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A Complete Real-Variable Theory of Hardy Spaces on Spaces of Homogeneous Type

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Abstract

Let (X, d, μ) be a space of homogeneous type, with the upper dimension ω, in the sense of Coifman and Weiss. Assume that η is the smoothness index of the wavelets on X constructed by Auscher and Hytönen. In this article, when $p \in (\omega/(\omega+\eta), 1]$, for the atomic Hardy spaces $H^p_{\mathrm{cw}}(X)$ introduced by Coifman and Weiss, the authors establish their various real-variable characterizations, respectively, in terms of the grand maximal functions, the radial maximal functions, the non-tangential maximal functions, the various Littlewood–Paley functions and wavelet functions. This completely answers the question of Coifman and Weiss by showing that no additional (geometrical) condition is necessary to guarantee the radial maximal function characterization of $H^1_{\mathrm{cw}}(X)$ and even of $H^p_{\mathrm{cw}}(X)$ with p as above. As applications, the authors obtain the finite atomic characterizations of $H^p_{\mathrm{cw}}(X)$, which further induce some criteria for the boundedness of sublinear operators on $H^p_{\mathrm{cw}}(X)$. Compared with the known results, the novelty of this article is that μ is not assumed to satisfy the reverse doubling condition and d is only a quasi-metric, moreover, the range $p \in (\omega/(\omega+\eta), 1]$ is natural and optimal.

Keywords Space of homogeneous type · Hardy space · Maximal function · Atom · Littlewood–Paley function · Wavelet

Mathematics Subject Classification Primary 42B30; Secondary 42B25 · 42B20 · 30L99

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1 Introduction

The real-variable theory of Hardy spaces plays a fundamental role in harmonic analysis. The classical Hardy space on the n-dimensional Euclidean space $\mathbb{R}^n$ was initially developed by Stein and Weiss [50] and later by Fefferman and Stein [11]. Hardy spaces $H^p(\mathbb{R}^n)$ have proved a suitable substitute of Lebesgue spaces $L^p(\mathbb{R}^n)$ with $p \in (0, 1]$ in the study of the boundedness of operators. Indeed, any element in the Hardy space can be decomposed into a sum of some basic elements (which are called *atoms*); see Coifman [5] for $n = 1$ and Latter [36] for general $n \in \mathbb{N}$. Characterizations of Hardy spaces via Littlewood–Paley functions were due to Uchiyama [51]. For more study on classical Hardy spaces on $\mathbb{R}^n$, we refer the reader to the well-known monographs [16–18,41,49]. Modern developments regarding the real-variable theory of Hardy spaces are so deep and vast that we can only list a few literatures here, for example, the theory of Hardy spaces associated with operators (see [2,3,10,30]), Hardy spaces with variable exponents (see [44]), the real-variable theory of Musielak–Orlicz Hardy spaces (see [35,53]), and also Hardy spaces for ball quasi-Banach spaces (see [48]).

In this article, we focus on the real-variable theory of Hardy spaces on spaces of homogeneous type. It is known that the space of homogeneous type introduced by Coifman and Weiss [6,7] provides a natural setting for the study of both function spaces and the boundedness of operators. A *quasi-metric space* (X, d) is a non-empty set X equipped with a *quasi-metric* d, that is, a non-negative function defined on $X \times X$, satisfying that, for any $x,\ y,\ z \in X$,

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) there exists a constant $A_0 \in [1, \infty)$ such that $d(x, z) \le A_0[d(x, y) + d(y, z)]$.

The *ball* B on X centered at $x_0 \in X$ with radius $r \in (0, \infty)$ is defined by setting

$$B := B(x_0, r) := \{x \in X :\ d(x, x_0) < r\}.$$

For any ball B and $\tau \in (0, \infty)$, denote by τB the ball with the same center as that of B but of radius τ times that of B. Given a quasi-metric space (X, d) and a non-negative measure μ, we call (X, d, μ) a *space of homogeneous type* if μ satisfies the *doubling condition*: there exists a positive constant $C_{(\mu)} \in [1, \infty)$ such that, for any ball $B \subset X$,

$$\mu(2B) \le C_{(\mu)}\mu(B).$$

The above doubling condition is equivalent to that, for any ball B and $\lambda \in [1, \infty)$,

$$\mu(\lambda B) \le C_{(\mu)}\lambda^{\omega}\mu(B), \tag{1.1}$$

where $\omega := \log_2 C_{(\mu)}$ is called the *upper dimension* of X. If $A_0 = 1$, we call (X, d, μ) a *doubling metric measure space*.

According to [7, pp. 587–588], we *always make* the following assumptions throughout this article. For any point $x \in X$, assume that the balls $\{B(x, r)\}_{r\in(0,\infty)}$ form a *basis* of open neighborhoods of x; assume that μ is Borel regular, which means that open sets are measurable and every set $A \subset X$ is contained in a Borel set E satisfying that $\mu(A) = \mu(E)$; we also assume that $\mu(B(x,r)) \in (0,\infty)$ for any $x \in X$ and $r \in (0,\infty)$. For the presentation concision, we always assume that (X, d, μ) is non-atomic [namely, $\mu(\{x\}) = 0$ for any $x \in X$] and $\operatorname{diam}(X) := \sup\{d(x,y) : x, y \in X\} = \infty$. It is known that $\operatorname{diam}(X) = \infty$ implies that $\mu(X) = \infty$ (see, for example, [45, Lemma 5.1] or [1, Lemma 8.1]).

Let us recall the notion of the atomic Hardy space on spaces of homogeneous type introduced by Coifman and Weiss [7]. For any $\alpha \in (0,\infty)$, the *Lipschitz space* $\mathcal{L}_\alpha(X)$ is defined to be the collection of all measurable functions f such that

$$\|f\|_{\mathcal{L}_\alpha(X)} := \sup_{x\neq y} \frac{|f(x) - f(y)|}{[\mu(B(x, d(x,y)))]^\alpha} < \infty.$$

Denote by $(\mathcal{L}_\alpha(X))'$ the *dual space* of $\mathcal{L}_\alpha(X)$ equipped with the weak-$*$ topology.

Definition 1.1 Let $p \in (0,1]$ and $q \in (p,\infty] \cap [1,\infty]$. A function a is called a (p,q)-*atom* if

(i) $\operatorname{supp} a := \{x \in X : a(x) \neq 0\} \subset B(x_0, r)$ for some $x_0 \in X$ and $r \in (0,\infty)$;
(ii) $[\int_X |a(x)|^q \, d\mu(x)]^{\frac{1}{q}} \le [\mu(B(x_0,r))]^{\frac{1}{q}-\frac{1}{p}}$;
(iii) $\int_X a(x)\, d\mu(x) = 0$.

The *atomic Hardy space* $H^{p,q}_{\mathrm{cw}}(X)$ is defined as the subspace of $(\mathcal{L}_{1/p-1}(X))'$ when $p \in (0,1)$ or of $L^1(X)$ when $p = 1$, which consists of all the elements f admitting an atomic decomposition

$$f = \sum_{j=0}^{\infty} \lambda_j a_j, \tag{1.2}$$

where $\{a_j\}_{j=0}^\infty$ are (p,q)-atoms, $\{\lambda_j\}_{j=0}^\infty \subset \mathbb{C}$ satisfies $\sum_{j=0}^\infty |\lambda_j|^p < \infty$ and the series in (1.2) converges in $(\mathcal{L}_{1/p-1}(X))'$ when $p \in (0,1)$ or in $L^1(X)$ when $p = 1$. Define

$$\|f\|_{H^{p,q}_{\mathrm{cw}}(X)} := \inf\left\{\left(\sum_{j=0}^{\infty} |\lambda_j|^p\right)^{\frac{1}{p}}\right\},$$

where the infimum is taken over all the decompositions of f as in (1.2).

It was proved in [7] that the atomic Hardy space $H^{p,q}_{\mathrm{cw}}(X)$ is independent of the choice of q and hence we sometimes write $H^p_{\mathrm{cw}}(X)$ for short. It was also proved in [7] that the dual space of $H^p_{\mathrm{cw}}(X)$ is the Lipschitz space $\mathcal{L}_{1/p-1}(X)$ when $p \in (0,1)$, and the space $\mathrm{BMO}(X)$ of bounded mean oscillation when $p = 1$.

It is well known that the most basic result in the real-variable theory of Hardy spaces is their characterizations in terms of maximal functions. Coifman and Weiss [7, pp. 641–642] observed that a proof of the duality result between $H^1(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$ from Carleson [4] can be extended to the general setting of spaces of homogeneous type provided a certain additional geometrical assumption is added, from which one can then obtain a radial maximal function characterization of $H^1_{\mathrm{cw}}(X)$. Coifman and Weiss [7, p. 642] then asked that *to what extent their geometrical condition is necessary for the validity of the radial maximal function characterization of* $H^1_{\mathrm{cw}}(X)$. Since then, lots of efforts are made to build various real-variable characterizations of the atomic Hardy spaces on spaces of homogeneous type with few geometrical assumptions. In this article, we completely answer the aforementioned question of Coifman and Weiss by showing that no any additional (geometrical) condition is necessary to guarantee the radial maximal function characterization of $H^1_{\mathrm{cw}}(X)$ and even of $H^p_{\mathrm{cw}}(X)$ with $p \le 1$ but near to 1.

Recall that a triple (X, d, μ) is said to be *Ahlfors-n regular* if $\mu(B(x,r)) \sim r^n$ for any $x \in X$ and $r \in (0, \operatorname{diam} X)$ with positive equivalence constants independent of x and r. When (X, d, μ) is Ahlfors-n regular, upon assuming the quasi-metric d satisfying that there exists $\theta \in (0,1)$ such that, for any $x,\ x',\ y \in X$,

$$|d(x,y) - d(x',y)| \lesssim [d(x,x')]^\theta [d(x,y) + d(x',y)]^{1-\theta}, \tag{1.3}$$

Macías and Segovia [43] characterized Hardy spaces via the grand maximal functions, and Li [37] obtained another grand maximal function characterization via test functions introduced in [28]. Also, Duong and Yan [9] characterized Hardy spaces via the Lusin area function associated with certain semigroup.

Recall that an RD-*space* (X, d, μ) is a doubling metric measure space with the measure μ further satisfying the *reverse doubling condition*, that is, there exist a positive constant $\widetilde{C} \in (0,1]$ and $\kappa \in (0, \omega]$ such that, for any ball $B(x,r)$ with $x \in X$, $r \in (0, \operatorname{diam} X/2)$ and $\lambda \in [1, \operatorname{diam} X/[2r])$,

$$\widetilde{C} \lambda^\kappa \mu(B(x,r)) \le \mu(B(x, \lambda r)).$$

Indeed, any path connected doubling metric measure space is an RD-space (see [27, 57]). Characterizations of Hardy spaces on RD-spaces via various Littlewood–Paley functions were established in [26,27]. Also, characterizations of Hardy spaces on RD-spaces via various maximal functions can be found in [20,21,56]. It should be mentioned that local Hardy spaces can be used to characterize more general scale of function spaces like Besov and Triebel–Lizorkin spaces on RD-spaces (see [57]). For a systematic study of Besov and Triebel–Lizorkin spaces on RD-spaces, we refer the reader to [27]. More on analysis over Ahlfors-n regular metric measure spaces or RD-spaces can be found in [8,19,22,32–34,54,57,58].

The main motivation of studying the real-variable theory of function spaces and the boundedness of operators on spaces of homogeneous type comes from the celebrated work of Auscher and Hytönen [1], in which they constructed an orthonormal wavelet basis $\{\psi^k_\alpha : k \in \mathbb{Z},\ \alpha \in \mathcal{G}_k\}$ of $L^2(X)$ with Hölder continuity exponent $\eta \in (0,1)$ and exponential decay by using the system of random dyadic cubes. The first creative attempt of using the idea of [1] to investigate the real-variable theory of Hardy spaces on spaces of homogeneous type was due to Han et al. [23] (see also Han et al. [24]). Indeed, in [23], Hardy spaces via wavelets on spaces of homogeneous type were introduced and then these spaces proved to have atomic decompositions. The method used in

[23] is based on a new Calderón reproducing formula on spaces of homogeneous type (see [23, Proposition 2.5]). But there exists an *error* in the proof of [23, Proposition 2.5], namely, since the regularity exponent of the approximations of the identity in [23, p. 3438] is θ [indeed, θ is from the regularity of the quasi-metric d in (1.3)], it follows that the regularity exponent in [23, (2.6)] should be $\min\{\theta, \eta\}$ and hence the correct range of p in [23, Proposition 2.5] (indeed, all results of [23]) seems to be $(\omega/[\omega+\min\{\theta, \eta\}], 1]$ which is not optimal. Moreover, the criteria of the boundedness of Calderón–Zygmund operators on the dual of Hardy spaces were established in [23]. Also, Fu and Yang [14] obtained an unconditional basis of $H^1_{\rm cw}(X)$ and several equivalent characterizations of $H^1_{\rm cw}(X)$ in terms of wavelets.

Another motivation of this article comes from the Calderón reproducing formulae established in [29]. Indeed, the work of [29] was partly motivated by the wavelet theory of Auscher and Hytönen in [1] and a corresponding wavelet reproducing formula (which can converge in the distribution space) in [29]. The already existing works (see [20,26,27,56,57]) regarding Hardy spaces on RD-spaces show the feasibility of establishing various real-variable characterizations of the atomic Hardy spaces on spaces of homogeneous type via the Calderón reproducing formulae. It should be mentioned that a characterization of the atomic Hardy spaces via the Littlewood–Paley functions was established in [25] via the aforementioned wavelet reproducing formula; see also [25] for some corresponding conclusions of product Hardy spaces on spaces of homogeneous type.

In this article, motivated by [23,29], for the atomic Hardy spaces $H^p_{\rm cw}(X)$ with any $p \in (\omega/[\omega + \eta], 1]$, we establish their various real-variable characterizations, respectively, in terms of the grand maximal functions, the radial maximal functions, the non-tangential maximal functions, the various Littlewood–Paley functions and wavelets. Observe that these characterizations are true for $H^p_{\rm cw}(X)$ with $p \in (\omega/[\omega + \eta], 1]$ and X being any space of homogeneous type *without any additional (geometrical) conditions*, which completely answers the aforementioned question asked by Coifman and Weiss [7, p. 642]. As an application, we obtain the finite atomic characterizations of Hardy spaces, which further induce some criteria for the boundedness of sublinear operators on Hardy spaces. Compared with the known results, the novelty of this article is that μ is not assumed to satisfy the reverse doubling condition and d is only a quasi-metric. Moreover, the range of $p \in (\omega/(\omega + \eta), 1]$ for the various maximal function characterizations and the Littlewood–Paley function characterizations of the atomic Hardy spaces $H^p_{\rm cw}(X)$ is natural and optimal. The key tool used through this article is those Calderón reproducing formulae from [29].

In addition, we point out that, when X is a doubling metric measure space, the finite atomic characterizations of Hardy spaces are also useful in establishing the bilinear decomposition of the product space $H^1_{\rm cw}(X) \times {\rm BMO}(X)$ and $H^p_{\rm cw}(X) \times \mathcal{L}_{1/p-1}(X)$, with $p \in (\omega/[\omega + \eta], 1)$ in [13–15,40], and also in the study of the endpoint boundedness of commutators generated by Calderón-Zygmund operators and ${\rm BMO}(X)$ functions in [38,39].

The organization of this article is as follows.

In Sect. 2, we recall the notions of the space of test functions and the space of distributions introduced in [26], as well as the random dyadic cubes in [1] and the approximation of the identity with exponential decay introduced in [29]. Then we restate the Calderón reproducing formulae established in [29].

Section 3 concerns Hardy spaces defined via the grand maximal functions, the radial maximal functions and the non-tangential maximal functions. We show that these Hardy spaces are all equivalent to the Lebesgue space $L^p(X)$ when $p \in (1, \infty]$ (see Sect. 3.1), and they are all mutually equivalent when $p \in (\omega/(\omega+\eta), 1]$ (see Sect. 3.2), all in the sense of equivalent (quasi-)norms. The proof for the latter borrows some ideas from [56] and uses the Calderón reproducing formulae built in [29]. Moreover, we prove that the Hardy space $H^{*,p}(X)$ defined via the grand maximal function is independent of the choices of the distribution space $(\mathcal{G}_0^\eta(\beta, \gamma))'$ whenever $\beta,\ \gamma \in (\omega[1/p - 1], \eta)$; see Proposition 3.8 below.

Section 4 is devoted to the atomic characterizations of $H^{*,p}(X)$. Notice that, if a distribution has an atomic decomposition, then it belongs to $H^{*,p}(X)$ obviously by the definition of atoms; see Sect. 4.1. All we remain to do is to establish the converse relationship. In Sect. 4.2, by modifying the definition of the grand maximal function f^* to $f^\star$ so that the level set $\Omega_\lambda := \{x \in X : f^\star(x) > \lambda\}$ with $\lambda \in (0, \infty)$ is open, we then apply the partition of unity to the open set Ω_λ and obtain a Calderón–Zygmund decomposition of $f \in H^{*,p}(X)$. This is further used in Sect. 4.3 to construct an atomic decomposition of f. In Sect. 4.4, we compare the atomic Hardy spaces $H_{\mathrm{at}}^{p,q}(X)$ with $H_{\mathrm{cw}}^{p,q}(X)$ and prove that they are exactly the same space in the sense of equivalent (quasi-)norms.

Section 5 deals with the Littlewood–Paley theory of Hardy spaces. In Sect. 5.1, we show that the Hardy space $H^p(X)$, defined via the Lusin area function, is independent of the choices of exp-ATIs. In Sect. 5.2, we use the homogeneous continuous Calderón reproducing formula and the molecular characterizations of the atomic Hardy spaces (see [39]) to establish the atomic decompositions of elements in $H^p(X)$, and then we connect $H^p(X)$ with $H^{*,p}(X)$. In Sect. 5.3, we characterize Hardy spaces $H^p(X)$ via the Lusin area function with aperture, the Littlewood–Paley g-function and the Littlewood–Paley g_λ^*-function.

In Sect. 6, we consider the Hardy space $H_{\mathrm{w}}^p(X)$ defined via wavelets, which was introduced in [23]. We improve the result of [25, Theorem 4.3] and prove that $H_{\mathrm{w}}^p(X)$ coincides with $H^p(X)$ in the sense of equivalent (quasi-)norms.

In Sect. 7, as an application, we obtain criteria of the boundedness of the sublinear operators from Hardy spaces to quasi-Banach spaces. To this end, we first establish the finite atomic characterizations, namely, we show that, if $q \in (p, \infty) \cap [1, \infty)$, then $\|\cdot\|_{H_{\mathrm{fin}}^{p,q}(X)}$ and $\|\cdot\|_{H_{\mathrm{at}}^{p,q}(X)}$ are equivalent (quasi)-norms on a dense subspace $H_{\mathrm{fin}}^{p,q}(X)$ of $H_{\mathrm{at}}^{p,q}(X)$; the above equivalence also holds true on a dense subspace $H_{\mathrm{fin}}^{p,\infty}(X) \cap \mathrm{UC}(X)$ of $H_{\mathrm{at}}^{p,\infty}(X)$, where $\mathrm{UC}(X)$ denotes the space of all uniformly continuous functions on X.

At the end of this section, we make some conventions on notation. We *always assume* that ω is as in (1.1) and η the smoothness index of wavelets (see [1, Theorem 7.1] or Definition 2.4 below). We assume that δ is a very small positive number, for example, $\delta \le (2A_0)^{-10}$ in order to construct the dyadic cube system and the wavelet system on X (see [31, Theorem 2.2] or Lemma 2.3 below). For any $x,\ y \in X$ and $r \in (0, \infty)$, let

$$V_r(x) := \mu(B(x,r)) \quad \text{and} \quad V(x,y) := \mu(B(x,d(x,y))),$$

where $B(x,r) := \{y \in X : \ d(x,y) < r\}$. We always let $\mathbb{N} := \{1,2,\ldots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. For any $p \in [1,\infty]$, we use p' to denote its *conjugate index*, namely, $1/p + 1/p' = 1$. The symbol C denotes a positive constant which is independent of the main parameters, but it may vary from line to line. We also use $C_{(\alpha,\beta,\ldots)}$ to denote a positive constant depending on the indicated parameters α, β, ... The symbol $A \lesssim B$ means that there exists a positive constant C such that $A \le CB$. The symbol $A \sim B$ is used as an abbreviation of $A \lesssim B \lesssim A$. We also use $A \lesssim_{\alpha,\beta,\ldots} B$ to indicate that here the implicit positive constant depends on α, β, ... and, similarly, $A \sim_{\alpha,\beta,\ldots} B$. We also use the following convention: If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, *rather than* $f \lesssim g = h$ or $f \lesssim g \le h$. For any $s,\ t \in \mathbb{R}$, denote the *minimum* of s and t by $s \wedge t$. For any finite set $\mathcal{J}$, we use $\#\mathcal{J}$ to denote its *cardinality*. Also, for any set E of X, we use $\mathbf{1}_E$ to denote its characteristic function and $E^{\complement}$ the set $X \setminus E$.

2 Calderón Reproducing Formulae

This section is devoted to recalling Calderón reproducing formulae obtained in [29]. To this end, we first recall the notions of both the space of test functions and the distribution space.

Definition 2.1 Let $x_1 \in X$, $r \in (0,\infty)$, $\beta \in (0,1]$ and $\gamma \in (0,\infty)$. A function f defined on X is called a *test function of type* (x_1, r, β, γ), denoted by $f \in \mathcal{G}(x_1, r, \beta, \gamma)$, if there exists a positive constant C such that

(i) (the *size condition*) for any $x \in X$,

$$|f(x)| \le C \frac{1}{V_r(x_1) + V(x_1,x)} \left[\frac{r}{r + d(x_1,x)}\right]^\gamma;$$

(ii) (the *regularity condition*) for any $x,\ y \in X$ satisfying $d(x,y) \le (2A_0)^{-1}[r + d(x_1,x)]$,

$$|f(x) - f(y)| \le C \left[\frac{d(x,y)}{r + d(x_1,x)}\right]^\beta \frac{1}{V_r(x_1) + V(x_1,x)} \left[\frac{r}{r + d(x_1,x)}\right]^\gamma.$$

For any $f \in \mathcal{G}(x_1, r, \beta, \gamma)$, define the norm

$$\|f\|_{\mathcal{G}(x_1,r,\beta,\gamma)} := \inf\{C \in (0,\infty) : \ C \text{ satisfies (i) and (ii)}\}.$$

Define

$$\mathring{\mathcal{G}}(x_1, r, \beta, \gamma) := \left\{f \in \mathcal{G}(x_1, r, \beta, \gamma) : \ \int_X f(x)\, d\mu(x) = 0\right\}$$

equipped with the norm $\|\cdot\|_{\mathring{\mathcal{G}}(x_1,r,\beta,\gamma)} := \|\cdot\|_{\mathcal{G}(x_1,r,\beta,\gamma)}$.

Observe that the above version of $\mathcal{G}(x_1, r, \beta, \gamma)$ was originally introduced by Han et al. [27] (see also [26]).

Fix $x_0 \in X$. For any $x \in X$ and $r \in (0, \infty)$, we know that $\mathcal{G}(x, r, \beta, \gamma) = \mathcal{G}(x_0, 1, \beta, \gamma)$ with equivalent norms, but the positive equivalence constants depend on x and r. Obviously, $\mathcal{G}(x_0, 1, \beta, \gamma)$ is a Banach space. In what follows, we simply write $\mathcal{G}(\beta, \gamma) := \mathcal{G}(x_0, 1, \beta, \gamma)$ and $\mathring{\mathcal{G}}(\beta, \gamma) := \mathring{\mathcal{G}}(x_0, 1, \beta, \gamma)$.

Fix $\epsilon \in (0, 1]$ and $\beta,\ \gamma \in (0, \epsilon)$. Let $\mathcal{G}_0^\epsilon(\beta, \gamma)$ [resp., $\mathring{\mathcal{G}}_0^\epsilon(\beta, \gamma)$] be the completion of the set $\mathcal{G}(\epsilon, \epsilon)$ [resp., $\mathring{\mathcal{G}}(\epsilon, \epsilon)$] in $\mathcal{G}(\beta, \gamma)$, that is, if $f \in \mathcal{G}_0^\epsilon(\beta, \gamma)$ [resp., $f \in \mathring{\mathcal{G}}_0^\epsilon(\beta, \gamma)$], then there exists $\{\phi_j\}_{j=1}^\infty \subset \mathcal{G}(\epsilon, \epsilon)$ [resp., $\{\phi_j\}_{j=1}^\infty \subset \mathring{\mathcal{G}}(\epsilon, \epsilon)$] such that $\|\phi_j - f\|_{\mathcal{G}(\beta,\gamma)} \to 0$ as $j \to \infty$. If $f \in \mathcal{G}_0^\epsilon(\beta, \gamma)$ [resp., $f \in \mathring{\mathcal{G}}_0^\epsilon(\beta, \gamma)$], we then let

$$\|f\|_{\mathcal{G}_0^\epsilon(\beta,\gamma)} := \|f\|_{\mathcal{G}(\beta,\gamma)} \quad \text{[resp., } \|f\|_{\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)} := \|f\|_{\mathcal{G}(\beta,\gamma)}\text{]}.$$

The *dual space* $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$ [resp., $(\mathring{\mathcal{G}}_0^\epsilon(\beta, \gamma))'$] is defined to be the set of all continuous linear functionals on $\mathcal{G}_0^\epsilon(\beta, \gamma)$ [resp., $\mathring{\mathcal{G}}_0^\epsilon(\beta, \gamma)$] and equipped with the weak-$*$ topology. The spaces $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$ and $(\mathring{\mathcal{G}}_0^\epsilon(\beta, \gamma))'$ are called the *spaces of distributions*.

Let $L^1_{\rm loc}(X)$ be the space of all locally integrable functions on X. Denote by $\mathcal{M}$ the *Hardy–Littlewood maximal operator*, that is, for any $f \in L^1_{\rm loc}(X)$ and $x \in X$,

$$\mathcal{M}(f)(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y),$$

where the supremum is taken over all balls B of X that contain x. For any $p \in (0, \infty]$, the *Lebesgue space* $L^p(X)$ is defined to be the set of all μ-measurable functions f such that

$$\|f\|_{L^p(X)} := \left[\int_X |f(x)|^p\, d\mu(x)\right]^{\frac{1}{p}} < \infty$$

with the usual modification made when $p = \infty$; the *weak Lebesgue space* $L^{p,\infty}(X)$ is defined to be the set of all μ-measurable functions f such that

$$\|f\|_{L^{p,\infty}(X)} := \sup_{\lambda \in (0,\infty)} \lambda[\mu(\{x \in X : |f(x)| > \lambda\})]^{\frac{1}{p}} < \infty.$$

It is known (see [7]) that $\mathcal{M}$ is bounded on $L^p(X)$ when $p \in (1, \infty]$ and bounded from $L^1(X)$ to $L^{1,\infty}(X)$. Then we state some estimates from [27, Lemma 2.1], which are proved by using (1.1).

Lemma 2.2 *Let* $\beta,\ \gamma \in (0, \infty)$.

(i) *For any* $x,\ y \in X$ *and* $r \in (0, \infty)$, $V(x, y) \sim V(y, x)$ *and*

$$\begin{aligned} V_r(x) + V_r(y) + V(x, y) &\sim V_r(x) + V(x, y) \\ &\sim V_r(y) + V(x, y) \sim \mu(B(x, r + d(x, y))), \end{aligned}$$

where the positive equivalence constants are independent of x, y *and* r.

(ii) *There exists a positive constant* C *such that, for any* $x_1 \in X$ *and* $r \in (0, \infty)$,

$$\int_X \frac{1}{V_r(x_1) + V(x_1, x)} \left[\frac{r}{r + d(x_1, x)}\right]^\gamma d\mu(x) \le C.$$

(iii) *There exists a positive constant* C *such that, for any* $x \in X$ *and* $R \in (0, \infty)$,

$$\int_{d(x,y)\le R} \frac{1}{V(x, y)} \left[\frac{d(x, y)}{R}\right]^\beta d\mu(y) \le C \quad \text{and}$$
$$\int_{d(x,y)\ge R} \frac{1}{V(x, y)} \left[\frac{R}{d(x, y)}\right]^\beta d\mu(y) \le C.$$

(iv) *There exists a positive constant* C *such that, for any* $x_1 \in X$ *and* $R,\ r \in (0, \infty)$,

$$\int_{d(x,x_1)\ge R} \frac{1}{V_r(x_1) + V(x_1, x)} \left[\frac{r}{r + d(x_1, x)}\right]^\gamma d\mu(x) \ \le C \left(\frac{r}{r + R}\right)^\gamma.$$

(v) *There exists a positive constant* C *such that, for any* $r \in (0, \infty)$, $f \in L^1_{\mathrm{loc}}(X)$ *and* $x \in X$,

$$\int_X \frac{1}{V_r(x) + V(x, y)} \left[\frac{r}{r + d(x, y)}\right]^\gamma |f(y)|\, d\mu(y) \le C\mathcal{M}(f)(x).$$

Next we recall the system of dyadic cubes established in [31, Theorem 2.2] (see also [1]), which is restated in the following version.

Lemma 2.3 *Fix constants* $0 < c_0 \le C_0 < \infty$ *and* $\delta \in (0, 1)$ *such that* $12A_0^3C_0\delta \le c_0$. *Assume that a set of points,* $\{z_\alpha^k :\ k \in \mathbb{Z},\ \alpha \in \mathcal{A}_k\} \subset X$ *with* $\mathcal{A}_k$ *for any* $k \in \mathbb{Z}$ *being a countable set of indices, has the following properties: for any* $k \in \mathbb{Z}$,

(i) $d(z_\alpha^k, z_\beta^k) \ge c_0\delta^k$ *if* $\alpha \ne \beta$;

(ii) $\min_{\alpha\in\mathcal{A}_k} d(x, z_\alpha^k) \le C_0\delta^k$ *for any* $x \in X$.

Then there exists a family of sets, $\{Q_\alpha^k :\ k \in \mathbb{Z},\ \alpha \in \mathcal{A}_k\}$, *satisfying*

(iii) *for any* $k \in \mathbb{Z}$, $\bigcup_{\alpha\in\mathcal{A}_k} Q_\alpha^k = X$ *and* $\{Q_\alpha^k :\ \ \alpha \in \mathcal{A}_k\}$ *is disjoint;*

(iv) *if* $k,\ l \in \mathbb{Z}$ *and* $l \ge k$, *then either* $Q_\beta^l \subset Q_\alpha^k$ *or* $Q_\beta^l \cap Q_\alpha^k = \emptyset$;

(v) *for any* $k \in \mathbb{Z}$ *and* $\alpha \in \mathcal{A}_k$, $B(z_\alpha^k, c_\natural\delta^k) \subset Q_\alpha^k \subset B(z_\alpha^k, C^\natural\delta^k)$, *where* $c_\natural := (3A_0^2)^{-1}c_0$, $C^\natural := 2A_0C_0$ *and* z_α^k *is called "the center" of* Q_α^k.

Throughout this article, we keep the notation used in Lemma 2.3. Moreover, for any $k \in \mathbb{Z}$, let

$$\mathcal{X}^k := \{z_\alpha^k\}_{\alpha\in\mathcal{A}_k}, \qquad \mathcal{G}_k := \mathcal{A}_{k+1} \setminus \mathcal{A}_k \qquad \text{and} \qquad \mathcal{Y}^k := \{z_\alpha^{k+1}\}_{\alpha\in\mathcal{G}_k} =: \{y_\alpha^k\}_{\alpha\in\mathcal{G}_k}.$$

Next we recall the notion of approximations of the identity with exponential decay introduced in [29].

Definition 2.4 A sequence $\{Q_k\}_{k\in\mathbb{Z}}$ of bounded linear integral operators on $L^2(X)$ is called an *approximation of the identity with exponential decay* (for short, exp-ATI) if there exist constants $C,\ \nu\in(0,\infty)$, $a\in(0,1]$ and $\eta\in(0,1)$ such that, for any $k\in\mathbb{Z}$, the kernel of operator Q_k, which is still denoted by Q_k, satisfying

(i) (the *identity condition*) $\sum_{k=-\infty}^{\infty} Q_k = I$ in $L^2(X)$, where I is the identity operator on $L^2(X)$;

(ii) (the *size condition*) for any $x,\ y\in X$,

$$|Q_k(x,y)| \le C\frac{1}{\sqrt{V_{\delta^k}(x)\,V_{\delta^k}(y)}}\exp\left\{-\nu\left[\frac{d(x,y)}{\delta^k}\right]^a\right\} \times\exp\left\{-\nu\left[\frac{\max\{d(x,\mathcal{Y}^k),\ d(y,\mathcal{Y}^k)\}}{\delta^k}\right]^a\right\}; \tag{2.1}$$

(iii) (the *regularity condition*) for any $x,\ x',\ y\in X$ with $d(x,x')\le\delta^k$,

$$\begin{aligned}&|Q_k(x,y)-Q_k(x',y)|+|Q_k(y,x)-Q_k(y,x')|\\ &\quad\le C\left[\frac{d(x,x')}{\delta^k}\right]^\eta\frac{1}{\sqrt{V_{\delta^k}(x)\,V_{\delta^k}(y)}}\exp\left\{-\nu\left[\frac{d(x,y)}{\delta^k}\right]^a\right\}\\ &\quad\times\exp\left\{-\nu\left[\frac{\max\{d(x,\mathcal{Y}^k),\ d(y,\mathcal{Y}^k)\}}{\delta^k}\right]^a\right\};\end{aligned} \tag{2.2}$$

(iv) (the *second difference regularity condition*) for any $x,\ x',\ y,\ y'\in X$ with $d(x,x')\le\delta^k$ and $d(y,y')\le\delta^k$, then

$$\begin{aligned}&|[Q_k(x,y)-Q_k(x',y)]-[Q_k(x,y')-Q_k(x',y')]|\\ &\quad\le C\left[\frac{d(x,x')}{\delta^k}\right]^\eta\left[\frac{d(y,y')}{\delta^k}\right]^\eta\frac{1}{\sqrt{V_{\delta^k}(x)\,V_{\delta^k}(y)}}\exp\left\{-\nu\left[\frac{d(x,y)}{\delta^k}\right]^a\right\}\\ &\quad\times\exp\left\{-\nu\left[\frac{\max\{d(x,\mathcal{Y}^k),\ d(y,\mathcal{Y}^k)\}}{\delta^k}\right]^a\right\};\end{aligned} \tag{2.3}$$

(v) (the *cancelation condition*) for any $x,\ y\in X$,

$$\int_X Q_k(x,y')\,d\mu(y') = 0 = \int_X Q_k(x',y)\,d\mu(x').$$

Remark 2.5 By [29, Remark 2.8], we know that the factor $\frac{1}{\sqrt{V_{\delta^k}(x)V_{\delta^k}(y)}}$ in (2.1), (2.2) and (2.3) can be replaced by $\frac{1}{V_{\delta^k}(x)}$ or $\frac{1}{V_{\delta^k}(y)}$, and $\max\{d(x,\mathcal{Y}^k),d(y,\mathcal{Y}^k)\}$ by $d(x,\mathcal{Y}^k)$ or by $d(y,\mathcal{Y}^k)$, with $\exp\{-\nu[\frac{d(x,y)}{\delta^k}]^a\}$ replaced by $\exp\{-\nu'[\frac{d(x,y)}{\delta^k}]^a\}$, where $\nu'\in(0,\nu)$ only depends on a and A_0. Moreover, the condition in Definition 2.4(iii) [resp., (iv)] can be replaced by $d(x,x')\le(2A_0)^{-1}[\delta^k+d(x,y)]$ (resp.,

$d(x,x') \le (2A_0)^{-2}[\delta^k + d(x,y)]$ and $d(y,y') \le (2A_0)^{-2}[\delta^k + d(x,y)]$). For their proofs, see [29, Proposition 2.9].

