operator. We also refer to [4–6,8–11,19,23–25,27–34].

To be more precise, let us recall the results for maximal operators associated to the Schrödinger equation in the flat space $\mathbb{R}^d$

$$\Big\| \sup_{|t|<1} |e^{-it\Delta} f(x)| \Big\|_{L^2(B(0,1))} \le C\|f(x)\|_{H^s(\mathbb{R}^d)}, \tag{1.5}$$

where $B(0, 1) \subset \mathbb{R}^d$ is the unit ball centered at zero. Carleson first raised such problem in [8], where he answered the $d = 1$ case with $s \ge \frac{1}{4}$. This result was shown to be optimal by Dahlberg and Kenig [11]. In dimension $d \ge 2$, Sjölin [29] and Vega [33] established (1.5) with $s > \frac{1}{2}$ independently. In particular, the result can be strengthened to $s = \frac{1}{2}$ by Sjölin [29] when $d = 2$. Meanwhile, Vega [33] gave a counterexample to show that (1.5) fails if $s < \frac{1}{4}$.

The maximal inequality (1.5) for dimension two is closely related to the development of the Fourier restriction theory. The first breakthrough for $s < \frac{1}{2}$ was achieved by Bourgain [4,5], where he proved that there exists $s < \frac{1}{2}$ such that (1.5) holds true. Thereafter, Moyua–Vargas–Vega [25] further developed Tomas-Stein $X_{p,4}$-space to obtain that (1.5) holds if $s > s_0$ for some $s_0 \in \left(\frac{20}{41}, \frac{40}{81}\right)$. By making use of the bilinear Restriction estimate for paraboloid, Tao–Vargas [32] and Tao [31] improved the result to $s > \frac{15}{32}$ and $s > \frac{2}{5}$ respectively. Recently, observing the localization properties of Schrödinger waves, Lee [19] obtained the result for $s > \frac{3}{8}$. Shao [30] gave an alternative proof by using the method of stationary phase and wave packet decomposition. In [7], Bourgain gave a counterexample to show that $s \ge \frac{1}{3}$ is necessary for (1.5) with $d = 2$. By using polynomial partitioning and decoupling method [16], Du–Guth–Li [13] got the result for $s > \frac{1}{3}$, which is sharp up to the endpoint $s = \frac{1}{3}$.

Previous to [6], the results about $d \geq 3$ remained $s > \frac{1}{2}$, and $s \geq \frac{1}{4}$ was still believed to be the correct condition for (1.5) in every dimension. The study on this problem stagnated for several years until the recent work [6], where the $\frac{1}{2}-$barrier was broken for *all* dimensions. More precisely, Bourgain [6] proved that (1.5) holds if $s > \frac{1}{2} - \frac{1}{4d}$. More surprisingly, Bourgain also discovered some counterexamples to disprove the widely believed assertion on the $\frac{1}{4}-$threshold. Specifically, he showed that $s \geq \frac{1}{2} - \frac{1}{d}$ is necessary for (1.5) if $d \geq 5$. These examples originated essentially from an observation on arithmetical progressions. Recently, R. Luca and M. Rogers [20] showed that $s \geq \frac{1}{2} - \frac{1}{d+2}$ is necessary for (1.5) if $d \geq 3$. More recently, Bourgain [7] gave a counterexample to see that $s < \frac{1}{2} - \frac{1}{d+2}$ is necessary for $d \geq 3$. Up to the endpoint, Du–Zhang [14] proved the sharp result for (1.5) in higher dimensions $d \geq 3$. We remark that their result [14] also gives improved results on the size of divergence set of Schrödinger solutions, the Falconer distance set problem and the spherical average Fourier decay rates of fractal measures.

Pointwise convergence

			Counterexample
$d = 1$	$s \geq \frac{1}{4}$	Carleson [8]	Dahlberg–Kenig [11]$s < \frac{1}{4}$
$d = 2$	$s = \frac{3}{8}+$	Bouragin [7], Lee [19], Shao [30]	
	$s > \frac{1}{3}$	Du–Guth–Li [13]	Bourgain [7]$s < \frac{1}{3}$
$d \geq 3$	$s > \frac{1}{2}$	Sjölin [29], Vega [33]	
	$s > \frac{1}{2} - \frac{1}{4d}$	Bourgain [6]	$s < \frac{1}{2} - \frac{1}{d}, d \geq 5$
		Luca-Rogers [20,21], Demeter–Guo [12]	$s < \frac{1}{2} - \frac{1}{d+2}, d \geq 3$
		Bourgain [7]	$s < \frac{1}{2} - \frac{1}{2(d+1)}$
	$s > \frac{1}{2} - \frac{1}{2(d+1)}$	Du–Zhang [14]	
$d \geq 1$	$s \geq \frac{1}{4}$	Gigante–Soria [15]	Radial initial data

Now, we list our main results.

Theorem 1.1 *Let* $s > \frac{1}{2}$. *There holds*

$$\Big\| \sup_{|t|<1} |S(t)f| \Big\|_{L^2(B((x_0,y_0),1)\cap\Omega)} \leq C\|f\|_{H^s_D(\Omega)}, \quad \forall\, f \in H^s_D(\Omega) \tag{1.6}$$

for any $(x_0, y_0) \in \Omega$, *where* $B((x_0, y_0), 1)$ *is the unit ball centered at* (x_0, y_0), *with the metric inherited from* Δ_D.

As a consequence of Theorem 1.1, we obtain the point-wise convergence result.

