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Support Theorems and an Injectivity Result for Integral Moments of a Symmetric *m*-Tensor Field

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Abstract

In this work, we show an injectivity result and support theorems for integral moments of a symmetric m-tensor field on a simple, real analytic, Riemannian manifold. Integral moments of symmetric m-tensor fields were first introduced by Sharafutdinov. First we generalize a Helgason type support theorem proven by Krishnan and Stefanov (Inverse Probl Imaging 3(3):453–464, 2009). We use this extended result along with the first integral moments of a symmetric m-tensor field to prove the aforementioned results.

Keywords Integral moments · Analytic microlocal analysis · Support theorems

Mathematics Subject Classification 47G10 · 47G30 · 53B21

1 Introduction

Let (Ω, g) be a compact, simple, real-analytic Riemannian manifold of dimension n with smooth boundary $\partial\Omega$. We parametrize the maximal geodesics in Ω with endpoints on $\partial\Omega$ by their starting points and directions.

Set

$$\Gamma_- := \left\{(x, \xi) \in T\Omega |\ x \in \partial\Omega, |\xi| = 1, \langle \xi, \nu(x)\rangle < 0\right\},$$

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where $\nu(x)$ is the outer unit normal to $\partial\Omega$ at x. Then we define the q-th integral moment of a symmetric m-tensor field f, $I^q f$ as a function on Γ_- by

$$I^q f(x,\xi) = \int_0^{l(\gamma_{x,\xi})} t^q \langle f(\gamma_{x,\xi}(t)), \gamma^m_{x,\xi}(t)\rangle dt$$
$$= \int_0^{l(\gamma_{x,\xi})} t^q f_{i_1\ldots i_m}(\gamma_{x,\xi}(t))\dot{\gamma}^{i_1}_{x,\xi}(t)\cdots\dot{\gamma}^{i_m}_{x,\xi}(t)dt.$$

where $\gamma_{x,\xi}(t)$ is the geodesic starting from x in the direction ξ and $l(\gamma_{x,\xi})$ is the value of the parameter t at which this geodesic intersects the boundary again.

The above definition of integral moments for symmetric m-tensor fields was first introduced by Sharafutdinov in the context of $\mathbb{R}^n$, see [16]. In the same paper, he proved that if the first $(m+1)$ integral moments $I^q f$ for $q = 0, 1, \ldots, m$ of a compactly supported symmetric m-tensor field f are known along all straight lines, then f can be uniquely recovered.
The zeroth integral moment coincides with the usual geodesic ray transform of a symmetric m-tensor field. In this work, we are interested in injectivity results and support theorem for integral moments defined above. Microlocal techniques play a very crucial role in proving such results. Guillemin first introduced the microlocal approach in the Radon transform setting, see [7]. Analytic microlocal techniques were used by Boman and Quinto in [3] to prove support theorems for Radon transforms with positive real-analytic weights. For more literature on such support theorems, we refer to the reader [1,2,4–6,13,14,23] and references therein. For the analytic microlocal techniques used in this paper, we will mainly refer to [11,18–21].

The geodesic ray transform of a symmetric 2-tensor field which in our notation will be denoted by $I^0(f)$, arises naturally in the context of lens and boundary rigidity problems and has been studied in e.g. [15,17,19,20]. Support theorems for such transforms are of independent interest among mathematicians. In [20], the authors prove an s-injectivity result for symmetric 2-tensor fields. The same proof works for a symmetric tensor field of any order. That is, if $I^0(f) = 0$ for all geodesics of Ω then its solenoidal part vanishes. A question arises as to what data is sufficient for us to conclude such an injectivity result for the tensor field f itself. Using the result stated above, we show that if $I^q f = 0$ for $q = 0, 1, \ldots, m$ for all the geodesics of Ω, then $f = 0$. Injectivity results for the local geodesic ray transform of a function have been proved in [22] using new techniques. We also treat the case in which the integral moments are known for the open set of geodesics that do not intersect a given geodesically convex set. We do so using the techniques laid out in [11], where the authors prove a Helgason type support theorem for symmetric 2-tensor fields over simple, real analytic Riemannian manifolds. We first extend the result in [11] for symmetric m tensor fields. Using this new result, we prove a stronger version of such support type theorems, i.e. if we know $I^q f = 0$ for $q = 0, 1, \ldots, m$ over the open set of geodesics not intersecting a convex set, then it implies that the support of f lies in the convex set. We would also like to mention that Krishnan already proved such a support theorem for the case of functions in [9].

The paper is organized as follows. In Sect. 2, we give the definitions and our main theorems. Section 3 has some preliminary propositions and lemmas that are needed for the proof of the main theorems. In Sect. 4, we will prove a Helgason type support theorem which we state in Sect. 2 and prove the support theorem. In Sect. 5, we prove the s-injectivity result mentioned above and use it to prove the injectivity of integral moments. Finally in the "Appendix", we provide the proof of some lemmas and inequalities.

2 Definitions and Statements of the Theorems

Definition 1 (*Simple manifold*) A compact Riemannian manifold (Ω, g) with boundary is said to be simple if

(i) The boundary $\partial\Omega$ is strictly convex: $\langle \nabla_\xi \nu(x), \xi\rangle > 0$ for each $\xi \in T_x(\partial\Omega)$ where $\nu(x)$ is the unit outward normal to the boundary.
(ii) The map $\exp_x : \exp_x^{-1}(\Omega) \to \Omega$ is a diffeomorphism for each $x \in \Omega$.

The second condition ensures that any two points x, y in Ω are connected by a unique geodesic in Ω that depends smoothly on x, y. Any simple manifold Ω is necessarily diffeomorphic to a ball in $\mathbb{R}^n$, see [17]. Therefore, in the analysis of simple manifolds, we can assume that Ω is a domain $\Omega \subset \mathbb{R}^n$. We are going to work on a fixed simple Riemannian manifold (Ω, g) with a fixed real analytic atlas. Let $S^m(\Omega)$ be the collection of symmetric m-tensor fields defined on Ω and $C^\infty(S^m(\Omega))$ be the space of symmetric m-tensor fields whose components are in $C^\infty(\Omega)$. We will assume the Einstein summation convention and raise and lower indexes using the metric tensor. The tensors $f_{i_1\dots i_m}$ and $f^{i_1\dots i_m} = f_{j_1\dots j_m} g^{i_1 j_1} \cdots g^{i_m j_m}$ will be thought of as the same tensors with different representations.

For $0 \le k \in \mathbb{Z}$, we define the real Hilbert space $H^k(S^m(\Omega))$ as a completion of $C^\infty(S^m(\Omega))$ with respect to the Sobolev norm $||\cdot||_k$ corresponding to the following inductively defined inner product $(\cdot,\cdot)_k$:

$$(f, g)_k = (\nabla f, \nabla g)_{k-1} + (f, g)_{L_2(S^m(\Omega))},$$

where

$$(f, g)_{L_2(S^m(\Omega))} = \int_\Omega f_{i_1\dots i_m}(x) g^{i_1\dots i_m}(x)\mathrm{d}x$$

and $L_2(S^m(\Omega)) = \{f \in S^m(\Omega) : ||f||_{L_2(S^m(\Omega))} = (f, f)_{L_2(S^m(\Omega))} < \infty\} = H^0(S^m(\Omega))$.

It is well known from [17] that any symmetric m-tensor field can be decomposed uniquely in the following way:

Theorem 1 [17, Theorem 3.3.2] *Let Ω be a compact Riemannian manifold with boundary; let $k \ge 1$ and $m \ge 0$ be integers. For every field $f \in H^k(S^m(\Omega))$, there exist uniquely determined $f^s \in H^k(S^m(\Omega))$ and $v \in H^{k+1}(S^{m-1}(\Omega))$ such that*

$$f = f^s + \mathrm{d}v, \qquad \delta f^s = 0, \qquad v|_{\partial\Omega} = 0$$

where d *is symmetrized covariant derivative and* δ *is the formal dual of* $-$ d*. The fields* f^s *and* dv *are known as the solenoidal and the potential parts of* f *respectively.*

Let $\widetilde{\Omega}$ be an open, real analytic extension of Ω such that g can also be extended to a real analytic metric in $\widetilde{\Omega}$. We will also extend all symmetric tensor fields f defined on Ω by 0 in $\widetilde{\Omega} \setminus \Omega$. We will think of each maximal geodesic in Ω as a restriction of a geodesic with distinct endpoints in $\widetilde{\Omega} \setminus \Omega$ to Ω. Let $\gamma_{[x,y]}$ be the geodesic connecting x and y.