With the above exp-ATI, we have the following homogeneous continuous Calderón reproducing formula established in [29].

Theorem 2.6 *Let $\{Q_k\}_{k\in\mathbb{Z}}$ be an* exp-ATI *and $\beta, \gamma \in (0,\eta)$. Then there exists a sequence $\{\widetilde{Q}_k\}_{k\in\mathbb{Z}}$ of bounded linear operators on $L^2(X)$ such that, for any $f \in (\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$,*

$$f = \sum_{k=-\infty}^{\infty} \widetilde{Q}_k Q_k f,$$

where the series converges in $(\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$. Moreover, there exists a positive constant C such that, for any $k \in \mathbb{Z}$, the kernel of $\widetilde{Q}_k$ satisfies the following conditions:

(i) *for any $x, y \in X$,*

$$\left|\widetilde{Q}_k(x,y)\right| \le C \frac{1}{V_{\delta^k}(x) + V(x,y)} \left[\frac{\delta^k}{\delta^k + d(x,y)}\right]^\gamma;$$

(ii) *for any $x, x', y \in X$ with $d(x,x') \le (2A_0)^{-1}[\delta^k + d(x,y)]$,*

$$\begin{aligned}&\left|\widetilde{Q}_k(x,y) - \widetilde{Q}_k(x',y)\right| \\ &\quad \le C \left[\frac{d(x,x')}{\delta^k + d(x,y)}\right]^\beta \frac{1}{V_{\delta^k}(x) + V(x,y)} \left[\frac{\delta^k}{\delta^k + d(x,y)}\right]^\gamma;\end{aligned}$$

(iii) *for any $x \in X$,*

$$\int_X \widetilde{Q}_k(x,y)\,d\mu(y) = 0 = \int_X \widetilde{Q}_k(y,x)\,d\mu(y).$$

Next, we recall the homogeneous discrete Calderón reproducing formulae established in [29]. To this end, let $j_0 \in \mathbb{N}$ be a sufficiently large integer such that $\delta^{j_0} \le (2A_0)^{-4} C^\natural$, where $C^\natural$ is as in Lemma 2.3. Based on Lemma 2.3, for any $k \in \mathbb{Z}$ and $\alpha \in \mathcal{A}_k$, we let

$$\mathcal{N}(k,\alpha) := \{\tau \in \mathcal{A}_{k+j_0} : Q_\tau^{k+j_0} \subset Q_\alpha^k\}$$

and $N(k,\alpha)$ be the cardinality of the set $\mathcal{N}(k,\alpha)$. For any $k \in \mathbb{Z}$ and $\alpha \in \mathcal{A}_k$, we rearrange the set $\{Q_\tau^{k+j_0} : \tau \in \mathcal{N}(k,\alpha)\}$ as $\{Q_\alpha^{k,m}\}_{m=1}^{N(k,\alpha)}$, whose centers are denoted, respectively, by $\{z_\alpha^{k,m}\}_{m=1}^{N(k,\alpha)}$.

Theorem 2.7 *Let $\{Q_k\}_{k\in\mathbb{Z}}$ be an* exp-ATI *and $\beta, \gamma \in (0,\eta)$. For any $k \in \mathbb{Z}$, $\alpha \in \mathcal{A}_k$ and $m \in \{1,\dots,N(k,\alpha)\}$, suppose that $y_\alpha^{k,m}$ is an arbitrary point in $Q_\alpha^{k,m}$. Then, for any $i \in \{1,2\}$, there exists a sequence $\{\widetilde{Q}_k^{(i)}\}_{k=-\infty}^\infty$ of bounded linear operators on $L^2(X)$ such that, for any $f \in (\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$,*

$$f(\cdot) = \sum_{k=-\infty}^{\infty} \sum_{\alpha \in \mathcal{A}_k} \sum_{m=1}^{N(k,\alpha)} \widetilde{Q}_k^{(1)}\left(\cdot, y_\alpha^{k,m}\right) \int_{Q_\alpha^{k,m}} Q_k f(y)\, d\mu(y)$$
$$= \sum_{k=-\infty}^{\infty} \sum_{\alpha \in \mathcal{A}_k} \sum_{m=1}^{N(k,\alpha)} \mu\left(Q_\alpha^{k,m}\right) \widetilde{Q}_k^{(2)}\left(\cdot, y_\alpha^{k,m}\right) Q_k f\left(y_\alpha^{k,m}\right),$$

where the equalities converge in $(\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$. *Moreover, for any* $k \in \mathbb{Z}$, *the kernels of* $\widetilde{Q}_k^{(1)}$ *and* $\widetilde{Q}_k^{(2)}$ *satisfy (i), (ii) and (iii) of Theorem 2.6.*

To recall the inhomogeneous discrete Calderón reproducing formulae established in [29], we introduce the following 1-exp-ATI and exp-IATI.

Definition 2.8 A sequence $\{P_k\}_{k=-\infty}^{\infty}$ of bounded linear operators on $L^2(X)$ is called an *approximation of the identity with exponential decay and integration* 1 (for short, 1-exp-ATI) if $\{P_k\}_{k=-\infty}^{\infty}$ has the following properties:

(i) for any $k \in \mathbb{Z}$, P_k satisfies (ii), (iii) and (iv) of Definition 2.4 but without the exponential decay factor

$$\exp\left\{-\nu\left[\frac{\max\{d(x,\mathcal{Y}^k), d(y,\mathcal{Y}^k)\}}{\delta^k}\right]^a\right\};$$

(ii) for any $k \in \mathbb{Z}$ and $x \in X$, $\int_X P_k(x,y)\, d\mu(y) = 1 = \int_X P_k(y,x)\, d\mu(y)$;

(iii) for any $k \in \mathbb{Z}$, letting $Q_k := P_k - P_{k-1}$, then $\{Q_k\}_{k\in\mathbb{Z}}$ is an exp-ATI.

Remark 2.9 The existence of the 1-exp-ATI is guaranteed by [1, Lemma 10.1]. Moreover, by the proofs of [29, Proposition 2.9] and [27, Proposition 2.7(iv)], we know that, for any $f \in L^2(X)$, $\lim_{k\to\infty} P_k f = f$ in $L^2(X)$.

Definition 2.10 A sequence $\{Q_k\}_{k=0}^{\infty}$ of bounded linear operators on $L^2(X)$ is called an *inhomogeneous approximation of the identity with exponential decay* (for short, exp-IATI) if there exists a 1-exp-ATI $\{P_k\}_{k=-\infty}^{\infty}$ such that $Q_0 = P_0$ and $Q_k = P_k - P_{k-1}$ for any $k \in \mathbb{N}$.

Next we recall the following inhomogeneous discrete Calderón reproducing formula established in [29].

Theorem 2.11 *Let* $\{Q_k\}_{k=0}^{\infty}$ *be an* exp-IATI *and* $\beta,\ \gamma \in (0,\eta)$. *Then there exists a sequence* $\{\widetilde{Q}_k\}_{k=0}^{\infty}$ *of bounded linear operators on* $L^2(X)$ *such that, for any* $f \in (\mathcal{G}_0^\eta(\beta,\gamma))'$,

$$f(\cdot) = \sum_{k=0}^{N} \sum_{\alpha \in \mathcal{A}_k} \sum_{m=1}^{N(0,\alpha)} \int_{Q_\alpha^{k,m}} \widetilde{Q}_k(\cdot, y)\, d\mu(y) Q_{\alpha,1}^{k,m}(f)$$
$$+ \sum_{k=N+1}^{\infty} \sum_{\alpha \in \mathcal{A}_k} \sum_{m=1}^{N(k,\alpha)} \mu\left(Q_\alpha^{k,m}\right) \widetilde{Q}_k\left(\cdot, y_\alpha^{k,m}\right) Q_k f\left(y_\alpha^{k,m}\right),$$

where the equality converges in $(\mathcal{G}_0^\eta(\beta,\gamma))'$, *every* $y_\alpha^{k,m}$ *is an arbitrary point in* $Q_\alpha^{k,m}$ *and, for any* $k \in \{0, \ldots, N\}$,

$$Q_{\alpha,1}^{k,m}(f) := \frac{1}{\mu(Q_\alpha^{k,m})} \int_{Q_\alpha^{k,m}} Q_k f(u)\, d\mu(u).$$

Moreover, for any $k \in \mathbb{Z}_+$, $\widetilde{Q}_k$ *satisfies (i) and (ii) of Theorem 2.6 and, for any* $x \in X$,

$$\int_X \widetilde{Q}_k(x,y)\, d\mu(y) = \int_X \widetilde{Q}_k(y,x)\, d\mu(y) = \begin{cases} 1 & \text{if } k \in \{0, \ldots, N\}, \\ 0 & \text{if } k \in \{N+1, N+2, \ldots\}, \end{cases}$$

where $N \in \mathbb{N}$ *is some fixed constant independent of* f *and* $y_\alpha^{k,m}$.

3 Hardy Spaces via Various Maximal Functions

Let $\beta,\ \gamma \in (0,\eta)$ and $f \in (\mathcal{G}_0^\eta(\beta,\gamma))'$. Let $\{P_k\}_{k\in\mathbb{Z}}$ be a 1-exp-ATI as in Definition 2.8. Define the *radial maximal function* $\mathcal{M}^+(f)$ of f by setting

$$\mathcal{M}^+(f)(x) := \sup_{k\in\mathbb{Z}} |P_k f(x)|, \quad \forall\, x \in X.$$

Define the *non-tangential maximal function* $\mathcal{M}_\theta(f)$ *of* f *with aperture* $\theta \in (0,\infty)$ by setting

$$\mathcal{M}_\theta(f)(x) := \sup_{k\in\mathbb{Z}} \sup_{y\in B(x,\theta\delta^k)} |P_k f(y)|, \quad \forall\, x \in X.$$

Also, define the *grand maximal function* f^* of f by setting

$$f^*(x) := \sup\left\{|\langle f, \varphi\rangle| :\ \varphi \in \mathcal{G}_0^\eta(\beta,\gamma) \text{ and } \|\varphi\|_{\mathcal{G}(x,r_0,\beta,\gamma)} \le 1 \text{ for some } r_0 \in (0,\infty)\right\}, \quad \forall\, x \in X.$$

Correspondingly, for any $p \in (0,\infty]$, the *Hardy spaces* $H^{+,p}(X)$, $H_\theta^p(X)$ with $\theta \in (0,\infty)$ and $H^{*,p}(X)$ are defined, respectively, by setting

$$H^{+,p}(X) := \left\{f \in \left(\mathcal{G}_0^\eta(\beta,\gamma)\right)' :\ \|f\|_{H^{+,p}(X)} := \|\mathcal{M}^+(f)\|_{L^p(X)} < \infty\right\},$$

$$H_\theta^p(X) := \left\{f \in \left(\mathcal{G}_0^\eta(\beta,\gamma)\right)' :\ \|f\|_{H_\theta^p(X)} := \|\mathcal{M}_\theta(f)\|_{L^p(X)} < \infty\right\}$$

and

$$H^{*,p}(X) := \left\{f \in \left(\mathcal{G}_0^\eta(\beta,\gamma)\right)' :\ \|f\|_{H^{*,p}(X)} := \|f^*\|_{L^p(X)} < \infty\right\}.$$

Based on [20, Remark 2.9(ii)], we easily observe that, for any $f \in (\mathcal{G}_0^\eta(\beta,\gamma))'$ and $x \in X$,

$$\mathcal{M}^+(f)(x) \le \mathcal{M}_\theta(f)(x) \le Cf^*(x), \tag{3.1}$$

where C is a positive constant only depending on θ.

The aim of this section is to prove that the Hardy spaces $H^{+,p}(X)$, $H_\theta^p(X)$ and $H^{*,p}(X)$ are mutually equivalent when $p \in (\omega/(\omega+\eta), \infty]$ in the sense of equivalent (quasi-)norms (see Sect. 3.2); in particular, they all are equivalent to the Lebesgue space $L^p(X)$ when $p \in (1,\infty]$ in the sense of equivalent norms (see Sect. 3.1). Moreover, we prove that $H^{*,p}(X)$ is independent of the choices of the distribution space $(\mathcal{G}_0^\eta(\beta,\gamma))'$ whenever $\beta,\ \gamma \in (\omega(1/p-1),\eta)$; see Proposition 3.8 below.

3.1 Equivalence to the Lebesgue Space $L^p(X)$ When $p \in (1,\infty]$

In this section, we show that the Hardy spaces $H^{+,p}(X)$, $H_\theta^p(X)$ and $H^{*,p}(X)$ are all equivalent to the Lebesgue space $L^p(X)$, when $p \in (1,\infty]$, in the sense of both representing the same distributions and equivalent norms. First we give some basic properties of $H^{*,p}(X)$.

Proposition 3.1 *Let $p \in (0,\infty]$. Then $H^{*,p}(X)$ is a (quasi-)Banach space, which is continuously embedded into $(\mathcal{G}_0^\eta(\beta,\gamma))'$, where $\beta,\ \gamma \in (0,\eta)$.*

Proof Let $f \in H^{*,p}(X)$ and $\varphi \in \mathcal{G}_0^\eta(\beta,\gamma)$ with $\|\varphi\|_{\mathcal{G}(\beta,\gamma)} \le 1$. For any $x \in B(x_0,1)$, by Definition 2.1, we easily know that $\|\varphi\|_{\mathcal{G}(x,1,\beta,\gamma)} \lesssim 1$ with the implicit positive constant independent of x and hence $|\langle f,\varphi\rangle| \lesssim f^*(x)$. Therefore, for any $\varphi \in \mathcal{G}_0^\eta(\beta,\gamma)$ with $\beta,\ \gamma \in (0,\eta)$, we have

$$|\langle f,\varphi\rangle|^p \lesssim \frac{1}{V_1(x_0)}\int_{B(x_0,1)} [f^*(x)]^p\,d\mu(x) \lesssim \|f^*\|_{L^p(X)}^p \sim \|f\|_{H^{*,p}(X)}^p.$$

This implies that $H^{*,p}(X)$ is continuously embedded into $(\mathcal{G}_0^\eta(\beta,\gamma))'$.

To see that $H^{*,p}(X)$ is a (quasi-)Banach space, we only prove its completeness. Indeed, suppose that $\{f_k\}_{k=1}^\infty$ in $H^{*,p}(X)$ is a Cauchy sequence, which is also a Cauchy sequence in $(\mathcal{G}_0^\eta(\beta,\gamma))'$ with $\beta,\ \gamma \in (0,\eta)$. By the completeness of $(\mathcal{G}_0^\eta(\beta,\gamma))'$, the sequence $\{f_k\}_{k=1}^\infty$ converges to some element $f \in (\mathcal{G}_0^\eta(\beta,\gamma))'$ as $k \to \infty$. If $\varphi \in \mathcal{G}_0^\eta(\beta,\gamma)$ satisfies $\|\varphi\|_{\mathcal{G}(x,r_0,\beta,\gamma)} \le 1$ for some $x \in X$ and $r_0 \in (0,\infty)$, then $|\langle f_{k+l}-f_k,\varphi\rangle| \le (f_{k+l}-f_k)^*(x)$ for any $k,\ l \in \mathbb{N}$. Letting $l \to \infty$, we obtain

$$|\langle f-f_k,\varphi\rangle| \le \liminf_{l\to\infty}(f_{k+l}-f_k)^*(x),$$

which further implies that, for any $x \in X$,

$$(f-f_k)^*(x) \le \liminf_{l\to\infty}(f_{k+l}-f_k)^*(x).$$

By the Fatou lemma, we conclude that

$$\|(f-f_k)^*\|_{L^p(X)} \le \liminf_{l\to\infty}\|(f_{k+l}-f_k)^*\|_{L^p(X)} \to 0$$

as $k \to \infty$, which, together with the sublinearity of $\|\cdot\|_{H^{*,p}(X)}$, further implies that $f \in H^{*,p}(X)$ and $\lim_{k\to\infty} \|f - f_k\|_{H^{*,p}(X)} = 0$. Therefore, $H^{*,p}(X)$ is complete. This finishes the proof of Proposition 3.1. □

To show the equivalence of $H^{+,p}(X)$, $H^p_\theta(X)$ and $H^{*,p}(X)$ to the Lebesgue space $L^p(X)$ when $p \in (1, \infty]$ in the sense of both representing the same distributions and equivalent norms, we need the following technical lemma.

Lemma 3.2 *Let* $\{P_k\}_{k\in\mathbb{Z}}$ *be a* 1-exp-ATI *as in Definition* 2.8. *Assume that* $\beta, \gamma \in (0, \eta)$. *Then the following statements hold true:*

- (i) *there exists a positive constant* C *such that, for any* $k \in \mathbb{Z}$ *and* $\varphi \in \mathcal{G}(\beta, \gamma)$, $\|P_k\varphi\|_{\mathcal{G}(\beta,\gamma)} \le C\|\varphi\|_{\mathcal{G}(\beta,\gamma)}$;
- (ii) *for any* $f \in \mathcal{G}(\beta, \gamma)$ *and* $\beta' \in (0, \beta)$, $\lim_{k\to\infty} P_k f = f$ *in* $\mathcal{G}(\beta', \gamma)$;
- (iii) *if* $f \in \mathcal{G}^\eta_0(\beta, \gamma)$ *[resp.,* $f \in (\mathcal{G}^\eta_0(\beta, \gamma))'$*], then* $\lim_{k\to\infty} P_k f = f$ *in* $\mathcal{G}^\eta_0(\beta, \gamma)$ *[resp.,* $(\mathcal{G}^\eta_0(\beta, \gamma))'$*].*

Proof The proof of (i) can be obtained by the method used in the proof of [29, Lemma 4.14]. The proof of (ii) is given in [20, Lemma 3.6], whose proof does not rely on the reverse doubling condition of μ and the metric d. We obtain (iii) directly by (i), (ii) and a standard duality argument. This finishes the proof of Lemma 3.2. □

We have the following proposition.

Proposition 3.3 *Let* $p \in [1, \infty]$, $\beta, \gamma \in (0, \eta)$ *and* $\{P_k\}_{k\in\mathbb{Z}}$ *be a* 1-exp-ATI. *If* $f \in (\mathcal{G}^\eta_0(\beta, \gamma))'$ *belongs to* $H^{+,p}(X)$, *then there exists* $\widetilde{f} \in L^p(X)$ *such that, for any* $\varphi \in \mathcal{G}^\eta_0(\beta, \gamma)$,

$$\langle f, \varphi\rangle = \int_X \widetilde{f}(x)\varphi(x)\, d\mu(x) \tag{3.2}$$

and $\|\widetilde{f}\|_{L^p(X)} \le \|\mathcal{M}^+(f)\|_{L^p(X)}$*; moreover, if* $p \in [1, \infty)$, *then, for almost every* $x \in X$, $|\widetilde{f}(x)| \le \mathcal{M}^+(f)(x)$.

Proof Let $f \in (\mathcal{G}^\eta_0(\beta, \gamma))'$ and $\mathcal{M}^+(f) := \sup_{k\in\mathbb{Z}} |P_k f| \in L^p(X)$, where $\{P_k\}_{k\in\mathbb{Z}}$ is a 1-exp-ATI as in Definition 2.8. Then $\{P_k f\}_{k\in\mathbb{Z}}$ is uniformly bounded in $L^p(X)$. If $p \in (1, \infty]$, then $p' \in [1, \infty)$ and $L^{p'}(X)$ is separable. Thus, by the Banach–Alaoglu theorem (see, for example, [46, Theorem 3.17]), we find a function $\widetilde{f} \in L^p(X)$ and a sequence $\{k_j\}_{j=1}^\infty \subset \mathbb{Z}$ such that $k_j \to \infty$ and $P_{k_j} f \to \widetilde{f}$ as $j \to \infty$ in the weak-$*$ topology of $L^p(X)$. By this and the Hölder inequality, for any $g \in L^{p'}(X)$, we have

$$\left|\int_X \widetilde{f}(x)g(x)\, d\mu(x)\right| = \lim_{j\to\infty} \left|\int_X P_{k_j} f(x)g(x)\, d\mu(x)\right| \le \|\mathcal{M}^+(f)\|_{L^p(X)}\|g\|_{L^{p'}(X)},$$

which further implies that $\|\widetilde{f}\|_{L^p(X)} \le \|\mathcal{M}^+(f)\|_{L^p(X)}$.

If $p = 1$, notice that $\|\sup_{k\in\mathbb{Z}} |P_k f|\|_{L^1(X)} = \|\mathcal{M}^+(f)\|_{L^1(X)} < \infty$. Then, by the proof of [52, Theorem III.C.12], $\{P_k f\}_{k\in\mathbb{Z}}$ is relatively compact in $L^1(X)$. Therefore, by the Eberlin–Šmulian theorem (see [52, II.C]), we know that $\{P_k f\}_{k\in\mathbb{Z}}$ is weakly sequentially compact, that is, there exist a function $\widetilde{f} \in L^1(X)$ and a subsequence $\{P_{k_j} f\}_{j=1}^{\infty}$ such that $\{P_{k_j} f\}_{j=1}^{\infty}$ converges to $\widetilde{f}$ weakly in $L^1(X)$. As the arguments for the case $p \in (1, \infty]$, we still have $\|\widetilde{f}\|_{L^1(X)} \le \|\mathcal{M}^+(f)\|_{L^1(X)}$.

Moreover, for any $\varphi \in \mathcal{G}_0^{\eta}(\beta, \gamma)$, by the fact $\mathcal{G}_0^{\eta}(\beta, \gamma) \subset L^p(X)$ for any $p \in [1, \infty]$ and Lemma 3.2(iii), we conclude that

$$\langle f, \varphi\rangle = \lim_{k\to\infty} \langle P_{k_j} f, \varphi\rangle = \lim_{j\to\infty} \int_X P_{k_j} f(x)\varphi(x)\, d\mu(x) = \int_X \widetilde{f}(x)\varphi(x)\, d\mu(x). \tag{3.3}$$

Let $p \in [1, \infty)$. For any $j \in \mathbb{N}$ and $x \in X$, we have $P_{k_j}(x, \cdot) \in \mathcal{G}(\eta, \eta)$ (see the proof of [29, Proposition 2.10]), which, together with (3.3), implies that

$$P_{k_j} f(x) = \langle f, P_{k_j}(x, \cdot)\rangle = \int_X P_{k_j}(x, y)\widetilde{f}(y)\, d\mu(y) = P_{k_j}\widetilde{f}(x).$$

From this and [27, Proposition 2.7(iv)], we deduce that $\{P_{k_j} f\}_{j\in\mathbb{N}}$ converges to $\widetilde{f}$ in the sense of $\|\cdot\|_{L^p(X)}$. Then, by the Riesz theorem, we find a subsequence of $\{P_{k_j} f\}_{j\in\mathbb{N}}$, still denoted by $\{P_{k_j} f\}_{j\in\mathbb{N}}$, such that $P_{k_j} f(x) \to \widetilde{f}(x)$ as $k_j \to \infty$ for almost every $x \in X$. Therefore, $|\widetilde{f}(x)| \le \mathcal{M}^+(f)(x)$ for almost every $x \in X$. This finishes the proof of Proposition 3.3. □

Finally, we show the following main result of this section.

Theorem 3.4 *Let $p \in (1, \infty]$ and $\beta,\ \gamma \in (0, \eta)$. Then the following hold true:*

(i) *if $f \in (\mathcal{G}_0^{\eta}(\beta, \gamma))'$ belongs to $H^{+,p}(X)$, then there exists $\widetilde{f} \in L^p(X)$ such that (3.2) holds true and $\|\widetilde{f}\|_{L^p(X)} \le \|f\|_{H^{+,p}(X)}$;*
(ii) *any $f \in L^p(X)$ induces a distribution on $\mathcal{G}_0^{\eta}(\beta, \gamma)$ as in (3.2), still denoted by f, such that $f \in H^{*,p}(X)$ and $\|f\|_{H^{*,p}(X)} \le C\|f\|_{L^p(X)}$, where C is a positive constant independent of f.*

Consequently, for any fixed $\theta \in (0, \infty)$, $H^{+,p}(X) = H_{\theta}^p(X) = H^{*,p}(X) = L^p(X)$ in the sense of both representing the same distributions and equivalent norms.

Proof We obtain (i) directly by Proposition 3.3. Now we prove (ii). Suppose that $p \in (1, \infty]$ and $f \in L^p(X)$. Clearly, f induces a distribution on $\mathcal{G}_0^{\eta}(\beta, \gamma)$ as in (3.2). By [20, Proposition 3.9], we find that, for almost every $x \in X$, $f^*(x) \lesssim \mathcal{M}(f)(x)$, with the implicit positive constant independent of f and x. Therefore, from the boundedness of $\mathcal{M}$ on $L^p(X)$, we deduce that $\|f^*\|_{L^p(X)} \lesssim \|\mathcal{M}(f)\|_{L^p(X)} \lesssim \|f\|_{L^p(X)}$. This finishes the proof of (ii).

By (i), (ii) and (3.1), we obtain $H^{+,p}(X) = H_{\theta}^p(X) = H^{*,p}(X) = L^p(X)$, which completes the proof of Theorem 3.4. □

3.2 Equivalence of Hardy Spaces Defined via Various Maximal Functions

The main aim of this section concerns the equivalence of Hardy spaces defined via various maximal functions for the case $p \in (\omega/(\omega + \eta), 1]$. Indeed, our goal is to show the following equivalence theorem.

Theorem 3.5 *Assume that* $p \in (\omega/(\omega+\eta), 1]$ *and* $\theta \in (0,\infty)$. *Then, for any* $f \in (\mathcal{G}_0^\eta(\beta,\gamma))'$ *with* $\beta,\ \gamma \in (\omega(1/p-1), \eta)$,

$$\|f\|_{H^{+,p}(X)} \sim \|f\|_{H^p_\theta(X)} \sim \|f\|_{H^{*,p}(X)},$$

with positive equivalence constants independent of f. *In other words,* $H^{+,p}(X) = H^p_\theta(X) = H^{*,p}(X)$ *with equivalent (quasi-)norms.*

To prove Theorem 3.5, we borrow some ideas from [56]. To this end, we need the following two technical lemmas.

Lemma 3.6 *Assume that* $\phi \in \mathcal{G}_0^\eta(\beta,\gamma)$ *with* $\beta,\ \gamma \in (0,\eta)$. *Let* $\sigma := \int_X \phi(x)\,d\mu(x)$. *If* $\psi \in \mathcal{G}(\eta,\eta)$ *with* $\int_X \psi(x)\,d\mu(x) = 1$, *then* $\phi - \sigma\psi \in \mathring{\mathcal{G}}_0^\eta(\beta,\gamma)$.

Proof Since $\phi \in \mathcal{G}_0^\eta(\beta,\gamma)$ with $\beta,\ \gamma \in (0,\eta)$, it follows that there exists $\{\phi_n\}_{n=1}^\infty \subset \mathcal{G}(\eta,\eta)$ such that $\lim_{n\to\infty}\|\phi-\phi_n\|_{\mathcal{G}(\beta,\gamma)} = 0$. Letting $\sigma_n := \int_X \phi_n(x)\,d\mu(x)$ for any $n \in \mathbb{N}$, by Definition 2.1 and Lemma 2.2(ii), we conclude that $\lim_{n\to\infty}|\sigma-\sigma_n| = 0$, where $\sigma := \int_X \phi(x)\,d\mu(x)$. Let $\varphi_n := \phi_n - \sigma_n\psi$ for any $n \in \mathbb{N}$. Then $\varphi_n \in \mathring{\mathcal{G}}(\eta,\eta)$ and

$$\|\phi - \sigma\psi - \varphi_n\|_{\mathcal{G}(\beta,\gamma)} \le \|\phi-\phi_n\|_{\mathcal{G}(\beta,\gamma)} + |\sigma-\sigma_n|\|\psi\|_{\mathcal{G}(\beta,\gamma)} \to 0 \quad \text{as } n\to\infty.$$

Thus, $\phi - \sigma\psi \in \mathring{\mathcal{G}}_0^\eta(\beta,\gamma)$. This finishes the proof of Lemma 3.6. □

The next lemma comes from [27, Lemma 5.3], whose proof remains true for a quasi-metric d and also does not rely on the reverse doubling condition of μ.

Lemma 3.7 *Let all the notation be as in Theorem* 2.7. *Let* $k,\ k' \in \mathbb{Z}$, $\{a_\alpha^{k,m}\}_{k\in\mathbb{Z},\ \alpha\in\mathcal{A}_k,\ m\in\{1,\ldots,N(k,\alpha)\}} \subset \mathbb{C}$, $\gamma \in (0,\eta)$ *and* $r \in (\omega/(\omega+\gamma), 1]$. *Then there exists a positive constant* C, *independent of* k, k', $y_\alpha^{k,m} \in Q_\alpha^{k,m}$ *and* $a_\alpha^{k,m}$ *with* $k \in \mathbb{Z}$, $\alpha \in \mathcal{A}_k$ *and* $m \in \{1,\ldots,N(k,\alpha)\}$, *such that, for any* $x \in X$,

$$\sum_{\alpha\in\mathcal{A}_k}\sum_{m=1}^{N(k,\alpha)} \mu\left(Q_\alpha^{k,m}\right) \frac{1}{V_{\delta^{k\wedge k'}}(x) + V(x, y_\alpha^{k,m})} \left[\frac{\delta^{k\wedge k'}}{\delta^{k\wedge k'} + d(x,y_\alpha^{k,m})}\right]^\gamma \left|a_\alpha^{k,m}\right|$$
$$\le C\delta^{[k-(k\wedge k')]\omega(1-\frac{1}{r})}\left[\mathcal{M}\left(\sum_{\alpha\in\mathcal{A}_k}\sum_{m=1}^{N(k,\alpha)} \left|a_\alpha^{k,m}\right|^r \mathbf{1}_{Q_\alpha^{k,m}}\right)(x)\right]^{\frac{1}{r}}.$$

Now we show Theorem 3.5 by using the above two technical lemmas. In what follows, the *symbol* $\epsilon \to 0^+$ means that $\epsilon \in (0,\infty)$ and $\epsilon \to 0$.

Proof of Theorem 3.5 Let $f \in (\mathcal{G}_0^\eta(\beta,\gamma))'$ with $\beta,\ \gamma \in (\omega(1/p-1),\eta)$. Fix $\theta \in (0,\infty)$. By (3.1), we have

$$\|\mathcal{M}^+(f)\|_{L^p(X)} \le \|\mathcal{M}_\theta(f)\|_{L^p(X)} \lesssim \|f^*\|_{L^p(X)}.$$

Thus, the proof of Theorem 3.5 is reduced to showing

$$\|f^*\|_{L^p(X)} \lesssim \|\mathcal{M}^+(f)\|_{L^p(X)}. \tag{3.4}$$

To obtain (3.4), it suffices to prove that, for some $r \in (0, p)$ and any $x \in X$,

$$f^*(x) \lesssim \mathcal{M}^+(f)(x) + \left\{\mathcal{M}\left(\left[\mathcal{M}^+(f)\right]^r\right)(x)\right\}^{\frac{1}{r}}. \tag{3.5}$$

If (3.5) holds true, then, by the boundedness of $\mathcal{M}$ on $L^{p/r}(X)$, we conclude that

$$\|f^*\|_{L^p(X)} \lesssim \|\mathcal{M}^+(f)\|_{L^p(X)} + \left\|\mathcal{M}\left(\left[\mathcal{M}^+(f)\right]^r\right)\right\|_{L^{p/r}(X)}^{\frac{1}{r}} \sim \|\mathcal{M}^+(f)\|_{L^p(X)},$$

which proves (3.4).

We now fix $x \in X$ and show (3.5). Let $\{P_k\}_{k\in\mathbb{Z}}$ be a 1-exp-ATI. For any $k \in \mathbb{Z}$, define $Q_k := P_k - P_{k-1}$. Then $\{Q_k\}_{k\in\mathbb{Z}}$ is an exp-ATI. Assume for the moment that, for any $\varphi \in \mathring{\mathcal{G}}_0^\eta(\beta,\gamma)$ with $\|\varphi\|_{\mathcal{G}(x,\delta^l,\beta,\gamma)} \le 1$ for some $l \in \mathbb{Z}$,

$$|\langle f, \varphi\rangle| \lesssim \left\{\mathcal{M}\left(\left[\mathcal{M}^+(f)\right]^r\right)(x)\right\}^{\frac{1}{r}}. \tag{3.6}$$

We now use (3.6) to show (3.5). For any $\phi \in \mathcal{G}_0^\eta(\beta,\gamma)$ with $\|\phi\|_{\mathcal{G}(x,r_0,\beta,\gamma)} \le 1$ for some $r_0 \in (0,\infty)$, choose $l \in \mathbb{Z}$ such that $\delta^{l+1} \le r_0 < \delta^l$. Clearly, $\|\phi\|_{\mathcal{G}(x,\delta^l,\beta,\gamma)} \lesssim 1$. Let $\sigma := \int_X \phi(y)\,d\mu(y)$ and $\varphi := \phi - \sigma P_l(x,\cdot)$. Notice that $\int_X P_l(x,y)\,d\mu(y) = 1$ and $P_l(x,\cdot) \in \mathcal{G}(\eta,\eta)$ (see the proof of [29, Proposition 2.10]). From Lemma 3.6, it follows that $\varphi \in \mathring{\mathcal{G}}_0^\eta(\beta,\gamma)$. Moreover, $\|\varphi\|_{\mathcal{G}(x,\delta^l,\beta,\gamma)} \lesssim \|\phi\|_{\mathcal{G}(x,\delta^l,\beta,\gamma)} + |\sigma|\|P_l(x,\cdot)\|_{\mathcal{G}(x,\delta^l,\beta,\gamma)} \lesssim 1$. By (3.6), we know that

$$\begin{aligned}|\langle f,\phi\rangle| &\le |\langle f,\varphi\rangle| + |\sigma||\langle f, P_l(x,\cdot)\rangle|\\ &\lesssim \left\{\mathcal{M}\left(\left[\mathcal{M}^+(f)\right]^r\right)(x)\right\}^{\frac{1}{r}} + |P_l f(x)|\\ &\lesssim \left\{\mathcal{M}\left(\left[\mathcal{M}^+(f)\right]^r\right)(x)\right\}^{\frac{1}{r}} + \mathcal{M}^+(f)(x),\end{aligned}$$

which is exactly (3.5).