Corollary 1.2 *Let* $s > \frac{1}{2}$. *Then, we have*

$$\lim_{t\to 0} S(t)f(x, y) = f(x, y), \ \ a.e.\ (x, y) \in \Omega, \tag{1.7}$$

for any function $f \in H^s_D(\Omega)$.

Theorem 1.3 *The convergence property* (1.4) *requires* $s \geq \frac{1}{4}$.

Remark 1.4 As I know, this is the first result to consider the point-wise convergence result in convex domain. In the future, we will try to utilize the polynomial partitioning and decoupling method to improve the result in Theorem 1.1, and describe the size of divergence set of Schrödinger solutions in convex domain.

We conclude this section by giving some notations which will be used throughout this paper. To simplify the expression of our inequalities, we introduce some symbols $\lesssim, \sim$. If X, Y are nonnegative quantities, we use $X \lesssim Y$ or $X = O(Y)$ to denote the estimate $X \leq CY$ for some absolute constant C, and $X \sim Y$ to denote the estimate $X \lesssim Y \lesssim X$.

2 Preliminaries

2.1 Airy Function

First, we recall a few well-known facts about Airy functions. For $z \in \mathbb{C}$, $\mathrm{Ai}(z)$ is defined by

$$\mathrm{Ai}(z) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(s^3/3 - sz)}\, ds = \frac{1}{2\pi} \int_{\mathbb{R}} \cos\left(\frac{s^3}{3} - sz\right) ds. \tag{2.1}$$

This integral is not absolutely convergent, but is well-defined as the Fourier transform of a temperate distribution. And it is easy to see that $\mathrm{Ai}(z)$ satisfies the Airy equation

$$\mathrm{Ai}''(z) - z\mathrm{Ai}(z) = 0. \tag{2.2}$$

For positive $z > 0$, as $z \to \infty$, we have $\mathrm{Ai}(z) = O(z^{-\infty})$, while for negative values

$$\mathrm{Ai}(-z) = e^{-i\frac{\pi}{3}} \mathrm{Ai}(e^{-i\frac{\pi}{3}} z) + e^{i\frac{\pi}{3}} \mathrm{Ai}(e^{i\frac{\pi}{3}} z) =: A_+(z) + A_-(z).$$

Notice that $A_-(z) = \overline{A_+(\bar{z})}$. We also have asymptotic expansion (e.g. [26])

$$A_-(z) = \frac{1}{2\sqrt{\pi} z^{\frac{1}{4}}} e^{i\frac{\pi}{4}} e^{-\frac{2}{3} i z^{\frac{3}{2}}} e^{\Upsilon(z^{\frac{3}{2}})} = z^{-\frac{1}{4}} e^{i\frac{\pi}{4}} e^{-\frac{2}{3} i z^{\frac{3}{2}}} \Psi_-(z), \tag{2.3}$$

with $e^{\Upsilon(z^{\frac{3}{2}})} \sim \left(1 + \sum_{k \geq 1} c_k z^{-\frac{3k}{2}}\right) \sim 2\sqrt{\pi}\Psi_-(z)$ as $z \to +\infty$, and the corresponding expansion for A_+, where we define $\Psi_+(z) := \overline{\Psi_-(\bar{z})}$.

Next, we recall some basic properties of $Ai(z)$.

Proposition 2.1 [22] *All the zeros of* $\mathrm{Ai}(z)$ *are real and negative, say*

$$\mathrm{Ai}(-w_k) = 0, \quad -1 > -w_1 > -w_2 > \cdots \to -\infty. \tag{2.4}$$

Moreover,

$$w_k = \left(\tfrac{3}{2}\big(k-\tfrac{1}{4}\big)\pi\right)^{\frac{2}{3}} \simeq k^{\frac{2}{3}}, \quad \forall\, k \geq 1. \tag{2.5}$$

Lemma 2.2 [17] *There exists C_0 such that for $L \geq 1$, the following holds true:*

$$\sup_{b\in\mathbb{R}} \left(\sum_{1\leq k\leq L} k^{-\frac{1}{3}} |\mathrm{Ai}(b-w_k)|^2 \right) \leq C_0 L^{\frac{1}{3}}. \tag{2.6}$$

2.2 Eigenfunctions and Sobolev Spaces

In this subsection, we recall some basic properties of Sobolev spaces in the Friedlander model case as in [17,18]. Let $\Omega := \{(x, y) \in \mathbb{R}^2 \mid x > 0,\ y \in \mathbb{R}\}$ denote the half-space $\mathbb{R}^2_+ := \mathbb{R}^+ \times \mathbb{R}$ with the Laplacian given by $\Delta_D = \partial_x^2 + (1+x)\partial_y^2$ with Dirichlet boundary condition on $\partial\Omega$. Taking the Fourier transformation in the $y-$variable gives

$$-\Delta_{D,\eta} = -\partial_x^2 + (1+x)\eta^2.$$

For $\eta \neq 0,\ -\Delta_{D,\eta}$ is a self-adjoint, positive operator on $L^2([0,\infty))$ with compact resolvent. In fact, the potential $V(x,\eta) = (1+x)\eta^2$ is bounded from below, it is continuous and $\lim_{x\to\infty} V(x,\eta) = \infty$. Thus, we can consider the form associated with $-\partial_x^2 + V(x,\eta)$,