Let $\mathcal{A}$ be an open set of geodesics with endpoints in $\widetilde{\Omega} \setminus \Omega$ such that any geodesic in $\mathcal{A}$ is homotopic, within the set $\mathcal{A}$, to a geodesic lying outside Ω. Set of points lying on the geodesics in $\mathcal{A}$ is denoted by $\Omega_{\mathcal{A}}$ i.e. $\Omega_{\mathcal{A}} = \bigcup\limits_{\gamma \in \mathcal{A}} \gamma$ and $\partial_{\mathcal{A}}\Omega = \Omega_{\mathcal{A}} \cap \partial\Omega$. Now we will define what we mean by a geodesically convex subset.

Definition 2 A subset K of the Riemannian manifold (Ω, g) is said to be geodesically convex if for any two points $x \in K$ and $y \in K$, the geodesic connecting them lies entirely in the set K.

Finally, let $\mathcal{E}'(\widetilde{\Omega})$ be the space of compactly supported tensor fields with distributional components. We can then extend the definition of I by duality on tensor fields which are distributions in $\widetilde{\Omega}$ supported in Ω, see [11]. Now we are ready to state the main theorems that we will prove in this article.

Theorem 2 *Let* f *be a symmetric* m*-tensor field on a simple, real analytic Riemannian manifold* (Ω, g) *with components in* $\mathcal{E}'(\widetilde{\Omega})$ *and supported in* Ω*. Let* K *be a closed geodesically convex subset of* Ω*. If for each geodesic* γ *not intersecting* K*, we have that* $I^0 f(\gamma) = 0$ *then we can find a* $(m-1)$*-tensor field* v *with components in* $\mathcal{D}'(int(\widetilde{\Omega}) \setminus K)$ *such that* $f = \mathrm{d}v$ *in* $int(\widetilde{\Omega}) \setminus K$ *and* $v = 0$ *in* $int(\widetilde{\Omega}) \setminus \Omega$*.*

Here we would like to mention that this theorem has been shown to be true for the case $m = 2$ in [11].

Theorem 3 *Let* f *be a symmetric* m*-tensor field on a simple, real analytic Riemannian manifold* (Ω, g) *with components in* $\mathcal{E}'(\widetilde{\Omega})$ *and supported in* Ω *and* K *be a closed geodesically convex subset of* Ω*. If for each geodesic* γ *not intersecting* K*, we have that* $I^q f(\gamma) = 0$ *for* $q = 0, 1, \dots, m$ *then* $supp(f) \subset K$*.*

Theorem 4 *Let* (Ω, g) *be a simple, real analytic Riemannian manifold and suppose that* g *is real analytic in a neighborhood of* $cl(\Omega)$*. If for a symmetric* m*-tensor field* f *with components in* $L^2(\Omega)$*, we have that* $I^q f = 0$ *for* $q = 0, 1, \dots, m$*, then* $f = 0$*.*

Here we would like to comment that the Theorem 4 also follows as a corollary of Theorem 3 when f is supported in Ω, however as we show in this paper that it can also be proved independently using s-injectivity of ray transform where we say $I^0 = I$ is s-injective if $If = 0$ implies $f^s = 0$. In the next section we will prove a proposition and some lemmas that will be needed for the proofs of our main theorems.

3 Preliminaries

We will now prove some results which are analogue of some results already proved for the case of symmetric 2-tensor fields in [11]. These will be needed later in the proof of our main theorems.

Fix a maximal geodesic γ_0 connecting $x_0 \neq y_0$ in the closure of $\widetilde{\Omega}$. We construct normal coordinates $x = (x', x^n)$ at x_0 in $\widetilde{\Omega}$ so that x^n is the distance to x_0, and $\frac{\partial}{\partial x^n}$ is normal to $\frac{\partial}{\partial x^\alpha}$, $\alpha < n$, see [21, Section 2]. In these coordinates, the metric g satisfies $g_{ni} = \delta_{ni}$, for all i, and the Christoffel symbols satisfy $\Gamma^i_{nn} = \Gamma^n_{in} = 0$. Under these coordinates lines of the type $x' =$ constant are now geodesics with x^n as the arc length parameter.

Let U be a tubular neighborhood of γ_0 in Ω, $U = \{(x', x^n) : |x'| < \epsilon, a(x') \leq x^n \leq b(x')\}$, where $\partial\Omega$ is locally given by $x^n = a(x')$ and $x^n = b(x')$. In the next proposition, we prove that for a symmetric m-tensor field f, one can always construct an $(m-1)$-tensor field v in U such that for

$$h := f - \mathrm{d}v$$

one has

$$h_{i_1\dots i_{m-1}n} = 0, \qquad \text{for all possible values of } i_j \text{ and } v(x', a(x')) = 0.$$

We use the notation $\widetilde{U}$ to denote the tubular neighborhood of γ_0 of the same type but in $\widetilde{\Omega}$.

Remark 1 Numbers of n in the suffix of the tensor $v_{n\dots ni_1\dots i_k}$ will be clear from the order of the tensor v. For example, if v is a m-tensor then

$$v_{n\dots ni_1\dots i_k} = v_{\underbrace{n\dots n}_{(m-k)}i_1\dots i_k}.$$

Proposition 1 *Let f be a smooth symmetric m-tensor field then there exists a unique $(m-1)$-tensor field v such that $v(x', a(x')) = 0$ and for $h = f - \mathrm{d}v$, we have*

$$h_{i_1\dots i_{m-1}n} = 0, \quad \textit{for all possible values of } i_j.$$

To prove this proposition, we need the following lemma for which we provide a proof in the "Appendix":

Lemma 1 *Let v be a symmetric $(m-1)$-tensor field. Then for any $0 \leq k \leq m$, we have*

$$(\mathrm{d}v)_{n\dots ni_k\dots i_1} = \frac{(m-k)}{m}\frac{\partial v_{n\dots ni_k\dots i_1}}{\partial x^n} - \frac{2(m-k)}{m}\sum_{l=1}^{k}\Gamma^p_{i_l n} v_{n\dots ni_k\dots\hat{i_l}\dots i_1 p}$$
$$+ \frac{1}{m}\sum_{l=1}^{k}\frac{\partial v_{n\dots ni_k\dots\hat{i_l}\dots i_1}}{\partial x^{i_l}} - \frac{2}{m}\sum_{l,q=1,l\neq q}^{k}\Gamma^p_{i_l i_q} v_{n\dots ni_k\dots\hat{i_l}\dots\hat{i_q}\dots i_1 p}.$$

Now, let us come back to the proof of Proposition 1.

Proof of Proposition 1 Let us first recall the following definition:

$$(\mathrm{d}v)_{i_1\dots i_m} = \sigma(i_1,\dots,i_m)\left(\frac{\partial v_{i_1\dots i_{m-1}}}{\partial x^{i_m}} - \sum_{l=1}^{m-1}\Gamma^p_{i_m i_l} v_{i_1,\dots i_{l-1}pi_{l+1}\dots i_{m-1}}\right)$$

where σ is a the symmetrization operator.

Proving

$$h_{i_1\dots i_{m-1}n} = 0$$

is equivalent to proving the existence of an $(m-1)-$tensor field v such that

$$(\mathrm{d}v)_{i_1\dots i_{m-1}n} = f_{i_1\dots i_{m-1}n}.$$

First, we consider

$$\frac{\partial v_{n\dots n}}{\partial x^n} = f_{n\dots n}.$$

We can solve this equation together with the initial condition $v_{n\dots n}(x', a(x')) = 0$ to get $v_{n\dots n}$.

Now, we use this $v_{n\dots n}$ to get $v_{n\dots ni}$ for $1 \le i \le (n-1)$ by solving the following system of equation:

$$\begin{aligned} (\mathrm{d}v)_{n\dots ni} &= f_{n\dots ni} \\ \Rightarrow \qquad \frac{\partial v_{n\dots ni}}{\partial x^n}(x) - 2\Gamma^p_{in} v_{n\dots np}(x) &= \frac{m}{m-1} f_{n\dots ni}(x) - \frac{1}{m-1}\frac{\partial v_{n\dots n}}{\partial x^i}(x) \end{aligned}$$

together with initial conditions $v_{n\dots ni}(x', a(x')) = 0$.