It remains to prove (3.6). For any $\epsilon \in (0,\infty)$, choose $y_\alpha^{k,m} \in Q_\alpha^{k,m}$ such that

$$\left|Q_k f\left(y_\alpha^{k,m}\right)\right| \le \inf_{z\in Q_\alpha^{k,m}} |Q_k f(z)| + \epsilon \le 2 \inf_{z\in Q_\alpha^{k,m}} \mathcal{M}^+(f)(z) + \epsilon.$$

Let $g := f|_{\mathring{\mathcal{G}}_0^\eta(\beta,\gamma)}$ be the restriction of f on $\mathring{\mathcal{G}}_0^\eta(\beta,\gamma)$. Obviously, $g \in (\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$ and $\|g\|_{(\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'} \le \|f\|_{(\mathcal{G}_0^\eta(\beta,\gamma))'}$. By Theorem 2.7, we conclude that

$$\langle f,\varphi\rangle = \langle g,\varphi\rangle = \sum_{k=-\infty}^{\infty}\sum_{\alpha\in\mathcal{A}_k}\sum_{m=1}^{N(k,\alpha)} \mu\left(Q_\alpha^{k,m}\right)\widetilde{Q}_k^*\varphi\left(y_\alpha^{k,m}\right)Q_k g\left(y_\alpha^{k,m}\right)$$

$$= \sum_{k=-\infty}^{\infty} \sum_{\alpha \in \mathcal{A}_k} \sum_{m=1}^{N(k,\alpha)} \mu\left(Q_\alpha^{k,m}\right) \widetilde{Q}_k^* \varphi\left(y_\alpha^{k,m}\right) Q_k f\left(y_\alpha^{k,m}\right),$$

where $\widetilde{Q}_k^*$ denotes the *dual operator* of $\widetilde{Q}_k$. By the proof of [27, (3.2)], which remains true for a quasi-metric d and does not rely on the reverse doubling condition of μ, we find that, for any fixed $\beta' \in (0, \beta \wedge \gamma)$ and any $k \in \mathbb{Z}$,

$$\left|\widetilde{Q}_k^* \varphi\left(y_\alpha^{k,m}\right)\right| \lesssim \delta^{|k-l|\beta'} \frac{1}{V_{\delta^{k\wedge l}}(x) + V(x, y_\alpha^{k,m})} \left[\frac{\delta^{k\wedge l}}{\delta^{k\wedge l} + d(x, y_\alpha^{k,m})}\right]^\gamma. \tag{3.7}$$

Choose $\beta' \in (0, \beta \wedge \gamma)$ such that $\omega/(\omega + \beta') < p$. From this and Lemma 3.7, we deduce that, for any fixed $r \in (\omega/(\omega + \beta'), p)$,

$$\begin{aligned}
|\langle f, \varphi\rangle| &\lesssim \sum_{k=-\infty}^{\infty} \delta^{|k-l|\beta'} \sum_{\alpha \in \mathcal{A}_k} \sum_{m=1}^{N(k,\alpha)} \mu\left(Q_\alpha^{k,m}\right) \\
&\quad \times \frac{\inf_{z \in Q_\alpha^{k,m}} \mathcal{M}^+(f)(z) + \epsilon}{V_{\delta^{k\wedge l}}(x) + V(x, y_\alpha^{k,m})} \left[\frac{\delta^{k\wedge l}}{\delta^{k\wedge l} + d(x, y_\alpha^{k,m})}\right]^\gamma \\
&\lesssim \sum_{k=-\infty}^{\infty} \delta^{|k-l|\beta'} \delta^{[k-(k\wedge l)]\omega\left(1-\frac{1}{r}\right)} \\
&\quad \times \left\{\mathcal{M}\left(\sum_{\alpha \in \mathcal{A}_k} \sum_{m=1}^{N(k,\alpha)} \left[\inf_{z \in Q_\alpha^{k,m}} \mathcal{M}^+(f)(z) + \epsilon\right]^r \mathbf{1}_{Q_\alpha^{k,m}}\right)(x)\right\}^{\frac{1}{r}} \\
&\lesssim \sum_{k=-\infty}^{\infty} \delta^{|k-l|\beta'} \delta^{[k-(k\wedge l)]\omega\left(1-\frac{1}{r}\right)} \left\{\mathcal{M}\left(\left[\mathcal{M}^+(f) + \epsilon\right]^r\right)(x)\right\}^{\frac{1}{r}} \\
&\lesssim \left\{\mathcal{M}\left(\left[\mathcal{M}^+(f)\right]^r\right)(x) + \epsilon^r\right\}^{\frac{1}{r}} \to \left\{\mathcal{M}\left(\left[\mathcal{M}^+(f)\right]^r\right)(x)\right\}^{\frac{1}{r}} \quad \text{as } \epsilon \to 0^+.
\end{aligned} \tag{3.8}$$

This proves (3.6) and hence finishes the proof of Theorem 3.5. □

To conclude this section, we show that the Hardy space $H^{*,p}(X)$ is independent of the choices of $(\mathcal{G}_0^\eta(\beta, \gamma))'$ whenever $\beta, \gamma \in (\omega(1/p - 1), \eta)$.

Proposition 3.8 *Let $p \in (\omega/(\omega + \eta), 1]$ and $\beta_1, \beta_2, \gamma_1, \gamma_2 \in (\omega(1/p - 1), \eta)$. If $f \in (\mathcal{G}_0^\eta(\beta_1, \gamma_1))'$ and $f \in H^{*,p}(X)$, then $f \in (\mathcal{G}_0^\eta(\beta_2, \gamma_2))'$.*

Proof Let $f \in (\mathcal{G}_0^\eta(\beta_1, \gamma_1))'$ with $\|f\|_{H^{*,p}(X)} < \infty$. We first prove that there exists $\theta \in (0, \infty)$ such that, for any $\varphi \in \mathcal{G}(\eta, \eta)$ with $\|\varphi\|_{\mathcal{G}(\beta_2,\gamma_2)} \le 1$,

$$|\langle f, \varphi\rangle| \lesssim \|\mathcal{M}_\theta(f)\|_{L^p(X)}. \tag{3.9}$$

Notice that $\varphi \in \mathcal{G}(\eta,\eta) \subset \mathcal{G}_0^\eta(\beta_1,\gamma_1)$ and $f \in (\mathcal{G}_0^\eta(\beta_1,\gamma_1))'$. With all the notation involved as in Theorem 2.11, we have

$$\langle f,\varphi\rangle = \sum_{k=0}^{N}\sum_{\alpha\in\mathcal{A}_0}\sum_{m=1}^{N(k,\alpha)}\int_{Q_\alpha^{k,m}} \widetilde{Q}_k^*\varphi(y)\,d\mu(y)\,Q_{\alpha,1}^{k,m}(f)$$
$$+\sum_{k=N+1}^{\infty}\sum_{\alpha\in\mathcal{A}_k}\sum_{m=1}^{N(k,\alpha)}\mu\left(Q_\alpha^{k,m}\right)\widetilde{Q}_k^*\varphi\left(y_\alpha^{k,m}\right)Q_kf\left(y_\alpha^{k,m}\right) =: \mathrm{Z}_1+\mathrm{Z}_2.$$

Choose $\theta := 2A_0C^\natural$ with $C^\natural$ as in Lemma 2.3(v). By the definition of $Q_\alpha^{k,m}$ and Lemma 2.3(v), we have $Q_\alpha^{k,m} \subset B(z_\alpha^{k,m}, C^\natural\delta^{k+j_0}) \subset B(z, 2A_0C^\natural\delta^k) = B(z,\theta\delta^k)$ for any $z \in Q_\alpha^{k,m}$.

Fix $x \in B(x_0,1)$. Then $\|\varphi\|_{\mathcal{G}(x,1,\beta_2,\gamma_2)} \sim \|\varphi\|_{\mathcal{G}(x_0,1,\beta_2,\gamma_2)} \lesssim 1$. If $k \in \{0,\ldots,N\}$, then we have $\|\varphi\|_{\mathcal{G}(x,\delta^k,\beta_2,\gamma_2)} \sim \|\varphi\|_{\mathcal{G}(x,1,\beta_2,\gamma_2)} \lesssim 1$, where the implicit positive constants are independent of x but can depend on N. Let $\beta_- := \min\{\beta_1,\gamma_1,\beta_2,\gamma_2\}$. By [27, (3.2)], we conclude that, for any $y \in Q_\alpha^{k,m}$,

$$|\widetilde{Q}_k^*\varphi(y)| \lesssim \frac{1}{V_1(x)+V(x,y)}\left[\frac{1}{1+d(x,y)}\right]^{\beta_-} \sim \frac{1}{V_1(x)+V(x,y_\alpha^{k,m})}\left[\frac{1}{1+d(x,y_\alpha^{k,m})}\right]^{\beta_-}.$$

Moreover, for any $k \in \{0,\ldots,N\}$ and $z \in Q_\alpha^{k,m}$, we have

$$\left|Q_{\alpha,1}^{k,m}(f)\right| \le \frac{1}{\mu(Q_\alpha^{k,m})}\int_{Q_\alpha^{k,m}}[|P_kf(y)|+|P_{k-1}f(y)|]\,d\mu(y) \le 2\mathcal{M}_\theta(f)(z).$$

Thus, we obtain

$$|\mathrm{Z}_1| \lesssim \sum_{k=0}^{N}\sum_{\alpha\in\mathcal{A}_k}\sum_{m=1}^{N(k,\alpha)}\mu\left(Q_\alpha^{k,m}\right)\frac{1}{V_1(x)+V(x,y_\alpha^{k,m})}$$
$$\times\left[\frac{1}{1+d(x,y_\alpha^{k,m})}\right]^{\beta_-}\inf_{z\in Q_\alpha^{k,m}}\mathcal{M}_\theta(f)(z). \tag{3.10}$$

If $k \in \{N+1,N+2,\ldots\}$, then $|Q_kf(y_\alpha^{k,m})| \le 2\inf_{z\in Q_\alpha^{k,m}}\mathcal{M}_\theta(f)(z)$. Again, by $\|\varphi\|_{\mathcal{G}(x,1,\beta_2,\gamma_2)} \lesssim 1$ and [27, (3.2)], we find that, for any fixed $\beta' \in (0,\beta_-)$,

$$\left|\widetilde{Q}_k^*\varphi\left(y_\alpha^{k,m}\right)\right| \lesssim \delta^{k\beta'}\frac{1}{V_1(x)+V(x,y_\alpha^{k,m})}\left[\frac{1}{1+d(x,y_\alpha^{k,m})}\right]^{\beta_-},$$

because now $k \in \mathbb{Z}_+$ and we do not need the cancelation of φ. Therefore, we have

$$|Z_2| \lesssim \sum_{k=N+1}^{\infty} \delta^{k\beta'} \sum_{\alpha \in \mathcal{A}_k} \sum_{m=1}^{N(k,\alpha)} \mu\left(Q_\alpha^{k,m}\right) \times \frac{1}{V_1(x) + V(x, y_\alpha^{k,m})} \left[\frac{1}{1 + d(x, y_\alpha^{k,m})}\right]^{\beta_-} \inf_{z \in Q_\alpha^{k,m}} \mathcal{M}_\theta(f)(z). \tag{3.11}$$

Following the estimation of (3.8), from (3.10) and (3.11), we deduce that, for some $r \in (\omega/(\omega + \eta), p)$,

$$|\langle f, \varphi\rangle| \lesssim \left\{\mathcal{M}\left(\left[\mathcal{M}_\theta(f)\right]^r\right)(x)\right\}^{\frac{1}{r}}.$$

Notice that the above inequality holds true for any $x \in B(x_0, 1)$. Then, by the boundedness of $\mathcal{M}$ on $L^{p/r}(X)$, we further conclude that

$$|\langle f, \varphi\rangle|^p \lesssim \frac{1}{V_1(x_0)} \int_X \left\{\mathcal{M}\left(\left[\mathcal{M}_\theta(f)\right]^r\right)(x)\right\}^{\frac{p}{r}} d\mu(x) \lesssim \|\mathcal{M}_\theta(f)\|_{L^p(X)}^p,$$

which is exactly (3.9).

Combining (3.9) and (3.1), we find that, for any $\varphi \in \mathcal{G}(\eta, \eta)$,

$$|\langle f, \varphi\rangle| \lesssim \|\mathcal{M}_\theta(f)\|_{L^p(X)} \|\varphi\|_{\mathcal{G}(\beta_2,\gamma_2)} \lesssim \|f\|_{H^{*,p}(X)} \|\varphi\|_{\mathcal{G}(\beta_2,\gamma_2)}. \tag{3.12}$$

Now let $g \in \mathcal{G}_0^\eta(\beta_2, \gamma_2)$. By the definition of $\mathcal{G}_0^\eta(\beta_2, \gamma_2)$, we know that there exist $\{\varphi_j\}_{j=1}^\infty \subset \mathcal{G}(\eta, \eta)$ such that $\|g - \varphi_j\|_{\mathcal{G}(\beta_2,\gamma_2)} \to 0$ as $j \to \infty$, which implies that $\{\varphi_j\}_{j=1}^\infty$ is a Cauchy sequence in $\mathcal{G}(\beta_2, \gamma_2)$. By (3.12), we find that, for any $j,\ k \in \mathbb{N}$,

$$|\langle f, \varphi_j - \varphi_k\rangle| \lesssim \|f\|_{H^{*,p}(X)} \|\varphi_j - \varphi_k\|_{\mathcal{G}(\beta_2,\gamma_2)}.$$

Therefore, $\lim_{j\to\infty}\langle f, \varphi_j\rangle$ exists and the limit is independent of the choice of $\{\varphi_j\}_{j=1}^\infty$. Thus, it is reasonable to define $\langle f, g\rangle := \lim_{j\to\infty}\langle f, \varphi_j\rangle$. Moreover, by (3.12), we conclude that

$$\begin{aligned}|\langle f, g\rangle| = \lim_{j\to\infty} |\langle f, \varphi_j\rangle| &\lesssim \|f\|_{H^{*,p}(X)} \liminf_{j\to\infty} \|\varphi_j\|_{\mathcal{G}(\beta_2,\gamma_2)} \\ &\sim \|f\|_{H^{*,p}(X)} \|g\|_{\mathcal{G}_0^\eta(\beta_2,\gamma_2)}.\end{aligned}$$

This implies $f \in (\mathcal{G}_0^\eta(\beta_2, \gamma_2))'$ and $\|f\|_{(\mathcal{G}_0^\eta(\beta_2,\gamma_2))'} \lesssim \|f\|_{H^{*,p}(X)}$, which completes the proof of Proposition 3.8. □

4 Grand Maximal Function Characterizations of Atomic Hardy Spaces

In this section, we establish the atomic characterizations of $H^{*,p}(X)$ with $p \in (\omega/(\omega+\eta), 1]$.

Definition 4.1 Let $p \in (\omega/(\omega+\eta), 1]$, $q \in (p, \infty] \cap [1, \infty]$ and $\beta, \gamma \in (\omega(1/p-1), \eta)$. The *atomic Hardy space* $H^{p,q}_{\rm at}(X)$ is defined to be the set of all $f \in (\mathcal{G}^\eta_0(\beta,\gamma))'$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$, where $\{a_j\}_{j=1}^\infty$ is a sequence of (p, q)-atoms and $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ satisfies $\sum_{j=1}^\infty |\lambda_j|^p < \infty$. Moreover, let

$$\|f\|_{H^{p,q}_{\rm at}(X)} := \inf\left\{\left(\sum_{j=1}^\infty |\lambda_j|^p\right)^{\frac{1}{p}}\right\},$$

where the infimum is taken over all the decompositions of f as above.

Observe that the difference between $H^{p,q}_{\rm cw}(X)$ and $H^{p,q}_{\rm at}(X)$ mainly lies on the choices of distribution spaces. When (X, d, μ) is a doubling metric measure space, it was proved in [40, Theorem 4.4] that $H^{p,q}_{\rm cw}(X)$ and $H^{p,q}_{\rm at}(X)$ coincide with equivalent (quasi-)norms. Since now d is a quasi-metric, for the completeness of this article, we include a proof of their equivalence in Sect. 4.4 below.

The main aim in this section is to prove the following conclusion.

Theorem 4.2 *Let $p \in (\omega/(\omega+\eta), 1]$, $q \in (p, \infty] \cap [1, \infty]$ and $\beta, \gamma \in (\omega(1/p-1), \eta)$. As subspaces of $(\mathcal{G}^\eta_0(\beta,\gamma))'$, $H^{*,p}(X) = H^{p,q}_{\rm at}(X)$ with equivalent (quasi-) norms.*

We divide the proof of Theorem 4.2 into three sections. In Sect. 4.1, we prove that $H^{p,q}_{\rm at}(X) \subset H^{*,p}(X)$ directly by the definition of $H^{p,q}_{\rm at}(X)$. The next two sections mainly deal with the proof of $H^{*,p}(X) \subset H^{p,q}_{\rm at}(X)$. In Sect. 4.2, we obtain a Calderón–Zygmund decomposition for any $f \in H^{*,p}(X)$. Then, in Sect. 4.3, we show that any $f \in H^{*,p}(X)$ has a (p,∞)-atomic decomposition. In Sect. 4.4, we reveal the equivalent relationship between $H^{p,q}_{\rm at}(X)$ and $H^{p,q}_{\rm cw}(X)$.

4.1 Proof of $H^{p,q}_{\rm at}(X) \subset H^{*,p}(X)$

In this section, we prove $H^{p,q}_{\rm at}(X) \subset H^{*,p}(X)$, as subspaces of $(\mathcal{G}^\eta_0(\beta,\gamma))'$ with $\beta, \gamma \in (\omega(1/p-1), \eta)$. To do this, we need the following technical lemma.

Lemma 4.3 *Let $p \in (\omega/(\omega+\eta), 1]$ and $q \in (p, \infty] \cap [1, \infty]$. Then there exists a positive constant C such that, for any (p, q)-atom a supported on $B := B(x_B, r_B)$, with $x_B \in X$ and $r_B \in (0, \infty)$, and any $x \in X$,*

$$a^*(x) \le C\mathcal{M}(a)(x)\mathbf{1}_{B(x_B,2A_0r_B)}(x) + C\left[\frac{r_B}{d(x_B,x)}\right]^\beta \frac{[\mu(B)]^{1-\frac{1}{p}}}{V(x_B,x)}\mathbf{1}_{[B(x_B,2A_0r_B)]^\complement}(x) \tag{4.1}$$

and

$$\|a^*\|_{L^p(X)} \le C, \tag{4.2}$$

where the atom a is viewed as a distribution on $\mathcal{G}^\eta_0(\beta,\gamma)$ with $\beta, \gamma \in (\omega(1/p-1), \eta)$.

Proof First, we show (4.1). Let $\varphi \in \mathcal{G}_0^\eta(\beta,\gamma)$ be such that $\|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)} \le 1$ for some $r \in (0,\infty)$, where $\beta,\ \gamma \in (\omega(1/p-1),\eta)$. When $x \in B(x_B, 2A_0r_B)$, by Lemma 2.2(v), we find that

$$|\langle a,\varphi\rangle| = \left|\int_X a(y)\varphi(y)\,d\mu(y)\right| \le \int_X |a(y)|\frac{1}{V_r(x)+V(x,y)}\left[\frac{r}{r+d(x,y)}\right]^\gamma d\mu(y)$$
$$\lesssim \mathcal{M}(a)(x),$$

which consequently implies that $a^*(x) \lesssim \mathcal{M}(a)(x)$.

Let $x \notin B(x_B, 2A_0r_B)$. Then, for any $y \in B$, we have $d(x,x_B) \ge 2A_0r_B > 2A_0d(x_B,y)$. Therefore, by the definition of (p,q)-atoms and Definition 2.1(ii), we conclude that

$$\begin{aligned}|\langle a,\varphi\rangle| &= \left|\int_B a(y)\varphi(y)\,d\mu(y)\right| \le \int_B |a(y)||\varphi(y)-\varphi(x_B)|\,d\mu(y)\\ &\le \int_B |a(y)|\left[\frac{d(x_B,y)}{r+d(x,x_B)}\right]^\beta \frac{1}{V_r(x)+V(x,x_B)}\left[\frac{r}{r+d(x,x_B)}\right]^\gamma d\mu(y)\\ &\le \left[\frac{r_B}{d(x_B,x)}\right]^\beta \frac{1}{V(x,x_B)}\|a\|_{L^1(X)} \lesssim \left[\frac{r_B}{d(x_B,x)}\right]^\beta \frac{[\mu(B)]^{1-\frac{1}{p}}}{V(x_B,x)}.\end{aligned}$$

Taking the supremum over all such $\varphi \in \mathcal{G}_0^\eta(\beta,\gamma)$ satisfying $\|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)} \le 1$ for some $r \in (0,\infty)$, we obtain (4.1).

Now, we use (4.1) to show (4.2). When $q \in (1,\infty]$, from the Hölder inequality and the boundedness of $\mathcal{M}$ on $L^q(X)$, we deduce that

$$\begin{aligned}\int_{B(x_B,2A_0r_B)} [\mathcal{M}(a)(x)]^p\,d\mu(x) &\le [\mu(B(x_B,2A_0r_B))]^{1-p/q}\|\mathcal{M}(a)\|_{L^q(X)}^p\\ &\lesssim [\mu(B)]^{1-p/q}\|a\|_{L^q(X)}^p \lesssim 1.\end{aligned}$$

If $q=1$, then, by $p \in (\omega/(\omega+\eta),1)$ and the boundedness of $\mathcal{M}$ from $L^1(X)$ to $L^{1,\infty}(X)$, we conclude that

$$\begin{aligned}&\int_{B(x_B,2A_0r_B)} [\mathcal{M}(a)(x)]^p\,d\mu(x)\\ &\quad= \int_0^\infty \mu(\{x \in B(x_B,2A_0r_B):\ \mathcal{M}(a)(x) > \lambda\})\,d\lambda^p\\ &\quad\lesssim \int_0^\infty \min\left\{\mu(B), \frac{\|a\|_{L^1(X)}}{\lambda}\right\} d\lambda^p\\ &\quad\lesssim \int_0^{\|a\|_{L^1(X)}/\mu(B)} \mu(B)\,d\lambda^p + \int_{\|a\|_{L^1(X)}/\mu(B)}^\infty \|a\|_{L^1(X)}\lambda^{-1}\,d\lambda^p\\ &\quad\lesssim \|a\|_{L^1(X)}^p[\mu(B)]^{1-p} \lesssim 1.\end{aligned}$$

By the fact $\beta > \omega(1/p - 1)$ and the doubling condition (1.1), we have

$$\int_{d(x,x_B)\geq 2A_0r_B} \left[\frac{r_B}{d(x_B,x)}\right]^{\beta p} \left[\frac{1}{\mu(B)}\right]^{1-p} \left[\frac{1}{V(x_B,x)}\right]^{p} d\mu(x)$$
$$\lesssim \sum_{k=1}^{\infty} 2^{-k\beta p} 2^{k\omega(1-p)} \int_{2^k A_0 r_B \leq d(x,x_B)<2^{k+1}A_0 r_B} \frac{1}{V(x_B,x)}\, d\mu(x) \lesssim 1.$$

Combining the last three formulae with (4.1), we obtain (4.2), which then completes the proof of Lemma 4.3. □

Proof of $H^{p,q}_{\mathrm{at}}(X) \subset H^{*,p}(X)$ Assume that $f \in (\mathcal{G}^{\eta}_0(\beta,\gamma))'$ is non-zero and it belongs to $H^{p,q}_{\mathrm{at}}(X)$ with $\beta,\ \gamma \in (\omega(1/p-1),\eta)$. Then $f = \sum_{j=1}^{\infty} \lambda_j a_j$, where $\{a_j\}_{j=1}^{\infty}$ are (p,q)-atoms and $\{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C}$ satisfy $\sum_{j=1}^{\infty} |\lambda_j|^p \sim \|f\|^p_{H^{p,q}_{\mathrm{at}}(X)}$. By the definition of the grand maximal function, we conclude that, for any $x \in X$,

$$f^*(x) \leq \sum_{j=1}^{\infty} |\lambda_j| a_j^*(x).$$

From this and (4.2), we deduce that

$$\|f^*\|^p_{L^p(X)} \lesssim \sum_{j=1}^{\infty} |\lambda_j|^p \left\|a_j^*\right\|_{L^p(X)} \lesssim \sum_{j=1}^{\infty} |\lambda_j|^p \sim \|f\|^p_{H^{p,q}_{\mathrm{at}}(X)}.$$

This finishes the proof of $H^{p,q}_{\mathrm{at}}(X) \subset H^{*,p}(X)$. □

4.2 Calderón–Zygmund Decomposition of Distributions from $H^{*,p}(X)$

In this section, we obtain a Calderón–Zygmund decomposition of any $f \in H^{*,p}(X)$. First we establish a partition of unity for an open set Ω with $\mu(\Omega) < \infty$.

Proposition 4.4 *Suppose $\Omega \subset X$ is a proper open set with $\mu(\Omega) \in (0,\infty)$ and $A \in [1,\infty)$. For any $x \in \Omega$, let*

$$r(x) := \frac{d(x,\Omega^{\complement})}{2AA_0} \in (0,\infty).$$

Then there exist $L_0 \in \mathbb{N}$ and a sequence $\{x_k\}_{k\in I} \subset \Omega$, where I is a countable index set, such that

(i) *$\{B(x_k, r_k/(5A_0^3))\}_{k\in I}$ is disjoint. Here and hereafter, $r_k := r(x_k)$ for any $k \in I$;*
(ii) *$\bigcup_{k\in I} B(x_k,r_k) = \Omega$ and $B(x_k, Ar_k) \subset \Omega$;*
(iii) *for any $x \in \Omega$, $Ar_k \leq d(x,\Omega^{\complement}) \leq 3AA_0^2 r_k$ whenever $x \in B(x_k,r_k)$ and $k \in I$;*
(iv) *for any $k \in I$, there exists $y_k \notin \Omega$ such that $d(x_k,y_k) < 3AA_0 r_k$;*

(v) *for any given $k \in I$, the number of balls $B(x_j, Ar_j)$ that intersect $B(x_k, Ar_k)$ is at most L_0;*
(vi) *if, in addition, Ω is bounded, then, for any $\sigma \in (0, \infty)$, the set $\{k \in I : r_k > \sigma\}$ is finite.*

Proof We show this proposition by borrowing some ideas from [49, pp. 15–16]. Let $\epsilon := (5A_0^3)^{-1}$ and $\{B(x, \epsilon r(x))\}_{x\in\Omega}$ be a covering of Ω. Now we pick the maximal disjoint subcollection of $\{B(x, \epsilon r(x))\}_{x\in\Omega}$, denoted by $\{B_k\}_{k\in I}$, which is at most countable, because of (1.1) and $\mu(\Omega) \in (0, \infty)$. For any $k \in I$, denote the center of B_k by x_k and $r(x_k)$ by r_k. Then we obtain (i).

Properties (iii) and (iv) can be shown by the definition of r_k, the details being omitted. Now we show (ii). Obviously, $B(x_k, Ar_k) \subset \Omega$ for any $k \in I$. It suffices to prove that $\Omega \subset \bigcup_{k\in I} B(x_k, r_k)$. For any $x \in \Omega$, since $\{B_k\}_{k\in I}$ is maximal, it then follows that there exists $k \in I$ such that $B(x_k, \epsilon r_k) \cap B(x, \epsilon r(x)) \neq \emptyset$. We claim that $r_k \geq r(x)/(4A_0^2)$. If not, then $r_k < r(x)/(4A_0^2)$. Suppose that $x_0 \in B(x_k, \epsilon r_k) \cap B(x, \epsilon r(x))$. Then, for any $y \in B(x_k, 3AA_0r_k)$, we have

$$\begin{aligned} d(y, x) &\leq A_0[d(y, x_0) + d(x_0, x)] \leq A_0^2[d(y, x_k) + d(x_k, x_0)] + A_0 d(x_0, x) \\ &\leq 6AA_0^3 r_k + A_0\epsilon r(x) \\ &\leq \frac{3}{2}AA_0 r(x) + \frac{1}{5}AA_0 r(x) = \frac{17}{10}AA_0 r(x) \end{aligned}$$

and hence $B(x_k, 3AA_0r_k) \subset B(x, \frac{17}{10}AA_0r(x)) \subset \Omega$, which contradicts to (iv). This proves the claim. Further, by the fact that $r(x) \leq 4A_0^2 r_k$, we have

$$d(x, x_k) \leq A_0[d(x, x_0) + d(x_0, x_k)] < A_0\epsilon r(x) + A_0\epsilon r_k \leq 5A_0^3\epsilon r_k = r_k,$$

that is, $x \in B(x_k, r_k)$. This finishes the proof of (ii).

Now we prove (v). Fix $k \in I$. Suppose that $B(x_j, Ar_j) \cap B(x_k, Ar_k) \neq \emptyset$. We claim that $r_j \leq 8A_0^2 r_k$. If not, then $r_j > 8A_0^2 r_k$. Choose $y_0 \in B(x_j, Ar_j) \cap B(x_k, Ar_k)$. For any $y \in B(x_k, 3AA_0r_k)$, we have

$$\begin{aligned} d(y, x_j) &\leq A_0[d(y, y_0) + d(y_0, x_j)] \leq A_0^2[d(y, x_k) + d(x_k, y_0)] + A_0 d(y_0, x_j) \\ &\leq 3AA_0^3 r_k + AA_0^2 r_k + AA_0 r_j \leq \frac{3}{2}AA_0 r_j, \end{aligned}$$

which further implies that $y \in B(x_j, \frac{3}{2}AA_0r_j)$. Therefore, $B(x_k, 3AA_0r_k) \subset B(x_j, \frac{3}{2}AA_0r_j) \subset \Omega$, which contradicts to (iv), Thus, we have $r_j \leq 8A_0^2 r_k$. By symmetry, we also have $r_k \leq 8A_0^2 r_j$. Let

$$\mathcal{J} := \{j \in I : B(x_j, Ar_j) \cap B(x_k, Ar_k) \neq \emptyset\}.$$

Then, for any $j \in \mathcal{J}$, $d(x_j, x_k) < AA_0(r_j + r_k) \leq 9AA_0^3 r_k$, which further implies that

$$B\left(x_j, (5A_0^3)^{-1}r_j\right) \subset B\left(x_k, A_0\left[d(x_j, x_k) + (5A_0^3)^{-1}r_j\right]\right) \subset B(x_k, 11AA_0^4 r_k).$$

Then, from the fact $d(x_j, x_k) \lesssim \min\{r_j, r_k\}$ and (1.1), we deduce that

$$\mu\left(B\left(x_j, (5A_0^3)^{-1}r_j\right)\right) \sim \mu(B(x_j, r_j)) \sim \mu(B(x_k, r_k)) \sim \mu(B(x_k, 11AA_0^4 r_k))$$

with the positive equivalence constants depending on A. Thus, we obtain (v) by (i).

Finally we prove (vi). Since Ω is bounded, it follows that there exist $x_0 \in X$ and $R \in (0, \infty)$ such that $\Omega \subset B(x_0, R)$. If (vi) fails, then there exists $\sigma_0 \in (0, \infty)$ such that $\mathcal{K} := \{k \in I : r_k > \sigma_0 R\}$ is infinite. Then, for any $k \in \mathcal{K}$,

$$\mu(B(x_k, r_k/(5A_0^3))) \sim \mu(B(x_k, \sigma_0 R)) \gtrsim \mu(B(x_0, R)) \gtrsim \mu(\Omega) > 0.$$

By this and (i), we have $\mu(\Omega) \ge \sum_{k\in\mathcal{K}} \mu(B(x_k, r_k/(5A_0^3))) = \infty$. That is a contradiction. This proves (vi) and hence finishes the proof of Proposition 4.4. □

Proposition 4.5 *Let $\Omega \subset X$ be an open set and $\mu(\Omega) < \infty$. Suppose that sequences $\{x_k\}_{k\in I}$ and $\{r_k\}_{k\in I}$ are as in Proposition 4.4 with $A := 16A_0^4$. Then there exist non-negative functions $\{\phi_k\}_{k\in I}$ such that*

(i) *for any $k \in I$, $0 \le \phi_k \le 1$ and* supp $\phi_k \subset B(x_k, 2A_0 r_k)$*;*
(ii) $\sum_{k\in I} \phi_k = \mathbf{1}_\Omega$*;*
(iii) *for any $k \in I$, $\phi_k \ge L_0^{-1}$ in $B(x_k, r_k)$, where L_0 is as in Proposition 4.4;*
(iv) *there exists a positive constant C such that, for any $k \in I$, $\|\phi_k\|_{\mathcal{G}(x_k,r_k,\eta,\eta)} \le C V_{r_k}(x_k)$.*

Proof By [1, Corollary 4.2], for any $k \in I$, we find a function ψ_k such that $\mathbf{1}_{B(x_k,r_k)} \le \psi_k \le \mathbf{1}_{B(x_k,2A_0 r_k)}$ and $\|\psi_k\|_{\dot{C}^\eta(X)} \lesssim r_k^{-\eta}$. Here and hereafter, for any $s \in (0, \eta]$ and a measurable function f, define

$$\|f\|_{\dot{C}^s(X)} := \sup_{x\neq y} \frac{|f(x) - f(y)|}{[d(x, y)]^\beta}.$$

Since $A \ge 2A_0$, from (ii) and (v) of Proposition 4.4, it follows that, for any $x \in \Omega$, $1 \le \sum_{k\in I} \psi_k(x) \le L_0$. For any $k \in I$ and $x \in X$, let

$$\phi_k(x) := \begin{cases} \psi_k(x)\left[\sum_{j\in I} \psi_j(x)\right]^{-1} & \text{when } x \in \Omega, \\ 0, & \text{when } x \notin \Omega. \end{cases}$$

Then, for any $k \in I$, we have $0 \le \phi_k \le 1$, supp $\phi_k \subset B(x_k, 2A_0 r_k)$ and $\sum_{k\in I} \phi_k(x) = 1$ when $x \in \Omega$. Moreover, for any $k \in I$, we have $\phi_k \ge L_0^{-1}$ in $B(x_k, r_k)$. Thus, we prove (i), (ii) and (iii).