$$\begin{aligned} Q(v) &= \int_0^\infty \big(|\partial_x v|^2 + V(x,\eta)|v|^2\big)dx, \\ D(Q) &= H_0^1(\mathbb{R}_+) \cap \big\{v \in L^2(\mathbb{R}_+),\, (1+x)^{1/2} v \in L^2(\mathbb{R}_+)\big\}, \end{aligned}$$

which is clearly symmetric, closed and bounded from below by a positive constant c. If $c \gg 1$ is chosen such that $-\Delta_{D,\eta} + c$ is invertible, then $(-\Delta_{D,\eta} + c)^{-1}$ sends $L^2([0,\infty))$ in $D(Q)$ and we deduce that $(-\Delta_{D,\eta} + c)^{-1}$ is also a self-adjoint and compact operator. The last assertion follows from the compact inclusion

$$D(Q) = \big\{v |\ \partial_x v,\ (1+x)^{1/2} v \in L^2([0,\infty)),\, v(0) = 0\big\} \hookrightarrow\ L^2([0,\infty)).$$

Thus, we derive from classical spectral theory that there exists a base of eigenfunctions v_k of $-\Delta_{D,\eta}$ associated to a sequence of eigenvalues $\lambda_k(\eta) \to \infty$. From $-\Delta_{D,\eta} v = \lambda v$, we get

$$\begin{cases} \partial_x^2 v = (\eta^2 - \lambda + x\eta^2)v, \\ v(0,\eta) = 0. \end{cases}$$

and after a suitable change of variables, we find that an orthonormal basis of $L^2([0, +\infty))$ is given by eigenfunctions

$$e_k(x, \eta) = f_k \frac{\eta^{1/3}}{k^{1/6}} \mathrm{Ai}(\eta^{\frac{2}{3}} x - w_k) \tag{2.7}$$

where f_k are constants so that

$$\|e_k(\cdot, \eta)\|_{L^2((0,\infty))} = 1,$$

and e_k satisfies

$$-\Delta_{D,\eta} e_k(x, \eta) = \lambda_k(\eta) e_k(x, \eta), \quad \lambda_k(\eta) = \eta^2 + w_k \eta^{\frac{4}{3}}, \tag{2.8}$$

with $\{w_k\}_k$ being the zeros of Airy's function in decreasing order, see (2.5).

Remark 2.3 (1) As Remark 3.1 in [17], if we denote $\delta_{x=a}$ to be the Dirac distribution on $\mathbb{R}^+$, $a > 0$, then it reads as follows

$$\delta_{x=a} = \sum_{k \geq 1} e_k(x, \eta) e_k(a, \eta). \tag{2.9}$$

(2) f_k as in (2.7) has uniform upper bound and lower bound with respect to k. Indeed, we get by $\|e_k(x, \eta)\|^2_{L^2_x(R^+)} = 1$

$$f_k^2 \frac{\eta^{2/3}}{k^{1/3}} \int_0^\infty |\mathrm{Ai}(\eta^{\frac{2}{3}} x - w_k)|^2 dx = 1.$$

By scaling, we have

$$\frac{f_k^2}{k^{1/3}} \int_0^\infty |\mathrm{Ai}(x - w_k)|^2 dx = 1.$$

Hence,

$$f_k^2 = k^{\frac{1}{3}} \Big(\int_0^\infty |\mathrm{Ai}(x - w_k)|^2 dx \Big)^{-1}.$$

We are reduced to show that there exist constants C_1 and C_2 such that

$$C_1 k^{\frac{1}{3}} \leq \int_0^\infty |\mathrm{Ai}(x - w_k)|^2 \, dx \leq C_2 k^{\frac{1}{3}}. \tag{2.10}$$

Using the asymptotic behavior of Airy function: $\mathrm{Ai}(-z) = O(z^{-\frac{1}{4}})$ as $z \to +\infty$, and $w_k \simeq k^{\frac{2}{3}}$, we obtain for k sufficiently large

$$\int_0^{w_k - w_k^{\frac{1}{3}}} |\mathrm{Ai}(x - w_k)|^2 \, dx \simeq \int_0^{w_k - w_k^{\frac{1}{3}}} (w_k - x)^{-\frac{1}{2}} dx$$
$$\simeq 2\sqrt{w_k} - 2 \simeq \sqrt{w_k} \simeq k^{\frac{1}{3}},$$

By $|\mathrm{Ai}(z)| \le C(1 + |z|)^{-\frac{1}{4}}$, we estimate

$$\int_{w_k - w_k^{\frac{1}{3}}}^{w_k + w_k^{\frac{1}{3}}} |\mathrm{Ai}(x - w_k)|^2 \, dx \lesssim w_k^{\frac{1}{3}} \lesssim k^{\frac{2}{9}}.$$

On the other hand, by the asymptotic behavior of Airy function: $\mathrm{Ai}(z) = O(z^{-\infty})$ as $z \to +\infty$, we get

$$\int_{w_k + w_k^{\frac{1}{3}}}^{\infty} |\mathrm{Ai}(x - w_k)|^2 \, dx \lesssim \int_{w_k^{\frac{1}{3}}}^{\infty} x^{-6} \, dx \lesssim 1.$$

And so (2.10) follows.