Proceeding in a similar manner let us assume that for a given k such that $0 \le (k-1) \le (m-1)$, we have already found $v_{n\dots ni_{k-1}\dots i_1}$ for which $h_{n\dots ni_{k-1}\dots i_1} = f_{n\dots ni_{k-1}\dots i_1} - (\mathrm{d}v)_{n\dots ni_{k-1}\dots i_1} = 0$. If $(k-1) = (m-1)$ then we are done and if not then we can find $v_{n\dots ni_k\dots i_1}$ in the following manner. Using Lemma 1, we can construct the following system of equations for $h_{n\dots ni_k\dots i_1} = 0$.

$$\begin{aligned} &\frac{\partial v_{n\dots ni_k\dots i_1}}{\partial x^n}(x) - 2\sum_{l=1}^{k}\Gamma^p_{i_l n} v_{n\dots ni_k\dots\hat{i_l}\dots i_1 p}(x) \\ &\quad = \frac{1}{(m-k)}\left\{ m f_{n\dots ni_k\dots i_1}(x) - \sum_{l=1}^{k}\frac{\partial v_{n\dots ni_k\dots\hat{i_l}\dots i_1}}{\partial x^{i_l}}(x) \right. \\ &\quad\quad \left. + 2\sum_{l,q=1, l\ne q}^{k}\Gamma^p_{i_l i_q} v_{n\dots ni_k\dots\hat{i_l}\dots\hat{i_q}\dots i_1 p}(x) \right\}. \end{aligned}$$

Finally, we solve the above system of equations with the initial conditions $v_{n\ldots ni_k\ldots i_1}(x', a(x')) = 0$ to get $v_{n\ldots ni_k\ldots i_1}$ uniquely. We repeat the same process till $k = (m-1)$ to prove the proposition. □

Lemma 2 *Let f be supported in Ω, and $I^0 f(\gamma) = 0$ for all maximal geodesics in $\widetilde{U}$ belonging to some neighborhood of the geodesics $x_0 = const$. Then $v = 0$ in $int(\widetilde{U}) \setminus \Omega$.*

Proof First let $f \in C^\infty(\widetilde{\Omega})$ with support in Ω. We will give another invariant definition of v and use it to conclude our lemma. For any $x \in \widetilde{U}$ and any $\xi \in T_x\widetilde{U} \setminus \{0\}$ so that $\gamma_{x,\xi}$ stays in $\widetilde{U}$, we set

$$u(x,\xi) = \int_{\tau_-(x,\xi)}^{0} f_{i_1\ldots i_m}(\gamma_{x,\xi}(t))\dot{\gamma}^{i_1}_{x,\xi}(t)\cdots\dot{\gamma}^{i_m}_{x,\xi}(t)dt. \tag{1}$$

where $\tau_-(x,\xi) \le 0$ is defined by tracing back the geodesic, such that $\gamma_{x,\xi}(\tau_-(x,\xi)) \in \partial M$.

Extend the definition of $\gamma_{x,\xi}$ for $\xi \neq 0$ as a solution of the geodesic equation. Then $u(x,\xi)$ is positive homogeneous of order $(m-1)$ in ξ. Consider

$$\begin{aligned}
&u(x,\lambda\xi) = \lambda^{m-1}u(x,\xi)\\
&\qquad\Rightarrow\qquad \xi^{j_1}\cdots\xi^{j_{m-1}}\frac{\partial^{m-1}}{\partial\xi^{j_1}\cdots\partial\xi^{j_{m-1}}}u(x,\lambda\xi) = (m-1)!\,u(x,\xi),\\
&\qquad\qquad\quad \text{diff. } (m-1) \text{ times w.r.t } \lambda\\
&\qquad\Rightarrow\qquad \xi^{j_1}\cdots\xi^{j_{m-1}}\frac{\partial^{m-1}}{\partial\xi^{j_1}\cdots\partial\xi^{j_{m-1}}}u(x,\xi) = (m-1)!\,u(x,\xi), \text{ for } \lambda = 1.
\end{aligned}$$

Now, we shall define a symmetric $(m-1)$-tensor field v as the following:

$$v_{i_1\ldots i_{m-1}}(x) = \frac{1}{(m-1)!}\left.\frac{\partial^{m-1}}{\partial\xi^{i_1}\cdots\partial\xi^{i_{m-1}}}u(x,\xi)\right|_{\xi=e_n}. \tag{2}$$

Consider for any $0 \le l \le (m-1)$

$$\begin{aligned}
v_{i_1\ldots i_{m-1-l}n\ldots n}(x) &= \frac{1}{(m-1)!}\left.\frac{\partial^{m-1}}{\partial\xi^{i_1}\cdots\partial\xi^{i_{m-1-l}}\partial\xi^{n}\cdots\partial\xi^{n}}u(x,\xi)\right|_{\xi=e_n}\\
&= \frac{1}{(m-1)!}\,\xi^{j_1}\cdots\xi^{j_l}\left.\frac{\partial^{m-1}}{\partial\xi^{i_1}\cdots\partial\xi^{i_{m-1-l}}\partial\xi^{j_1}\cdots\partial\xi^{j_l}}u(x,\xi)\right|_{\xi=e_n}\\
&= \frac{l!}{(m-1)!}\left.\frac{\partial^{m-1-l}}{\partial\xi^{i_1}\cdots\partial\xi^{i_{m-1-l}}}u(x,\xi)\right|_{\xi=e_n} \quad \text{(using homogenity of } u\text{)}.
\end{aligned}$$

Then, we have

$$v_{n\ldots n}(x) = u(x, e_n).$$

We will now show that with this definition of v, for $h = f - \mathrm{d}v$, one has

$$h_{i_1\ldots i_{m-1}n} = 0, \quad \text{for all possible values of } i_j.$$

Define

$$w(x,\xi) = \int_{\tau_-(x,\xi)}^{0} h_{i_1\ldots i_m}(\gamma_{x,\xi}(t))\dot{\gamma}_{x,\xi}^{i_1}(t)\cdots\dot{\gamma}_{x,\xi}^{i_m}(t)dt. \tag{3}$$

□

Claim 1 Let $0 \le l \le (m-1)$ and $w(x,\xi)$ be defined as above. Then

$$\left.\frac{\partial^l}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}} w(x,\xi)\right|_{\xi=e_n} = 0. \tag{4}$$

Proof of Claim 1 Consider for any $0 \le l \le (m-1)$,

$$\begin{aligned}
\frac{\partial^l}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}} w(x,\xi) &= \frac{\partial^l}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}} u(x,\xi) \\
&\quad - \frac{\partial^l}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}} \\
&\quad \int_{\tau_-(x,\xi)}^{0} (\mathrm{d}v)_{i_1\ldots i_m}(\gamma_{x,\xi}(t))\dot{\gamma}_{x,\xi}^{i_1}(t)\cdots\dot{\gamma}_{x,\xi}^{i_m}(t)dt \\
&= \frac{\partial^l}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}} u(x,\xi) - \frac{\partial^l}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}} \\
&\quad \int_{\tau_-(x,\xi)}^{0} \frac{d}{dt}\left(v_{i_1\ldots i_{m-1}}(\gamma_{x,\xi}(t))\dot{\gamma}_{x,\xi}^{i_1}(t)\cdots\dot{\gamma}_{x,\xi}^{i_{m-1}}(t)\right)dt \\
&= \frac{\partial^l}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}} u(x,\xi) - \frac{\partial^l}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}}\left(v_{i_1\ldots i_{m-1}}(x)\xi^{i_1}\cdots\xi^{i_{m-1}}\right) \\
&= \frac{\partial^l}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}} u(x,\xi) - \frac{(m-1)!}{(m-l-1)!} \\
&\quad \left(v_{j_1\ldots j_l i_1\ldots i_{m-l-1}}(x)\xi^{i_1}\cdots\xi^{i_{m-l-1}}\right) \\
\Rightarrow \quad \left.\frac{\partial^{m-1}}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}} w(x,\xi)\right|_{\xi=e_n} &= \left.\frac{\partial^l}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}} u(x,\xi)\right|_{\xi=e_n} - \frac{(m-1)!}{(m-l-1)!}\, v_{j_1\ldots j_l n\ldots n}(x) \\
&= \frac{(m-1)!}{(m-l-1)!}\, v_{j_1\ldots j_l n\ldots n}(x) - \frac{(m-1)!}{(m-l-1)!}\, v_{j_1\ldots j_l n\ldots n}(x) \\
&= 0.
\end{aligned}$$

Thus the Claim 1 is proved. □

Now let us recall the following relation from [17, Section 1.2]:

$$Gw(x,\xi) = h_{i_1\dots i_m}(x)\xi^{i_1}\cdots\xi^{i_m} \tag{5}$$

where $G = \xi^i\partial_{x^i} - \Gamma^k_{ij}\xi^i\xi^j\partial_{\xi^k}$ is the generator of the geodesic flow. After differentiating (5) $(m-1)$ times w.r.t. ξ, we get

$$\begin{aligned}
&\frac{\partial^{m-1}}{\partial\xi^{j_1}\dots\partial\xi^{j_{m-1}}}Gw(x,\xi) = m!\, h_{j_1\dots j_{m-1}i}(x)\xi^i\\
\Rightarrow\quad &\frac{\partial^{m-1}}{\partial\xi^{j_1}\dots\partial\xi^{j_{m-1}}}Gw(x,\xi)\bigg|_{\xi=e_n} = m!\, h_{j_1\dots j_{m-1}n}(x).
\end{aligned}$$