It remains to prove (iv). Fix $k \in I$. For any $y \in X$, we have

$$|\phi_k(y)| \le \mathbf{1}_{B(x_k,2A_0 r_k)}(y) \lesssim V_{r_k}(x_k) \frac{1}{V_{r_k}(x_k) + V(x_k, y)} \left[\frac{r_k}{r_k + d(x_k, y)}\right]^\eta.$$

Now we prove that ϕ_k satisfies the regularity condition. Suppose that $d(y,y') \le (2A_0)^{-1}[r_k + d(x_k,y)]$. If $|\phi_k(y)-\phi_k(y')| \ne 0$, then $d(x_k,y) < (3A_0)^2 r_k$. If not, then $d(x_k,y) \ge (3A_0)^2 r_k$, so that $\phi_k(y)=0$ and

$$d(y',x_k) \ge A_0^{-1} d(x_k,y) - d(y,y') \ge (2A_0)^{-1} d(x_k,y) - (2A_0)^{-1} r_k > 2A_0 r_k$$

and hence $\phi_k(y') = 0$, which contradicts to $|\phi_k(y)-\phi_k(y')| \ne 0$. Notice that $\psi_k(y')|\psi_j(y)-\psi_j(y')| \ne 0$ implies that $y' \in B(x_k, 2A_0 r_k)$ and also y or y' belongs to $B(x_j, 2A_0 r_j)$, which further implies that $B(x_k, Ar_k) \cap B(x_j, Ar_j) \ne \emptyset$. Then, by the proof of Proposition 4.4(v), the number of j satisfying $\psi_k(y')|\psi_j(y)-\psi_j(y')| \ne 0$ is at most L_0 and $r_j \sim r_k$. Therefore,

$$\begin{aligned}
|\phi_k(y)-\phi_k(y')| &= \left| \frac{\psi_k(y)}{\sum_{j\in I}\psi_j(y)} - \frac{\psi_k(y')}{\sum_{j\in I}\psi_j(y')} \right| \\
&\le \frac{|\psi_k(y)-\psi_k(y')|}{\sum_{j\in I}\psi_j(y)} + \frac{\psi_k(y')\sum_{j\in I}|\psi_j(y)-\psi_j(y')|}{[\sum_{j\in I}\psi_j(y)][\sum_{j\in I}\psi_j(y')]} \\
&\lesssim \left[\frac{d(y,y')}{r_k}\right]^\eta + \sum_{\{j\in I:\ B(x_k,Ar_k)\cap B(x_j,Ar_j)\ne\emptyset\}} \left[\frac{d(y,y')}{r_j}\right]^\eta \\
&\lesssim \left[\frac{d(y,y')}{r_k}\right]^\eta \\
&\sim V_{r_k}(x_k)\left[\frac{d(y,y')}{r_k+d(x_k,y)}\right]^\eta \frac{1}{V_{r_k}(x_k)+V(x_k,y)}\left[\frac{r_k}{r_k+d(x_k,y)}\right]^\eta .
\end{aligned}$$

Then we obtain the desired regularity condition of ϕ_k. This finishes the proof of (iv) and hence of Proposition 4.5. □

Assume that $f \in (\mathcal{G}_0^\eta(\beta,\gamma))'$ belongs to $f \in H^{*,p}(X)$, where $p \in (\omega/(\omega+\eta), 1]$ and $\beta,\ \gamma \in (\omega(1/p-1),\eta)$. To obtain the Calderón–Zygmund decomposition of f, we apply Propositions 4.4 and 4.5 to the level set $\{x \in X:\ f^*(x) > \lambda\}$ with $\lambda \in (0,\infty)$. The encountering problem is that such a level set may not be open even in the case that d is a metric. To solve this problem in the case that d is a metric, a variant of the notion of the space of test functions is adopted in [20, Definition 2.5] so that to ensure that the level set is open (see [20, Remark 2.9]). Here, we borrow some ideas from [20].

By the proof of [42, Theorem 2], we know that there exist $\theta \in (0,1)$ and a metric d' such that $d' \sim d^\theta$. For any $x \in X$ and $r \in (0,\infty)$, define the *d'-ball* $B'(x,r) := \{y \in X:\ d'(x,y) < r\}$. Then (X, d', μ) is a doubling metric measure space. Moreover, for any $x,\ y \in X$ and $r \in (0,\infty)$, we have

$$\mu(B(y, r+d(x,y))) \sim \mu\left(B'\left(y, [r+d(x,y)]^\theta\right)\right) \sim \mu\left(B'\left(y, r^\theta + d'(x,y)\right)\right),$$

where the positive equivalence constants are independent of x and r. Using the metric d', we introduce a variant of the space of test functions as follows.

Definition 4.6 For any $x \in X$, $\rho \in (0,\infty)$ and $\beta',\ \gamma' \in (0,\infty)$, define $G(x,\rho,\beta',\gamma')$ to be the set of all functions f satisfying that there exists a positive constant C such that

(i) (the *size condition*) for any $y \in X$,

$$|f(y)| \le C \frac{1}{\mu(B'(y,\rho+d'(x,y)))}\left[\frac{\rho}{\rho+d'(x,y)}\right]^{\gamma'};$$

(ii) (the *regularity condition*) for any $y,\ y' \in X$ satisfying $d(y,y') \le [\rho + d'(x,y)]/2$, then

$$|f(y)-f(y')| \le C\left[\frac{d'(y,y')}{\rho+d'(y,y')}\right]^{\beta'} \frac{1}{\mu(B'(y,\rho+d'(x,y)))}\left[\frac{\rho}{\rho+d'(x,y)}\right]^{\gamma'}.$$

Also, define

$$\|f\|_{G(x,\rho,\beta',\gamma')} := \inf\{C \in (0,\infty) :\ \text{(i) and (ii) hold true}\}.$$

By the previous argument, we find that $\mathcal{G}(x,r,\beta,\gamma) = G(x,r^\theta,\beta/\theta,\gamma/\theta)$ with equivalent norms, where the positive equivalence constants are independent of x and r. For any $\beta,\ \gamma \in (0,\eta)$ and $f \in (\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$, define the *modified grand maximal function* of f by setting, for any $x \in X$,

$$f^\star(x) := \sup\left\{|\langle f,\varphi\rangle| :\ \varphi \in \mathring{\mathcal{G}}_0^\eta(\beta,\gamma) \text{ with } \|\varphi\|_{G(x,r^\theta,\beta/\theta,\gamma/\theta)} \le 1 \text{ for some } r \in (0,\infty)\right\}.$$

Then $f^\star \sim f^*$ pointwisely on X. For any $\lambda \in (0,\infty)$ and $j \in \mathbb{Z}$, define

$$\Omega_\lambda := \{x \in X :\ f^\star(x) > \lambda\} \quad \text{and} \quad \Omega^j := \Omega_{2^j}.$$

By the argument used in [20, Remark 2.9(ii)], we find that Ω_λ is *open* under the topology induced by d', so is it under the topology induced by d.

Now suppose that $p \in (\omega/(\omega+\eta),1]$, $\beta,\gamma \in (\omega(1/p-1),\eta)$ and $f \in H^{*,p}(X)$. Then $f^\star \in L^p(X)$ and every Ω^j with $j \in \mathbb{Z}$ has finite measure. Consequently, there exist $\{x_k^j\}_{k\in I_j} \subset X$ with I_j being a countable index set, $\{r_k^j\}_{k\in I_j} \subset (0,\infty)$, $L_0 \in \mathbb{N}$ and a sequence $\{\phi_k^j\}_{k\in I_j}$ of non-negative functions satisfying all the conclusions of Propositions 4.4 and 4.5. For any $j \in \mathbb{Z}$ and $k \in I_j$, define Φ_k^j by setting, for any $\varphi \in \mathring{\mathcal{G}}_0^\eta(\beta,\gamma)$ and $x \in X$,

$$\Phi_k^j(\varphi)(x) := \phi_k^j(x)\left[\int_X \phi_k^j(z)\,d\mu(z)\right]^{-1}\int_X [\varphi(x)-\varphi(z)]\phi_k^j(z)\,d\mu(z).$$

It can be seen that Φ_k^j is bounded on $\mathring{\mathcal{G}}_0^\eta(\beta,\gamma)$ with operator norm depending on j and k; see [20, Lemma 4.9]. Thus, it makes sense to define a distribution b_k^j on $\mathring{\mathcal{G}}_0^\eta(\beta,\gamma)$ by setting, for any $\varphi \in \mathring{\mathcal{G}}_0^\eta(\beta,\gamma)$,

$$\left\langle b_k^j, \varphi \right\rangle := \left\langle f, \Phi_k^j(\varphi) \right\rangle. \tag{4.3}$$

To estimate $(b_k^j)^*$, we have the following result. For its proof, see, for example, [37, Lemma 3.7].

Proposition 4.7 *For any $j \in \mathbb{Z}$ and $k \in I_j$, b_k^j is defined as in (4.3). Then there exists a positive constant C such that, for any $j \in \mathbb{Z}$, $k \in I_j$ and $x \in X$,*

$$\left(b_k^j\right)^*(x) \le C2^j \frac{\mu(B(x_k^j, r_k^j))}{\mu(B(x_k^j, r_k^j)) + V(x_k^j, x)} \left[\frac{r_k^j}{r_k^j + d(x_k^j, x)}\right]^\beta \mathbf{1}_{[B(x_k^j, 16A_0^4 r_k^j)]^\complement}(x)$$
$$+ Cf^*(x)\mathbf{1}_{B(x_k^j, 16A_0^4 r_k^j)}(x).$$

The next lemma is exactly [20, Lemma 4.10]. The proof remains true if d is a quasi-metric and μ does not satisfy the reverse doubling condition.

Lemma 4.8 *Let $\beta \in (0, \infty)$, $p \in (\omega/(\omega+\beta), \infty)$, $L_0 \in \mathbb{N}$ and I be a countable index set. Then there exists a positive constant C such that, for any sequences $\{x_k\}_{k\in I} \subset X$ and $\{r_k\}_{k\in I} \subset (0, \infty)$ satisfying $\sum_{k\in I} \mathbf{1}_{B(x_k, r_k)} \le L_0$,*

$$\int_X \left\{\sum_{k\in I} \frac{V_{r_k}(x_k)}{V_{r_k}(x_k) + V(x_k, x)} \left[\frac{r_k}{r_k + d(x_k, x)}\right]^\beta\right\}^p d\mu(x) \le C\mu\left(\bigcup_{k\in I} B(x_k, r_k)\right).$$

Then, by Proposition 4.7 and Lemma 4.8, we have the following result.

Proposition 4.9 *Let $p \in (\omega/(\omega + \eta), 1]$. For any $j \in \mathbb{Z}$ and $k \in I_j$, let b_k^j be as in (4.3). Then there exists a positive constant C such that, for any $j \in \mathbb{Z}$,*

$$\int_X \sum_{k\in I_j} \left[\left(b_k^j\right)^*(x)\right]^p d\mu(x) \le C \left\| f^*\mathbf{1}_{\Omega^j} \right\|_{L^p(X)}^p; \tag{4.4}$$

moreover, there exists $b^j \in H^{*,p}(X)$ such that $b^j = \sum_{k\in I_j} b_k^j$ in $H^{*,p}(X)$ and, for any $x \in X$,

$$(b^j)^*(x) \le C2^j \sum_{k\in I_j} \frac{\mu(B(x_k^j, r_k^j))}{\mu(B(x_k^j, r_k^j)) + V(x_k^j, x)} \left[\frac{r_k^j}{r_k^j + d(x_k^j, x)}\right]^\beta + Cf^*(x)\mathbf{1}_{\Omega^j}(x); \tag{4.5}$$

if $g^j := f - b^j$ for any $j \in \mathbb{Z}$, then, for any $x \in X$,

$$(g^j)^*(x) \le C2^j \sum_{k\in I_j} \frac{\mu(B(x_k^j, r_k^j))}{\mu(B(x_k^j, r_k^j)) + V(x_k^j, x)} \left[\frac{r_k^j}{r_k^j + d(x_k^j, x)}\right]^\beta + Cf^*(x)\mathbf{1}_{(\Omega^j)^\complement}(x). \tag{4.6}$$

Proof Fix $j \in \mathbb{Z}$. We first prove (4.4). Indeed, by Proposition 4.7, we find that

$$\begin{aligned}
&\int_X \sum_{k\in I_j} \left[\left(b_k^j\right)^*(x)\right]^p \, d\mu(x) \\
&\quad \lesssim 2^{jp} \int_X \sum_{k\in I_j} \left\{ \frac{\mu(B(x_k^j, r_k^j))}{\mu(B(x_k^j, r_k^j)) + V(x_k^j, x)} \left[\frac{r_k^j}{r_k^j + d(x_k^j, x)}\right]^\beta \right\}^p \, d\mu(x) \\
&\qquad + \int_{\bigcup_{k\in I_j} B(x_k^j, 16A_0^4 r_k^j)} [f^*(x)]^p \, d\mu(x).
\end{aligned}$$

By Proposition 4.4(ii), we have $\Omega^j = \bigcup_{k\in I_j} B(x_k^j, 16A_0^4 r_k^j)$. Applying this and Lemma 4.8, the first term in the right-hand side of the above formula is bounded by a harmless positive constant multiple of $2^{jp}\mu(\Omega^j)$. Combining this with $f^* \sim f^\star$ implies that

$$\int_X \sum_{k\in I_j} \left[\left(b_k^j\right)^*(x)\right]^p \, d\mu(x) \lesssim 2^{jp}\mu\left(\Omega^j\right) + \int_{\Omega^j} [f^*(x)]^p \, d\mu(x) \lesssim \left\| f^* \mathbf{1}_{\Omega^j} \right\|_{L^p(X)}^p,$$

which proves (4.4).

Next we prove (4.5). By (4.4), the dominated convergence theorem and the completeness of $H^{*,p}(X)$ (see Proposition 3.1), we know that there exists $b^j \in H^{*,p}(X)$ such that $b^j = \sum_{k\in I_j} b_k^j$ in $H^{*,p}(X)$. Moreover, from Proposition 4.7 and $\Omega^j = \bigcup_{k\in I_j} B(x_k^j, 16A_0^4 r_k^j)$, we deduce that, for any $x \in X$,

$$\begin{aligned}
(b^j)^*(x) &\le \sum_{k\in I_j} \left(b_k^j\right)^*(x) \\
&\lesssim 2^j \sum_{k\in I_j} \frac{\mu(B(x_k^j, r_k^j))}{\mu(B(x_k^j, r_k^j)) + V(x_k^j, x)} \left[\frac{r_k^j}{r_k^j + d(x_k^j, x)}\right]^\beta + f^*(x)\mathbf{1}_{\Omega^j}(x).
\end{aligned}$$

This finishes the proof of (4.5).

It remains to prove (4.6). If $x \in (\Omega^j)^\complement$, then, by (4.5), we conclude that

$$\begin{aligned}
(g^j)^*(x) &\le f^*(x) + (b^j)^*(x) \\
&\lesssim 2^j \sum_{k\in I_j} \frac{\mu(B(x_k^j, r_k^j))}{\mu(B(x_k^j, r_k^j)) + V(x_k^j, x)} \left[\frac{r_k^j}{r_k^j + d(x_k^j, x)}\right]^\beta + f^*(x),
\end{aligned}$$

as desired.

Now we consider the case $x \in \Omega^j$. According to Proposition 4.4(v), for any $n \in I_j$, we choose a point $y_n^j \notin \Omega^j$ satisfying $32A_0^5 r_n^j \le d(x_n^j, y_n^j) < 48A_0^5 r_n^j$. Since $x \in \Omega^j$, it follows that there exists $k_0 \in I_j$ such that $x \in B(x_{k_0}^j, r_{k_0}^j)$. Let $\mathcal{J}$ be the set of all $n \in I_j$ such that $B(x_n^j, 16A_0^4 r_n^j) \cap B(x_{k_0}^j, 16A_0^4 r_{k_0}^j) \neq \emptyset$. Then, by the proof of Proposition 4.4(v), $\#\mathcal{J} \le L_0$ and $r_n^j \sim r_{k_0}^j$ whenever $n \in \mathcal{J}$.

Suppose that $\varphi \in \mathcal{G}_0^\eta(\beta,\gamma)$ with $\|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)} \le 1$ for some $r \in (0,\infty)$. We then estimate $\langle g^j, \varphi\rangle$ by considering the cases $r \le r_{k_0}^j$ and $r > r_{k_0}^j$, respectively.

Case 1) $r \le r_{k_0}^j$. In this case, we write

$$\begin{aligned}\langle g^j, \varphi\rangle &= \langle f, \varphi\rangle - \sum_{n\in I_j} \langle b_n^j, \varphi\rangle = \langle f, \varphi\rangle - \sum_{n\in \mathcal{J}} \langle b_n^j, \varphi\rangle - \sum_{n\notin \mathcal{J}} \langle b_n^j, \varphi\rangle \\ &= \langle f, \widetilde{\varphi}\rangle - \sum_{n\in\mathcal{J}} \langle f, \widetilde{\varphi}_n\rangle - \sum_{n\notin\mathcal{J}} \langle b_n^j, \varphi\rangle,\end{aligned}$$

where $\widetilde{\varphi} := (1 - \sum_{n\in\mathcal{J}} \phi_n^j)\varphi$ and, for any $n \in \mathcal{J}$,

$$\widetilde{\varphi}_n := \phi_n^j \left[\int_X \phi_n^j(z)\, d\mu(z)\right]^{-1} \int_X \varphi(z)\phi_n^j(z)\, d\mu(z).$$

We first consider the term $\sum_{n\notin\mathcal{J}} \langle b_n^j, \varphi\rangle$. Indeed, from $x \in B(x_{k_0}^j, r_{k_0}^j)$, it follows that $x \notin B(x_n^j, 16A_0^4 x_n^j)$ when $n \notin \mathcal{J}$. Applying Proposition 4.7 implies that

$$\left|\left\langle b_n^j, \varphi\right\rangle\right| \le \left|\left(b_n^j\right)^*(x)\right| \lesssim 2^j \frac{\mu(B(x_n^j, r_n^j))}{\mu(B(x_n^j, r_n^j)) + V(x_n^j, x)} \left[\frac{r_n^j}{r_n^j + d(x_n^j, x)}\right]^\beta,$$

and hence

$$\sum_{n\notin\mathcal{J}} \left|\left\langle b_n^j, \varphi\right\rangle\right| \lesssim 2^j \sum_{n\notin\mathcal{J}} \frac{\mu(B(x_n^j, r_n^j))}{\mu(B(x_n^j, r_n^j)) + V(x_n^j, x)} \left[\frac{r_n^j}{r_n^j + d(x_n^j, x)}\right]^\beta,$$

as desired.

Next we consider the term $\sum_{n\in\mathcal{J}} \langle f, \widetilde{\varphi}_n\rangle$. Notice that $\|\widetilde{\varphi}_n\|_{\mathcal{G}(x_n^j, r_n^j, \beta,\gamma)} \lesssim 1$. By $d(x_n^j, y_n^j) \sim r_n^j$, we then have $\|\widetilde{\varphi}_n\|_{\mathcal{G}(y_n^j, r_n^j,\beta,\gamma)} \lesssim 1$. Therefore,

$$|\langle f, \widetilde{\varphi}_n\rangle| \lesssim f^*\left(y_n^j\right) \sim f^\star\left(y_n^j\right) \lesssim 2^j \sim 2^j \frac{\mu(B(x_n^j, r_n^j))}{\mu(B(x_n^j, r_n^j)) + V(x_n^j, x)} \left[\frac{r_n^j}{r_n^j + d(x_n^j, x)}\right]^\beta,$$

where, in the last step, we used the facts that $x \in B(x_{k_0}^j, r_{k_0}^j)$ and $d(x_n^j, x_{k_0}^j) \lesssim r_n^j + r_{k_0}^j \sim r_n^j$ whenever $n \in \mathcal{J}$. Then, summing all $n \in \mathcal{J}$, we obtain the desired estimate.

Finally, we consider the term $\langle f, \widetilde{\varphi}\rangle$. Since $\varphi \in \mathcal{G}_0^\eta(\beta,\gamma)$, it is easy to see that $\widetilde{\varphi} \in \mathcal{G}_0^\eta(\beta,\gamma)$. Once we have proved that

$$\|\widetilde{\varphi}\|_{\mathcal{G}(y_{k_0}^j, r_{k_0}^j, \beta,\gamma)} \lesssim 1, \tag{4.7}$$

then

$$|\langle f, \widetilde{\varphi}\rangle| \lesssim f^*\left(y_{k_0}^j\right) \sim f^\star\left(y_{k_0}^j\right) \lesssim 2^j \sim 2^j \frac{\mu(B(x_{k_0}^j, r_{k_0}^j))}{\mu(B(x_{k_0}^j, r_{k_0}^j)) + V(x_{k_0}^j, x)} \left[\frac{r_{k_0}^j}{r_{k_0}^j + d(x_{k_0}^j, x)}\right]^\beta,$$

as desired.

To prove (4.7), we first consider the size condition. For any $z \in B(x_{k_0}^j, 16A_0^4 r_{k_0}^j)$, by Proposition 4.5, we have $\sum_{n\in\mathcal{J}} \phi_n^j(z) = \sum_{n\in I_j} \phi_n^j(z) = 1$ and hence $\widetilde{\varphi}(z) = 0$. When $d(z, x_{k_0}^j) \geq 16A_0^4 r_{k_0}^j$, by the fact $d(x_{k_0}^j, z) \geq 2A_0 d(x, x_{k_0}^j)$, we have

$$\begin{aligned} r_{k_0}^j + d\left(z, y_{k_0}^j\right) &\leq r_{k_0}^j + A_0\left[d\left(z, x_{k_0}^j\right) + d\left(x_{k_0}^j, y_{k_0}^j\right)\right] \leq (2A_0)^6\left[r_{k_0}^j + d\left(z, x_{k_0}^j\right)\right] \\ &\leq (2A_0)^7 d\left(z, x_{k_0}^j\right) \leq (2A_0)^8 d(x, z) \leq (2A_0)^8[r + d(x, z)] \end{aligned} \tag{4.8}$$

and hence $\mu(B(y_{k_0}^j, r_{k_0}^j)) + V(y_{k_0}^j, z) \lesssim V_r(x) + V(x, z)$, which, together with the size condition of φ and the fact that $r \leq r_{k_0}^j$, further implies that

$$\begin{aligned} |\widetilde{\varphi}(z)| \leq |\varphi(z)| &\leq \frac{1}{V_r(x) + V(x, z)}\left[\frac{r}{r + d(x, z)}\right]^\gamma \\ &\lesssim \frac{1}{\mu(B(y_{k_0}^j, r_{k_0}^j)) + V(y_{k_0}^j, z)}\left[\frac{r_{k_0}^j}{r_{k_0}^j + d(y_{k_0}^j, z)}\right]^\gamma. \end{aligned}$$

This finishes the proof of the size condition.

Now we consider the regularity of $\widetilde{\varphi}$. Suppose that $z,\ z' \in X$ with $d(z, z') \leq (2A_0)^{-1}[r_{k_0}^j + d(z, y_{k_0}^j)]$. Due to the size condition, we only need to consider the case $d(z, z') \leq (2A_0)^{-9}[r_{k_0}^j + d(z, y_{k_0}^j)]$. If $\widetilde{\varphi}(z) - \widetilde{\varphi}(z') \neq 0$, then either $d(z, x_{k_0}^j) \geq 16A_0^4 r_{k_0}^j$ or $d(z', x_{k_0}^j) \geq 16A_0^4 r_{k_0}^j$, which always implies that $d(z, x_{k_0}^j) \geq 8A_0^3 r_{k_0}^j$. Indeed, if $d(z, x_{k_0}^j) < 8A_0^3 r_{k_0}^j$, then $d(z, y_{k_0}^j) \leq A_0[d(z, x_{k_0}^j) + d(x_{k_0}^j, y_{k_0}^j)] < (2A_0)^6 r_{k_0}^j$ and hence $d(z, z') \leq (2A_0)^{-2} r_{k_0}^j$, which further implies that $d(z', x_{k_0}^j) \leq A_0[d(z', z) + d(z, x_{k_0}^j)] < 16A_0^4 r_{k_0}^j$ and it is a contraction.

Notice that $d(z, x_{k_0}^j) \geq 8A_0^3 r_{k_0}^j$, which, together with an argument as in the estimation of (4.8), implies $r_{k_0}^j + d(z, y_{k_0}^j) \leq (2A_0)^8[r + d(z, x)]$, so that $d(z, z') \leq (2A_0)^{-1}[r + d(z, x)]$. By the definition of $\widetilde{\varphi}$, we find that

$$\left|\widetilde{\varphi}(z) - \widetilde{\varphi}(z')\right| \leq \left[1 - \sum_{n\in\mathcal{J}} \phi_n^j(z)\right] |\varphi(z) - \varphi(z')| + |\varphi(z')| \sum_{n\in\mathcal{J}} \left|\phi_n^j(z) - \phi_n^j(z')\right|.$$

Using the regularity condition of φ and the fact $d(z, z') \leq (2A_0)^{-1}[r + d(z, x)]$, we obtain

$$\left[1-\sum_{n\in\mathcal{J}}\phi_n^j(z)\right]|\varphi(z)-\varphi(z')|$$
$$\lesssim\left[\frac{d(z,z')}{r+d(z,x)}\right]^{\beta}\frac{1}{V_r(x)+V(x,z)}\left[\frac{r}{r+d(x,z)}\right]^{\gamma}$$
$$\lesssim\left[\frac{d(z,z')}{r_{k_0}^j+d(z,y_{k_0}^j)}\right]^{\beta}\frac{1}{\mu(B(y_{k_0}^j,r_{k_0}^j))+V(y_{k_0}^j,z)}\left[\frac{r_{k_0}^j}{r_{k_0}^j+d(y_{k_0}^j,z)}\right]^{\gamma},$$

where, in the last step, we used $r_{k_0}^j+d(z,y_{k_0}^j)\lesssim r+d(z,x)$, $r\le r_{k_0}^j$, $x\in B(x_{k_0}^j,r_{k_0}^j)$ and $d(y_{k_0}^j,x_{k_0}^j)\sim r_{k_0}^j$.

We now estimate $|\varphi(z')|\sum_{n\in\mathcal{J}}|\phi_n^j(z)-\phi_n^j(z')|$. If $\varphi(z')|\phi_n^j(z)-\phi_n^j(z')|\neq 0$, then $z'\notin B(x_{k_0}^j,16A_0^4r_{k_0}^j)$ and z or z' belongs to $B(x_n^j,2A_0r_n^j)$. When $n\in\mathcal{J}$, we have $r_n^j\sim r_{k_0}^j\sim r_{k_0}^j+d(y_{k_0}^j,z)$. Also, $r_{k_0}^j+d(z,y_{k_0}^j)\lesssim r+d(z,x)\sim r+d(z',x)$. By these, $\#\mathcal{J}\le L_0$ and $r\le r_{k_0}^j$, we conclude that

$$|\varphi(z')|\sum_{n\in\mathcal{J}}\left|\phi_n^j(z)-\phi_n^j(z')\right|$$
$$\lesssim\frac{1}{V_r(x)+V(x,z')}\left[\frac{r}{r+d(z,x)}\right]^{\gamma}\sum_{n\in\mathcal{J}}\left[\frac{d(z,z')}{r_n^j}\right]^{\eta}$$
$$\lesssim\left[\frac{d(z,z')}{r_{k_0}^j+d(y_{k_0}^j,z)}\right]^{\beta}\frac{1}{\mu(B(y_{k_0}^j,r_{k_0}^j))+V(y_{k_0}^j,z)}\left[\frac{r_{k_0}^j}{r_{k_0}^j+d(y_{k_0}^j,z)}\right]^{\gamma}.$$

This finishes the proof of the regularity condition and hence of (4.7). Thus, we complete the proof of Case 1).

Case 2) $r>r_{k_0}^j$. In this case, we write

$$\left|\left\langle g^j,\varphi\right\rangle\right|\le|\langle f,\varphi\rangle|+\sum_{n\in\mathcal{J}}\left|\left\langle b_n^j,\varphi\right\rangle\right|+\sum_{n\notin\mathcal{J}}\left|\left\langle b_n^j,\varphi\right\rangle\right|.$$

The estimation of $\sum_{n\notin\mathcal{J}}|\langle b_n^j,\varphi\rangle|$ has already been given in Case 1).

From $x\in B(x_{k_0}^j,r_{k_0}^j)$ and $d(y_{k_0}^j,x_{k_0}^j)\sim r_{k_0}^j\lesssim r$, it follows that $\|\varphi\|_{\mathcal{G}(y_{k_0}^j,r,\beta,\gamma)}\lesssim 1$ and hence

$$|\langle f,\varphi\rangle|\lesssim f^*\left(y_{k_0}^j\right)\lesssim 2^j\sim 2^j\frac{\mu(B(x_{k_0}^j,r_{k_0}^j))}{\mu(B(x_{k_0}^j,r_{k_0}^j))+V(x_{k_0}^j,x)}\left[\frac{r_{k_0}^j}{r_{k_0}^j+d(x_{k_0}^j,x)}\right]^{\beta}.$$

If $n\in\mathcal{J}$, then $r_n^j\sim r_{k_0}^j$ and hence $d(y_n^j,x_{k_0}^j)\lesssim r_{k_0}^j$. This, together with the fact $r_{k_0}^j<r$ and $x\in B(x_{k_0}^j,r_{k_0}^j)$, implies that $\|\varphi\|_{\mathcal{G}(y_n^j,r,\beta,\gamma)}\lesssim 1$. Thus, by Proposition 4.7, we have

$$\sum_{n\in\mathcal{J}}\left|\left\langle b_n^j,\varphi\right\rangle\right| \lesssim \sum_{n\in\mathcal{J}}\left(b_n^j\right)^*\left(y_n^j\right)$$
$$\lesssim 2^j\sum_{n\in\mathcal{J}}\frac{\mu(B(x_n^j,r_n^j))}{\mu(B(x_n^j,r_n^j))+V(x_n^j,x)}\left[\frac{r_n^j}{r_n^j+d(x_n^j,x)}\right]^\beta.$$

Then we obtain the desired estimate for $\langle g^j,\varphi\rangle$ in the case $r > r_{k_0}^j$.

Combining the two cases above, we find that, for any $x\in\Omega^j$,

$$(g^j)^*(x)\lesssim 2^j\sum_{k\in I_j}\frac{\mu(B(x_k^j,r_k^j))}{\mu(B(x_k^j,r_k^j))+V(x_k^j,x)}\left[\frac{r_k^j}{r_k^j+d(x_k^j,x)}\right]^\beta.$$

Thus, (4.6) holds true. This finishes the proof of Proposition 4.9. □

4.3 Atomic Characterizations of $H^{*,p}(X)$

In this section, we prove $H^{*,p}(X)\subset H^{p,q}_{\mathrm{at}}(X)$ and complete the proof of Theorem 4.2. First, we obtain dense subspaces of $H^{*,p}(X)$ as follows.

Lemma 4.10 ([20, Proposition 4.12]) *Let $p\in(\omega/(\omega+\eta),1]$, $\beta,\ \gamma\in(\omega(1/p-1),\eta)$ and $q\in[1,\infty)$. If regard $H^{*,p}(X)$ as a subspace of $(\mathcal{G}_0^\eta(\beta,\gamma))'$, then $L^q(X)\cap H^{*,p}(X)$ is dense in $H^{*,p}(X)$.*

In the next two lemmas, we suppose that $f\in L^2(X)\cap H^{*,p}(X)$. Based on Proposition 3.3 and (3.1), we may follow [20, Remark 4.14] and assume that there exists a positive constant C such that, for any $x\in X$, $|f(x)|\le Cf^*(x)$. With all the notation as in the previous section, for any $j\in\mathbb{Z}$ and $k\in I_j$, define

$$m_k^j:=\frac{1}{\|\phi_k^j\|_{L^1(X)}}\int_X f(\xi)\phi_k^j(\xi)\,d\mu(\xi)\quad\text{and}\quad b_k^j:=\left(f-m_k^j\right)\phi_k^j. \tag{4.9}$$

Then we have the following technical lemma.

Lemma 4.11 ([20, Proposition 4.13]) *For any $j\in\mathbb{Z}$ and $k\in I_j$, let m_k^j and b_k^j be as in (4.9). Then*

(i) *there exists a positive constant C, independent of j and $k\in I_j$, such that $|m_k^j|\le C2^j$;*
(ii) *b_k^j induces the same distribution as defined in (4.3);*
(iii) *$\sum_{k\in I_j} b_k^j$ converges to some function b^j in $L^2(X)$, which induces a distribution that coincides with b^j as in Proposition 4.9;*
(iv) *let $g^j:=f-b^j$. Then $g^j=f\mathbf{1}_{(\Omega^j)^\complement}+\sum_{k\in I_j}m_k^j\phi_k^j$. Moreover, there exists a positive constant C, independent of j, such that, for any $x\in X$, $|g^j(x)|\le C2^j$.*

For any $j \in \mathbb{Z}$, $k \in I_j$ and $l \in I_{j+1}$, define

$$L_{k,l}^{j+1} := \frac{1}{\|\phi_l^{j+1}\|_{L^1(X)}} \int_X \left[f(\xi) - m_l^{j+1}\right] \phi_k^j(\xi)\phi_l^{j+1}(\xi)\, d\mu(\xi). \tag{4.10}$$

Then $L_{k,l}^{j+1}$ has the following properties.

Lemma 4.12 *For any $j \in \mathbb{Z}$, $k \in I_j$ and $l \in I_{j+1}$, let $L_{k,l}^{j+1}$ be as in (4.10). Then*

(i) *there exists a positive constant C, independent of j, k and l, such that*

$$\sup_{x\in X} \left|L_{k,l}^{j+1}\phi_l^{j+1}(x)\right| \le C2^j;$$

(ii) $\sum_{k\in I_j}\sum_{l\in I_{j+1}} L_{k,l}^{j+1}\phi_l^{j+1} = 0$ *both in $(\mathcal{G}_0^\eta(\beta,\gamma))'$ and everywhere.*

Proof We first show (i). Indeed, for any $j \in \mathbb{Z}$, $k \in I_j$, $l \in I_{j+1}$ and $x \in X$,

$$\left|L_{k,l}^{j+1}\phi_l^{j+1}(x)\right| \le \left|m_l^{j+1}\right|\phi_l^{j+1}(x) + \phi_l^{j+1}(x)\left|\int_X f(\xi)\frac{\phi_k^j(\xi)\phi_l^{j+1}(\xi)}{\|\phi_l^{j+1}\|_{L^1(X)}}\, d\mu(\xi)\right| =: \mathrm{Y}_1 + \mathrm{Y}_2.$$

By Lemma 4.11(i) and the definition of ϕ_l^{j+1}, it is easy to obtain $\mathrm{Y}_1 \lesssim 2^j$.