(3) We have by (2.6)

$$\sup_{x \in \mathbb{R}^+} \Big(\sum_{1 \le k \le L} |e_k(x, \eta)|^2 \Big) \le C_0 \eta^{\frac{2}{3}} L^{\frac{1}{3}}. \tag{2.11}$$

For each function $f(x, y) \in L^2(\Omega)$, taking the Fourier transformation in the y-variable, and using the fact that $\{e_k(x, \eta)\}_{k \ge 1}$ forms an orthonormal basis of $L^2([0, +\infty))$, we have the expansion formula

$$\hat{f}(x, \eta) = \sum_{k=1}^{\infty} \hat{f}_k(\eta) e_k(x, \eta), \quad \hat{f}_k(\eta) = \int_0^{\infty} \hat{f}(x, \eta) e_k(x, \eta) \, dx.$$

By orthogonality, it gives

$$\|\hat{f}(x, \eta)\|_{L^2_x(\mathbb{R}^+)} = \Big(\sum_{k=1}^{\infty} |\hat{f}_k(\eta)|^2 \Big)^{\frac{1}{2}}.$$

Therefore, we have by Plancherel theorem

$$\|f(x, y)\|_{L^2(\Omega)} = \|\hat{f}(x, \eta)\|_{L^2(\mathbb{R}^+ \times \mathbb{R})} = \Big(\sum_{k=1}^{\infty} \|\hat{f}_k(\eta)\|^2_{L^2(\mathbb{R})} \Big)^{\frac{1}{2}}. \tag{2.12}$$

Similarly,

$$\|f(x,y)\|_{\dot{H}^s_D(\Omega)} = \Big(\sum_{k=1}^{\infty} \big\|\lambda_k(\eta)^{\frac{s}{2}} \hat{f}_k(\eta)\big\|^2_{L^2(\mathbb{R})}\Big)^{\frac{1}{2}}. \tag{2.13}$$

3 Proof of Main Theorem

3.1 Proof of Theorem 1.1

First, we recall the dyadic partition of unity, see Proposition 2.10 in Bahouri-Chemin-Danchin [3].

Proposition 3.1 (Dyadic partition of unity, [3]) *Let $\mathcal{C}$ be the annulus $\big\{\xi \in \mathbb{R} : \frac{3}{4} \le |\xi| \le \frac{8}{3}\big\}$. There exist even functions φ and ψ, valued in the interval $[0,1]$, and $\mathrm{supp}\varphi \subset B(0,4/3)$, $\mathrm{supp}\psi \subset \mathcal{C}$, and such that*

$$\varphi(\xi) + \sum_{j\ge 0} \psi(2^{-j}\xi) = 1, \qquad \forall\, \xi \in \mathbb{R}. \tag{3.1}$$

Applying the dyadic partition of unity (3.1), we estimate

$$\begin{aligned} &\Big\| \sup_{|t|<1} |S(t)f| \Big\|_{L^2(B((x_0,y_0),1)\bigcap\Omega)} \\ &\quad \le \Big\| \sup_{|t|<1} |S(t)\varphi(\sqrt{-\Delta_D})f| \Big\|_{L^2(B((x_0,y_0),1)\bigcap\Omega)} \\ &\quad + \sum_{j=0}^{\infty} \Big\| \sup_{|t|<1} |S(t)\psi(2^{-j}\sqrt{-\Delta_D})f| \Big\|_{L^2(B((x_0,y_0),1)\bigcap\Omega)}. \end{aligned} \tag{3.2}$$

Then, we are reduced to show

$$\|S(t)\varphi(\sqrt{-\Delta_D})f\|_{L^\infty_{t,x,y}} \le C\|f\|_{L^2(\Omega)}, \tag{3.3}$$

$$\Big\| \sup_{|t|<1} |S(t)\psi(2^{-j}\sqrt{-\Delta_D})f| \Big\|_{L^2(B((x_0,y_0),1)\bigcap\Omega)} \le C2^{\frac{j}{2}} \big\|\psi(2^{-j}\sqrt{-\Delta_D})f\big\|_{L^2(\Omega)}. \tag{3.4}$$

Indeed, plugging (3.3) and (3.4) into (3.2), and using the Cauchy–Schwartz inequality, we obtain for any $s > \frac{1}{2}$

$$
\begin{aligned}
&\Big\| \sup_{|t|<1} |S(t)f| \Big\|_{L^2(B((x_0,y_0),1)\bigcap\Omega)} \\
&\lesssim \|f\|_{L^2(\Omega)} + \Big(\sum_{j=0}^{\infty} 2^j \|\psi(2^{-j}\sqrt{-\Delta_D})f\|^2_{L^2(\Omega)} \Big)^{\frac12} \\
&\lesssim \|f\|_{L^2(\Omega)} + \Big(\sum_{j=0}^{\infty} 2^j \sum_{k=1}^{\infty} \big\|\psi\big(2^{-j}\sqrt{\lambda_k(\eta)}\big)\hat{f}_k(\eta)\big\|^2_{L^2(\mathbb{R})} \Big)^{\frac12} \\
&\lesssim \|f\|_{L^2(\Omega)} + \Big(\sum_{k=1}^{\infty} \big\|\lambda_k(\eta)^{\frac{s}{2}}\hat{f}_k(\eta)\big\|^2_{L^2(\mathbb{R})} \Big)^{\frac12} \\
&\lesssim \|f\|_{H^s_D(\Omega)}.
\end{aligned}
$$

First, we consider the contribution from the lower frequency term, i.e. (3.3). Taking the Fourier transformation in the y-variable, we have

$$
\varphi(\sqrt{-\Delta_{D,\eta}})\hat{f}(x,\eta) = \sum_{k=1}^{\infty} \varphi\big(\sqrt{\lambda_k(\eta)}\big)\hat{f}_k(\eta)e_k(x,\eta).
$$