We will prove that the L.H.S. of the above equation is 0 which will complete the proof our lemma. Consider

$$\begin{aligned}
\frac{\partial Gw(x,\xi)}{\partial\xi^{j_1}} &= \frac{\partial}{\partial\xi^{j_1}}\left(\xi^i\frac{\partial}{\partial x^i}w(x,\xi)\right) - \Gamma^k_{ij}\frac{\partial}{\partial\xi^{j_1}}\left(\xi^i\xi^j\frac{\partial}{\partial\xi^k}w(x,\xi)\right)\\
&= \frac{\partial w(x,\xi)}{\partial x^{j_1}} + \xi^i\frac{\partial^2 w(x,\xi)}{\partial\xi^{j_1}\partial x^i}\\
&\quad - \Gamma^k_{ij}\frac{\partial}{\partial\xi^{j_1}}\left(\xi^i\xi^j\right)\frac{\partial w(x,\xi)}{\partial\xi^k} - \Gamma^k_{ij}\xi^i\xi^j\frac{\partial^2 w(x,\xi)}{\partial\xi^{j_1}\partial\xi^k}\\
\Rightarrow\quad \frac{\partial^2 Gw(x,\xi)}{\partial\xi^{j_1}\partial\xi^{j_2}} &= \frac{\partial^2 w(x,\xi)}{\partial x^{j_1}\partial\xi^{j_2}} + \frac{\partial^2 w(x,\xi)}{\partial\xi^{j_1}\partial x^{j_2}} + \xi^i\frac{\partial^3 w(x,\xi)}{\partial\xi^{j_1}\partial\xi^{j_2}\partial x^i}\\
&\quad - \Gamma^k_{ij}\frac{\partial^2}{\partial\xi^{j_1}\partial\xi^{j_2}}\left(\xi^i\xi^j\right)\frac{\partial w(x,\xi)}{\partial\xi^k} - \Gamma^k_{ij}\frac{\partial}{\partial\xi^{j_1}}(\xi^i\xi^j)\frac{\partial^2 w(x,\xi)}{\partial\xi^{j_2}\partial\xi^k}\\
&\quad - \Gamma^k_{ij}\frac{\partial}{\partial\xi^{j_2}}(\xi^i\xi^j)\frac{\partial^2 w(x,\xi)}{\partial\xi^{j_1}\partial\xi^k} - \Gamma^k_{ij}\xi^i\xi^j\frac{\partial^3 w(x,\xi)}{\partial\xi^{j_1}\partial\xi^{j_2}\partial\xi^k}\\
&= \frac{\partial^2 w(x,\xi)}{\partial x^{j_1}\partial\xi^{j_2}} + \frac{\partial^2 w(x,\xi)}{\partial\xi^{j_1}\partial x^{j_2}} + \xi^i\frac{\partial^3 w(x,\xi)}{\partial\xi^{j_1}\partial\xi^{j_2}\partial x^i} - 2\Gamma^k_{j_1j_2}\frac{\partial w(x,\xi)}{\partial\xi^k}\\
&\quad - 2\Gamma^k_{ij_1}\xi^i\frac{\partial^2 w(x,\xi)}{\partial\xi^{j_2}\partial\xi^k} - 2\Gamma^k_{ij_2}\xi^i\frac{\partial^2 w(x,\xi)}{\partial\xi^{j_1}\partial\xi^k} - \Gamma^k_{ij}\xi^i\xi^j\frac{\partial^3 w(x,\xi)}{\partial\xi^{j_1}\partial\xi^{j_2}\partial\xi^k}.
\end{aligned}$$

Using similar calculations, we get

$$\begin{aligned}
\frac{\partial^{m-1}Gw(x,\xi)}{\partial\xi^{j_1}\dots\partial\xi^{j_{m-1}}} &= \xi^i\frac{\partial^m w(x,\xi)}{\partial\xi^{j_1}\dots\partial\xi^{j_{m-1}}\partial x^i}\\
&\quad - \sum_{l,k=1,l\neq k}^{m-1} 2\Gamma^k_{j_lj_p}\frac{\partial^{m-2}w(x,\xi)}{\partial\xi^k\partial\xi^{j_1}\dots\partial\hat{\xi^{j_l}}\dots\partial\hat{\xi^{j_p}}\dots\partial\xi^{j_{m-1}}}\\
&\quad + \sum_{l=1}^{m-1}\frac{\partial^{m-1}w(x,\xi)}{\partial x^{j_l}\partial\xi^{j_1}\dots\partial\hat{\xi^{j_l}}\dots\partial\xi^{j_{m-1}}}
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{l=1}^{m-1} 2\Gamma_{i j_l}^{k} \xi^{i} \frac{\partial^{m-1} w(x,\xi)}{\partial \xi^{k} \partial \xi^{j_1} \dots \partial \xi^{\hat{j_l}} \dots \partial \xi^{j_{m-1}}} \\
&- \Gamma_{ij}^{k} \xi^{i} \xi^{j} \frac{\partial^{m} w(x,\xi)}{\partial \xi^{k} \partial \xi^{j_1} \dots \partial \xi^{j_{m-1}}}.
\end{aligned}
$$

From the relation above it follows:

$$
\left.\frac{\partial^{m-1} G w(x,\xi)}{\partial \xi^{j_1} \dots \partial \xi^{j_{m-1}}}\right|_{\xi = e_n} = 0, \qquad \text{(Using Claim 1 and } \Gamma_{nn}^{k} = 0).
$$

Now that we have proved the proposition for the case when f is smooth, it can be extended to the case when f is a distribution by exactly the same reasoning as in [11, Lemma 3.1]. □

4 Proofs of Theorems 2 and 3

We will start with proving some lemmas and propositions required to prove our main theorems.

From now on, we will work with $\widetilde{\Omega}$ and we also note that the analytic wavefront set of f, $WF_A(f)$, is contained in the interior of $T^*\widetilde{\Omega}$. The canonical projection form $T^*\widetilde{\Omega}$ onto the base $\widetilde{\Omega}$ will be denoted by π.

Lemma 3 *Let f be a symmetric m-tensor field as above. Let γ_0 be a geodesic of $\widetilde{\Omega}$ and U, as in the previous section, be a neighborhood of γ_0 in $\widetilde{\Omega}$. Assume that $WF_A(f) \bigcap \pi^{-1}(U)$ does not contain co-vectors of the type $(\xi', 0)$, then the analytic wavefront of $h = f - \mathrm{d}v$, i.e. $WF_A(h)$ also does not contain such co-vectors.*

Proof Since v and $\mathrm{d}v$ have the same analytic wavefront set, so we will prove the lemma for v. We will prove this by induction by proving it for $v_{n\dots n i_k \dots i_1}$ for every $k \le (m-1)$. Let us first do the analysis for $v_{n\dots n}$. Note that $v_{n\dots n}$ can be rewritten as a convolution with the Heaviside function in the following manner:

$$
\begin{aligned}
v_{n\dots n}(x) &= \int_{-\infty}^{x^n} f_{n\dots n}(x', y^n) dy^n \\
&= \int_{-\infty}^{\infty} f_{n\dots n}(x', y^n) H(x^n - y^n) dy^n.
\end{aligned}
$$

The analytic wavefront set of the convolution can be found by applying [8, 8.2.16]. Since we have assumed that $\mathrm{WF}_A(f) \bigcap \pi^{-1}(\mathrm{U})$ does not contain co-vectors of the type $(\xi', 0)$, hence it will be true for $v_{n\dots n}(x)$ as well. Now let us assume that the lemma holds for any $0 \le k-1 < (m-1)$ i.e. $v_{n\dots n i_{k-1} \dots i_1}$ satisfies the same wavefront conditions. We will show that this implies that the Lemma 3 is true for k. For this consider the system of ODEs from Lemma 1,

$$
\frac{\partial v_{n\dots n i_k \dots i_1}}{\partial x^n}(x) - 2 \sum_{l=1}^{k} \Gamma_{i_l n}^{p} v_{n\dots n i_k \dots \hat{i_l} \dots i_1 p}(x)
$$

$$= \frac{1}{(m-k)} \left\{ m f_{n\ldots n i_k \ldots i_1}(x) - \sum_{l=1}^{k} \frac{\partial v_{n\ldots n i_k \ldots \hat{i_l} \ldots i_1}}{\partial x^{i_l}}(x) + 2 \sum_{l,q=1, l\neq q}^{k} \Gamma^{p}_{i_l i_q} v_{n\ldots n i_k \ldots \hat{i_l} \ldots \hat{i_q} \ldots i_1 p}(x) \right\},$$

$$v_{n\ldots n i_k \ldots i_1}(x', a(x')) = 0.$$

This can be rewritten as :

$$\partial_n(\tilde{v}) - A(x', x^n)\tilde{v} = w,$$
$$\tilde{v}|_{x^n << 0} = 0$$

where A is an analytic matrix, $\tilde{v} = v_{n\ldots n i_k \ldots i_1}$ and $\mathrm{WF}_A(w) \bigcap \pi^{-1}(\mathrm{U})$ does not have covectors of the type $(\xi', 0)$.