Now we consider Y_2. If $\phi_k^j\phi_l^{j+1}$ is a non-zero function, then $B(x_k^j, 2A_0r_k^j) \cap B(x_l^{j+1}, 2A_0r_l^{j+1}) \neq \emptyset$, which further implies that $r_l^{j+1} \le 3A_0r_k^j$. Otherwise, if $r_l^{j+1} > 3A_0r_k^j$, then, for any $y \in B(x_k^j, 48A_0^5r_k^j)$,

$$\begin{aligned} d\left(y, x_l^{j+1}\right) &\le A_0\left[d\left(y, x_k^j\right) + d\left(x_k^j, x_l^{j+1}\right)\right] < 48A_0^6r_k^j + A_0^2\left(2A_0r_k^j + 2A_0r_l^{j+1}\right)\\ &< 16A_0^5r_l^{j+1} + \frac{2}{3}A_0^2r_l^{j+1} + 2A_0^3r_l^{j+1} < 20A_0^5r_l^{j+1}, \end{aligned}$$

which implies that $B(x_k^j, 48A_0^5r_k^j) \subset B(x_l^{j+1}, 20A_0^5r_l^{j+1}) \subset \Omega^{j+1} \subset \Omega^j$ and hence contradicts to Proposition 4.4(v).

Define $\varphi := \phi_k^j\phi_l^{j+1}/\|\phi_l^{j+1}\|_{L^1(X)}$. According to Proposition 4.4(iv) with $A := 16A_0^4$, we can choose $y_l^{j+1} \in (\Omega^{j+1})^\complement$ such that $d(y_l^{j+1}, x_l^{j+1}) \le 48A_0^5r_l^{j+1}$. We now show $\varphi \in \mathcal{G}(y_l^{j+1}, r_l^{j+1}, \eta, \eta)$ and $\|\varphi\|_{\mathcal{G}(y_l^{j+1}, r_l^{j+1}, \eta,\eta)} \lesssim 1$. Notice that supp $\varphi \subset B(x_l^{j+1}, 2A_0r_l^{j+1})$. Moreover, by this and the choice of y_l^{j+1}, we conclude that, for any $x \in B(x_l^{j+1}, 2A_0r_l^{j+1})$,

$$\begin{aligned} |\varphi(x)| \lesssim |\phi_l^{j+1}(x)| &\lesssim \frac{1}{\mu(B(x_l^{j+1}, r_l^{j+1})) + V(x_l^{j+1}, x)}\left[\frac{r_l^{j+1}}{r_l^{j+1} + d(x_l^{j+1}, x)}\right]^\eta\\ &\sim \frac{1}{\mu(B(y_l^{j+1}, r_l^{j+1})) + V(y_l^{j+1}, x)}\left[\frac{r_l^{j+1}}{r_l^{j+1} + d(y_l^{j+1}, x)}\right]^\eta. \end{aligned}$$

This shows the size condition of φ.

To consider the regularity condition of φ, we suppose that $x,\ x' \in X$ satisfying $d(x, x') \le (2A_0)^{-1}[r_l^{j+1} + d(y_l^{j+1}, x)]$. Due to the size condition, we may assume $d(x, x') \le (2A_0)^{-3}[r_l^{j+1} + d(y_l^{j+1}, x)]$. We claim that $\varphi(x) - \varphi(x') \ne 0$ implies that $d(x, x_l^{j+1}) \le 96A_0^6 r_l^{j+1}$.

Indeed, if $d(x, x_l^{j+1}) > 96A_0^6 r_l^{j+1}$, then $\varphi(x) = 0$. By $d(x_l^{j+1}, y_l^{j+1}) \le 48A_0^5 r_l^{j+1}$, we find that $d(x, y_l^{j+1}) > 48A_0^5 r_l^{j+1}$ and hence $d(x, x') \le (2A_0)^{-2} d(x, y_l^{j+1}) \le (2A_0)^{-1} d(x, x_l^{j+1})$. Consequently, $d(x', x_l^{j+1}) \ge A_0^{-1} d(x, x_l^{j+1}) - d(x, x') > 48A_0^5 r_l^{j+1}$ and $\varphi(x') = 0$. This contradicts to $\varphi(x) - \varphi(x') \ne 0$.

By the above claim, $r_l^{j+1} \le 3A_0 r_k^j$ and $d(y_l^{j+1}, x_j^{j+1}) \sim r_l^{j+1}$, we know that

$$
\begin{aligned}
&|\varphi(x) - \varphi(x')| \\
&\quad \lesssim \frac{1}{\mu(B(x_l^{j+1}, r_l^{j+1}))} \left[\phi_k^j(x) \left|\phi_l^{j+1}(x) - \phi_l^{j+1}(x')\right| + \left|\phi_k^j(x) - \phi_k^j(x')\right| \phi_l^{j+1}(x')\right] \\
&\quad \lesssim \frac{1}{\mu(B(x_l^{j+1}, r_l^{j+1}))} \left\{\left[\frac{d(x, x')}{r_l^{j+1}}\right]^\eta + \left[\frac{d(x, x')}{r_k^j}\right]^\eta\right\} \\
&\quad \sim \left[\frac{d(x, x')}{r_l^{j+1} + d(y_l^{j+1}, x)}\right]^\eta \frac{1}{\mu(B(y_l^{j+1}, r_l^{j+1})) + V(y_l^{j+1}, x)} \left[\frac{r_l^{j+1}}{r_l^{j+1} + d(y_l^{j+1}, x)}\right]^\eta.
\end{aligned}
$$

Thus, we obtain $\varphi \in \mathcal{G}(y_l^{j+1}, r_l^{j+1}, \eta, \eta)$ and $\|\varphi\|_{\mathcal{G}(y_l^{j+1}, r_l^{j+1}, \eta, \eta)} \lesssim 1$, which further implies that $\|\varphi\|_{\mathcal{G}(y_l^{j+1}, r_l^{j+1}, \beta, \gamma)} \lesssim 1$ and hence

$$
\Upsilon_2 = |\langle f, \varphi\rangle| \lesssim f^*\left(y_l^{j+1}\right) \lesssim 2^j.
$$

This finishes the proof of (i).

Next we prove (ii). If $L_{k,l}^{j+1} \ne 0$, then the proof in (i) implies $B(x_k^j, 2A_0 r_k^j) \cap B(x_l^{j+1}, 2A_0 r_l^{j+1}) \ne \emptyset$ and $r_l^{j+1} \le 3A_0 r_k^j$. Further, for any $y \in B(x_l^{j+1}, 2A_0 r_l^{j+1})$, we have

$$
\begin{aligned}
d\left(y, x_k^j\right) &\le A_0\left[d\left(y, x_l^{j+1}\right) + d\left(x_k^j, x_l^{j+1}\right)\right] < 2A_0^2 r_l^{j+1} + A_0^2\left(2A_0 r_k^j + 2A_0 r_l^{j+1}\right) \\
&< 6A_0^3 r_k^j + 2A_0^3 r_k^j + 6A_0^4 r_k^j \le 14A_0^4 r_k^j < 16A_0^4 r_k^j,
\end{aligned}
$$

which implies that $B(x_l^{j+1}, 2A_0 r_l^{j+1}) \subset B(x_k^j, 16A_0^4 r_k^j) \subset \Omega^j$ by Proposition 4.4(v). Thus, for any $k \in I_j$ and $x \in X$, we find that

$$
\sum_{l \in I_{j+1}} \left|L_{k,l}^{j+1} \phi_l^{j+1}\right| \lesssim 2^j \mathbf{1}_{B(x_k^j, 16A_0^4 r_k^j)}(x) \tag{4.11}
$$

and hence

$$\sum_{k\in I_j}\sum_{l\in I_{j+1}}\left|L_{k,l}^{j+1}\phi_l^{j+1}(x)\right|\lesssim 2^j\sum_{k\in I_j}\mathbf{1}_{B(x_k^j,16A_0^4r_k^j)}(x)\lesssim 2^j\mathbf{1}_{\Omega^j}(x).$$

Consequently,

$$\begin{aligned}\sum_{k\in I_j}\sum_{l\in I_{j+1}}L_{k,l}^{j+1}\phi_l^{j+1}&=\sum_{l\in I_{j+1}}\left(\sum_{k\in I_j}L_{k,l}^{j+1}\right)\phi_l^{j+1}\\&=\sum_{l\in I_{j+1}}\frac{\phi_l^{j+1}}{\|\phi_l^{j+1}\|_{L^1(X)}}\int_X\left[f(\xi)-m_l^{j+1}\right]\phi_l^{j+1}(\xi)\sum_{k\in I_j}\phi_k^j(\xi)\,d\mu(\xi)\\&=\sum_{l\in I_{j+1}}\frac{\phi_l^{j+1}}{\|\phi_l^{j+1}\|_{L^1(X)}}\int_X\left[f(\xi)-m_l^{j+1}\right]\phi_l^{j+1}(\xi)\,d\mu(\xi)\\&=\sum_{l\in I_{j+1}}\frac{\phi_l^{j+1}}{\|\phi_l^{j+1}\|_{L^1(X)}}\int_X b_l^{j+1}(\xi)\,d\mu(\xi)=0.\end{aligned}$$

By the fact that $\sum_{k\in I_j}\sum_{l\in I_{j+1}}\int_X|L_{k,l}^{j+1}\phi_l^{j+1}(\xi)|\,d\mu(\xi)\lesssim 2^j\mu(\Omega^j)<\infty$ and the dominated convergence theorem, we find that $\sum_{k\in I_j}\sum_{l\in I_{j+1}}L_{k,l}^{j+1}\phi_l^{j+1}=0$ in $L^1(X)$ and hence in $(\mathcal{G}_0^\eta(\beta,\gamma))'$. This finishes the proof of Lemma 4.12. □

Now we show the other side of Theorem 4.2.

Proof of $H^{*,p}(X)\subset H^{p,q}_{\mathbf{at}}(X)$ By Lemma 4.10, we first suppose $f\in L^2(X)\cap H^{*,p}(X)$. We may also assume $|f(x)|\lesssim f^*(x)$ for any $x\in X$. We use the same notation as in Lemmas 4.11 and 4.12. For any $j\in\mathbb{N}$, let $h^j:=g^{j+1}-g^j=b^j-b^{j+1}$. Then $f-\sum_{j=-m}^m h^j=b^{m+1}+g^{-m}$. For any $m\in\mathbb{Z}$, by Lemma 4.11, we conclude that $\|g^{-m}\|_{L^\infty(X)}\lesssim 2^{-m}$. Moreover, by (4.5), we find that $\|(b^{m+1})^*\|_{L^p(X)}\lesssim\|f^*\mathbf{1}_{(\Omega^{m+1})^\complement}\|_{L^p(X)}\to 0$ as $m\to\infty$. Thus, $f=\sum_{j=-\infty}^\infty h^j$ in $(\mathcal{G}_0^\eta(\beta,\gamma))'$. Besides, by the definition of b_k^m, we know that supp $b^{m+1}\subset\Omega^{m+1}$, which then implies that $\sum_{j=-\infty}^\infty h^j$ converges almost everywhere. Notice that, by Lemma 4.12(ii), for any $j\in\mathbb{Z}$, we have

$$\begin{aligned}h^j=b^j-b^{j+1}&=\sum_{k\in I_j}b_k^j-\sum_{l\in I_{j+1}}b_l^{j+1}+\sum_{k\in I_j}\sum_{l\in I_{j+1}}L_{k,l}^{j+1}\phi_l^{j+1}\\&=\sum_{k\in I_j}\left[b_k^j-\sum_{l\in I_{j+1}}\left(b_l^{j+1}\phi_k^j-L_{k,l}^{j+1}\phi_l^{j+1}\right)\right]=:\sum_{k\in I_j}h_k^j,\end{aligned}\tag{4.12}$$

which converges in $(\mathcal{G}_0^\eta(\beta,\gamma))'$ and almost everywhere. Moreover, for any $j\in\mathbb{Z}$ and $k\in\mathbb{N}$,

$$
\begin{aligned}
h_k^j &= b_k^j - \sum_{l\in I_{j+1}} \left(b_l^{j+1}\phi_k^j - L_{k,l}^{j+1}\phi_l^{j+1}\right)\\
&= \left(f - m_k^j\right)\phi_k^j - \sum_{l\in I_{j+1}} \left[\left(f - m_l^{j+1}\right)\phi_k^j - L_{k,l}^{j+1}\right]\phi_l^{j+1}\\
&= f\phi_k^j \mathbf{1}_{(\Omega^{j+1})^\complement} - m_k^j\phi_k^j + \phi_k^j \sum_{l\in I_{j+1}} m_l^{j+1}\phi_l^{j+1} + \sum_{l\in I_{j+1}} L_{k,l}^{j+1}\phi_l^{j+1}.
\end{aligned}
$$

The fourth term is supported on $B_k^j := B(x_k^j, 16A_0^4 r_k^j)$, which is deduced from (4.11). Thus, supp $h_k^j \subset B_k^j$. Moreover, by Lemmas 4.11(i) and 4.12(i), we conclude that there exists a positive constant C, independent of j and k, such that $\|h_k^j\|_{L^\infty(X)} \le C2^j$. Now, let

$$
\lambda_k^j := C2^j \left[\mu\left(B_k^j\right)\right]^{\frac{1}{p}} \quad \text{and} \quad a_k^j := \left(\lambda_k^j\right)^{-1} h_k^j. \tag{4.13}
$$

Then a_k^j is a (p,∞)-atom supported on B_k^j and $f = \sum_{j=-\infty}^{\infty}\sum_{k\in I_j}\lambda_k^j a_k^j$ in $(\mathcal{G}_0^\eta(\beta,\gamma))'$. Moreover, we have

$$
\begin{aligned}
\sum_{j=-\infty}^{\infty}\sum_{k\in I_j}\left|\lambda_k^j\right|^p &\lesssim \sum_{j=-\infty}^{\infty} 2^{-jp}\sum_{k\in I_j}\mu\left(B_k^j\right)\\
&\lesssim \sum_{j=-\infty}^{\infty} 2^{-jp}\mu\left(\Omega^j\right) \sim \|f^\star\|_{L^p(X)}^p \sim \|f^*\|_{L^p(X)}^p,
\end{aligned}
$$

which further implies that $\|f\|_{H_{\mathrm{at}}^{p,\infty}(X)} \lesssim \|f\|_{H^{*,p}(X)}$.

When $f \in H^{*,p}(X)$, using Lemma 4.10 and a standard density argument and following the proof in [43, pp. 301–302], we obtain the atomic decomposition of f, the details being omitted. This finishes the proof of $H^{*,p}(X) \subset H_{\mathrm{at}}^{p,q}(X)$ and hence of Theorem 4.2. □

Remark 4.13 By the argument used in the proof of $H^{*,p}(X) \subset H_{\mathrm{at}}^{p,q}(X)$, we find that, if $f \in L^q(X) \cap H^{*,p}(X)$ with $q \in [1,\infty]$, then $f = \sum_{j=1}^{\infty}\sum_{k\in I_j} h_k^j$ in $(\mathcal{G}_0^\eta(\beta,\gamma))'$ and almost everywhere, where, for any $j \in \mathbb{Z}$ and $k \in I_j$, h_k^j is as in (4.12).

4.4 Relationship Between $H_{\mathrm{at}}^{p,q}(X)$ and $H_{\mathrm{cw}}^{p,q}(X)$

In this section, we consider the relationship between $H_{\mathrm{at}}^{p,q}(X)$ and $H_{\mathrm{cw}}^{p,q}(X)$. To see this, we need the following two technical lemmas.

Lemma 4.14 ([7, p. 592]) *Let $p \in (0,1)$, $q \in (p,\infty] \cap [1,\infty]$ and a be a (p,q)-atom. Then, for any $\varphi \in \mathcal{L}_{1/p-1}(X)$, $|\langle a,\varphi\rangle| \le \|\varphi\|_{\mathcal{L}_{1/p-1}(X)}$.*

Lemma 4.15 *Let $\beta \in (0,\eta]$ and $\gamma \in (0,\infty)$. If $\varphi \in \mathcal{G}(\beta,\gamma)$, then $\varphi \in \mathcal{L}_{\beta/\omega}(X)$ and there exists a positive constant C, independent of φ, such that $\|\varphi\|_{\mathcal{L}_{\beta/\omega}(X)} \le C\|\varphi\|_{\mathcal{G}(\beta,\gamma)}$.*

Proof Suppose that $\|\varphi\|_{\mathcal{G}(\beta,\gamma)} \le 1$. If $d(x,y) \le (2A_0)^{-1}[1+d(x_0,x)]$, then, by the regularity condition of φ and (1.1), we have

$$|\varphi(x)-\varphi(y)| \le \left[\frac{d(x,y)}{1+d(x_0,x)}\right]^{\beta} \frac{1}{V_1(x_0)+V(x_0,x)} \left[\frac{1}{1+d(x_0,x)}\right]^{\gamma}$$
$$\lesssim \left[\frac{\mu(B(x,d(x,y)))}{\mu(B(x,1+d(x_0,x)))}\right]^{\beta/\omega} \lesssim [V(x,y)]^{\beta/\omega}.$$

If $d(x,y) > (2A_0)^{-1}[1+d(x_0,x)]$, then, from the size condition of φ, we deduce that

$$|\varphi(x)-\varphi(y)| \lesssim 1 \sim [\mu(B(x_0,1))]^{\beta/\omega} \lesssim [\mu(B(x_0,1+d(x_0,x)))]^{\beta/\omega}$$
$$\sim [\mu(B(x,1+d(x_0,x)))]^{\beta/\omega} \lesssim [V(x,y)]^{\beta/\omega}.$$

Thus, for any $x,\ y \in X$, we always have $|\varphi(x)-\varphi(y)| \lesssim \|\varphi\|_{\mathcal{G}(\beta,\gamma)}[V(x,y)]^{\beta/\omega}$. This implies $\varphi \in \mathcal{L}_{\beta/\omega}(X)$ and $\|\varphi\|_{\mathcal{L}_{\beta/\omega}(X)} \lesssim \|\varphi\|_{\mathcal{G}(\beta,\gamma)}$, which completes the proof of Lemma 4.15. □

Now we establish the relationship between two kinds of atomic Hardy spaces.

Theorem 4.16 *Let $p \in (\omega/(\omega+\eta), 1]$, $q \in (p,\infty] \cap [1,\infty]$ and $\beta,\ \gamma \in (\omega(1/p-1),\eta)$. If regard $H^{p,q}_{\rm at}(X)$ as a subspace of $(\mathcal{G}^{\eta}_0(\beta,\gamma))'$, then $H^{p,q}_{\rm cw}(X) = H^{p,q}_{\rm at}(X)$ with equal (quasi-)norms.*

Proof We only consider the case $p \in (\omega/(\omega+\eta), 1)$. The proof of $p=1$ is similar and the details are omitted.

We first prove $H^{p,q}_{\rm cw}(X) \subset H^{p,q}_{\rm at}(X)$. By Lemma 4.15, we have $\mathcal{G}^{\eta}_0(\beta,\gamma) \subset \mathcal{G}(\omega(1/p-1),\gamma) \subset \mathcal{L}_{1/p-1}(X)$ and hence $(\mathcal{L}_{1/p-1}(X))' \subset (\mathcal{G}^{\eta}_0(\beta,\gamma))'$. For any $f \in H^{p,q}_{\rm cw}(X)$, by Definition 1.1, we know that there exist (p,q)-atoms $\{a_j\}_{j=1}^{\infty}$ and $\{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C}$ with $\sum_{j=1}^{\infty}|\lambda_j|^p < \infty$ such that $f = \sum_{j=1}^{\infty}\lambda_j a_j$ in $(\mathcal{L}_{1/p-1}(X))'$ and hence in $(\mathcal{G}^{\eta}_0(\beta,\gamma))'$. Let $g := f|_{\mathcal{G}^{\eta}_0(\beta,\gamma)}$. Then, for any $\varphi \in \mathcal{G}^{\eta}_0(\beta,\gamma) \subset \mathcal{L}_{1/p-1}(X)$, we have

$$\langle g,\varphi\rangle = \langle f,\varphi\rangle = \sum_{j=1}^{\infty}\lambda_j\langle a_j,\varphi\rangle.$$

Thus, $g = \sum_{j=1}^{\infty}\lambda_j a_j$ in $(\mathcal{G}^{\eta}_0(\beta,\gamma))'$ and $\|g\|_{H^{p,q}_{\rm at}(X)} \le (\sum_{j=1}^{\infty}|\lambda_j|^p)^{\frac{1}{p}}$. If we take the infimum over all the atomic decompositions of f as above, we obtain $\|g\|_{H^{p,q}_{\rm at}(X)} \le \|f\|_{H^{p,q}_{\rm cw}(X)}$. Thus, $H^{p,q}_{\rm cw}(X) \subset H^{p,q}_{\rm at}(X)$.

To show $H^{p,q}_{\rm cw}(X) \supset H^{p,q}_{\rm at}(X)$, following the proof of [7, p. 593, Theorem B], we conclude that the dual space of $H^{p,q}_{\rm at}(X)$ is $\mathcal{L}_{1/p-1}(X)$ in the following sense: every bounded linear functional on $H^{p,q}_{\rm at}(X)$ is a mapping of the form

$$f \mapsto \sum_{j=1}^{\infty}\lambda_j\int_X a_j(x)g(x)\,d\mu(x),$$

where $g \in \mathcal{L}_{1/p-1}(X)$ and f has an atomic decomposition

$$f = \sum_{j=1}^{\infty} \lambda_j a_j \tag{4.14}$$

in $(\mathcal{G}_0^\eta(\beta,\gamma))'$ with (p,q)-atoms $\{a_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ satisfying $\sum_{j=1}^\infty |\lambda_j|^p < \infty$. Therefore, it is reasonable to define the pair $\langle f, g\rangle$ as follows:

$$\langle f, g\rangle := \sum_{j=1}^{\infty} \lambda_j \int_X a_j(x) g(x)\, d\mu(x).$$

In this way, we find that (4.14) also converges in $(\mathcal{L}_{1/p-1}(X))'$, and hence $f \in H_{\mathrm{cw}}^{p,q}(X)$ and $\|f\|_{H_{\mathrm{cw}}^{p,q}(X)} \leq (\sum_{j=1}^\infty |\lambda_j|^p)^{\frac{1}{p}}$. Taking the infimum over all the atomic decompositions of f as above, we obtain $\|f\|_{H_{\mathrm{cw}}^{p,q}(X)} \leq \|f\|_{H_{\mathrm{at}}^{p,q}(X)}$. Thus, $H_{\mathrm{at}}^{p,q}(X) \subset H_{\mathrm{cw}}^{p,q}(X)$, which completes the proof of Theorem 4.16. □

5 Littlewood–Paley Function Characterizations of Atomic Hardy Spaces

In this section, we consider the Littlewood–Paley function characterizations of Hardy spaces. Differently from Sects. 3 and 4, we use $(\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$ as underlying spaces to introduce Hardy spaces. Let $p \in (\omega/(\omega+\eta), 1]$, $\beta, \gamma \in (\omega(1/p-1), \eta)$, $f \in (\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$ and $\{Q_k\}_{k\in\mathbb{Z}}$ be an exp-ATI. For any $\theta \in (0,\infty)$, define the *Lusin area function of* f*, with aperture* θ, $\mathcal{S}_\theta(f)$, by setting, for any $x \in X$,

$$\mathcal{S}_\theta(f)(x) := \left[\sum_{k=-\infty}^{\infty} \int_{B(x,\theta\delta^k)} |Q_k f(y)|^2 \frac{d\mu(y)}{V_{\theta\delta^k}(x)}\right]^{\frac{1}{2}}. \tag{5.1}$$

In particular, when $\theta = 1$, we write $\mathcal{S}_\theta$ simply as $\mathcal{S}$. Define the *Hardy space* $H^p(X)$ via the Lusin area function by setting

$$H^p(X) := \left\{ f \in \left(\mathring{\mathcal{G}}_0^\eta(\beta,\gamma)\right)' :\ \|f\|_{H^p(X)} := \|\mathcal{S}(f)\|_{L^p(X)} < \infty \right\}.$$

In Sect. 5.1, we show that $H^p(X)$ is independent of the choices of exp-ATIs. In Sect. 5.2, we connect $H^p(X)$ with $H^{*,p}(X)$ by considering the molecular and the atomic characterizations of elements in $H^p(X)$. Sect. 5.3 deals with equivalent characterizations of $H^p(X)$ via the *Littlewood–Paley g-function*

$$g(f)(x) := \left[\sum_{k=-\infty}^{\infty} |Q_k f(x)|^2\right]^{\frac{1}{2}} \tag{5.2}$$

and the *Littlewood–Paley g_λ^*-function*

$$g_\lambda^*(f)(x) := \left\{ \sum_{k=-\infty}^{\infty} \int_X |Q_k f(y)|^2 \left[\frac{\delta^k}{\delta^k + d(x,y)} \right]^\lambda \frac{d\mu(y)}{V_{\delta^k}(x) + V_{\delta^k}(y)} \right\}^{\frac{1}{2}}, \quad (5.3)$$

where $f \in (\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$ with $\beta, \ \gamma \in (\omega(1/p-1), \eta)$, $x \in X$ and $\lambda \in (0,\infty)$.

5.1 Independence of exp-ATIs

In this section, we show that $H^p(X)$ is independent of the choices of exp-ATIs. If $\mathcal{E} := \{E_k\}_{k\in\mathbb{Z}}$ and $\mathcal{Q} := \{Q_k\}_{k\in\mathbb{Z}}$ are two exp-ATIs, then we denote by $\mathcal{S}_\mathcal{E}$ and $\mathcal{S}_\mathcal{Q}$ the Lusin area functions via $\mathcal{E}$ and $\mathcal{Q}$, respectively.

Theorem 5.1 *Let $\mathcal{E} := \{E_k\}_{k\in\mathbb{Z}}$ and $\mathcal{Q} := \{Q_k\}_{k\in\mathbb{Z}}$ be two* exp-ATIs. *Suppose that $p \in (\omega/(\omega+\eta), 1]$ and $\beta, \ \gamma \in (\omega(1/p-1), \eta)$. Then there exists a positive constant C such that, for any $f \in (\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$,*

$$C^{-1}\|\mathcal{S}_\mathcal{Q}(f)\|_{L^p(X)} \le \|\mathcal{S}_\mathcal{E}(f)\|_{L^p(X)} \le C\|\mathcal{S}_\mathcal{Q}(f)\|_{L^p(X)}.$$

To show Theorem 5.1, the Fefferman–Stein vector-valued maximal inequality is necessary.

Lemma 5.2 ([22, Theorem 1.2] or [47, Theorem 1.3]) *Suppose that $p \in (1,\infty)$ and $u \in (1,\infty]$. Then there exists a positive constant C such that, for any sequence $\{f_j\}_{j=1}^\infty$ of measurable functions,*

$$\left\| \left\{ \sum_{j=1}^\infty [\mathcal{M}(f_j)]^u \right\}^{\frac{1}{u}} \right\|_{L^p(X)} \le C \left\| \left(\sum_{j=1}^\infty |f_j|^u \right)^{\frac{1}{u}} \right\|_{L^p(X)}$$

with the usual modification made when $u = \infty$.

Proof of Theorem 5.1 By symmetry, we only need to prove

$$\|\mathcal{S}_\mathcal{E}(f)\|_{L^p(X)} \lesssim \|\mathcal{S}_\mathcal{Q}(f)\|_{L^p(X)}.$$

For any $k \in \mathbb{Z}$, $f \in (\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$ with $\beta, \ \gamma$ as in Theorem 5.1, and $z \in X$, define

$$m_k(f)(z) := \left[\frac{1}{V_{\delta^k}(z)} \int_{B(z,\delta^k)} |Q_k f(u)|^2 \, d\mu(u) \right]^{\frac{1}{2}}.$$

Now suppose that $l \in \mathbb{Z}$, $x \in X$ and $y \in B(x, \delta^l)$. By Theorem 2.7, we conclude that

$$E_l f(y) = \sum_{k=-\infty}^{\infty} \sum_{\alpha \in \mathcal{A}_k} \sum_{m=1}^{N(k,\alpha)} E_l \widetilde{Q}_k \left(y, y_\alpha^{k,m}\right) \int_{Q_\alpha^{k,m}} Q_k f(u)\, d\mu(u),$$

where all the notation is as in Theorem 2.7 and $\{\widetilde{Q}_k\}_{k=-\infty}^{\infty}$ satisfies the conditions of Theorem 2.7. Notice that, if $z \in Q_\alpha^{k,m}$, then $Q_\alpha^{k,m} \subset B(z, \delta^k)$ and $\mu(Q_\alpha^{k,m}) \sim V_{\delta^k}(z)$. Therefore, we have

$$\left| \frac{1}{\mu(Q_\alpha^{k,m})} \int_{Q_\alpha^{k,m}} Q_k f(u)\, d\mu(u) \right| \lesssim \left[\frac{1}{V_{\delta^k}(z)} \int_{B(z,\delta^k)} |Q_k f(u)|^2\, d\mu(y) \right]^{\frac{1}{2}} \sim m_k(f)(z),$$

which further implies that

$$\left| \frac{1}{\mu(Q_\alpha^{k,m})} \int_{Q_\alpha^{k,m}} Q_k f(u)\, d\mu(u) \right| \lesssim \inf_{z \in Q_\alpha^{k,m}} m_k(f)(z).$$

Moreover, by the proof of (3.7), we find that, for any fixed $\beta' \in (0, \beta)$,

$$\begin{aligned} \left| E_l \widetilde{Q}_k \left(y, y_\alpha^{k,m}\right) \right| &\lesssim \delta^{|k-l|\beta'} \frac{1}{V_{\delta^{k\wedge l}}(y) + V(y, y_\alpha^{k,m})} \left[\frac{\delta^{k\wedge l}}{\delta^{k\wedge l} + d(y, y_\alpha^{k,m})} \right]^\gamma \\ &\sim \delta^{|k-l|\beta'} \frac{1}{V_{\delta^{k\wedge l}}(x) + V(x, y_\alpha^{k,m})} \left[\frac{\delta^{k\wedge l}}{\delta^{k\wedge l} + d(x, y_\alpha^{k,m})} \right]^\gamma, \end{aligned}$$

where only the regularity condition of $\widetilde{Q}_k$ on the first variable is used. Therefore, by Lemma 3.7, for any fixed $r \in (\omega/(\omega+\gamma), 1]$, we have

$$\begin{aligned} |E_l f(y)| &\lesssim \sum_{k=-\infty}^{\infty} \delta^{|k-l|\beta'} \sum_{\alpha \in \mathcal{A}_k} \sum_{m=1}^{N(k,\alpha)} \mu\left(Q_\alpha^{k,m}\right) \frac{1}{V_{\delta^{k\wedge l}}(x) + V(x, y_\alpha^{k,m})} \\ &\quad \times \left[\frac{\delta^{k\wedge l}}{\delta^{k\wedge l} + d(x, y_\alpha^{k,m})} \right]^\gamma \inf_{z \in Q_\alpha^{k,m}} m_k(f)(z) \\ &\lesssim \sum_{k=-\infty}^{\infty} \delta^{|k-l|\beta'} \delta^{[k-(k\wedge l)]\omega(1-\frac{1}{r})} \\ &\quad \times \left\{ \mathcal{M}\left(\sum_{\alpha \in \mathcal{A}_k} \sum_{m=1}^{N(k,\alpha)} \inf_{z \in Q_\alpha^{k,m}} [m_k(f)(z)]^r \mathbf{1}_{Q_\alpha^{k,m}} \right)(x) \right\}^{\frac{1}{r}}. \end{aligned}$$

Choose β' and r such that $r \in (\omega/(\omega+\beta'), p)$. Then, by the Hölder inequality, we conclude that

$$\begin{aligned}
\left[\mathcal{S}_{\mathcal{E}}(f)(x)\right]^2 &= \sum_{l=-\infty}^{\infty} \int_{B(x,\delta^l)} |E_l f(y)|^2 \frac{dy}{V_{\delta^l}(x)} \\
&\lesssim \sum_{l=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} \delta^{|k-l|\beta'} \delta^{[k-(k\wedge l)]\omega(1-\frac{1}{r})} \right. \\
&\quad \left. \times \left\{ \mathcal{M}\left(\sum_{\alpha\in\mathcal{A}_k} \sum_{m=1}^{N(k,\alpha)} \inf_{z\in Q_\alpha^{k,m}} [m_k(f)(z)]^r \mathbf{1}_{Q_\alpha^{k,m}} \right)(x) \right\}^{\frac{1}{r}} \right]^2 \\
&\lesssim \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta^{|k-l|\beta'} \delta^{[k-(k\wedge l)]\omega(1-\frac{1}{r})} \\
&\quad \times \left\{ \mathcal{M}\left(\sum_{\alpha\in\mathcal{A}_k} \sum_{m=1}^{N(k,\alpha)} \inf_{z\in Q_\alpha^{k,m}} [m_k(f)(z)]^r \mathbf{1}_{Q_\alpha^{k,m}} \right)(x) \right\}^{\frac{2}{r}} \\
&\lesssim \sum_{k=-\infty}^{\infty} \left\{ \mathcal{M}\left(\sum_{\alpha\in\mathcal{A}_k} \sum_{m=1}^{N(k,\alpha)} \inf_{z\in Q_\alpha^{k,m}} [m_k(f)(z)]^r \mathbf{1}_{Q_\alpha^{k,m}} \right)(x) \right\}^{\frac{2}{r}} \\
&\lesssim \sum_{k=-\infty}^{\infty} \left\{ \mathcal{M}\left([m_k(f)]^r\right)(x) \right\}^{\frac{2}{r}}.
\end{aligned}$$

Therefore, from Lemma 5.2, we deduce that

$$\begin{aligned}
\|\mathcal{S}_{\mathcal{E}}(f)\|_{L^p(X)} &\lesssim \left\| \left(\sum_{k=-\infty}^{\infty} \left\{ \mathcal{M}\left([m_k(f)]^r\right) \right\}^{\frac{2}{r}} \right)^{\frac{r}{2}} \right\|_{L^{p/r}(X)}^{\frac{1}{r}} \\
&\lesssim \left\| \left\{ \sum_{k=-\infty}^{\infty} [m_k(f)]^2 \right\}^{\frac{1}{2}} \right\|_{L^p(X)} \sim \|\mathcal{S}_Q(f)\|_{L^p(X)}.
\end{aligned}$$

This finishes the proof of Theorem 5.1. □

5.2 Atomic Characterizations of $H^p(X)$

The main aim of this section is to obtain the atomic characterizations of $H^p(X)$ when $p \in (\omega/(\omega+\eta), 1]$.

For any $p \in (\omega/(\omega+\eta), 1]$, $q \in (p, \infty] \cap [1, \infty]$ and $\beta, \gamma \in (\omega(1/p-1), \eta)$, we define the *homogeneous atomic Hardy space* $\mathring{H}^{p,q}_{\mathrm{at}}(X)$ in the same way as $H^{p,q}_{\mathrm{at}}(X)$, but with the distribution space $(\mathcal{G}_0^\eta(\beta,\gamma))'$ replaced by $(\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$. Then the following relationship between $H^{p,q}_{\rm at}(X)$ and $\mathring{H}^{p,q}_{\rm at}(X)$ can be found in [20, Theorem 5.4].