By the support property of φ, we know that $|\eta| \lesssim 1$. Then, we obtain by Bernstein's inequality in y-variable and (2.11) with $L \sim |\eta|^{-2}$

$$
\begin{aligned}
\|S(t)\varphi(\sqrt{-\Delta_D})f\|_{L^\infty_y(\mathbb{R})} &\lesssim \|S(t)\varphi(\sqrt{-\Delta_D})f\|_{L^2_y(\mathbb{R})} \\
&\lesssim \big\|e^{it\Delta_{D,\eta}}\varphi\big(\sqrt{-\Delta_{D,\eta}}\big)\hat{f}(x,\eta)\big\|_{L^2_\eta(\mathbb{R})} \\
&\lesssim \Big\| \sum_{k=1}^{\infty} e^{-it\lambda_k(\eta)}\varphi\big(\sqrt{\lambda_k(\eta)}\big)\hat{f}_k(\eta)e_k(x,\eta) \Big\|_{L^2_\eta(\mathbb{R})} \\
&\lesssim \Big\| \Big(\sum_{k=1}^{\infty} |\hat{f}_k(\eta)|^2 \Big)^{\frac12} \Big(\sum_{k=1}^{|\eta|^{-2}} |e_k(x,\eta)|^2 \Big)^{\frac12} \Big\|_{L^2_\eta(\mathbb{R})} \\
&\lesssim \Big(\sum_{k=1}^{\infty} \|\hat{f}_k(\eta)\|^2_{L^2(\mathbb{R})} \Big)^{\frac12} \\
&\lesssim \|f\|_{L^2(\Omega)}.
\end{aligned}
$$

And so (3.3) follows.

Next, we turn to prove (3.4). Notice that

$$
\begin{aligned}
&S(t)\psi(2^{-j}\sqrt{-\Delta_D})f(x,y) \\
&= \int_{\mathbb{R}} e^{iy\eta}e^{it\Delta_{D,\eta}}\psi\big(2^{-j}\sqrt{-\Delta_{D,\eta}}\big)\hat{f}(x,\eta)\,d\eta \\
&= \int_{\mathbb{R}} e^{iy\eta}\sum_{k=1}^{\infty} e^{-it\lambda_k(\eta)}\psi(2^{-j}\sqrt{\lambda_k(\eta)})\hat{f}_k(\eta)e_k(x,\eta)\,d\eta
\end{aligned}
$$

$$= 2\int_0^\infty \cos(y\rho)\sum_{k=1}^\infty e^{-it\lambda_k(\rho)}\psi(2^{-j}\sqrt{\lambda_k(\rho)})\hat{f}_k(\rho)e_k(x,\rho)\,d\rho$$
$$= \int_0^\infty e^{-it\tau}\Big[2\sum_{k=1}^\infty \cos\big(y\rho(\tau,k)\big)\psi(2^{-j}\sqrt{\tau})\hat{f}_k\big(\rho(\tau,k)\big)\frac{\partial\rho(\tau,k)}{\partial\tau}e_k\big(x,\rho(\tau,k)\big)\Big]\,d\tau$$
$$= \mathcal{F}^{-1}_{t\mapsto\tau}\Big[2\chi_{[0,\infty)}(\tau)\sum_{k=1}^\infty \cos\big(y\rho(\tau,k)\big)\psi(2^{-j}\sqrt{\tau})\hat{f}_k\big(\rho(\tau,k)\big)\frac{\partial\rho(\tau,k)}{\partial\tau}e_k\big(x,\rho(\tau,k)\big)\Big](t),$$

where we use a change of variables: $\tau = \lambda_k(\rho) = \rho^2 + w_k\rho^{\frac{4}{3}} \sim 2^{2j}$. We utilize Bernstein's inequality and Plancherel theorem in time t to get

$$\sup_{|t|<1}\big|S(t)\psi(2^{-j}\sqrt{-\Delta_D})f(x,y)\big|^2$$
$$\le 2^{2j}\|S(t)\psi(2^{-j}\sqrt{-\Delta_D})f(x,y)\|^2_{L^2_t(\mathbb{R})}$$
$$\le 2^{2j+1}\int_0^\infty\Big|\sum_{k=1}^\infty \cos\big(y\rho(\tau,k)\big)\psi(2^{-j}\sqrt{\tau})\hat{f}_k\big(\rho(\tau,k)\big)\frac{\partial\rho(\tau,k)}{\partial\tau}e_k\big(x,\rho(\tau,k)\big)\Big|^2\,d\tau.$$