By Duhamel's principle the solution to the above equation is given by:

$$\tilde{v}(x', x^n) = \int_{-\infty}^{x^n} \Phi(x', x^n, y^n) w(x', y^n) dy^n$$

where Φ is analytic. The expression given above for $\tilde{v}(x', x^n)$ can be rewritten as:

$$\tilde{v}(x', x^n) = \int_{R^n} \Phi(x', x^n, y^n) H(x^n - y^n) \delta(x' - y') w(y', y^n) dy' dy^n.$$

The kernel of the integral operator is given by: $\Phi(x', x^n, y^n)H(x^n - y^n)\delta(x' - y')$. Note that the frequency set of the analytic wavefront set of the Heaviside and delta distributions here are perpendicular to each other and hence satisfy Hörmander's non cancellation condition [8, Theorem 8.5.3]. The lemma then follows from the argument in [11, Lemma 3.2]. ◻

4.1 Analyticity Along Conormal Directions

Before moving further, we will need the following proposition which is an analogue of Proposition 2 from [21] and generalizes that proposition for the case when f is a symmetric m-tensor field. We will mimic the proof for the case when $m = 2$ as given in that paper and adapt the arguments wherever necessary to make it work for a symmetric tensor field of any order.

Proposition 2 *Let Ω and f be as above. Let γ_0 be a fixed geodesic through x_0 normal to ξ_0 where $(x_0, \xi_0) \in T^*\Omega \backslash 0$. Assume $(I^0 f)(\gamma) = 0$ for all γ in a neighborhood of γ_0 and g is analytic in this neighborhood. Let $\delta f = 0$ near x_0. Then*

$$(x_0, \xi_0) \notin WF_A(f).$$

Proof For the given geodesic γ_0 that passes through x_0 and is normal to ξ_0, let us consider a tubular neighborhood U of γ_0 endowed with analytic semi-geodesic coordinates $x = (x', x^n)$ on it. Without loss of generality, assume that $x_0 = 0$. Furthermore, $\forall x \in \gamma_0$, $x' = 0$. Note that $U = \{(x', x^n) : |x'| < \epsilon$ and $l^- < x_n < l^+; 0 < \epsilon << 1\}$ in this co-ordinate system. Choose ϵ such that $\{x : x_n = l^-, l^+$ and $|x'| < \epsilon\}$ lies outside Ω. Clearly $\xi_0 = (\xi_0', 0)$. Hence our goal is now to show:

$$(0, \xi_0) \notin \mathrm{WF}_A(f).$$

As stated earlier, we will reproduce the arguments from [21] here for the sake of completeness. Consider $Z = \{|x| < \frac{7\epsilon}{8} : |x_n| = 0\}$ and let the x' variable be denoted on Z by z'. Then (z', θ') are local co-ordinates in $\mathrm{nbd}(\gamma_0)$ (in the set of geodesics) given by $(z', \theta') \to \gamma_{(z',0),(\theta',1)}$. Here, $|\theta'| << 1$ (where, the geodesic is in the direction $(\theta', 1)$). By following their arguments verbatim, we get

$$\int e^{i\lambda z'(x,\theta').\xi'} a_N(x, \theta') f_{i_1\dots i_m}(x) b^{i_1}(x, \theta') \dots b^{i_m}(x, \theta') dx = 0. \tag{6}$$

Here, $(x, \theta') \mapsto a_N$ (a sequence of functions indexed by N) is analytic and satisfies

$$|\partial^\alpha a_N| \le (CN)^{|\alpha|}, \quad \alpha \le N, \tag{7}$$

see [21, Equation (38)]. Also, note that $b(0, \theta') = \theta$ and $a_N(0, \theta') = 1$.

Further, let us choose $\theta(\xi)$ to be a vector depending analytically on ξ near $\xi = \xi_0$ and satisfying the following conditions:

$$\theta(\xi) \dots \xi = 0, \quad \theta^n(\xi) = 1 \qquad \text{and}$$
$$\theta(\xi_0) = (0, \dots, 1) = e_n$$

Now, we can rewrite (6) using the above mapping in the following form:

$$\int e^{i\lambda\phi(x,\xi)} \tilde{a_N}(x, \xi) f_{i_1\dots i_m}(x) \tilde{b}^{i_1}(x, \xi) \dots \tilde{b}^{i_m}(x, \xi) dx = 0. \tag{8}$$

Here $\phi(x, \xi) = z' \dots \xi'$. This phase function has been shown in [21] to be non-degenerate in a neighborhood of $(0, \xi_0)$ by showing $\phi_{x\xi}(0, \xi) = \mathrm{Id}$. This also implies that $x \mapsto \phi_\xi(x, \xi)$ is a diffeomorphism in this neighborhood.

To establish the above condition in a neighborhood of the geodesic γ_0, one chooses the co-normal vector

$$\xi_0 = e_{n-1}, \quad \text{i.e. the covector } (0, 0, \dots, 0, 1, 0) \tag{9}$$

and defines

$$\theta(\xi) = \left(\xi_1, \dots, \xi_{n-2}, -\frac{\xi_1^2 + \dots + \xi_{n-2}^2 + \xi_n}{\xi_{n-1}}, 1\right).$$

This definition of θ is consistent with the requirement put on $\theta(\xi)$ as above. One can then show that the differential of the map $\xi \mapsto \theta(\xi)$ where $\xi \in \mathbb{S}^{n-1}$ is invertible at $\xi_0 = e_{n-1}$, see [21, Equation (44)].

Lemma 4 [21, Lemma 5] *Let, $\theta(\xi)$ and $\phi(x,\xi)$ be as above. Then, $\exists\, \delta > 0$ such that if*

$$\phi_\xi(x,\xi) = \phi_\xi(y,\xi)$$

for some $x \in U$, $|y| < \delta$, $|\xi - \xi_0| < \delta$ where ξ is complex, then $y = x$.

We study the analytic wavefront set of f using Sjöstrand's complex stationary phase method. For this assume x, y as in Lemma 4 and $|\xi_0 - \eta| < \frac{\delta}{\widetilde{C}}$ with $\widetilde{C} >> 2$ and $\delta << 1$. Multiply (8) by

$$\tilde{\chi}(\xi - \eta) e^{i\lambda\left(i\frac{(\xi-\eta)^2}{2} - \phi(y,\xi)\right)}$$

where $\tilde{\chi}$ is the characteristic function of the ball $B(0,\delta) \subset \mathbb{C}^n$ and then integrate w.r.t. ξ to get:

$$\iint e^{i\lambda\Phi(y,x,\xi,\eta)} \tilde{\tilde{a}}_N(x,\xi) f_{i_1\ldots i_m}(z) \tilde{b}^{i_1}(x,\xi)\ldots\tilde{b}^{i_m}(x,\xi)\,dx\,d\xi = 0. \qquad (10)$$

In the above equation, $\tilde{\tilde{a}}_N = \tilde{\chi}(\xi - \eta)\tilde{a}_N$ is another analytic and elliptic amplitude for x close to zero and $|\xi - \eta| < \frac{\delta}{\widetilde{C}}$ and

$$\Phi = -\phi(y,\xi) + \phi(x,\xi) + \frac{i}{2}(\xi - \eta)^2.$$

Furthermore,

$$\Phi_\xi = \phi_\xi(x,\xi) - \phi_\xi(y,\xi) + i(\xi - \eta).$$

To apply the stationary phase method we need to know the critical points of $\xi \mapsto \Phi$. For x and y as in Lemma 4 above, we have:

(i) If $y = x$, $\exists$ a unique real critical point $\xi_c = \eta$
(ii) If $y \neq x$, there are no real critical points
(iii) Also by Lemma 4, if $y \neq x$, there is a unique complex critical point if $|x - y| < \delta/C_1$ and no critical points for $|x - y| > \delta/C_0$ for some constants C_0 and C_1 with $C_1 > C_0$.