Proposition 5.3 *Suppose* $p\in(\omega/(\omega+\eta),1]$, $\beta,\ \gamma\in(\omega(1/p-1),\eta)$ *and* $q\in(p,\infty]\cap[1,\infty]$. *Then* $\mathring{H}^{p,q}_{\rm at}(X)=H^{p,q}_{\rm at}(X)$ *with equivalent (quasi)-norms. More precisely, if* $f\in H^{p,q}_{\rm at}(X)$, *then the restriction of* f *on* $\mathring{\mathcal{G}}_0^\eta(\beta,\gamma)$ *belongs to* $\mathring{H}^{p,q}_{\rm at}(X)$; *Conversely, if* $f\in\mathring{H}^{p,q}_{\rm at}(X)$, *then there exists a unique* $\widetilde{f}\in H^{p,q}_{\rm at}(X)$ *such that* $\widetilde{f}=f$ *in* $(\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$.

Due to the fact that the kernels $\widetilde{Q}_k$ in the homogeneous continuous Calderón formula in Theorem 2.6 has no compact support, we can only use Theorem 2.6 to decompose an element of $H^p(X)$ into a linear combination of the following *molecules*.

Definition 5.4 Suppose that $p\in(0,1]$, $q\in(p,\infty]\cap[1,\infty]$ and $\vec{\epsilon}:=\{\epsilon_m\}_{m=1}^\infty\subset[0,\infty)$ satisfying

$$\sum_{m=1}^\infty m[\epsilon_m]^p<\infty. \tag{5.4}$$

A function $M\in L^q(X)$ is called a $(p,q,\vec{\epsilon})$-*molecule* centered at a ball $B:=B(x_0,r_0)$ for some $x_0\in X$ and $r\in(0,\infty)$ if M has the following properties:

(i) $\|M\mathbf{1}_B\|_{L^q(X)}\le[\mu(B)]^{\frac1q-\frac1p}$;

(ii) for any $m\in\mathbb{N}$, $\|M\mathbf{1}_{B(x_0,\delta^{-m}r_0)\setminus B(x_0,\delta^{-m+1}r_0)}\|_{L^q(X)}\le\epsilon_m[\mu(B(x_0,\delta^{-m}r_0))]^{\frac1q-\frac1p}$;

(iii) $\int_X M(x)\,d\mu(x)=0$.

By (i) and (ii) of Definition 5.4, the Hölder inequality, (5.4) and the fact $p\in(0,1]$, we find that, if M satisfies (i) and (ii) of Definition 5.4, then $M\in L^1(X)$ and hence Definition 5.4(iii) makes sense.

After carefully checking the proof of [39, Theorem 3.4], we obtain the following molecular characterization of the atomic Hardy space $H^{p,q}_{\rm cw}(X)$ of Coifman and Weiss [7], the details being omitted.

Proposition 5.5 *Suppose that* $p\in(0,1]$, $q\in(p,\infty]\cap[1,\infty]$ *and* $\vec{\epsilon}:=\{\epsilon_l\}_{l=1}^\infty$ *satisfies* (5.4). *Then* $f\in(\mathcal{G}_0^\eta(\beta,\gamma))'$ *belonging to* $H^{p,q}_{\rm cw}(X)$ *if and only if there exist* $(p,q,\vec{\epsilon})$-*molecules* $\{M_j\}_{j=1}^\infty$ *and* $\{\lambda_j\}_{j=1}^\infty\subset\mathbb{C}$, *with* $\sum_{j=1}^\infty|\lambda_j|^p<\infty$, *such that*

$$f=\sum_{j=1}^\infty\lambda_jM_j \tag{5.5}$$

converges in $(\mathcal{L}_{1/p-1}(X))'$ *when* $p\in(0,1)$ *or in* $L^1(X)$ *when* $p=1$. *Moreover, there exists a positive constant* C, *independent of* f, *such that, for any* $f\in H^{p,q}_{\rm cw}(X)$,

$$C^{-1}\|f\|_{H^{p,q}_{\rm cw}(X)}\le\inf\left\{\left(\sum_{j=1}^\infty|\lambda_j|^p\right)^{\frac1p}\right\}\le C\|f\|_{H^{p,q}_{\rm cw}(X)},$$

where the infimum is taken over all the molecular decompositions of f *as in* (5.5).

Let $p \in (\omega/(\omega+\eta), 1]$ and $q \in (p, \infty] \cap [1, \infty]$. By Proposition 5.3, $\mathring{H}^{p,q}_{\rm at}(X) = H^{p,q}_{\rm cw}(X)$ and the already known fact that $H^{p,q}_{\rm cw}(X)$ is independent of the choice of $q \in (p, \infty] \cap [1, \infty]$, we know that $\mathring{H}^{p,q}_{\rm at}(X) = \mathring{H}^{p,2}_{\rm at}(X)$. With this observation, we show $\mathring{H}^{p,q}_{\rm at}(X) \subset H^p(X)$ as follows.

Proposition 5.6 *Let $p \in (\omega/(\omega+\eta), 1]$, $\beta,\ \gamma \in (\omega(1/p-1), \eta)$, $q \in (p, \infty] \cap [1, \infty]$ and $\{Q_k\}_{k\in\mathbb{Z}}$ be an* exp-ATI. *Let $\theta \in (0, \infty)$ and $\mathcal{S}_\theta$ be as in (5.1). Then there exists a positive constant C, independent of θ, such that, for any distribution $f \in (\mathring{\mathcal{G}}^\eta_0(\beta, \gamma))'$ belonging to $\mathring{H}^{p,2}_{\rm at}(X)$,*

$$\|\mathcal{S}_\theta(f)\|_{L^p(X)} \le C \max\left\{\theta^{-\omega/2}, \theta^{\omega/p}\right\} \|f\|_{\mathring{H}^{p,2}_{\rm at}(X)}. \tag{5.6}$$

In particular, $\mathring{H}^{p,q}_{\rm at}(X) = \mathring{H}^{p,2}_{\rm at}(X) \subset H^p(X)$.

Proof Let $\beta,\ \gamma \in (\omega(1/p-1), \eta)$. It suffices to show (5.6) for the case $\theta \in [1, \infty)$, because both (5.6) with $\theta = 1$ and $\mathcal{S}_\theta(f) \lesssim \theta^{-\omega/2}\mathcal{S}(f)$ for any $f \in (\mathring{\mathcal{G}}^\eta_0(\beta, \gamma))'$ whenever $\theta \in (0, 1)$ imply that (5.6) also holds true for any $\theta \in (0, 1)$.

We start with the proof of the fact that the Littlewood–Paley g-function as in (5.2) is bounded on $L^2(X)$. Indeed, for any $h \in L^2(X)$, we write

$$\|g(h)\|^2_{L^2(X)} = \sum_{k=-\infty}^{\infty} \int_X |Q_k h(z)|^2\, d\mu(z) = \sum_{k=-\infty}^{\infty} \left\langle Q_k^* Q_k h, h\right\rangle.$$

By Theorem 2.6 and the proof of [27, (3.2)], we find that, for any fixed $\beta' \in (0, \beta\wedge\gamma)$, any $k_1,\ k_2 \in \mathbb{Z}$ and $x,\ y \in X$, we have

$$\left|Q_{k_1} Q^*_{k_2}(x, y)\right| \lesssim \delta^{|k_1-k_2|\beta'} \frac{1}{V_{\delta^{k_1\wedge k_2}}(x) + V(x, y)} \left[\frac{\delta^{k_1\wedge k_2}}{\delta^{k_1\wedge k_2} + V(x, y)}\right]^\gamma. \tag{5.7}$$

Notice that, in (5.7), only the regularity of Q_k with respect to the second variable is used. Thus, by Lemma 2.2(v) and the boundedness of $\mathcal{M}$ on $L^2(X)$, we conclude that, for any $k_1,\ k_2 \in \mathbb{Z}$,

$$\left\|\left(Q^*_{k_1} Q_{k_1}\right)\left(Q^*_{k_2} Q_{k_2}\right)\right\|_{L^2(X)\to L^2(X)} \lesssim \left\|Q_{k_1} Q^*_{k_2}\right\|_{L^2(X)\to L^2(X)} \lesssim \delta^{|k_1-k_2|\beta'}.$$

Therefore, by the fact that $Q_k^* Q_k$ is self-adjoint and the Cotlar–Stein lemma (see [49, pp. 279–280] and [29, Lemma 4.5]), we obtain the boundedness of $\sum_{k=-\infty}^{\infty} Q_k^* Q_k$ on $L^2(X)$ and hence the boundedness of g on $L^2(X)$.

Suppose that a is a $(p, 2)$-atom supported on a ball $B := B(x_0, r_0)$ with $x_0 \in X$ and $r_0 \in (0, \infty)$. By the Fubini theorem and the boundedness of g on $L^2(X)$, we find that

$$\|\mathcal{S}_\theta(a)\|_{L^2(X)} \lesssim \left\|\left\{\sum_{k\in\mathbb{Z}} |Q_k a|^2\right\}^{1/2}\right\|_{L^2(X)} \sim \|g(a)\|_{L^2(X)} \lesssim \|a\|_{L^2(X)} \lesssim [\mu(B)]^{\frac12-\frac1p},$$

which further implies that

$$\int_{B(x_0,4A_0^2\theta r_0)} [\mathcal{S}_\theta(a)(x)]^p\, d\mu(x) \le \|\mathcal{S}_\theta(a)\|_{L^2(X)}^p \left[\mu\left(B\left(x_0, 4A_0^2\theta r_0\right)\right)\right]^{1-\frac{p}{2}} \lesssim \theta^{\omega(1-\frac{p}{2})}. \tag{5.8}$$

Let $x \notin B(x_0, 4A_0^2\theta r_0)$ and $y \in B(x, \theta\delta^k)$. Since now $\theta \in [1, \infty)$, for any $u \in B = B(x_0, r_0)$, we have $d(u, x_0) < (4A_0^2\theta)^{-1} d(x_0, x) < (2A_0)^{-1}[\delta^k + d(x_0, y)]$ and hence

$$\begin{aligned}
|Q_k a(y)| &= \left|\int_X Q_k(y,u)a(u)\, d\mu(u)\right| \le \int_B |Q_k(y,u) - Q_k(y,x_0)||a(u)|\, d\mu(u) \\
&\lesssim \int_B \left[\frac{d(x_0,u)}{\delta^k + d(x_0,y)}\right]^\eta \frac{1}{V_{\delta^k}(x_0) + V(x_0,y)} \left[\frac{\delta^k}{\delta^k + d(x_0,y)}\right]^\gamma |a(u)|\, d\mu(u) \\
&\lesssim [\mu(B)]^{1-\frac{1}{p}} \left[\frac{r_0}{\delta^k + d(x_0,y)}\right]^\eta \frac{1}{V_{\delta^k}(x_0) + V(x_0,y)} \left[\frac{\delta^k}{\delta^k + d(x_0,y)}\right]^\gamma.
\end{aligned}$$

On another hand, if $\delta^k < (4A_0^2\theta)^{-1} d(x_0, x)$, then $d(x_0, y) \ge (4A_0)^{-1} d(x_0, x)$ and hence

$$|Q_k a(y)| \lesssim [\mu(B)]^{1-\frac{1}{p}} \left[\frac{r_0}{d(x_0,x)}\right]^\eta \frac{1}{V(x_0,x)} \left[\frac{\delta^k}{d(x_0,x)}\right]^\gamma,$$

which further implies that

$$\begin{aligned}
&\sum_{\delta^k < (4A_0^2\theta)^{-1} d(x_0,x)} \int_{B(y,\theta\delta^k)} |Q_k a(y)|^2 \frac{d\mu(y)}{V_{\theta\delta^k}(x)} \\
&\quad\lesssim [\mu(B)]^{2-\frac{2}{p}} \left[\frac{r_0}{d(x_0,x)}\right]^{2\eta} \left[\frac{1}{V(x_0,x)}\right]^2 \sum_{\delta^k < (4A_0^2\theta)^{-1} d(x_0,x)} \left[\frac{\delta^k}{d(x_0,x)}\right]^{2\gamma} \\
&\quad\lesssim [\mu(B)]^{2-\frac{2}{p}} \left[\frac{r_0}{d(x_0,x)}\right]^{2\eta} \left[\frac{1}{V(x_0,x)}\right]^2.
\end{aligned}$$

If $\delta^k \ge (4A_0^2\theta)^{-1} d(x_0, x)$, then $V(x_0, x) \lesssim \mu(B(x_0, \theta\delta^k)) \lesssim \theta^\omega V_{\delta^k}(x_0)$ and

$$|Q_k a(y)| \lesssim \theta^\omega [\mu(B)]^{1-\frac{1}{p}} \left(\frac{r_0}{\delta^k}\right)^\eta \frac{1}{V(x_0,x)},$$

which further implies that

$$\sum_{\delta^k \geq (4A_0^2\theta)^{-1}d(x_0,x)} \int_{B(y,\theta\delta^k)} |Q_k a(y)|^2 \frac{d\mu(y)}{V_{\theta\delta^k}(x)}$$
$$\lesssim \theta^{2\omega}[\mu(B)]^{2-\frac{2}{p}} \left[\frac{1}{V(x_0,x)}\right]^2 \sum_{\delta^k \geq (4A_0^2\theta)^{-1}d(x_0,x)} \left(\frac{r_0}{\delta^k}\right)^{2\eta}$$
$$\sim \theta^{2\omega+2\eta}[\mu(B)]^{2-\frac{2}{p}} \left[\frac{r_0}{d(x_0,x)}\right]^{2\eta} \left[\frac{1}{V(x_0,x)}\right]^2.$$

Therefore, when $x \notin B(x_0, 4A_0^2\theta r_0)$, we have

$$\mathcal{S}_\theta(a)(x) \lesssim \theta^{\omega+\eta}[\mu(B)]^{1-\frac{1}{p}} \left[\frac{r_0}{d(x_0,x)}\right]^{\eta} \frac{1}{V(x_0,x)}.$$

Consequently, using $p \in (\eta/(\omega+\eta), 1]$, $B = B(x_0, r_0)$ and (1.1), we obtain

$$\int_{[B(x_0,4A_0^2\theta r_0)]^\complement} [\mathcal{S}_\theta(a)(x)]^p \, d\mu(x)$$
$$\lesssim \theta^{(\omega+\eta)p}[\mu(B)]^{p-1} \int_{[B(x_0,4A_0^2\theta r_0)]^\complement} \left[\frac{r_0}{d(x_0,x)}\right]^{p\eta} \left[\frac{1}{V(x_0,x)}\right]^p d\mu(x)$$
$$\lesssim \theta^{p\omega}[\mu(B)]^{p-1} \sum_{j=2}^{\infty} 2^{-jp\eta}$$
$$\times \int_{(2A_0)^j\theta r_0 \leq d(x_0,x) < (2A_0)^{j+1}\theta r_0} \left[\frac{1}{\mu(B(x_0,(2A_0)^j\theta r_0))}\right]^p d\mu(x)$$
$$\lesssim \theta^{\omega} \sum_{j=2}^{\infty} 2^{-j[p\eta-(1-p)\omega]} \lesssim \theta^{\omega}. \tag{5.9}$$

Combining (5.8) and (5.9) implies that, when $\theta \in [1,\infty)$,

$$\|\mathcal{S}_\theta(a)\|_{L^p(X)} \lesssim \theta^{\omega/p}. \tag{5.10}$$

Let $f \in \mathring{H}^{p,2}_{\mathrm{at}}(X)$. By the definition of $\mathring{H}^{p,2}_{\mathrm{at}}(X)$, we know that, for any $\epsilon \in (0,\infty)$, there exist $(p,2)$-atoms $\{a_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $(\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$ and $\sum_{j=1}^\infty |\lambda_j|^p \leq \|f\|^p_{\mathring{H}^{p,2}_{\mathrm{at}}(X)} + \epsilon$. By (5.10) and the fact $\mathcal{S}_\theta(f) \leq \sum_{j=1}^\infty |\lambda_j| \mathcal{S}_\theta(a_j)$, we conclude that

$$\|\mathcal{S}_\theta(f)\|^p_{L^p(X)} \leq \sum_{j=1}^\infty |\lambda_j|^p \|\mathcal{S}_\theta(a_j)\|^p_{L^p(X)} \lesssim \theta^\omega \sum_{j=1}^\infty |\lambda_j|^p$$
$$\lesssim \theta^\omega[\|f\|^p_{\mathring{H}^{p,2}_{\mathrm{at}}(X)} + \epsilon] \to \theta^\omega \|f\|^p_{\mathring{H}^{p,2}_{\mathrm{at}}(X)}$$

as $\epsilon \to 0^+$. This finishes the proof of (5.6) and hence of Proposition 5.6. □

Next, we use Proposition 5.5 to show the following converse of Proposition 5.6.

Proposition 5.7 *Let $p \in (\omega/(\omega+\eta), 1]$, $\beta, \gamma \in (\omega(1/p-1), \eta)$ and $f \in (\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$ belong to $H^p(X)$. Then there exist a sequence $\{a_j\}_{j=1}^\infty$ of $(p,2)$-atoms and $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $(\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$ and $\sum_{j=1}^\infty |\lambda_j|^p \le C\|f\|_{H^p(X)}^p$, where C is a positive constant independent of f. Consequently, $H^p(X) \subset \mathring{H}_{\mathrm{at}}^{p,2}(X)$.*

Proof Assume that $f \in (\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$ belongs to $H^p(X)$. In this proof, to avoid the confusion of notation between the exp-ATI $\{Q_k\}_{k\in\mathbb{Z}}$ and $Q \in \mathcal{D}$, we use $\{E_k\}_{k\in\mathbb{Z}}$ to denote an exp-ATI and then define $\mathcal{S}(f)$ as in (5.1) but with Q_k therein replaced by E_k. Denote by $\mathcal{D}$ the set of all dyadic cubes. For any $k \in \mathbb{Z}$, we define $\Omega_k := \{x \in X:\ \mathcal{S}(f)(x) > 2^k\}$ and

$$\mathcal{D}_k := \left\{Q \in \mathcal{D}:\ \mu(Q\cap\Omega_k) > \frac{1}{2}\mu(Q) \text{ and } \mu(Q\cap\Omega_{k+1}) \le \frac{1}{2}\mu(Q)\right\}.$$

It is easy to see that, for any $Q \in \mathcal{D}$, there exists a unique $k \in \mathbb{Z}$ such that $Q \in \mathcal{D}_k$. A dyadic cube $Q \in \mathcal{D}_k$ is called a *maximal cube in* $\mathcal{D}_k$ if $Q' \in \mathcal{D}$ and $Q' \supset Q$, then $Q' \notin \mathcal{D}_k$. Denote the set of all maximal cubes in $\mathcal{D}_k$ at level $j \in \mathbb{Z}$ by $\{Q_{\tau,k}^j\}_{\tau\in I_{j,k}}$, where $I_{j,k} \subset \mathcal{A}_j$ may be empty. The center of $Q_{\tau,k}^j$ is denoted by $z_{\tau,k}^j$. Then $\mathcal{D} = \bigcup_{j,\,k\in\mathbb{Z}} \bigcup_{\tau\in I_{j,k}} \{Q \in \mathcal{D}_k:\ Q \subset Q_{\tau,k}^j\}$.

From now on, we adopt the notation $E_Q := E_l$ and $\widetilde{E}_Q := \widetilde{E}_l$ whenever $Q = Q_\alpha^{l+1}$ for some $l \in \mathbb{Z}$ and $\alpha \in \mathcal{A}_{l+1}$. Then, by Theorem 2.6, we find that

$$\begin{aligned}
f(\cdot) &= \sum_{l=-\infty}^{\infty} \widetilde{E}_l E_l f(\cdot)\\
&= \sum_{l=-\infty}^{\infty} \sum_{\alpha\in\mathcal{A}_{l+1}} \int_{Q_\alpha^{l+1}} \widetilde{E}_l(\cdot,y) E_l f(y)\,d\mu(y)\\
&= \sum_{Q\in\mathcal{D}} \int_Q \widetilde{E}_Q(\cdot,y) E_Q f(y)\,d\mu(y)\\
&= \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{\tau\in I_{j,k}} \sum_{Q\in\mathcal{D}_k,\ Q\subset Q_{\tau,k}^j} \int_Q \widetilde{E}_Q(\cdot,y) E_Q f(y)\,d\mu(y)\\
&=: \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{\tau\in I_{j,k}} \lambda_{\tau,k}^j b_{\tau,k}^j(\cdot),
\end{aligned} \tag{5.11}$$

where all the equalities converge in $(\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$,

$$\lambda_{\tau,k}^j := \left[\mu\left(Q_{\tau,k}^j\right)\right]^{\frac{1}{p}-\frac{1}{2}} \left[\sum_{Q\in\mathcal{D}_k,\ Q\subset Q_{\tau,k}^j} \int_Q |E_Q f(y)|^2\,d\mu(y)\right]^{\frac{1}{2}}$$

and

$$b^j_{\tau,k}(\cdot) := \frac{1}{\lambda^j_{\tau,k}} \sum_{Q\in\mathcal{D}_k,\ Q\subset Q^j_{\tau,k}} \int_Q \widetilde{E}_Q(\cdot, y) E_Q f(y)\, d\mu(y). \tag{5.12}$$

For any $Q \in \mathcal{D}_k$ and $Q \subset Q^j_{\tau,k}$, assume that $Q = Q^{l+1}_\alpha$ for some $l \in \mathbb{Z}$ and $\alpha \in \mathcal{A}_{l+1}$. Since δ is assumed to satisfy $\delta < (2A_0)^{-10}$, it then follows that $2A_0 C^\natural \delta < 1$ so that $Q = Q^{l+1}_\alpha \subset B(y, \delta^l)$ for any $y \in Q$. By this and the fact that $\mu(Q \cap \Omega_{k+1}) \leq \frac{1}{2}\mu(Q)$, we obtain

$$\begin{aligned}\mu(B(y,\delta^l) \cap [Q^j_{\tau,k} \setminus \Omega_{k+1}]) &\geq \mu(B(y,\delta^l) \cap [Q \setminus \Omega_{k+1}])\\ &= \mu(Q \setminus \Omega_{k+1}) \geq \frac{1}{2}\mu(Q) \sim V_{\delta^l}(y).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\sum_{Q\in\mathcal{D}_k,\ Q\subset Q^j_{\tau,k}} \int_Q |E_Q f(y)|^2\, d\mu(y)\\ &\lesssim \sum_{l=j-1}^{\infty} \sum_{\alpha\in\mathcal{A}_{l+1},\ \mathcal{D}_k \ni Q^{l+1}_\alpha \subset Q^j_{\tau,k}} \int_{Q^{l+1}_\alpha} \frac{\mu(B(y,\delta^l) \cap (Q^j_{\tau,k} \setminus \Omega_{k+1}))}{V_{\delta^l}(y)} |E_l f(y)|^2\, d\mu(y)\\ &\lesssim \sum_{l=j-1}^{\infty} \int_{Q^j_{\tau,k}} \frac{\mu(B(y,\delta^l) \cap (Q^j_{\tau,k} \setminus \Omega_{k+1}))}{V_{\delta^l}(y)} |E_l f(y)|^2\, d\mu(y)\\ &\sim \int_X \sum_{l=j-1}^{\infty} \int_{B(y,\delta^l)\cap(Q^j_{\tau,k}\setminus\Omega_{k+1})} |E_l f(y)|^2 \frac{d\mu(x)}{V_{\delta^l}(y)}\, d\mu(y)\\ &\lesssim \int_{Q^j_{\tau,k}\setminus\Omega_{k+1}} [\mathcal{S}(f)(x)]^2\, d\mu(x) \lesssim 2^{2k} \mu\left(Q^j_{\tau,k}\right).\end{aligned}$$

From this and the fact $\mu(Q^j_{\tau,k}) < 2\mu(Q^j_{\tau,k} \cap \Omega_k)$, it follows that

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{\tau\in I_{j,k}} \left(\lambda^j_{\tau,k}\right)^p &\lesssim \sum_{k=-\infty}^{\infty} 2^{kp} \sum_{j=-\infty}^{\infty} \sum_{\tau\in I_{j,k}} \mu\left(Q^j_{\tau,k}\right)\\ &\lesssim \sum_{k=-\infty}^{\infty} 2^{kp} \sum_{j=-\infty}^{\infty} \sum_{\tau\in I_{j,k}} \mu\left(Q^j_{\tau,k} \cap \Omega_k\right)\\ &\lesssim \sum_{k=-\infty}^{\infty} 2^{kp} \mu\left(\Omega_k\right) \sim \|\mathcal{S}(f)\|^p_{L^p(X)}.\end{aligned} \tag{5.13}$$

Choose $\gamma' \in (\omega(1/p-1), \gamma)$ and let $\vec{\epsilon} := \{\delta^{m[\gamma' - \omega(1/p-1)]}\}_{m\in\mathbb{N}}$. Assume for the moment that every $b^j_{\tau,k}$ as in (5.12) is a $(p, 2, \vec{\epsilon})$-molecule centered at a ball

$B^j_{\tau,k} := B(z^j_{\tau,k}, 4A_0^2\delta^{j-1})$, whose proof is given in Lemma 5.8 below. Further, applying Proposition 5.5, we conclude that $\|b^j_{\tau,k}\|_{H^{p,2}_{\rm cw}(X)} \lesssim 1$. Thus, $b^j_{\tau,k}$ can be written as a linear combination of $(p,2)$-atoms, which converges in $(\mathcal{L}_{1/p-1}(X))'$ when $p \in (\omega/(\omega+\eta), 1)$ or in $L^1(X)$ when $p = 1$, and hence converges in $(\mathring{\mathcal{G}}^\eta_0(\beta,\gamma))'$ because $\mathring{\mathcal{G}}^\eta_0(\beta,\gamma) \subset \mathcal{L}_{1/p-1}(X)$ (see Lemma 4.15). Invoking this, (5.11) and (5.13), we find that $f \in \mathring{H}^{p,2}_{\rm at}(X)$ and $\|f\|_{\mathring{H}^{p,2}_{\rm at}(X)} \lesssim \|\mathcal{S}(f)\|_{L^p(X)}$. This finishes the proof of Proposition 5.7 modulo the proof of Lemma 5.8. □

Lemma 5.8 *Let all the notation be as in the proof of Proposition 5.7. Then every $b^j_{\tau,k}$ as in (5.12) is a harmless positive constant multiple of a $(p,2,\vec{\epsilon})$-molecule centered at the ball $B^j_{\tau,k} := B(z^j_{\tau,k}, 4A_0^2\delta^{j-1})$, where $\vec{\epsilon} := \{\delta^{m[\gamma'-\omega(1/p-1)]}\}_{m\in\mathbb{N}}$ and $\gamma' \in (\omega(1/p-1),\gamma)$.*

Proof Let $b^j_{\tau,k}$ be as in (5.12). For any $h \in L^2(X)$ with $\|h\|_{L^2(X)} \le 1$, by the Fubini theorem and the Hölder inequality, we conclude that

$$\begin{aligned}
&\left|\int_X b^j_{\tau,k}(x)h(x)\,d\mu(x)\right|\\
&\quad\le \frac{1}{\lambda^j_{\tau,k}} \sum_{Q\in\mathcal{D}_k,\ Q\subset Q^j_{\tau,k}} \int_Q |E_Q f(y)| \left|\int_X \widetilde{E}_Q(x,y)h(x)\,d\mu(x)\right| d\mu(y)\\
&\quad\le \frac{1}{\lambda^j_{\tau,k}} \left[\sum_{Q\in\mathcal{D}_k,\ Q\subset Q^j_{\tau,k}} \int_Q |E_Q f(y)|^2\,d\mu(y)\right]^{\frac12}\\
&\qquad\times \left[\sum_{Q\in\mathcal{D}_k,\ Q\subset Q^j_{\tau,k}} \int_X \left|\widetilde{E}^*_Q h(y)\right|^2\,d\mu(y)\right]^{\frac12}\\
&\quad\le \left[\mu\left(Q^j_{\tau,k}\right)\right]^{\frac12-\frac1p} \|\widetilde{g}(h)\|_{L^2(X)},
\end{aligned}$$

where $\widetilde{g}(h) := [\sum_{l=-\infty}^\infty |\widetilde{E}^*_l h|^2]^{1/2}$. Noticing that the kernel of $\widetilde{E}^*_l$ has the regularity with respect to the second variable, we follow the argument used in the beginning of the proof of Proposition 5.6 to deduce that $\widetilde{g}$ is bounded on $L^2(X)$. Thus, we have

$$\left|\int_X b^j_{\tau,k}(x)h(x)\,d\mu(x)\right| \lesssim \left[\mu\left(Q^j_{\tau,k}\right)\right]^{\frac12-\frac1p} \|h\|_{L^2(X)} \lesssim \left[\mu\left(B^j_{\tau,k}\right)\right]^{\frac12-\frac1p}.$$

Taking supremum over all $h \in L^2(X)$ with $\|h\|_{L^2(X)} \le 1$, we further find that

$$\left\|b^j_{\tau,k}\right\|_{L^2(X)} \lesssim \left[\mu\left(B^j_{\tau,k}\right)\right]^{\frac12-\frac1p}.$$

Let $\gamma' \in (\omega(1/p-1), \gamma)$. Fix $m \in \mathbb{N}$ and let $R_m := (\delta^{-m} B^j_{\tau,k}) \setminus (\delta^{-m+1} B^j_{\tau,k})$. Then, for any $x \in R_m$, by the Hölder inequality and the size condition of $\{\widetilde{E}_l\}_{l\in\mathbb{Z}}$, we conclude that

$$\begin{aligned}
\left|b^j_{\tau,k}(x)\right| &\le \frac{1}{\lambda^j_{\tau,k}} \sum_{Q\in\mathcal{D}_k,\ Q\subset Q^j_{\tau,k}} \int_Q \left|\widetilde{E}_Q(x,y)E_Q f(y)\right| d\mu(y) \\
&\lesssim \frac{1}{\lambda^j_{\tau,k}} \sum_{l=j-1}^{\infty} \sum_{\alpha\in\mathcal{A}_{l+1},\ \mathcal{D}_k\ni Q^{l+1}_\alpha \subset Q^j_{\tau,k}} \int_{Q^{l+1}_\alpha} \frac{1}{V_{\delta^l}(x)+V(x,y)} \\
&\quad \times \left[\frac{\delta^l}{\delta^l+d(x,y)}\right]^{\gamma} |E_l f(y)|\, d\mu(y) \\
&\lesssim \frac{1}{\lambda^j_{\tau,k}} \left\{\sum_{l=j-1}^{\infty} \sum_{\alpha\in\mathcal{A}_{l+1},\ Q^{l+1}_\alpha \subset Q^j_{\tau,k}} \int_{Q^{l+1}_\alpha} \frac{1}{V_{\delta^l}(x)+V(x,y)} \right.\\
&\quad \left. \times \left[\frac{\delta^l}{\delta^l+d(x,y)}\right]^{2\gamma'} d\mu(y)\right\}^{\frac12} \\
&\quad \times \left\{\sum_{l=j-1}^{\infty} \sum_{\substack{\alpha\in\mathcal{A}_{l+1}\\ \mathcal{D}_k\ni Q^{l+1}_\alpha \subset Q^j_{\tau,k}}} \int_{Q^{l+1}_\alpha} \frac{1}{V_{\delta^l}(x)+V(x,y)} \right.\\
&\quad \left. \times \left[\frac{\delta^l}{\delta^l+d(x,y)}\right]^{2(\gamma-\gamma')} |E_l f(y)|^2\, d\mu(y)\right\}^{\frac12} \\
&=: \frac{1}{\lambda^j_{\tau,k}} \mathrm{Y}(x)\mathrm{Z}(x).
\end{aligned}$$

Notice that, for any $x \in R_m$, we have $4A_0^2\delta^{j-m-1} \le d(x, z^j_{\tau,k}) < 4A_0^2\delta^{j-m-2}$ and, for any $y \in Q^{l+1}_\alpha \subset Q^j_{\tau,k}$, we have $\delta^l + d(x,y) \sim d(x,y) \sim \delta^{-m+j}$ and hence

$$\begin{aligned}
\mathrm{Y}(x) &\lesssim \left[\sum_{l=j-1}^{\infty} \sum_{\alpha\in\mathcal{A}_{l+1},\ Q^{l+1}_\alpha \subset Q^j_{\tau,k}} \int_{Q^{l+1}_\alpha} \frac{1}{\mu(B(y,\delta^{-m+j}))} \left(\frac{\delta^l}{\delta^{-m+j}}\right)^{2\gamma'} d\mu(y)\right]^{\frac12} \\
&\lesssim \left[\sum_{l=j-1}^{\infty} \left(\frac{\delta^l}{\delta^{-m+j}}\right)^{2\gamma'} \int_{Q^j_{\tau,k}} \frac{1}{\mu(B(z^j_{\tau,k},\delta^{-m+j}))} d\mu(y)\right]^{\frac12} \\
&\lesssim \delta^{m\gamma'} \left[\frac{\mu(B^j_{\tau,k})}{\mu(\delta^{-m} B^j_{\tau,k})}\right]^{\frac12}.
\end{aligned}$$

Thus, for any $x \in R_m$, we have

$$\left|b^j_{\tau,k}(x)\right| \lesssim \frac{1}{\lambda^j_{\tau,k}} \delta^{m\gamma'} \left[\frac{\mu(B^j_{\tau,k})}{\mu(\delta^{-m} B^j_{\tau,k})}\right]^{\frac{1}{2}} \mathrm{Z}(x),$$

which, together with the Fubini theorem and Lemma 2.2(ii), implies that

$$\begin{aligned}\left\|b^j_{\tau,k}\mathbf{1}_{R_m}\right\|_{L^2(X)} &\lesssim \frac{1}{\lambda^j_{\tau,k}} \delta^{m\gamma'} \left[\frac{\mu(B^j_{\tau,k})}{\mu(\delta^{-m} B^j_{\tau,k})}\right]^{\frac{1}{2}} \left\{\int_{R_m} [\mathrm{Z}(x)]^2 \, d\mu(x)\right\}^{\frac{1}{2}} \\ &\lesssim \frac{1}{\lambda^j_{\tau,k}} \delta^{m\gamma'} \left[\frac{\mu(B^j_{\tau,k})}{\mu(\delta^{-m} B^j_{\tau,k})}\right]^{\frac{1}{2}} \left\{\sum_{Q\in\mathcal{D}_k,\ Q\subset Q^j_{\tau,k}} \int_Q \left|E_Q f(y)\right|^2 \, d\mu(y)\right\}^{\frac{1}{2}} \\ &\lesssim \delta^{m\gamma'} \left[\frac{\mu(B^j_{\tau,k})}{\mu(\delta^{-m} B^j_{\tau,k})}\right]^{\frac{1}{2}} \left[\mu\left(B^j_{\tau,k}\right)\right]^{\frac{1}{2}-\frac{1}{p}} \\ &\lesssim \delta^{m[\gamma'-\omega(\frac{1}{p}-1)]} \left[\mu\left(\delta^{-m} B^j_{\tau,k}\right)\right]^{\frac{1}{2}-\frac{1}{p}}.\end{aligned}$$

The cancelation of $b^j_{\tau,k}$ follows directly from that of $\widetilde{E}_l$, the details being omitted.