By orthogonality, it gives

$$\Big\|\sup_{|t|<1}\big|S(t)\psi(2^{-j}\sqrt{-\Delta_D})f(x,y)\big|\Big\|^2_{L^2_x(\mathbb{R}^+)}$$
$$\le 2^{2j+1}\int_0^\infty\int_0^\infty\Big|\sum_{k=1}^\infty \cos\big(y\rho(\tau,k)\big)\psi(2^{-j}\sqrt{\tau})\hat{f}_k\big(\rho(\tau,k)\big)$$
$$\times\frac{\partial\rho(\tau,k)}{\partial\tau}e_k\big(x,\rho(\tau,k)\big)\Big|^2\,dx\,d\tau$$
$$\lesssim 2^{2j}\sum_{k=1}^\infty\int_0^\infty|\psi(2^{-j}\sqrt{\tau})|^2\big|\hat{f}_k\big(\rho(\tau,k)\big)\big|^2\Big|\frac{\partial\rho(\tau,k)}{\partial\tau}\Big|^2\,d\tau$$
$$\lesssim 2^{2j}\sum_{k=1}^\infty\int_\rho|\psi(2^{-j}\sqrt{\lambda_k(\rho)})|^2\big|\hat{f}_k(\rho)\big|^2\Big|\frac{\partial\rho(\tau,k)}{\partial\tau}\Big|\,d\rho$$
$$\lesssim 2^{2j}\sum_{k=1}^\infty\int_\rho|\psi(2^{-j}\sqrt{\lambda_k(\rho)})|^2\big|\hat{f}_k(\rho)\big|^2\frac{1}{2\rho+\frac{4}{3}w_k\rho^{\frac{1}{3}}}\,d\rho$$
$$\lesssim 2^{j}\sum_{k=1}^\infty\big\|\psi(2^{-j}\sqrt{\lambda_k(\eta)})\hat{f}_k(\eta)\big\|^2_{L^2(\mathbb{R})}$$
$$\lesssim 2^{j}\big\|\psi(2^{-j}\sqrt{-\Delta_D})f\big\|^2_{L^2(\Omega)},$$

where we use the changing variable $\tau = \lambda_k(\rho) = \rho^2 + w_k\rho^{\frac{4}{3}}$ and

$$\frac{1}{2\rho+\frac{4}{3}w_k\rho^{\frac{1}{3}}} \lesssim \frac{\rho}{\lambda_k(\rho)} \lesssim 2^{-j}.$$

Thus, using Hölder's inequality, we obtain

$$\begin{aligned}&\Big\| \sup_{|t|<1} |S(t)\psi(2^{-j}\sqrt{-\Delta_D})f| \Big\|_{L^2(B((x_0,y_0),1)\bigcap\Omega)}\\&\lesssim \sup_{y\in\mathbb{R}} \Big\| \sup_{|t|<1} |S(t)\psi(2^{-j}\sqrt{-\Delta_D})f| \Big\|_{L^2_x(\mathbb{R}^+)}\\&\lesssim 2^{\frac{j}{2}} \|\psi(2^{-j}\sqrt{-\Delta_D})f\|_{L^2(\Omega)}.\end{aligned}$$

This implies (3.4).

Therefore, we complete the proof of Theorem 1.1.

3.2 Proof of Corollary 1.2

Fixed $f(x,y) \in H^s_D(\Omega)$, we define

$$\Omega_* f(x,y) := \overline{\lim_{t\to 0}}\, S(t)f(x,y) - \underline{\lim_{t\to 0}}\, S(t)f(x,y) \quad and \quad \Omega^* f(x,y) := \sup_{|t|<1} \big|S(t)f\big|.$$

Then, $|\Omega_* f(x,y)| \le 2\Omega^* f(x,y)$. By the density, we get for each $f(x,y) \in H^s_D(\Omega)$,

$$\forall\, \varepsilon > 0,\ \exists\, g \in C_c^\infty(\Omega),\ s.t.\ \|f-g\|_{H^s_D(\Omega)} < \frac{\varepsilon}{3}. \tag{3.5}$$

Observing that

$$\|S(t)g - g\|_{L^\infty(\Omega)} \lesssim \|S(t)g-g\|_{H_D^{\frac{3}{2}}(\Omega)} \to 0, \quad \text{as} \quad t \to 0,$$

we obtain $\Omega_* g(x) \equiv 0,\ \forall\, g \in C_c^\infty(\Omega)$. Hence,

$$\Omega_* f = \Omega_*(f-g),\ \big|\Omega_* f\big| = \big|\Omega_*(f-g)\big| \le 2\Omega^*(f-g).$$

This together with (1.6) and (3.5) yields that

$$\big\|\Omega_* f\big\|_{L^2_{\mathrm{loc}}(\Omega)} \le 2\big\|\Omega^*(f-g)\big\|_{L^2_{\mathrm{loc}}(\Omega)} \lesssim \|f-g\|_{H^s_D(\Omega)} < \varepsilon. \tag{3.6}$$

We obtain $\Omega_* f = 0,\ a.e.\ (x,y) \in \Omega$, since ε is arbitrary. And so the limits $\lim_{t\to 0} S(t)f$ exists almost everywhere. On the other hand, by the orthonormal basis of $L^2([0,\infty))$, we have

$$\hat{f}(x,\eta) = \sum_{k=1}^{\infty} \hat{f}_k(\eta)e_k(x,\eta), \quad \hat{f}_k(\eta) = \int_0^\infty \hat{f}(x,\eta)e_k(x,\eta)\,dx,$$

and

$$\|f\|^2_{H^s_D(\Omega)} = \|\hat{f}(x,\eta)\|^2_{L^2(\mathbb{R}^+\times\mathbb{R})} + \big\|(-\Delta_{D,\eta})^{\frac{s}{2}}\hat{f}(x,\eta)\big\|^2_{L^2(\mathbb{R}^+\times\mathbb{R})}$$

$$= \sum_{k=1}^{\infty} \|\hat{f}_k(\eta)\|^2_{L^2(\mathbb{R})} + \sum_{k=1}^{\infty} \left\|\lambda_k(\eta)^{\frac{s}{2}} \hat{f}_k(\eta)\right\|^2_{L^2(\mathbb{R})},$$

with $e_k(x, \eta)$, $\lambda_k(\eta)$ and $\Delta_{D,\eta}$ defined in Sect. 2. Thus, for any $\varepsilon > 0$, there exists $N > 0$ and $R > 0$ such that

$$\sum_{k=N}^{\infty} \|\hat{f}_k(\eta)\|^2_{L^2(\mathbb{R})} + \sum_{k=1}^{\infty} \|\hat{f}_k(\eta)\|^2_{L^2(|\eta|>R)} < \frac{\varepsilon}{2}.$$