Define, $\psi(x,y,\eta) := \Phi(\xi_c)$. Then at $x = y$
(i) $\psi_y(x,x,\eta) = -\phi_x(x,\eta)$ (ii) $\psi_x(x,x,\eta) = \phi_x(x,\eta)$ (iii) $\psi(x,x,\eta) = 0$.

Now, we split the x integral in (10) into integration over $\{x : |x - y| > \delta/C_0\}$ for some $C_0 > 1$ and its complement. Since, $|\Phi_\xi|$ has a positive lower bound for $\{x : |x - y| > \delta/C_0\}$ and there are no critical points of $\xi \mapsto \Phi$ in this set, we can estimate that integral in the following manner: First note that, $e^{i\lambda\Phi(x,\xi)} = \frac{\Phi_\xi \partial_\xi}{i\lambda|\Phi_\xi|^2} e^{i\lambda\Phi(x,\xi)}$.

Integrating by parts N times repeatedly with respect to ξ together with (7) and using the fact that $|\xi-\eta|=\delta$ on the boundary, we get (please see "Appendix" for details)

$$\left|\iint_{|x-y|>\delta/C_0} e^{i\lambda\Phi(y,x,\xi,\eta)}\tilde{\tilde{a}}_N(x,\xi)f_{i_1\dots i_m}(x)\tilde{b}^{i_1}(x,\xi)\dots\tilde{b}^{i_m}(x,\xi)dxd\xi\right| \leq C\left(\frac{CN}{\lambda}\right)^N + CNe^{-\frac{\lambda}{C}}. \tag{11}$$

We choose $N\leq\lambda/Ce\leq N+1$ to get an exponential error on the right. Now in estimating the integral

$$\left|\int_{|x-y|\leq\delta/C_0} e^{i\lambda\Phi(y,x,\xi,\eta)}\tilde{\tilde{a}}_N(x,\xi)f_{i_1\dots i_m}(x)\tilde{b}^{i_1}(x,\xi)\dots\tilde{b}^{i_m}(x,\xi)dxd\xi\right|, \tag{12}$$

we use [18, Theorem 2.8] and [18, Remark 2.10] to conclude:

$$\int_{|x-y|\leq\delta/C_0} e^{i\lambda\psi(x,\alpha)}f_{i_1\dots i_m}(x)B^{i_1\dots i_m}(x,\alpha;\lambda)dx=\mathcal{O}(e^{-\lambda/C}) \tag{13}$$

where $\alpha=(y,\eta)$ and B is a classical analytical symbol with principal part $\tilde{b}\otimes\dots\otimes\tilde{b}$. See "Appendix" below for a proof of estimates (13).

Let, $\beta=(y,\mu)$ where, $\mu=\phi_y(y,\eta)=\eta+\mathcal{O}(\delta)$. At $y=0$, we have $\mu=\eta$. Also $\alpha\mapsto\beta$ is a diffeomorphism following similar analysis as in [21, Section 4]. If we write $\alpha=\alpha(\beta)$, then the above equation becomes:

$$\int_{|x-y|\leq\delta/C_0} e^{i\lambda\psi(x,\beta)}f_{i_1\dots i_m}(x)B^{i_1\dots i_m}(x,\beta;\lambda)dx=\mathcal{O}(e^{-\lambda/C}) \tag{14}$$

where ψ satisfies (i), (ii) and (iii), and B is a classical analytical symbol as before and $\psi_y(x,x,\eta)=-\mu,\quad \psi_x(x,x,\eta)=\mu\quad$ and $\quad\psi(x,x,\eta)=0.$

The symbols in (14) satisfy:

$$\sigma_P(B)(0,0,\mu)=\theta(\mu)\otimes\dots\otimes\theta(\mu)=\theta^{\otimes m}(\mu)$$

and in particular,

$$\sigma_P(B)(0,0,\xi_0)=e_n\otimes\dots\otimes e_n.$$

Let, $\theta_1=e_n,\theta_2,\dots,\theta_N$ be $N=\binom{n+m-2}{m}$ unit vectors at $x_0=0$ which lie in the hyperplane perpendicular to ξ_0. We will also assume that $\{\theta_i^{\odot m}\}_{i=1}^N$ are independent, where $\odot$ is a symmetrized product of vectors. The existence of such unit vectors in any open set can be shown. We can therefore assume that θ_p belongs in a small neighborhood around $\theta_1=e_n$. Then we can rotate the axes a little such that $\xi_0=e^{n-1}$

and $\theta_p = e_n$ and do the same construction as above. This gives us $N = \binom{n+m-2}{m}$ phase functions $\psi_{(p)}$, and as many number of analytic symbols for which (14) is true i.e.

$$\int_{|x-y|\leq \delta/C_0} e^{i\lambda\psi_{(p)}(x,\beta)} f_{i_1\dots i_m}(x) B^{i_1\dots i_m}_{(p)}(x,\beta;\lambda)dx = \mathcal{O}(e^{-\lambda/C}) \tag{15}$$

where

$$\sigma_P(B_p)(0,0,\mu) = \theta_p(\mu)\otimes\dots\otimes\theta_p(\mu), \quad p=1,\dots,N \quad \text{up to elliptic factors.}$$

Now we use the fact that $\delta f = 0$ near x_0. Let $\chi_0(x)$ be a smooth cutoff function near $x=0$ such that it is identically 1 in some neighborhood of $x=0$. On integrating

$$\frac{1}{\lambda}\exp(i\lambda\psi_{(1)}(x,\beta))\chi_0(x)\delta f = 0$$

w.r.t. x and after an integration by parts, we get

$$\begin{aligned}\int_{|x-y|\leq \delta/C_0} & e^{i\lambda\psi_{(1)}(x,\beta)} f_{i_1\dots i_m}(x) C^{i_m}(x,\beta;\lambda)dx \\ &= \mathcal{O}(e^{-\lambda/C}),\, i_j\in\{1,\dots,n\} \text{ and } j=1,\dots,(m-1)\end{aligned} \tag{16}$$

for $\beta_x = y$ small enough and where $\sigma_P(C^{i_m})(0,0,\xi_0) = (\xi_0)^{i_m}$. This gives us additional $\tilde{N} = \binom{n+m-2}{m-1}$ equations such that the system of $N+\tilde{N} = \binom{n+m-1}{m}$ Eqs. (15), (16) can be viewed as a tensor valued operator on f. We claim that the symbol for this operator is elliptic at $(0,0,\xi_0)$. Indeed, to show that the symbol is elliptic at $(0,0,\xi_0)$ amounts to showing that the only solution to the following system of equations is $f=0$:

$$\theta_p^{i_1}\dots\theta_p^{i_m} f_{i_1\dots i_m} = 0, \quad \text{for all } p=\{1,\dots,N\} \tag{17}$$

$$\xi_0^{i_m} f_{i_1\dots i_m} = 0, \quad \text{for } 1\leq i_1\leq\dots\leq i_{m-1}\leq n. \tag{18}$$

Using conditions on θ_p and ξ_0, it is proved in [10] that above system of equations will imply $f=0$. □

For the more general case, when δf is microlocally analytic at (x_0,ξ_0), we use the same arguments as above, except that we multiply (14) by an appropriate cut-off near (x_0,x_0,ξ_0) and use integration by parts as explained in [11, Section 4] to conclude the following proposition:

Proposition 3 *Let $\widetilde{\Omega}$, f and γ_0 be as in the statement of Proposition 2. If $(x_0,\xi_0)\notin WF_A(\delta f)$ (where ξ_0 is normal to the geodesic γ_0 at x_0), and $I^0 f(\gamma) = 0$ for all γ in a nbd. of γ_0, then $(x_0,\xi_0)\notin WF_A(f)$.*

The rest of the argument from [11] applies as it is and thereby we prove Theorem 2. We will briefly outline the ideas here for the sake of completeness. We will first need to show that the following analogue of [11, Theorem 2.2(a)] holds for the case of symmetric m-tensor fields as well:

Theorem 5 *Let f be as above. Then $I^0 f(\gamma) = 0$ for each geodesic γ in $\mathcal{A}$, if and only if for each geodesic $\gamma_0 \in \mathcal{A}$ there exists a neighborhood $\mathcal{U}$ of γ_0 and an $(m-1)$-tensor field $v \in \mathcal{D}'(\widetilde{\Omega}_{\mathcal{U}})$ such that $f = \mathrm{d}v$ in $\widetilde{\Omega}_{\mathcal{U}}$, and $v = 0$ outside Ω.*