Letting $\epsilon_m := \delta^{m[\gamma'-\omega(\frac{1}{p}-1)]}$ for any $m \in \mathbb{N}$, we find that $\{\epsilon_m\}_{m=1}^\infty$ satisfies (5.4) and $b^j_{\tau,k}$ is a harmless positive constant multiple of a $(p, 2, \vec{\epsilon})$-molecule. This finishes the proof of Lemma 5.8. □

Combining Propositions 5.6 and 5.7, we immediately obtain the following main result of this section, the details being omitted.

Theorem 5.9 *Suppose that $p \in (\omega/(\omega+\eta), 1]$, $\beta,\ \gamma \in (\omega(1/p-1), \eta)$ and $q \in (p, \infty] \cap [1, \infty]$. As subspaces of $(\mathring{\mathcal{G}}^\eta_0(\beta, \gamma))'$, it holds true that $\mathring{H}^{p,q}_{at}(X) = H^p(X)$ with equivalent (quasi-)norms.*

5.3 Hardy Spaces via Various Littlewood–Paley Functions

In this section, we characterize Hardy spaces $H^p(X)$ via the Lusin area functions with apertures, the Littlewood–Paley g-functions and the Littlewood–Paley g^*_λ-functions, respectively. We first consider the Littlewood–Paley g-function characterizations.

Theorem 5.10 *Let $p \in (\omega/(\omega+\eta), 1]$ and $\beta,\ \gamma \in (\omega(1/p-1), \eta)$. Then there exists a constant $C \in [1, \infty)$ such that, for any $f \in (\mathring{\mathcal{G}}^\eta_0(\beta, \gamma))'$,*

$$C^{-1}\|f\|_{H^p(X)} \le \|g(f)\|_{L^p(X)} \le C\|f\|_{H^p(X)}. \tag{5.14}$$

Proof Let $f \in (\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$ with $\beta,\ \gamma \in (\omega(1/p-1),\eta)$. With $\{Q_k\}_{k\in\mathbb{Z}}$ being an exp-ATI, we define $\mathcal{S}(f)$ and $g(f)$, respectively, as in (5.1) and (5.3). If $f \in H^p(X) = \mathring{H}_{\mathrm{at}}^{p,2}(X)$, then, following the proof of (5.6), we also obtain

$$\|g(f)\|_{L^p(X)} \lesssim \|f\|_{\mathring{H}_{\mathrm{at}}^{p,2}(X)} \sim \|f\|_{H^p(X)}.$$

To finish the proof of (5.14), it remains to prove $\|f\|_{H^p(X)} \lesssim \|g(f)\|_{L^p(X)}$. Indeed, for any $x \in X$, we have

$$\begin{aligned}\mathcal{S}(f)(x) &= \left[\sum_{k\in\mathbb{Z}}\sum_{\alpha\in\mathcal{A}_k}\sum_{m=1}^{N(k,\alpha)}\int_{B(x,\delta^k)}|Q_kf(y)|^2\mathbf{1}_{Q_\alpha^{k,m}}(x)\frac{d\mu(y)}{V_{\delta^k}(x)}\right]^{\frac12}\\ &\le \left\{\sum_{k\in\mathbb{Z}}\sum_{\alpha\in\mathcal{A}_k}\sum_{m=1}^{N(k,\alpha)}\left[\sup_{z\in B(z_\alpha^{k,m},\delta^{k-1})}|Q_kf(z)|^2\right]\mathbf{1}_{Q_\alpha^{k,m}}(x)\right\}^{\frac12}, \end{aligned}\tag{5.15}$$

where $Q_\alpha^{k,m}$ is as in Sect. 2 and $z_\alpha^{k,m}$ the center of $Q_\alpha^{k,m}$. With all the notation as in Theorem 2.7, we know that, for any $z \in B(z_\alpha^{k,m},\delta^{k-1})$,

$$Q_kf(z) = \sum_{k'\in\mathbb{Z}}\sum_{\alpha'\in\mathcal{A}_{k'}}\sum_{m'=1}^{N(k',\alpha')}\mu\left(Q_{\alpha'}^{k',m'}\right)Q_k\widetilde{Q}_{k'}\left(z,y_{\alpha'}^{k',m'}\right)Q_{k'}f\left(y_{\alpha'}^{k',m'}\right),$$

where $y_{\alpha'}^{k',m'}$ is an arbitrary point in $Q_{\alpha'}^{k',m'}$. Fix $\beta' \in (0,\beta\wedge\gamma)$. Then, similarly to the proof of (3.7) (see also [27, (3.2)]), we conclude that, for any $z \in B(z_\alpha^{k,m},\delta^{k-1})$

$$\left|Q_k\widetilde{Q}_{k'}\left(z,y_{\alpha'}^{k',m'}\right)\right| \lesssim \delta^{|k-k'|\beta'}\frac{1}{V_{\delta^{k\wedge k'}}(z)+V(z,y_{\alpha'}^{k',m'})}\left[\frac{1}{\delta^{k\wedge k'}+d(z,y_{\alpha'}^{k',m'})}\right]^\gamma. \tag{5.16}$$

The variable z in (5.16) can be replaced by any $x \in Q_\alpha^{k,m}$, because $\max\{d(z,x), d(z,z_\alpha^{k,m})\} \lesssim \delta^k \lesssim \delta^{k\wedge k'}$. Further, from Lemma 3.7, we deduce that, for any fixed $r \in (\omega/(\omega+\eta),1]$, any $k' \in \mathbb{Z}$ and $z \in B(z_\alpha^{k,m},\delta^{k-1})$,

$$\begin{aligned}&\left|\sum_{\alpha'\in\mathcal{A}_{k'}}\sum_{m'=1}^{N(k',\alpha')}\mu\left(Q_{\alpha'}^{k',m'}\right)Q_k\widetilde{Q}_{k'}\left(z,y_{\alpha'}^{k',m'}\right)Q_{k'}f\left(y_{\alpha'}^{k',m'}\right)\right|\\ &\quad\lesssim \delta^{(k\wedge k'-k)\omega(\frac1r-1)}\left[\mathcal{M}\left(\sum_{\alpha'\in\mathcal{A}_{k'}}\left|Q_{k'}f\left(y_{\alpha'}^{k',m'}\right)\right|^r\mathbf{1}_{Q_{\alpha'}^{k',m'}}\right)(x)\right]^{\frac1r}\end{aligned}$$

and hence

$$|Q_k f(z)| \lesssim \sum_{k'\in\mathbb{Z}} \delta^{|k-k'|\beta'} \delta^{(k\wedge k'-k)\omega(\frac{1}{r}-1)} \times \left[\mathcal{M}\left(\sum_{\alpha'\in\mathcal{A}_{k'}} \sum_{m'=1}^{N(k',\alpha')} \left|Q_{k'} f\left(y_{\alpha'}^{k',m'}\right)\right|^r \mathbf{1}_{Q_{\alpha'}^{k',m'}}\right)(x)\right]^{\frac{1}{r}}. \quad (5.17)$$

Combining (5.15) and (5.17), choosing r and β' such that $r \in (\omega/(\omega+\beta'), p)$ and applying the Hölder inequality, we further conclude that, for any $x \in X$,

$$\begin{aligned}
[\mathcal{S}(f)(x)]^2 &\lesssim \sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathcal{A}_k} \sum_{m=1}^{N(k,\alpha)} \left\{\sum_{k'\in\mathbb{Z}} \delta^{|k-k'|\beta'} \delta^{(k\wedge k'-k)\omega(\frac{1}{r}-1)}\right.\\
&\quad \left.\times \left[\mathcal{M}\left(\sum_{\alpha'\in\mathcal{A}_{k'}} \sum_{m'=1}^{N(k',\alpha')} \left|Q_{k'} f\left(y_{\alpha'}^{k',m'}\right)\right|^r \mathbf{1}_{Q_{\alpha'}^{k',m'}}\right)(x)\right]^{\frac{1}{r}} \mathbf{1}_{Q_\alpha^{k,m}}(x)\right\}^2\\
&\lesssim \sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathcal{A}_k} \sum_{m=1}^{N(k,\alpha)} \sum_{k'\in\mathbb{Z}} \delta^{|k-k'|\beta'} \delta^{(k\wedge k'-k)\omega(\frac{1}{r}-1)}\\
&\quad \times \left[\mathcal{M}\left(\sum_{\alpha'\in\mathcal{A}_{k'}} \sum_{m'=1}^{N(k',\alpha')} \left|Q_{k'} f\left(y_{\alpha'}^{k',m'}\right)\right|^r \mathbf{1}_{Q_{\alpha'}^{k',m'}}\right)(x)\right]^{\frac{2}{r}} \mathbf{1}_{Q_\alpha^{k,m}}(x)\\
&\lesssim \sum_{k\in\mathbb{Z}} \sum_{k'\in\mathbb{Z}} \delta^{|k-k'|[\beta'-\omega(\frac{1}{r}-1)]} \left[\mathcal{M}\left(\sum_{\alpha'\in\mathcal{A}_{k'}} \sum_{m'=1}^{N(k',\alpha')} \left|Q_{k'} f\left(y_{\alpha'}^{k',m'}\right)\right|^r \mathbf{1}_{Q_{\alpha'}^{k',m'}}\right)(x)\right]^{\frac{2}{r}}\\
&\lesssim \sum_{k'\in\mathbb{Z}} \left[\mathcal{M}\left(\sum_{\alpha'\in\mathcal{A}_{k'}} \sum_{m'=1}^{N(k',\alpha')} \left|Q_{k'} f\left(y_{\alpha'}^{k',m'}\right)\right|^r \mathbf{1}_{Q_{\alpha'}^{k',m'}}\right)(x)\right]^{\frac{2}{r}}.
\end{aligned}$$

From this and Lemma 5.2, we deduce that

$$\begin{aligned}
\|f\|_{H^p(X)} &= \|[\mathcal{S}(f)]^r\|_{L^{p/r}(X)}^{\frac{1}{r}}\\
&\lesssim \left\|\left\{\sum_{k'\in\mathbb{Z}} \left[\mathcal{M}\left(\sum_{\alpha'\in\mathcal{A}_{k'}} \sum_{m'=1}^{N(k',\alpha')} \left|Q_{k'} f\left(y_{\alpha'}^{k',m'}\right)\right|^r \mathbf{1}_{Q_{\alpha'}^{k',m'}}\right)\right]^{\frac{2}{r}}\right\}^{\frac{r}{2}}\right\|_{L^{p/r}(X)}^{\frac{1}{r}}
\end{aligned}$$

$$\lesssim \left\| \left\{ \sum_{k'\in\mathbb{Z}} \left[\sum_{\alpha'\in\mathcal{A}_{k'}} \sum_{m'=1}^{N(k',\alpha')} \left| Q_{k'} f\left(y_{\alpha'}^{k',m'}\right)\right|^r \mathbf{1}_{Q_{\alpha'}^{k',m'}} \right]^{\frac{2}{r}} \right\}^{\frac{r}{2}} \right\|_{L^{p/r}(X)}^{\frac{1}{r}}$$

$$\sim \left\| \left[\sum_{k'\in\mathbb{Z}} \sum_{\alpha'\in\mathcal{A}_{k'}} \sum_{m'=1}^{N(k',\alpha')} \left| Q_{k'} f\left(y_{\alpha'}^{k',m'}\right)\right|^2 \mathbf{1}_{Q_{\alpha'}^{k',m'}} \right]^{\frac{1}{2}} \right\|_{L^p(X)}.$$

By this and the arbitrariness of $y_{\alpha'}^{k',m'}$, we finally conclude that

$$\|f\|_{H^p(X)} \lesssim \left\| \left[\sum_{k'\in\mathbb{Z}} \sum_{\alpha'\in\mathcal{A}_{k'}} \sum_{m'=1}^{N(k',\alpha')} \inf_{z\in Q_{\alpha'}^{k',m'}} |Q_{k'} f(z)|^2 \mathbf{1}_{Q_{\alpha'}^{k',m'}} \right]^{\frac{1}{2}} \right\|_{L^p(X)} \lesssim \|g(f)\|_{L^p(X)}.$$

This finishes the proof of $\|f\|_{H^p(X)} \lesssim \|g(f)\|_{L^p(X)}$ and hence of Theorem 5.10. □

To consider the g_λ^*-function characterization of $H^p(X)$, we need a new kind of Littlewood–Paley functions. For any $\beta, \gamma \in (0, \eta)$, $\theta \in (0, \infty)$ and $f \in (\mathring{\mathcal{G}}_0^\eta(\beta, \gamma))'$, define the *Littlewood–Paley auxiliary function* $\mathcal{S}_\theta^{(1)}(f)$ *of* f *with aperture* θ by setting, for any $x \in X$,

$$\mathcal{S}_\theta^{(1)}(f)(x) := \left[\sum_{k=-\infty}^{\infty} \int_{B(y,\delta^k)} |Q_k f(y)|^2 \frac{d\mu(y)}{V_{\delta^k}(y)} \right]^{1/2}. \tag{5.18}$$

It is obvious that, for any $f \in (\mathring{\mathcal{G}}_0^\eta(\beta, \gamma))'$, $\mathcal{S}_1^{(1)}(f) \sim \mathcal{S}(f)$ and, for any $\theta \in [1, \infty)$, by the Fubini theorem, we have

$$\left\| \mathcal{S}_\theta^{(1)}(f) \right\|_{L^2(X)} \lesssim \theta^{\omega/2} \|\mathcal{S}(f)\|_{L^2(X)} \sim \theta^{\omega/2} \left\| \mathcal{S}_1^{(1)}(f) \right\|_{L^2(X)},$$

with the implicit positive constants independent of f. For the case $p \in (0, 2)$, we have the following lemma.

Lemma 5.11 *Let* $\beta, \gamma \in (0, \eta)$ *and* $p \in (0, 2)$. *Then there exists a positive constant* C *such that, for any* $\theta \in [1, \infty)$ *and* $f \in (\mathring{\mathcal{G}}_0^\eta(\beta, \gamma))'$,

$$\left\| \mathcal{S}_\theta^{(1)}(f) \right\|_{L^p(X)} \le C\theta^{\omega/p} \left\| \mathcal{S}_1^{(1)}(f) \right\|_{L^p(X)}. \tag{5.19}$$

Proof For any $\theta \in [1, \infty)$, any non-negative function g and any $x \in X$, define

$$\widetilde{\mathcal{M}}(g)(x) := \sup_{k\in\mathbb{Z}} \sup_{d(x,y)<\theta\delta^k} \frac{1}{V_{\delta^k}(y)} \int_{B(y,\delta^k)} g(z)\, d\mu(z).$$

Notice that, for any $k \in \mathbb{Z}$, $y \in B(x, \theta\delta^k)$ and $z \in B(y, \delta^k)$,

$$d(x, z) \le A_0[d(x, y) + d(y, z)] < 2A_0\theta\delta^k,$$

which further implies $B(y, \delta^k) \subset B(x, 2A_0\theta\delta^k)$. By this and (1.1), we find that, for any $k \in \mathbb{Z}$, $x \in X$ and $y \in B(x, \theta\delta^k)$,

$$\begin{aligned}\frac{1}{V_{\delta^k}(y)}\int_{B(y,\delta^k)} g(z)\,d\mu(z) &\le \frac{V_{2A_0\theta\delta^k}(x)}{V_{\delta^k}(y)}\frac{1}{V_{2A_0\theta\delta^k}(x)}\int_{B(x,2A_0\theta\delta^k)} g(z)\,d\mu(z)\\ &\lesssim \frac{V_{2A_0\theta\delta^k}(y)}{V_{\delta^k}(y)}\mathcal{M}(g)(x) \lesssim \theta^\omega \mathcal{M}(g)(x),\end{aligned}$$

which, together with the boundedness of $\mathcal{M}$ from $L^1(X)$ to $L^{1,\infty}(X)$, further implies that, for any $r \in (0, \infty)$,

$$\mu\left(\left\{x \in X : \widetilde{\mathcal{M}}(g)(x) > r\right\}\right) \lesssim \frac{\theta^\omega}{r}\|g\|_{L^1(X)}. \tag{5.20}$$

For any $t \in (0, \infty)$ and $f \in (\mathring{\mathcal{G}}_0^\eta(\beta, \gamma))'$ with $\beta,\ \gamma \in (0, \eta)$, define $E_t := \{x \in X : \mathcal{S}_1^{(1)}(f)(x) > t\}$ and $\widetilde{E}_t := \{x \in X : \widetilde{\mathcal{M}}(\mathbf{1}_{E_t})(x) > 1/2\}$. We claim that, for any $t \in (0, \infty)$,

$$\int_{\widetilde{E}_t^\complement}\left[\mathcal{S}_\theta^{(1)}(f)(x)\right]^2 d\mu(x) \lesssim \theta^\omega \int_{E_t^\complement}\left[\mathcal{S}_1^{(1)}(f)(x)\right]^2 d\mu(x). \tag{5.21}$$

Assuming this for the moment, we continue the proof of (5.19). Indeed, by the Chebyshev inequality, (5.20) and (5.21), we find that, for any $t \in (0, \infty)$,

$$\begin{aligned}\mu\left(\left\{x \in X : \mathcal{S}_\theta^{(1)}(f)(x) > t\right\}\right) &\le \mu\left(\widetilde{E}_t\right) + \mu\left(\left\{x \in \widetilde{E}_t^\complement : \mathcal{S}_\theta^{(1)}(f)(x) > t\right\}\right)\\ &\lesssim \theta^\omega\mu\left(E_t\right) + t^{-2}\int_{\widetilde{E}_t^\complement}\left[\mathcal{S}_\theta^{(1)}(f)(x)\right]^2 d\mu(x)\\ &\lesssim \theta^\omega\mu\left(E_t\right) + t^{-2}\theta^\omega\int_{E_t^\complement}\left[\mathcal{S}_1^{(1)}(f)(x)\right]^2 d\mu(x)\\ &\lesssim \theta^\omega\left[\mu\left(E_t\right) + t^{-2}\int_0^t s\mu(E_s)\,ds\right].\end{aligned}$$

By this, the Tonelli theorem and the fact $p \in (0, 2)$, we conclude that

$$\begin{aligned}\left\|\mathcal{S}_\theta^{(1)}(f)\right\|_{L^p(X)}^p &= p\int_0^\infty t^{p-1}\mu\left(\left\{x \in X : \mathcal{S}_\theta^{(1)}(f)(x) > t\right\}\right) dt\\ &\lesssim \theta^\omega\left[\int_0^\infty t^{p-1}\mu\left(E_t\right)\,dt + \int_0^\infty t^{p-3}\int_0^t s\mu(E_s)\,ds\,dt\right]\\ &\sim \theta^\omega\left\|\mathcal{S}_1^{(1)}(f)\right\|_{L^p(X)}^p + \theta^\omega\int_0^\infty s\mu(E_s)\int_s^\infty t^{p-3}\,dt\,ds\end{aligned}$$

$$\sim \theta^\omega \left\| \mathcal{S}_1^{(1)}(f) \right\|_{L^p(X)}^p + \theta^\omega \int_0^\infty s^{p-1} \mu(E_s)\, ds \sim \theta^\omega \left\| \mathcal{S}_1^{(1)}(f) \right\|_{L^p(X)}^p .$$

This finishes the proof of (5.19) under the assumption (5.21).

It remains to show (5.21). Fix $t \in (0,\infty)$ and, for any $y \in X$, let $\rho(y) := \inf_{x \in \widetilde{E}_t^{\complement}} d(x,y)$. Then, for any $k \in \mathbb{Z}$ and $x,\ y \in X$, $x \in \widetilde{E}_\lambda^{\complement} \cap B(y, \theta\delta^k)$ implies that $\rho(y) < \theta\delta^k$. By this and the Tonelli theorem, we find that

$$\begin{aligned}
\int_{\widetilde{E}_t^{\complement}} \left[\mathcal{S}_\theta^{(1)}(f)(x)\right]^2 d\mu(x) &= \int_{\widetilde{E}_t^{\complement}} \sum_{k=-\infty}^{\infty} \int_{B(x,\theta\delta^k)} |Q_k f(y)|^2 \frac{d\mu(y)}{V_{\delta^k}(y)}\, d\mu(x) \\
&= \sum_{k=-\infty}^{\infty} \int_{\rho(y)<\theta\delta^k} |Q_k f(y)|^2 \mu\left(\widetilde{E}_t^{\complement} \cap B(y,\theta\delta^k)\right) \frac{d\mu(y)}{V_{\delta^k}(y)} \\
&\lesssim \theta^\omega \sum_{k=-\infty}^{\infty} \int_{\rho(y)<\theta\delta^k} |Q_k f(y)|^2 \mu(B(y,\delta^k)) \frac{d\mu(y)}{V_{\delta^k}(y)}.
\end{aligned} \tag{5.22}$$

When $\rho(y) < \theta\delta^k$, we have $\widetilde{E}_t^{\complement} \cap B(y,\theta\delta^k) \neq \emptyset$. Choose $y_0 \in \widetilde{E}_t^{\complement} \cap B(y,\theta\delta^k)$. Then we have

$$\begin{aligned}
\mu(E_t \cap B(y,\delta^k)) &= \int_{B(y,\delta^k)} \mathbf{1}_{E_t}(z)\, d\mu(z) \le \mu(B(y,\delta^k)) \widetilde{\mathcal{M}}\left(\mathbf{1}_{E_t}\right)(y_0) \\
&\le \frac{1}{2}\mu(B(y,\delta^k)).
\end{aligned}$$

Thus, $\mu(E_t^{\complement} \cap B(y,\delta^k)) \ge \frac{1}{2}\mu(B(y,\delta^k))$. By this, (5.22) and the Tonelli theorem, we further conclude that

$$\begin{aligned}
\int_{\widetilde{E}_t^{\complement}} \left[\mathcal{S}_\theta^{(1)}(f)(x)\right]^2 d\mu(x) &\lesssim \theta^\omega \sum_{k=-\infty}^{\infty} \int_{\rho(y)<\theta\delta^k} |Q_k f(y)|^2 \mu(B(y,\delta^k)) \frac{d\mu(y)}{V_{\delta^k}(y)} \\
&\lesssim \theta^\omega \sum_{k=-\infty}^{\infty} \int_X |Q_k f(y)|^2 \mu\left(E_t^{\complement} \cap B(y,\delta^k)\right) \frac{d\mu(y)}{V_{\delta^k}(y)} \\
&\sim \sum_{k=-\infty}^{\infty} \int_{E_t^{\complement}} \int_{B(x,\delta^k)} |Q_k f(y)|^2 \frac{d\mu(y)}{V_{\delta^k}(y)}\, d\mu(x) \\
&\sim \theta^\omega \int_{E_t^{\complement}} \left[\mathcal{S}_1^{(1)}(f)(x)\right]^2 d\mu(x).
\end{aligned}$$

This finishes the proof of (5.21) and hence of Lemma 5.11. □

Using Lemma 5.11, we now establish the Littlewood–Paley g_λ^*-function characterization of $H^p(X)$ when $p \in (\omega/(\omega+\eta), 1]$.

Theorem 5.12 *Let* $p \in (\omega/(\omega+\eta), 1]$*,* $\beta,\ \gamma \in (\omega(1/p-1), \eta)$ *and* $\lambda \in (2\omega/p, \infty)$*. Then there exists a constant* $C \in [1, \infty)$ *such that, for any* $f \in (\mathring{\mathcal{G}}_0^\eta(\beta, \gamma))'$*,*

$$C^{-1}\|f\|_{H^p(X)} \le \|g_\lambda^*(f)\|_{L^p(X)} \le C\|f\|_{H^p(X)}. \tag{5.23}$$

Proof Fix $p \in (\omega/(\omega+\eta), 1]$ and $\lambda \in (2\omega/p, \infty)$. For any $f \in (\mathring{\mathcal{G}}_0^\eta(\beta, \gamma))'$ with $\beta,\ \gamma \in (\omega(1/p-1), \eta)$, by the definition of $g_\lambda^*(f)$ [see (5.3)], we find that

$$[\mathcal{S}(f)]^2 \lesssim \left[g_\lambda^*(f)\right]^2 \lesssim \sum_{j=0}^{\infty} 2^{(1-j)\lambda}\left[\mathcal{S}_{2^j}^{(1)}(f)\right]^2. \tag{5.24}$$

By (5.24), we easily obtain the first inequality of (5.23). For the second one, by (5.24), the fact $p < 2$, (5.19) and $\lambda > 2\omega/p$, we conclude that, for any $f \in (\mathring{\mathcal{G}}_0^\eta(\beta, \gamma))'$ with $\beta,\ \gamma \in (\omega(1/p-1), \eta)$,

$$\begin{aligned}\|g_\lambda^*(f)\|_{L^p(X)}^p &= \left\|\left[g_\lambda^*(f)\right]^2\right\|_{L^{p/2}(X)}^{p/2} \lesssim \sum_{j=0}^{\infty} 2^{-j\lambda p/2}\left\|\left[\mathcal{S}_{2^j}^{(1)}(f)\right]^2\right\|_{L^{p/2}(X)}^{p/2}\\ &\sim \sum_{j=0}^{\infty} 2^{-j\lambda p/2}\left\|\mathcal{S}_{2^j}^{(1)}(f)\right\|_{L^p(X)}^p \lesssim \left\|\mathcal{S}_1^{(1)}(f)\right\|_{L^p(X)}^p \sum_{j=0}^{\infty} 2^{-j\lambda p/2} 2^{j\omega}\\ &\sim \|\mathcal{S}(f)\|_{L^p(X)}^p.\end{aligned}$$

This finishes the proof of the second inequality of (5.23) and hence of Theorem 5.12. □

Remark 5.13 If X is a homogeneous group, Folland and Stein [12, Corollary 7.4] showed that, for any given $p \in (0, 2]$ and any $f \in \mathscr{S}'(X)$, $\|g_\lambda^*(f)\|_{L^p(X)} \le C\|\mathcal{S}(f)\|_{L^p(X)}$ whenever $\lambda \in (2\omega/p, \infty)$ with the positive constant C independent of f by observing that λ in (5.3) equals to 2λ with λ as in the Littlewood–Paley g_λ^*-function in [12], where $\mathscr{S}'(X)$ denotes the space of tempered distributions on X. Observe that Theorem 5.12 in this case coincides with Folland and Stein [12, Corollary 7.4], whose range of λ is the known best possible. Moreover, Lemma 5.11 and its proof have their own interest in dealing with Littlewood–Paley g_λ^*-functions on spaces of homogeneous type. For example, using this method, one can improve the range of λ in [26, Proposition 3.4(ii)] to $(2n/p, \infty)$, which then coincides with Theorem 5.12 and whose range of λ is then the known best possible.

6 Wavelet Characterizations of Hardy Spaces

In this section, we characterize the Hardy space $H^p(X)$ via the wavelet orthogonal system $\{\psi_\alpha^k : k \in \mathbb{Z},\ \alpha \in \mathcal{G}_k\}$ introduced in [1, Theorem 7.1]. The sequence $\{D_k\}_{k\in\mathbb{Z}}$ of operators on $L^2(X)$ associated with integral kernels

$$D_k(x, y) := \sum_{\alpha \in \mathcal{G}_k} \psi_\alpha^k(x)\psi_\alpha^k(y), \qquad \forall\, x,\ y \in X \tag{6.1}$$

turns out to be an exp-ATI; see [25,29]. Thus, all the conclusions in Sect. 5 hold true for $\{D_k\}_{k\in\mathbb{Z}}$.

For any $f \in (\mathring{\mathcal{G}}_0^\eta(\beta, \gamma))'$ with $\beta,\ \gamma \in (0, \eta)$, define the *wavelet Littlewood–Paley function* $S(f)$ by setting, for any $x \in X$,

$$S(f)(x) := \left\{ \sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathcal{G}_k} \left[\mu\left(Q_\alpha^{k+1}\right)\right]^{-1} \left|\left\langle \psi_\alpha^k, f\right\rangle\right|^2 \mathbf{1}_{Q_\alpha^{k+1}}(x) \right\}^{\frac{1}{2}}.$$

For any $p \in (0, \infty)$, define the corresponding *wavelet Hardy space* $H_{\mathrm{w}}^p(X)$ by

$$H_{\mathrm{w}}^p(X) := \left\{ f \in \left(\mathring{\mathcal{G}}_0^\eta(\beta, \gamma)\right)' : \ \|f\|_{H_{\mathrm{w}}^p(X)} := \|S(f)\|_{L^p(X)} < \infty \right\}.$$

For any $p \in (\omega/(\omega + \eta), \infty)$, the $L^p(X)$-norm equivalence between the wavelet Littlewood–Paley function $S(f)$ and the Littlewood–Paley g-function $g(f)$ was proved in [25, Theorem 4.3] whenever f is a distribution. The proof of [25, Theorem 4.3] seems *problematic* because the authors therein used an unknown fact that, when $f \in (\mathring{\mathcal{G}}(\beta, \gamma))'$ and $n \in \mathbb{N}$,

$$\sum_{|k|\le n} \sum_{\alpha\in\mathcal{G}_k} \left\langle f, \psi_\alpha^k \right\rangle \psi_\alpha^k \in L^2(X). \tag{6.2}$$

Although (6.2) may not be true for distributions, it is obviously true when $f \in L^2(X)$. Indeed, the argument used in the proof of [25, Theorem 4.3] proves the following result.

Theorem 6.1 *Suppose $p \in (\omega/(\omega + \eta), \infty)$ and $\beta,\ \gamma \in (0, \eta)$. Then there exists a positive constant C such that, for any $f \in (\mathring{\mathcal{G}}_0^\eta(\beta, \gamma))'$,*

$$\|\mathcal{G}(f)\|_{L^p(X)} \le C\|S(f)\|_{L^p(X)} \tag{6.3}$$

and, if $f \in L^2(X)$, then

$$C^{-1}\|S(f)\|_{L^p(X)} \le \|\mathcal{G}(f)\|_{L^p(X)} \le C\|S(f)\|_{L^p(X)}. \tag{6.4}$$

Here and hereafter, $\mathcal{G}(f)$ is defined as in (5.2), but with Q_k therein replaced by D_k in (6.1).

To show that (6.4) holds true for all distributions, we need the following basic property of $H_{\mathrm{w}}^p(X)$.

Proposition 6.2 *Let $p \in (\omega/(\omega + \eta), 1]$ and $\beta,\ \gamma \in (\omega(1/p - 1), \eta)$. Then $H_{\mathrm{w}}^p(X)$ is a (quasi-) Banach space that can be continuously embedded into $(\mathring{\mathcal{G}}_0^\eta(\beta, \gamma))'$.*

Proof Assume that $f\in(\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$ belongs to $H^p_{\rm w}(X)$. By (6.3), Theorems 5.10 and 5.9, we have $\|f\|_{\mathring{H}^{p,2}_{\rm at}(X)}\lesssim\|f\|_{H^p_{\rm w}(X)}$. Consequently, for any $\epsilon\in(0,\infty)$, there exist $(p,2)$-atoms $\{a_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty\subset\mathbb{C}$ satisfying $(\sum_{j=1}^\infty|\lambda_j|^p)^{\frac1p}\le\|f\|_{\mathring{H}^{p,2}_{\rm at}(X)}+\epsilon$ such that $f=\sum_{j=1}^\infty\lambda_ja_j$ in $(\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$. Combining this with Lemmas 4.14 and 4.15, we find that, for any $\varphi\in\mathring{\mathcal{G}}_0^\eta(\beta,\gamma)$,

$$\begin{aligned}|\langle f,\varphi\rangle|&\le\sum_{j=1}^\infty|\lambda_j||\langle a_j,\varphi\rangle|\lesssim\sum_{j=1}^\infty|\lambda_j|\|\varphi\|_{\mathcal{L}_{1/p-1}(X)}\lesssim\|\varphi\|_{\mathring{\mathcal{G}}_0^\eta(\beta,\gamma)}\left[\sum_{j=1}^\infty|\lambda_j|^p\right]^{1/p}\\&\lesssim\|\varphi\|_{\mathring{\mathcal{G}}_0^\eta(\beta,\gamma)}[\|f\|_{H^p_{\rm w}(X)}+\epsilon].\end{aligned}$$

Letting $\epsilon\to0^+$, we obtain $\|f\|_{(\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'}\lesssim\|f\|_{H^p_{\rm w}(X)}$. Thus, $H^p_{\rm w}(X)$ can be continuously embedded into $(\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$.

To prove that $H^p_{\rm w}(X)$ is a (quasi-)Banach space, we only prove its completeness. Let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $H^p_{\rm w}(X)$. Then $\{f_n\}_{n=1}^\infty$ is also a Cauchy sequence in $(\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$, so it converges to some element f in $(\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$. For any $n\in\mathbb{N}$ and $x\in X$, applying the Fatou lemma twice, we conclude that

$$\begin{aligned}S(f-f_n)(x)&=S\left(\lim_{m\to\infty}[f_m-f_n]\right)(x)\\&=\left[\sum_{k\in\mathbb{Z}}\sum_{\alpha\in\mathcal{G}_k}\left\{\mu(Q_\alpha^{k+1})\right\}^{-1}\left|\left\langle\psi_\alpha^k,\lim_{m\to\infty}[f_m-f_n]\right\rangle\right|^2\mathbf{1}_{Q_\alpha^{k+1}}(x)\right]^{\frac12}\\&=\left[\sum_{k\in\mathbb{Z}}\sum_{\alpha\in\mathcal{G}_k}\lim_{m\to\infty}\left\{\mu(Q_\alpha^{k+1})\right\}^{-1}\left|\left\langle\psi_\alpha^k,f_m-f_n\right\rangle\right|^2\mathbf{1}_{Q_\alpha^{k+1}}(x)\right]^{\frac12}\\&\le\liminf_{m\to\infty}\left[\sum_{k\in\mathbb{Z}}\sum_{\alpha\in\mathcal{G}_k}\left\{\mu(Q_\alpha^{k+1})\right\}^{-1}\left|\left\langle\psi_\alpha^k,f_m-f_n\right\rangle\right|^2\mathbf{1}_{Q_\alpha^{k+1}}(x)\right]^{\frac12}\\&=\liminf_{m\to\infty}S(f_m-f_n)(x)\end{aligned}$$

and hence

$$\begin{aligned}\|f-f_n\|^p_{H^p_{\rm w}(X)}&=\int_X[S(f-f_n)(x)]^p\,d\mu(x)\\&\le\int_X\liminf_{m\to\infty}[S(f_m-f_n)(x)]^p\,d\mu(x)\\&\le\liminf_{m\to\infty}\int_X[S(f_m-f_n)(x)]^p\,d\mu(x)=\liminf_{m\to\infty}\|f_m-f_n\|^p_{H^p_{\rm w}(X)}.\end{aligned}$$

Letting $n \to \infty$, we find that $f \in H^p_{\mathrm{w}}(X)$ and $\lim_{n\to\infty} \|f - f_n\|_{H^p_{\mathrm{w}}(X)} = 0$. Therefore, $H^p_{\mathrm{w}}(X)$ is complete. This finishes the proof of Proposition 6.2. □

Applying Theorem 6.1 and Proposition 6.2, we establish the following wavelet characterizations of Hardy spaces.