Then, for $|t| < \frac{c\varepsilon}{\|f\|^2_{L^2(\Omega)}(R^2+N^{\frac{2}{3}}R^{\frac{4}{3}})}$, we estimate

$$\begin{aligned}
\left\|S(t)f - f\right\|^2_{L^2(\Omega)} &= \left\|e^{it\Delta_{D,\eta}}\hat{f}(x,\eta) - \hat{f}(x,\eta)\right\|^2_{L^2(\mathbb{R}^+\times\mathbb{R})} \\
&= \sum_{k=1}^{\infty} \left\|\left(e^{it\lambda_k(\eta)} - 1\right)\hat{f}_k(\eta)\right\|^2_{L^2(\mathbb{R})} \\
&\le \sum_{k=1}^{N} \left\|\left(e^{it\lambda_k(\eta)} - 1\right)\hat{f}_k(\eta)\right\|^2_{L^2(\mathbb{R})} + \sum_{k=N}^{\infty} \|\hat{f}_k(\eta)\|^2_{L^2(\mathbb{R})} \\
&\le \sum_{k=1}^{N} \left\|\left(e^{it\lambda_k(\eta)} - 1\right)\hat{f}_k(\eta)\right\|^2_{L^2(|\eta|\le R)} + \sum_{k=1}^{N} \left\|\hat{f}_k(\eta)\right\|^2_{L^2(|\eta|>R)} \\
&\quad + \sum_{k=N}^{\infty} \|\hat{f}_k(\eta)\|^2_{L^2(\mathbb{R})} < \varepsilon,
\end{aligned}$$

where we use the fact that

$$\sum_{k=1}^{N} \left\|\left(e^{it\lambda_k(\eta)} - 1\right)\hat{f}_k(\eta)\right\|^2_{L^2(|\eta|\le R)} \le \sup_{\substack{1\le k\le N \\ |\eta|\le R}} \left|e^{it\lambda_k(\eta)} - 1\right| \sum_{k=1}^{\infty} \|\hat{f}_k(\eta)\|^2_{L^2(\mathbb{R})} < \varepsilon$$

in the last inequality. This implies

$$\lim_{t\to 0} \left\|S(t)f - f\right\|_{L^2} = 0.$$

Therefore, (1.7) follows by the fact that the limit $\lim_{t\to 0} S(t)f$ exists almost everywhere.

3.3 Proof of Theorem 1.3

Let $\phi_k \in C_c^{\infty}([2^k, 2^k + 2^{\frac{k}{2}}])$, $k = 1, 2, \ldots,$ and

$$0 \le \phi_k \le 1, \; \left|\frac{d^i}{dx^i}\phi_k(x)\right| \le C2^{-\frac{k}{2}i}.$$

Taking

$$g_k(x,y) = \mathcal{F}^{-1}\big(\phi_k(\eta)e_{2^{k+2}}(x,\eta)\big),$$

then, we have

$$e^{it\Delta_D}g_k(x,y) = \int_{\mathbb{R}} e^{i(y\eta - t\lambda_{2^{k+2}}(\eta))}\phi_k(\eta)e_{2^{k+2}}(x,\eta)\,d\eta.$$

Writing $\Phi(\eta) := y\eta - t\lambda_{2^{k+2}}(\eta) - \frac{2}{3}z(\eta,x)^{\frac{3}{2}}$ with $z(\eta,x) = w_{2^{k+2}} - \eta^{\frac{2}{3}}x$, then $\Phi'(\eta) = y - t\big(2\eta + \frac{4}{3}w_{2^{k+2}}\eta^{\frac{1}{3}}\big) + \frac{2}{3}z(\eta,x)^{\frac{1}{2}}\eta^{-\frac{1}{3}}x$. For $\frac{1}{2} \le x, y \le 1$, taking

$$t(x,y) = \frac{y + \frac{2}{3}z(\eta_0,x)^{\frac{1}{2}}\eta_0^{-\frac{1}{3}}x}{2\eta_0 + \frac{4}{3}w_{2^{k+2}}\eta_0^{\frac{1}{3}}},\ \eta_0 = 2^k + 2^{\frac{k}{2}-1},$$

we get $\Phi'(\eta_0) = 0$ and

$$\big|\Phi(\eta) - \Phi(\eta_0)\big| \le \sup_{\bar\eta}|\Phi'(\bar\eta)|\cdot|\eta-\eta_0| \le \tfrac{1}{2}, \quad \forall\,\eta \in [2^k, 2^k + 2^{\frac{k}{2}}]. \tag{3.7}$$