The "if" part follows from the Fundamental Theorem of Calculus. Note also that one can first prove the theorem for f such that $f = f^s$ in Ω and then use the decomposition theorem (see Theorem 1), to prove it for any general f. To prove the "only if" part of the theorem assume that γ_0 is a geodesic in the set $\mathcal{A}$, where $\mathcal{A}$ is defined in Sect. 2. This means γ_0 can be continuously deformed to a geodesic lying outside Ω and tangent to $\partial\Omega$. Hence by extending all geodesics in Ω to maximal geodesics in $\tilde{\Omega}$, we know that there must exist two continuous curves $a(t), b(t)$, $t \in [0, 1]$ such that $\gamma_{(a(0),b(0))}$ is tangent to $\partial\Omega$, $\gamma_{(a(t),b(t))} \in \mathcal{A}$ and $\gamma_{(a(1),b(1))}$ is γ_0. Using [12, Theorem A], one can show that the Theorem 5 is at least true in a small neighborhood of $\partial\Omega$ i.e. in some neighborhood of the geodesics $\gamma_{(a(t),b(t))}$ for $0 \le t \le 2t_0$ for some $t_0 << 1$. More precisely,

Lemma 5 [11, Lemma 5.1] *There exists a neighborhood V of $\partial\Omega$ such that $\forall x \in V$, $dist(x, \partial\Omega) < \epsilon_0$ for some $\epsilon_0 > 0$ and a unique v_0 such that $f = \mathrm{d}v_0$ in V, $v_0 = 0$ on $\partial\Omega$ and v_0 is analytic in V, up to the boundary $\partial\Omega$.*

Note that the above implies that in V, the tensor field $h = f - \mathrm{d}v$ as constructed in Proposition 1 is zero. We will now construct a sequence of neighborhoods beginning with a neighborhood of $\gamma_{(a(0),b(0))}$ and up to a neighborhood of $\gamma_{(a(1),b(1))}$ for which the locally defined tensor field $h = f - \mathrm{d}v$ is zero. However to implement this program we need the following theorem due to Sato–Kawai–Kashiwara, see e.g. [14] or [23]:

Lemma 6 [23, Lemma 3.1] *Let $f \in \mathcal{D}y'(\Omega)$. Let $x_0 \in \Omega$ and let U be a neighborhood of x_0. Assume that S is a C^2 submanifold of Ω and $x_0 \in supp(f) \cap S$. Furthermore, let S divide U into two open connected sets and assume that $f = 0$ on one of these open sets. Let $\xi \in N^*_{x_0}(S) \setminus 0$, then $(x_0, \xi) \in WF_A(f)$.*

Consider the cone of all vectors in $T_{a(t)}\widetilde{\Omega}$ at an angle less than ϵ with $\dot{\gamma}_{[a(t),b(t)]}$ for some small properly chosen ϵ. The cone $C_\epsilon(t)$ with its vertex at $a(t) \in \partial\tilde{\Omega}$ is then the image of the above cone of vectors under the exponential map. We choose $\epsilon > 0$ such that

1. $C_{2\epsilon}(t) \subset \widetilde{\Omega}_{\mathcal{A}}, \forall t \in [0, 1]$.
2. $C_\epsilon(t) \subset [\tilde{V}$ for $0 \le t \le t_0$ where $[\tilde{V} := V \cup (\widetilde{\Omega} \setminus \Omega)$.
3. No geodesic inside the cone $cl(C_{2\epsilon}(t))$, $t_0 < t < 1$, with vertex at $a(t)$ is tangent to $\partial\Omega$.

For any t, let us construct a tensor field h_t in $C_{2\epsilon}(t)$ just as in Proposition 1. Recall that the support of h_t lies in Ω. Since $C_\epsilon(t) \subset \tilde{V}$ for $0 \le t \le t_0$ then by Lemma 5 we have $h_t = 0$ in $C_\epsilon(t) \subset \tilde{V}$. Hence the set $\{t \in [0, 1] : h_t = 0 \text{ in } C_\epsilon(t)\}$ is non empty. Let $t^* = \sup\{t \in [0, 1] : h_t = 0 \text{ in } C_\epsilon(t)\}$. We will show: $t^* = 1$. This will imply that there exists a neighborhood $\mathcal{U}$ of γ_0 and a $(m-1)$ tensor field $v \in \mathcal{D}'(\widetilde{\Omega_{\mathcal{U}}})$ such that $h = f - \mathrm{d}v = 0$ there.

Assume $t^* < 1$. Then $h_{t^*} = 0$ in $C_\epsilon(t^*)$ because $h_{t^*} = 0$ outside Ω. If this were not true, one could find a cone $C_\epsilon(\tilde{t})$ for some $\tilde{t} < t^*$ such that $C_\epsilon(\tilde{t}) \cap \Omega \subset$

$C_{2\epsilon}(t^*) \cap \Omega$ and such that $h(\tilde{t})$ is not zero in $C_\epsilon(\tilde{t})$. Next we will show that $h_{t^*} = 0$ in $C_{2\epsilon}(t^*)$. This gives us a contradiction, because on increasing t^* slightly to t, we can get $C_\epsilon(t) \cap \Omega \subset C_{2\epsilon}(t^*) \cap \Omega$ such that h_t is zero in this $C_\epsilon(t)$. Here we would like to mention that h_t as constructed from a tensor field for which $f = f^s$ is locally unique in any open cone in which $h_t = 0$. (This follows from the fact that the solution of $\delta \mathrm{d}v = \delta f^s = 0$ and $v|_{\partial\Omega} = 0$ is unique in such cones, see also [11, section 5]). In particular, if $h_{t^*} = 0$ in $C_{2\epsilon}(t^*)$ and $C_\epsilon(t) \cap \Omega \subset C_{2\epsilon}(t^*) \cap \Omega$, then $h_t = 0$ in $C_\epsilon(t)$ which contradicts the choice for t^*. To fulfill our program, consider h_{t^*} in $C_{2\epsilon}(t^*)$. As stated earlier, $h_{t^*} = 0$ in $C_\epsilon(t^*)$. Let $\epsilon < \epsilon_0 \le 2\epsilon$ be such that $C_{\epsilon_0}(t^*)$ is the first cone whose boundary intersects $supp(h_{t^*})$. If no such ϵ_0 can be found then we are done. Let $q \in supp(h_{t^*}) \cap \partial C_{\epsilon_0}(t)$. Clearly $q \notin \partial\widetilde{\Omega}$, because $h_{t^*} = 0$ outside Ω. So q is an interior point of $\widetilde{\Omega}$. In $\widetilde{\Omega}$, $(\delta f)_{i_1\ldots i_{m-1}} = (\delta(f\chi))_{i_1\ldots i_{m-1}}$ where χ is the characteristic function of Ω. Recall that we are working with such tensor fields for which $f = f^s$, and, one knows that such a tensor field is analytic near $\partial\Omega$, up to $\partial\Omega$, see [11, Section 5]. Now,

$$\begin{aligned}
(\delta(f^s\chi))_{i_1\ldots i_{m-1}} &= \left(\nabla_k(f^s_{i_1\ldots i_{m-1}j}\chi)\right) g^{jk} \\
&= \left(\chi\nabla_k f^s_{i_1\ldots i_{m-1}j}\right) g^{jk} + f^s_{i_1\ldots i_{m-1}j} g^{jk}\nabla_k\chi \\
&= f^s_{i_1\ldots i_{m-1}j}\nabla^j\chi \\
&= -f^s_{i_1\ldots i_{m-1}j}\nu^j\delta_{\partial\Omega}, \quad \text{here } \delta_{\partial\Omega} \text{ represents surface measure on } \partial\Omega.
\end{aligned}$$

This shows that the analytic wavefront set of δf is in $N^*(\partial\Omega)$. Let $\tilde{\gamma}$ be the geodesic in Ω on the surface of $\partial C_{\epsilon_0}(t^*)$ that contains q. Because $N^*\tilde{\gamma}$ does not intersect $N^*\partial\Omega$, by Proposition 3 and by Lemma 3, h has no analytic singularities in $N^*\tilde{\gamma}$. Consider a small open set W containing q which is divided by the surface of $\partial C_{\epsilon_0}(t^*)$ into two open connected sets as in the statement of Lemma 6 and $h_{t^*} = 0$ in one of these open sets. Since the co-normals to $C_{\epsilon_0}(t^*)$ at q are not in $WF_A(h_{t^*})$, this implies $q \notin supp(h_{t^*})$ by the Sato–Kawai–Kashiwara theorem mentioned above. This shows that $h_{t^*} = 0$ in $C_{2\epsilon}(t^*)$ which in turn implies $t^* = 1$. This proves Lemma 5.