Theorem 6.3 *Suppose $p \in (\omega/(\omega+\eta), 1]$ and $\beta, \gamma \in (\omega(1/p - 1), \eta)$. As subspaces of $(\mathring{\mathcal{G}}^\eta_0(\beta,\gamma))'$, $H^p(X) = H^p_{\mathrm{w}}(X)$ with equivalent (quasi-)norms.*

Proof Due to (6.3), Theorems 5.10 and 5.9, we obtain $H^p_{\mathrm{w}}(X) \subset H^p(X)$ and

$$\|\cdot\|_{H^p(X)} \lesssim \|\cdot\|_{H^p_{\mathrm{w}}(X)}.$$

It remains to show $H^p(X) \subset H^p_w(X)$. To this end, by Theorem 5.9, we conclude that $L^2(X) \cap H^p(X)$ is dense in $H^p(X)$. Thus, for any $f \in H^p(X)$, there exist $\{f_n\}_{n=1}^\infty \subset L^2(X) \cap H^p(X)$ such that $\lim_{n\to\infty} \|f - f_n\|_{H^p(X)} = 0$. Obviously, $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence of $H^p(X)$. Noticing that $\{f_n\}_{n=1}^\infty \subset L^2(X)$, we use (6.4) and Theorem 5.10 to conclude that

$$\begin{aligned}\|f_m - f_n\|_{H^p_{\mathrm{w}}(X)} &= \|S(f_m - f_n)\|_{L^p(X)} \sim \|\mathcal{G}(f_m - f_n)\|_{L^p(X)}\\ &\sim \|f_m - f_n\|_{H^p(X)} \to 0\end{aligned}$$

as $m, n \to \infty$, so that $\{f_n\}_{n=1}^\infty$ is also a Cauchy sequence of $H^p_{\mathrm{w}}(X)$. By Proposition 6.2, there exists $\widetilde{f} \in H^p_{\mathrm{w}}(X)$ such that $f_n \to \widetilde{f}$ as $n \to \infty$ in $H^p_{\mathrm{w}}(X)$, also in $(\mathring{\mathcal{G}}^\eta_0(\beta,\gamma))'$. Meanwhile, $f_n \to f$ as $n \to \infty$ in $H^p(X)$, also in $(\mathring{\mathcal{G}}^\eta_0(\beta,\gamma))'$. Therefore, $\widetilde{f} = f$ in $(\mathring{\mathcal{G}}^\eta_0(\beta,\gamma))'$ and $f \in H^p_{\mathrm{w}}(X)$. Moreover,

$$\begin{aligned}\|f\|^p_{H^p_{\mathrm{w}}(X)} &\le \|f - f_n\|^p_{H^p_{\mathrm{w}}(X)} + \|f_n\|^p_{H^p_{\mathrm{w}}(X)} \sim \|f - f_n\|^p_{H^p_{\mathrm{w}}(X)} + \|f_n\|^p_{H^p(X)}\\ &\lesssim \|f\|^p_{H^p(X)}\end{aligned}$$

when n is sufficiently large. Thus, we obtain $H^p(X) \subset H^p_{\mathrm{w}}(X)$ and $\|\cdot\|_{H^p_{\mathrm{w}}(X)} \lesssim \|\cdot\|_{H^p(X)}$. This finishes the proof of Theorem 6.3. □

7 Criteria of the Boundedness of Sublinear Operators

Let $p \in (\omega/(\omega+\eta), 1]$. By the argument used in Sects. 3–6, we conclude that the Hardy spaces $H^{+,p}(X)$, $H^p_\theta(X)$ with $\theta \in (0,\infty)$, $H^{*,p}(X)$, $H^{p,q}_{\mathrm{at}}(X)$, $H^{p,q}_{\mathrm{cw}}(X)$, $\mathring{H}^{p,q}_{\mathrm{at}}(X)$ with $q \in (p,\infty] \cap [1,\infty]$ and $H^p_{\mathrm{w}}(X)$ are essentially the same space in the sense of equivalent (quasi-)norms. From now on, we simply use $H^p(X)$ to denote anyone of them if there is no confusion. In this section, we establish some criteria of the boundedness of sublinear operators on Hardy spaces via first establishing finite atomic characterizations of $H^p(X)$.

7.1 Finite Atomic Characterizations of Hardy Spaces

For any $p \in (\omega/(\omega+\eta), 1]$ and $q \in (p, \infty] \cap [1, \infty]$, we say $f \in H^{p,q}_{\mathrm{fin}}(X)$ if there exist $N \in \mathbb{N}$, a sequence $\{a_j\}_{j=1}^N$ of (p,q)-atoms and $\{\lambda_j\}_{j=1}^N \subset \mathbb{C}$ such that

$$f = \sum_{j=1}^N \lambda_j a_j.$$

Also, define

$$\|f\|_{H^{p,q}_{\mathrm{fin}}(X)} := \inf \left\{ \left(\sum_{j=1}^N |\lambda_j|^p \right)^{\frac{1}{p}} \right\},$$

where the infimum is taken over all the decompositions of f as above. It is easy to see that $H^{p,q}_{\mathrm{fin}}(X)$ is a dense subset of $H^{p,q}_{\mathrm{at}}(X)$ and $\|\cdot\|_{H^{p,q}_{\mathrm{at}}(X)} \le \|\cdot\|_{H^{p,q}_{\mathrm{fin}}(X)}$. Denote by the symbol $\mathrm{UC}(X)$ the *space of all uniformly continuous functions on* X, that is, a function $f \in \mathrm{UC}(X)$ if and only if, for any fixed $\epsilon \in (0,\infty)$, there exists $\sigma \in (0,\infty)$ such that $|f(x) - f(y)| < \epsilon$ whenever $d(x,y) < \sigma$. The next theorem characterizes $H^{p,q}_{\mathrm{at}}(X)$ via $H^{p,q}_{\mathrm{fin}}(X)$.

Theorem 7.1 *Suppose* $p \in (\omega/(\omega+\eta), 1]$. *Then the following statements hold true:*

(i) *if* $q \in (p,\infty) \cap [1,\infty)$, *then* $\|\cdot\|_{H^{p,q}_{\mathrm{fin}}(X)}$ *and* $\|\cdot\|_{H^{p,q}_{\mathrm{at}}(X)}$ *are equivalent (quasi)-norms on* $H^{p,q}_{\mathrm{fin}}(X)$*;*

(ii) $\|\cdot\|_{H^{p,\infty}_{\mathrm{fin}}(X)}$ *and* $\|\cdot\|_{H^{p,\infty}_{\mathrm{at}}(X)}$ *are equivalent (quasi)-norms on* $H^{p,q}_{\mathrm{fin}}(X) \cap \mathrm{UC}(X)$*;*

(iii) $H^{p,\infty}_{\mathrm{fin}}(X) \cap \mathrm{UC}(X)$ *is a dense subspace of* $H^{p,\infty}_{\mathrm{at}}(X)$.

Proof First, we prove (i). It suffices to show that $\|f\|_{H^{p,q}_{\mathrm{fin}}(X)} \lesssim \|f\|_{H^{p,q}_{\mathrm{at}}}$ for any $f \in H^{p,q}_{\mathrm{fin}}(X)$ with $q \in (p,\infty) \cap [1,\infty)$. We may as well assume that $\|f\|_{H^{*,p}(X)} = 1$. Let all the notation be as in the proof that $H^{*,p}(X) \subset H^{p,q}_{\mathrm{at}}(X)$ of Theorem 4.2. Then

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in I_j} \lambda_k^j a_k^j = \sum_{j\in\mathbb{Z}} \sum_{k\in I_j} h_k^j = \sum_{j\in\mathbb{Z}} h_j$$

both in $(\mathcal{G}_0^\eta(\beta,\gamma))'$ and almost everywhere. Here and hereafter, for any $j \in \mathbb{Z}$ and $k \in I_j$, the quantities h_j, h_k^j, λ_k^j and a_k^j are as in (4.12) and (4.13). Since $f \in H^{p,q}_{\mathrm{fin}}(X)$, it follows that there exist $x_1 \in X$ and $R \in (0,\infty)$ such that $\operatorname{supp} f \subset B(x_1, R)$. We claim that there exists a positive constant $\tilde{c}$ such that, for any $x \notin B(x_1, 16A_0^4 R)$,

$$f^\star(x) \le \tilde{c}[\mu(B(x_1,R))]^{-\frac{1}{p}}. \tag{7.1}$$

We admit (7.1) temporarily and use it to prove (i) and (ii). Let j' be the maximal integer such that $2^j \le \tilde{c}[\mu(B(x_1,R))]^{-\frac{1}{p}}$ and define

$$h := \sum_{j \le j'} \sum_{k \in I_j} \lambda_k^j a_k^j \quad \text{and} \quad \ell := \sum_{j > j'} \sum_{k \in I_j} \lambda_k^j a_k^j. \tag{7.2}$$

In what follows, for the sake of convenience, we elide the fact whether or not I_j is finite and simply write the summation $\sum_{k \in I_j}$ in (7.2) as $\sum_{k=1}^{\infty}$. If $j > j'$, then $\Omega^j = \{x \in X : f^\star(x) > 2^j\} \subset B(x_1, 16A_0^4 R)$, which implies that supp $\ell \subset B(x_1, 16A_0^4 R)$ because supp $a_k^j \subset \Omega^j$. From $f = h + \ell$, it then follows that supp $h \subset B(x_1, 16A_0^4 R)$. Noticing that

$$\|h\|_{L^\infty(X)} \le \sum_{j \le j'} \left\| h^j \right\|_{L^\infty(X)} \lesssim \sum_{j \le j'} 2^j \sim [\mu(B(x_1, R))]^{-\frac{1}{p}}$$

and $\int_X h(x)\, d\mu(x) = 0$, we conclude that h is a harmless constant multiple of a (p, ∞)-atom.

Next we deal with ℓ. For any $N := (N_1, N_2) \in \mathbb{N}^2$, define

$$\ell_N := \sum_{j=j'+1}^{N_1} \sum_{k=1}^{N_2} \lambda_k^j a_k^j = \sum_{j=j'+1}^{N_1} \sum_{k=1}^{N_2} h_k^j.$$

Then ℓ_N is a finite linear combination of (p, ∞)-atoms and $\sum_{j=j'+1}^{N_1} \sum_{k=1}^{N_2} |\lambda_k^j|^p \lesssim 1$. Notice that supp $(\ell - \ell_N) \subset B(x_1, 16A_0^4 R)$ and $\int_X [\ell(x) - \ell_N(x)]\, d\mu(x) = 0$. It suffices to show that $\|\ell - \ell_N\|_{L^q(X)} \to 0$ can be sufficiently small when N_1 and N_2 are big enough. Noticing that $\ell = \sum_{j=N_1+1}^{\infty} h^j + \sum_{j=j'+1}^{N_1} \sum_{k=1}^{\infty} h_k^j$, we have

$$\|\ell - \ell_N\|_{L^q(X)} \le \left\| \sum_{j=N_1+1}^{\infty} h^j \right\|_{L^q(X)} + \sum_{j=j'+1}^{N_1} \left\| \sum_{k=N_2+1}^{\infty} h_k^j \right\|_{L^q(X)}.$$

For any $j \in \mathbb{Z}$ and $k \in \mathbb{N}$, we recall that supp $h_k^j \subset B_k^j \subset \Omega^j$ and $\|h^j\|_{L^\infty(X)} \lesssim 2^j$. By $f = \sum_{j=-\infty}^{\infty} h^j$ and supp $(\sum_{j=N_1+1}^{\infty} h^j) \subset \Omega^{N_1}$, we conclude that, for any $z \in \Omega^{N_1}$,

$$\left| \sum_{j=N_1+1}^{\infty} h^j(z) \right| = \left| f(z) - \sum_{j \le N_1} h^j(z) \right| \le |f(z)| + \sum_{j \le N_1} \left| h^j(z) \right| \lesssim |f(z)| + 2^{N_1}.$$

Notice that, by [20, Proposition 3.9], there exists a constant $\widetilde{C} > 1$ such that $f^\star \le \widetilde{C}\mathcal{M}(f)$. With $f_1 := f \mathbf{1}_{\{x \in X: |f(x)| > 2^{N_1 - 1}/\widetilde{C}\}}$ and $f_2 := f - f_1$, we have

$$\begin{aligned} 2^{N_1 q} \mu\left(\Omega^{N_1}\right) &\le 2^{N_1 q} \mu\left(\left\{x \in X : \widetilde{C}\mathcal{M}(f)(x) > 2^{N_1}\right\}\right) \\ &\le 2^{N_1 q} \mu\left(\left\{x \in X : \widetilde{C}\mathcal{M}(f_1)(x) > 2^{N_1 - 1}\right\}\right) \lesssim \|f_1\|_{L^q(X)}^q \to 0 \end{aligned}$$

as $N_1 \to \infty$, because $\mathcal{M}$ is bounded from $L^q(X)$ to $L^{q,\infty}(X)$ and $f \in H^{p,q}_{\rm fin}(X) \subset L^q(X)$. Therefore,

$$\left\| \sum_{j=N_1+1}^{\infty} h^j \right\|_{L^q(X)}^q \lesssim \int_{\Omega^{N_1}} \left[|f(z)|^q + 2^{N_1 q} \right] d\mu(z) \lesssim \| f \mathbf{1}_{\Omega^{N_1}} \|_{L^q(X)}^q + 2^{N_1 q} \mu\left(\Omega^{N_1}\right) \to 0$$

as $N_1 \to \infty$. Then, for any $\epsilon \in (0,\infty)$, we choose $N_1 \in \mathbb{N}$ such that $\| \sum_{j=N_1+1}^{\infty} h^j \|_{L^q(X)} < \epsilon/2$.

If we fix $N_1 \in \mathbb{N}$ and $N_1 \geq j > j'$, then the fact $\sum_{k=1}^{\infty} |h_k^j| \lesssim 2^j \mathbf{1}_{\Omega^j} \in L^q(X)$ implies that

$$\lim_{N_2 \to 0} \left\| \sum_{k=N_2+1}^{\infty} h_k^j \right\|_{L^q(X)} = 0.$$

So, we further choose $N_2 \in \mathbb{N}$ such that $\sum_{j=j'+1}^{N_1} \| \sum_{k=N_2+1}^{\infty} h_k^j \|_{L^q(X)} < \epsilon/2$. In this way, we have $\|\ell - \ell_N\|_{L^q(X)} < \epsilon$ for large N. Then there exist a positive constant $C_\flat$, independent of N and ϵ, and a (p,q)-atom $a_{(N)}$ such that $\ell - \ell_N = C_\flat \epsilon a_{(N)}$. Therefore, we obtain $\|f\|_{H^{p,q}_{\rm fin}(X)} \lesssim 1 \sim \|f\|_{H^{p,q}_{\rm at}(X)}$ and complete the proof of (i) under the assumption (7.1).

To obtain (ii), we only need to prove that $\|f\|_{H^{p,\infty}_{\rm fin}(X)} \lesssim \|f\|_{H^{p,\infty}_{\rm at}}$ whenever $f \in H^{p,\infty}_{\rm fin}(X) \cap {\rm UC}(X)$. We may also assume that $\|f\|_{H^{*,p}(X)} = 1$. Notice that $f \in L^\infty(X)$ and $\|f^\star\|_{L^\infty(X)} \lesssim \|\mathcal{M}(f)\|_{L^\infty(X)} \leq c_0 \|f\|_{L^\infty(X)}$, where c_0 is a positive constant independent of f. Let $j'' > j'$ be the largest integer such that $2^j \leq c_0 \|f\|_{L^\infty(X)}$. We write $f = h + \ell$ with h as in (7.2) but now $\ell = \sum_{j'<j\leq j''} \sum_{k=1}^{\infty} h_k^j$. As in the proof of (i), we know that h is a harmless positive constant multiple of some (p,∞)-atom.

Now we consider ℓ. Notice that $f \in {\rm UC}(X)$. Then, for any $\epsilon \in (0,\infty)$, there exists $\sigma \in (0,\infty)$ such that $|f(x) - f(y)| \leq \epsilon$ whenever $d(x,y) \leq \sigma$. Split $\ell = \ell_1^\sigma + \ell_2^\sigma$ with

$$\ell_1^\sigma := \sum_{(j,k)\in G_1} h_k^j = \sum_{(j,k)\in G_1} \lambda_k^j a_k^j \quad \text{and} \quad \ell_2^\sigma := \sum_{(j,k)\in G_2} h_k^j,$$

where

$$G_1 := \{(j,k) :\ 12A_0^3 r_k^j \geq \sigma,\ j' < j \leq j''\} \quad \text{and}$$
$$G_2 := \{(j,k) :\ 12A_0^3 r_k^j < \sigma,\ j' < j \leq j''\}.$$

Notice that, for any $j' < j \leq j''$, Ω^j is bounded. Thus, by Proposition 4.4(vi), we find that G_1 is a finite set, which further implies that ℓ_1^σ is a finite linear combination of (p,∞)-atoms and

$$\sum_{(j,k)\in G_1} \left|\lambda_k^j\right|^p \lesssim 1.$$

To consider ℓ_2^σ, it is obvious that supp $\ell_2^\sigma \subset B(x_1, 16A_0^4R)$ and that $\int_X \ell_2^\sigma(x)\,d\mu(x) = 0$, so it remains to estimate $\|\ell_2^\sigma\|_{L^\infty(X)}$. For any $(j,k) \in G_2$, applying the definition of h_k^j in (4.12) implies that

$$\left|h_k^j\right| \le \left|b_k^j\right| + \sum_{l\in I_{j+1}} \left|b_l^{j+1}\phi_k^j\right| + \sum_{l\in I_{j+1}} \left|L_{k,l}^{j+1}\phi_l^{j+1}\right|.$$

By the definition of b_k^j, we have supp $b_k^j \subset B(x_k^j, 2A_0r_k^j)$. Moreover, for any $x \in B(x_k^j, 2A_0r_k^j)$,

$$\begin{aligned}\left|b_k^j(x)\right| &\le \left|f(x) - \frac{1}{\|\phi_k^j\|_{L^1(X)}}\int_{B(x_k^j,2A_0r_k^j)} f(\xi)\phi_k^j(\xi)\,d\mu(\xi)\right| \\ &\le \left|f(x) - f\left(x_k^j\right)\right| + \frac{1}{\|\phi_k^j\|_{L^1(X)}}\int_{B(x_k^j,2A_0r_k^j)} \left|f(\xi) - f\left(x_k^j\right)\right|\phi_k^j(\xi)\,d\mu(\xi) \lesssim \epsilon.\end{aligned} \tag{7.3}$$

If $b_l^{j+1}\phi_k^j \neq 0$, then $B(x_k^j, 2A_0r_k^j) \cap B(x_l^{j+1}, 2A_0r_l^{j+1}) \neq \emptyset$, which further implies that $r_l^{j+1} \le 6A_0^2r_k^j$. Thus, for any $x \in B(x_l^{j+1}, 2A_0r_l^{j+1})$, we have $d(x, x_l^{j+1}) < 12A_0^3r_k^j$ and hence an argument similar to the estimation of (7.3) gives

$$\begin{aligned}\left|b_l^{j+1}(x)\right| &= \left|f(x) - \frac{1}{\|\phi_l^{j+1}\|_{L^1(X)}}\int_{B(x_l^{j+1},2A_0r_l^{j+1})} f(\xi)\phi_l^{j+1}(\xi)\,d\mu(\xi)\right|\phi_l^{j+1}(x) \\ &\lesssim \epsilon\phi_l^{j+1}(x),\end{aligned}$$

so that

$$\sum_{l\in I_{j+1}} \left|b_l^{j+1}(x)\phi_k^j(x)\right| \lesssim \epsilon\phi_k^j(x)\sum_{l\in I_{j+1}} \phi_l^{j+1}(x) \sim \epsilon\phi_k^j(x) \lesssim \epsilon.$$

Using the definition of $L_{k,l}^{j+1}$ and arguing similarly to the estimation of (7.3), we conclude that, for any $x \in X$,

$$\sum_{l\in I_{j+1}} \left|L_{k,l}^{j+1}\phi_k^j(x)\right| \lesssim \epsilon,$$

where $L_{k,l}^{j+1}$ is as in (4.10). Summarizing all gives $\|h_k^j\|_{L^\infty(X)} \lesssim \epsilon$. Recalling that supp $h_k^j \subset B_k^j$ and $\sum_{k=1}^\infty \mathbf{1}_{B_k^j} \le L_0$, we obtain $\|\ell_2^\sigma\|_{L^\infty(X)} \lesssim \epsilon$. Therefore, there exist a positive constant $\widetilde{C}_\flat$, independent of σ and ϵ, and a (p, ∞)-atom $a_{(\sigma)}$ such that $\ell_2^\sigma = \widetilde{C}_\flat \epsilon a_{(\sigma)}$. This proves that $\|f\|_{H^{p,\infty}_{\rm fin}(X)} \lesssim 1$ and hence finishes the proof of (ii) under the assumption (7.1).

Now we prove (7.1). Let $x \notin B(x_1, 16A_0^4 R)$. Suppose that $\varphi \in \mathcal{G}_0^\eta(\beta, \gamma)$ with $\|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)} \lesssim 1$ for some $r \in (0, \infty)$. First we consider the case $r \geq 4A_0^2 d(x, x_1)/3$. For any $y \in B(x, d(x, x_1))$, we have $\|\varphi\|_{\mathcal{G}(y,r,\beta,\gamma)} \lesssim 1$, which implies that $|\langle f, \varphi\rangle| \lesssim f^*(y)$ and hence

$$|\langle f, \varphi\rangle| \lesssim \left\{\frac{1}{\mu(B(x, d(x, x_1)))} \int_{B(x,d(x,x_1))} \left[f^*(y)\right]^p \, d\mu(y)\right\}^{\frac{1}{p}} \lesssim [\mu(B(x_1, R))]^{-\frac{1}{p}}. \tag{7.4}$$

Next we consider the case $r < 4A_0^2 d(x_1, x)/3$. Choose a function ξ satisfying $\mathbf{1}_{B(x_1,(2A_0)^{-4}d(x_1,x))} \leq \xi \leq \mathbf{1}_{B(x_1,(2A_0)^{-3}d(x_1,x))}$ and $\|\xi\|_{\dot{C}^\eta(X)} \lesssim [d(x_1, x)]^{-\eta}$. Since $\operatorname{supp} f \subset B(x_1, R)$, it follows that $f\xi = f$. Let $\widetilde{\varphi} := \varphi\xi$. For any $y \in B(x, d(x, x_1))$, assuming for the moment that

$$\|\widetilde{\varphi}\|_{\mathcal{G}(y,r,\beta,\gamma)} \lesssim 1, \tag{7.5}$$

we obtain

$$|\langle f, \varphi\rangle| = \left|\int_X f(z)\varphi(z)\, d\mu(z)\right| = \left|\int_X f(z)\xi(z)\varphi(z)\, d\mu(z)\right| = |\langle f, \widetilde{\varphi}\rangle| \lesssim f^*(y),$$

which implies that (7.4) remains true in this case. Therefore, by the arbitrariness of φ and the fact that $f^* \sim f^\star$, we obtain (7.1).

Now we fix $y \in B(x_1, d(x_1, x))$ and prove (7.5). First we consider the size condition. Indeed, if $\widetilde{\varphi}(z) \neq 0$, then $d(z, x_1) < (2A_0)^{-3} d(x_1, x)$ and hence $d(z, y) < (16A_0^2/7) d(x, z)$, which implies that

$$\begin{aligned}|\widetilde{\varphi}(z)| \leq |\varphi(z)| &\leq \frac{1}{V_r(x) + V(x, z)} \left[\frac{r}{r + d(x, z)}\right]^\gamma \\ &\sim \frac{1}{V_r(y) + V(y, z)} \left[\frac{r}{r + d(y, z)}\right]^\gamma.\end{aligned}$$

To consider the regularity condition of $\widetilde{\varphi}$, we may assume that $d(z, z') \leq (2A_0)^{-10}[r + d(y, z)]$ due to the size condition. For the case $d(z, x_1) > (2A_0)^{-1} d(x_1, x)$, we have $\widetilde{\varphi}(z) = 0$ and, by $y \in B(x_1, d(x_1, x))$ and $r < 4A_0^2 d(x_1, x)/3$, we further obtain

$$\begin{aligned}d(z, z') &\leq (2A_0)^{-10}[r + d(y, z)] \leq (2A_0)^{-10}[r + A_0 d(y, x_1) + A_0 d(x_1, z)] \\ &\leq (2A_0)^{-10}[4A_0^2 d(x_1, x) + A_0 d(x_1, z)] \leq (2A_0)^{-2} d(x_1, z),\end{aligned}$$

which further implies that $d(z', x_1) \geq \frac{1}{A_0} d(x_1, z) - d(z, z') \geq (2A_0)^{-2} d(x_1, x)$ and hence $\widetilde{\varphi}(z') = 0$. So we only need to consider the case $d(z, x_1) \leq (2A_0)^{-1} d(x_1, x)$. Then we have $(2A_0)^{-1} d(x_1, x) \leq d(z, x) \leq 2A_0 d(x_1, x)$ and

$$d(y,z) \le A_0^2[d(y,x_1)+d(x_1,x)+d(x,z)] \le 2A_0^2 d(x_1,x) + A_0^2 d(x,z) \le (2A_0)^3 d(x,z),$$

which implies that $d(z,z') \le (2A_0)^{-1}[r+d(x,z)]$ and $r+d(y,z) \lesssim \min\{r+d(x,z), r+d(x,z'), d(x_1,x)\}$. Therefore, by the regularity of φ and the definition of ξ, we conclude that

$$\begin{aligned}\left|\widetilde{\varphi}(z)-\widetilde{\varphi}(z')\right| &\le \xi(z)|\varphi(z)-\varphi(z')| + |\varphi(z')||\xi(z)-\xi(z')| \\ &\lesssim \left[\frac{d(z,z')}{r+d(x,z)}\right]^\beta \frac{1}{V_r(x)+V(x,z)}\left[\frac{r}{r+d(x,z)}\right]^\gamma \\ &\quad + \frac{1}{V_r(x)+V(x,z')}\left[\frac{r}{r+d(x,z')}\right]^\gamma \left[\frac{d(z,z')}{d(x_1,x)}\right]^\beta \\ &\lesssim \left[\frac{d(z,z')}{r+d(y,z)}\right]^\beta \frac{1}{V_r(y)+V(y,z)}\left[\frac{r}{r+d(y,z)}\right]^\gamma .\end{aligned}$$

This proves (7.5) and hence finishes the proofs of (i) and (ii).

Now we prove (iii). According to [23, pp. 3347–3348] (see also [27, Theorem 2.6]), there exists a sequence $\{S_k\}_{k\in\mathbb{Z}}$ of bounded operators on $L^2(X)$ with their kernels satisfying the following conditions:

(i) $S_k(x,y)=0$ if $d(x,y)\ge C_\sharp\delta^k$ and, for any $x,\ y\in X$,

$$|S_k(x,y)| \lesssim \frac{1}{V_{\delta^k}(x)+V_{\delta^k}(y)},$$

where $C_\sharp$ is a fixed positive constant greater than 1;

(ii) for any $x,\ x',\ y\in X$ with $d(x,x')\le C_\sharp\delta^k$,

$$|S_k(x,y)-S_k(x',y)|+|S_k(y,x)-S_k(y,x')| \lesssim \left[\frac{d(x,x')}{\delta^k}\right]^\theta \frac{1}{V_{\delta^k}(x)+V_{\delta^k}(y)},$$

where θ is as in [23, Theorem 2.4];

(iii) for any $x\in X$, $\int_X S_k(x,y)\,d\mu(y)=1=\int_X S_k(y,x)\,d\mu(y)$.

For any $g\in\bigcup_{p\in[1,\infty]}L^p(X)$ and $x\in X$, define

$$S_k g(x) := \int_X S_k(x,y)g(y)\,d\mu(y).$$

Then, for any (p,∞)-atom a supported on $B(z,r)$ with $z\in X$ and $r\in(0,\infty)$, we observe that $S_k a$ has the following properties:

(a) $\|S_k a\|_{L^\infty(X)} \lesssim \|a\|_{L^\infty(X)}$ and $\lim_{k\to\infty}\|S_k a - a\|_{L^2(X)}=0$;
(b) when k is sufficiently large, supp $S_k(a)\subset B(z,2A_0 r)$;
(c) $\int_X S_k a(x)\,d\mu(x)=0$;
(d) $S_k a\in \mathrm{UC}(X)$.

Consequently, $S_k a$ is a harmless constant multiple of a (p,∞)-atom and hence of a $(p,2)$-atom. Thus, $\|S_k a - a\|_{H^{p,\infty}_{\rm at}(X)} \sim \|S_k a - a\|_{H^{p,2}_{\rm at}(X)} \to 0$ as $k \to \infty$. For any $f \in H^{p,\infty}_{\rm at}(X)$, there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset H^{p,\infty}_{\rm fin}(X)$ such that $\lim_{n\to\infty} \|f_n - f\|_{H^{p,q}_{\rm at}(X)} = 0$. Then, for any $n \in \mathbb{N}$, by the above (a)–(d), we find that $S_k(f_n) \in H^{p,\infty}_{\rm fin}(X) \cap {\rm UC}(X)$ and that $\lim_{k\to\infty} \|S_k f_n - f_n\|_{H^{p,\infty}_{\rm at}(X)} = 0$. This proves that $\|S_k f_n - f\|_{H^{p,\infty}_{\rm at}(X)} \to 0$ as $n,\ k \to \infty$, which completes the proof of (iii) and hence of Theorem 7.1. □

7.2 Criteria of the Boundedness of Sublinear Operators on Hardy Spaces

In this section, applying the finite atomic characterizations of Hardy spaces, we obtain two criteria on the boundedness of sublinear operators on Hardy spaces.

Recall that a complete vector space $\mathcal{B}$ is called a *quasi-Banach space* if its quasi-norm $\|\cdot\|_{\mathcal{B}}$ satisfies the following condition:

(i) for any $f \in \mathcal{B}$, $\|f\|_{\mathcal{B}} = 0$ if and only if f is the zero element in $\mathcal{B}$;
(ii) for any $\lambda \in \mathbb{C}$ and $f \in \mathcal{B}$, $\|\lambda f\|_{\mathcal{B}} = |\lambda|\|f\|_{\mathcal{B}}$;
(iii) there exists $C \in [1,\infty)$ such that, for any $f,\ g \in \mathcal{B}$, $\|f+g\|_{\mathcal{B}} \le C(\|f\|_{\mathcal{B}} + \|g\|_{\mathcal{B}})$.

Next we recall the definition of r-quasi-Banach spaces (see, for example, [20,35,53–55]).

Definition 7.2 Suppose that $r \in (0,1]$ and $\mathcal{B}_r$ is a quasi-Banach space with its quasi-norm $\|\cdot\|_{\mathcal{B}_r}$. The space $\mathcal{B}_r$ is called an *r-quasi-Banach space* if there exists $\kappa \in [1,\infty)$ such that, for any $m \in \mathbb{N}$ and $\{f_j\}_{j=1}^m \subset \mathcal{B}_r$,

$$\left\|\sum_{j=1}^m f_j\right\|_{\mathcal{B}_r}^r \le \kappa \sum_{j=1}^m \|f_j\|_{\mathcal{B}_r}^r.$$

Obviously, when $p \in (0,1]$, $L^p(X)$ and $H^{*,p}(X)$ are p-quasi-Banach-spaces. Let $\mathcal{Y}$ be a linear space and $\mathcal{B}_r$ an r-quasi-Banach space with $r \in (0,1]$. An operator $T:\ \mathcal{Y} \to \mathcal{B}_r$ is said to be $\mathcal{B}_r$-*sublinear* if there exists a positive constant $\kappa \in [1,\infty)$ such that

(i) for any $f,\ g \in \mathcal{Y}$, $\|T(f) - T(g)\|_{\mathcal{B}_r} \le \kappa \|T(f-g)\|_{\mathcal{B}_r}$;
(ii) for any $m \in \mathbb{N}$, $\{f_j\}_{j=1}^m \subset \mathcal{Y}$ and $\{\lambda_j\}_{j=1}^m \subset \mathbb{C}$,

$$\left\|T\left(\sum_{j=1}^m \lambda_j f_j\right)\right\|_{\mathcal{B}_r}^r \le \kappa \sum_{j=1}^m |\lambda_j|^r \|T(f_j)\|_{\mathcal{B}_r}^r$$

(see, for example, [35, Definition 2.5], [53, Definition 1.6.7], [55, Remark 1.1(3)], [54, Definition 1.6] and [20, Definition 5.8]).

The next theorem gives us some criteria for $\mathcal{B}_r$-sublinear operators that can be extended to bounded $\mathcal{B}_r$-sublinear operators from Hardy spaces to $\mathcal{B}_r$. It can be proved by following the proof of [20, Theorem 5.9] with some slight modifications, the details being omitted.

Theorem 7.3 *Let $p \in (\omega/(\omega+\eta), 1]$ and $r \in [p, 1]$. Suppose that $\mathcal{B}_r$ is an r-quasi-Banach space and either of the following holds true:*

(i) $q \in (p, \infty) \cap [1, \infty)$ *and* $T : H^{p,q}_{\mathrm{fin}}(X) \to \mathcal{B}_r$ *is a* $\mathcal{B}_r$*-sublinear operator with*

$$\sup\{\|T(a)\|_{\mathcal{B}_r} : a \text{ is any } (p,q)\text{-atom}\} < \infty;$$

(ii) $T : H^{p,\infty}_{\mathrm{fin}}(X) \cap \mathrm{UC}(X) \to \mathcal{B}_r$ *is a* $\mathcal{B}_r$*-sublinear operator with*

$$\sup\{\|T(a)\|_{\mathcal{B}_r} : a \text{ is any } (p,\infty)\text{-atom}\} < \infty.$$

Then T can be uniquely extended to a bounded $\mathcal{B}_r$-sublinear operator from $H^{p,q}_{\mathrm{at}}(X)$ to $\mathcal{B}_r$.

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