By (2.7), (2.3) and Remark 2.3, we get for all $\eta \in [2^k, 2^k + 2^{\frac{k}{2}}]$ and $|x| \le 1$

$$\begin{aligned} z :=& w_{2^{k+2}} - \eta^{\frac{2}{3}}x \simeq 2^{\frac{2}{3}k},\ e_{2^{k+2}}(x,\eta) = f_{2^{k+2}}\frac{\eta^{\frac{1}{3}}}{2^{\frac{k+2}{6}}}\mathrm{Ai}(-z),\\ \mathrm{Ai}(-z) =& A_-(z) + A_+(z),\ A_+(z) = \overline{A_-(z)},\\ A_-(z) =& z^{-\frac{1}{4}}e^{i\frac{\pi}{4}}e^{-\frac{2}{3}iz^{\frac{3}{2}}}\Psi_-(z),\ 2\sqrt{\pi}\Psi_-(z) \sim \left(1 + \sum_{k\ge1}c_k z^{-\frac{3k}{2}}\right) \end{aligned}$$

Thus,

$$\begin{aligned} &e^{it(y)\Delta_D}g_k(x,y)\\ &= \int_{\mathbb{R}} e^{i(y\eta - t\lambda_{2^{k+2}}(\eta))}\phi_k(\eta)f_{2^{k+2}}\frac{\eta^{\frac{1}{3}}}{2^{\frac{k+2}{6}}}\big[A_+ + A_-\big](z(\eta,x))\,d\eta\\ &= e^{i\frac{\pi}{4}}e^{i\Phi(\eta_0)}\int_{\mathbb{R}} e^{i[\Phi(\eta)-\Phi(\eta_0)]}\phi_k(\eta)f_{2^{k+2}}\frac{\eta^{\frac{1}{3}}}{2^{\frac{k+2}{3}}}\Psi_-(z(\eta,x))\,d\eta\\ &\quad + e^{-i\frac{\pi}{4}}\int_{\mathbb{R}} e^{i(y\eta - t\lambda_{2^{k+2}}(\eta) + \frac{2}{3}z(\eta,x)^{\frac{3}{2}})}f_{2^{k+2}}\phi_k(\eta)\frac{\eta^{\frac{1}{3}}}{2^{\frac{k+2}{3}}}\Psi_+(z(\eta,x))\,d\eta\\ &:= I_1 + I_2. \end{aligned}$$

From (3.7), we know that

$$|I_1| = \Big| \int_{\mathbb{R}} e^{i[\Phi(\eta)-\Phi(\eta_0)]} \phi_k(\eta) f_{2^{k+2}} \frac{\eta^{\frac{1}{3}}}{2^{\frac{k+2}{6}}} z(\eta, x)^{-\frac{1}{4}} \Psi_-(z(\eta, x))\, d\eta \Big| \gtrsim 2^{\frac{k}{2}}.$$

Since $|\partial_\eta[y\eta - t\lambda_{2^{k+2}}(\eta) + \frac{2}{3}z(\eta, x)^{\frac{3}{2}}]| \geq c_0 > 0$, and $|\partial_\eta^2[y\eta - t\lambda_{2^{k+2}}(\eta) + \frac{2}{3}z(\eta, x)^{\frac{3}{2}}]| \leq C2^{-k}$, and

$$\Big|\partial_\eta \Big[f_{2^{k+2}} \phi_k(\eta) \frac{\eta^{\frac{1}{3}}}{2^{\frac{k+2}{3}}} \Psi_+(z(\eta, x)) \Big]\Big| \leq C2^{-k},$$

we estimate by integrating by parts

$$|I_2| = \Big| \int_{\mathbb{R}} e^{i(y\eta - t\lambda_{2^{k+2}}(\eta) + \frac{2}{3}z(\eta,x)^{\frac{3}{2}})} f_{2^{k+2}} \phi_k(\eta) \frac{\eta^{\frac{1}{3}}}{2^{\frac{k+2}{3}}} \Psi_+(z(\eta, x))\, d\eta \Big| \lesssim 2^{-\frac{k}{2}}.$$

Hence,

$$\big|e^{it(y)\Delta_D} g_k(x, y)\big| \geq |I_1| - |I_2| \gtrsim 2^{\frac{k}{2}}, \quad \forall\, \tfrac{1}{2} \leq x, y \leq 1. \tag{3.8}$$

On the other hand,

$$\|g_k(x, y)\|_{H^s_D(\Omega)} \lesssim 2^{k(s+\frac{1}{4})}. \tag{3.9}$$

Now, we argue by contradiction. We assume that the convergence property (1.4) holds for any function $f \in H^s_D(\Omega)$ with $s < \frac{1}{4}$. Then, by the same argument as in [2, Lemma C.1], we have

$$\Big|\Big\{(x, y) \in \Omega:\ \tfrac{1}{2} \leq y \leq 1,\ |x| \leq 1,\ \sup_{0<t<1} |e^{it\Delta_D} f(x, y)| > \lambda \Big\}\Big| \leq C \Big(\frac{\|f\|_{H^s_D(\Omega)}}{\lambda} \Big)^2.$$

This together with (3.8) and (3.9) implies for $\lambda = 2^{\frac{k}{2}}$

$$\begin{aligned} \frac{1}{2} \leq & \Big|\Big\{(x, y) \in \Omega:\ \tfrac{1}{2} \leq y \leq 1,\ |x| \leq 1,\ \sup_{0<t<1} |e^{it\Delta_D} g_k(x, y)| > 2^{\frac{k}{2}} \Big\}\Big| \\ \leq & C \Big(\frac{\|g_k\|_{H^s_D(\Omega)}}{2^{\frac{k}{2}}} \Big)^2 \\ \leq & C 2^{2k(s-\frac{1}{4})}, \end{aligned}$$

which is a contradiction as $k \to +\infty$, as long as $s < \frac{1}{4}$. Therefore, we conclude Theorem 1.3.

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