Using the condition that any closed path with a base point on $\partial\Omega$ is homotopic to a point lying on $\partial\Omega$ and using the geometric arguments in Section 6 of [11] along with Lemma 5, we conclude the proof of Theorem 2.

Remark 2 The symmetric $(m-1)$-tensor field v also has components in $\mathcal{E}'(\widetilde{\Omega})$ and is supported in Ω just like the symmetric m-tensor field f.

4.2 Proof of Theorem 3

In order to prove Theorem 3, we need the following lemma:

Lemma 7 *For any* $1 \le k \le m$, *if* $f = \mathrm{d}v$ *with* $v|_{\partial\Omega} = 0$. *Then* $I^k f = -kI^{k-1}v$.

Proof Consider

$$I^k f(\gamma) = I^k(\mathrm{d}v)(\gamma)$$

$$
\begin{aligned}
&= \int_0^{l(\gamma)} t^k (\mathrm{d}v)_{i_1\dots i_m}(\gamma(t))\dots\gamma^{i_1}(t)\dots\dot{\gamma}^{i_m}(t)dt \\
&= \int_0^{l(\gamma)} t^k \frac{d}{dt}\{v_{i_1\dots i_{m-1}}(\gamma(t))\dots\gamma^{i_1}(t)\dots\dot{\gamma}^{i_{m-1}}(t)\}dt \\
&= \{t^k v_{i_1\dots i_{m-1}}(\gamma(t))\dot{\gamma}^{i_1}(t)\dots\dot{\gamma}^{i_{m-1}}(t)\}_0^{l(\gamma)} \\
&\quad - k\int_0^{l(\gamma)} t^{k-1} v_{i_1\dots i_{m-1}}(\gamma(t))\dots\gamma^{i_1}(t)\dots\dot{\gamma}^{i_{m-1}}(t)dt \\
&= -kI^{k-1}v(\gamma),
\end{aligned}
$$

where first term in the second last equality is 0 because of our assumption $v|_{\partial\Omega} = 0$. Thus, we have our lemma. □

Proof of Theorem 3 Let us come back to the proof of Theorem 3. As we know from Theorem 2 that if $I^0 f(\gamma) = If(\gamma) = 0$ for each geodesic γ not intersecting K then there exist $(m-1)$-tensor field v_1 which is 0 on the boundary $\partial\Omega$ such that $f = \mathrm{d}v_1$ on $\Omega \setminus K$. And from Lemma 7, we know

$$I^1 f(\gamma) = I^1(\mathrm{d}v_1)(\gamma) = -I^0 v_1(\gamma).$$

Again using Theorem 2 we conclude that there exist $(m-2)-$tensor field v_2 such that $v_1 = \mathrm{d}v_2$ and $v_2|_{\partial\Omega} = 0$. Using Theorem 2 along with Lemma 7 $(m-2)$ more times, we have

$$I^m f(\gamma) = m!(-1)^m I^0 v_m(\gamma) = 0$$

where v_m is 0-tensor i.e. a function. Now using [9, Theorem 1], we can conclude $v_m = 0$ on $\Omega \setminus K$. And since $f = \mathrm{d}^m v_m$ on an open connected set $\Omega \setminus K$ therefore f is also 0 on $\Omega \setminus K$. □

5 Proof of Theorem 4

As we mentioned in Sect. 2 that Theorem 4 also follows as a corollary of Theorem 3 when f is supported in Ω. However, we prove it here independently using s-injectivity of the geodesic ray transform.

To prove Theorem 4, we will need the s-injectivity of the ray transform for symmetric m-tensor fields. The proof of s-injectivity for symmetric 2-tensor fields is given in [20]. The same proof will also work for symmetric tensor fields of any order. For details, we will refer the reader to [20, Sections 2,3,4]. Hence we have,

Theorem 6 [20, Theorem 1.4] *Let (Ω, g) be a compact, simple real analytic Riemannian manifold with smooth boundary and f be a symmetric m-tensor field with components in $L^2(\Omega)$. If $I^0 f(\gamma) = 0$ for all γ which are geodesics in Ω, then $f^s = 0$ in Ω.*

Theorem 7 *Let Ω be a compact simple Riemannian manifold with boundary. Let $m \geq 0$ and $p \geq m$ be integers. Then for any $f \in L^2(S^m(\Omega))$, there exist uniquely determined $v_0, \ldots, v_m$ with $v_i \in H^i(S^{m-i}\Omega)$ for $i = 0, 1, \ldots, m$ such that*

$$f = \sum_{i=0}^{m} \mathrm{d}^i v_i, \quad \textit{with } v_i \textit{ solenoidal for } 0 \leq i \leq m-1$$

$$\textit{and for each } 0 \leq i \leq m-1, \quad \sum_{j=0}^{i} \mathrm{d}^j v_{m-i+j} = 0 \textit{ on } \partial\Omega.$$

Proof This follows from a repeated application of [17, Theorem 3.3.2]. □

Proof of Theorem 4 We have from Theorem 7 that

$$f = \sum_{i=0}^{m} \mathrm{d}^i v_i, \quad \text{with } v_i \text{ solenoidal for } 0 \leq i \leq m-1$$

$$\text{and for each } 0 \leq i \leq m-1, \quad \sum_{j=0}^{i} \mathrm{d}^j v_{m-i+j} = 0 \text{ on } \partial\Omega. \tag{19}$$

Using s-injectivity of I, we know that $v_0 = 0$, since it is solenoidal. Now consider

$$\begin{aligned} 0 = I^1 f(\gamma) &= I^1 \left(\sum_{i=0}^{m} \mathrm{d}^i v_i \right)(\gamma) \\ &= I^1 \left(\mathrm{d} \left(\sum_{i=1}^{m} \mathrm{d}^{i-1} v_i \right) \right)(\gamma), \qquad \text{since } v_0 = 0 \\ &= -I^0 \left(\sum_{i=1}^{m} \mathrm{d}^{i-1} v_i \right)(\gamma) \qquad \text{(using Lemma 7)}. \end{aligned}$$

From this, we can conclude v_1 is also 0 because it is the solenoidal part of tensor field $\sum_{i=1}^{m} \mathrm{d}^{i-1} v_i$.

Now suppose that $v_1, \ldots, v_k$ can be shown to be equal to 0 from the knowledge of $I^1 f, \ldots, I^k f$. Then

$$\begin{aligned} 0 = I^{k+1} \left(f - \sum_{0}^{k} \mathrm{d}^i v_i \right) &= I^{k+1} \left(\sum_{i=k+1}^{m} \mathrm{d}^i v_i \right) \\ \Rightarrow \qquad I^{k+1} \left(\sum_{i=k+1}^{m} \mathrm{d}^i v_i \right) &= 0 \\ \Rightarrow (-1)^{k+1}(k+1)! I^0 \left(\sum_{i=k+1}^{m} \mathrm{d}^{i-k-1} v_i \right) &= 0, \quad \text{(using Lemma 7, } (k+1) \text{ times)}. \end{aligned}$$

Therefore $v_{k+1} = 0$ because it is the solenoidal part of the tensor field $\left(\sum_{i=k+1}^{m} \mathrm{d}^{i-k-1} v_i\right)$. By induction, the proof is now complete. □

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Appendix

Proof of Lemma 1 First, let us recall that for a symmetric $(m-1)$-tensor field v,

$$(\mathrm{d}v)_{i_1\dots i_m} = \sigma(i_1,\dots,i_m)\left(\frac{\partial v_{i_1\dots i_{m-1}}}{\partial x^{i_m}} - \sum_{l=1}^{m-1}\Gamma^p_{i_m i_l} v_{i_1,\dots i_{l-1} p i_{l+1}\dots i_{m-1}}\right).$$

The idea here is to use an inductive argument for $0 \le k \le m$. We start by showing the result for $k = 0, 1$, and then for general $k \le m$.

$$(\mathrm{d}v)_{n\dots n} = \frac{\partial v_{n\dots n}}{\partial x^n}, \quad \text{for } k = 0$$
$$(\mathrm{d}v)_{n\dots ni} = \frac{m-1}{m}\frac{\partial v_{n\dots ni}}{\partial x^n} - \frac{2(m-1)}{m}\Gamma^p_{in} v_{n\dots np} + \frac{1}{m}\frac{\partial v_{n\dots n}}{\partial x^i}, \quad \text{for } k = 1.$$

From this, we see that the result is true for $k = 0$ and 1. Now, we are going to prove that the result is also true for $k \le m$. Consider

$$\begin{aligned}(\mathrm{d}v)_{n\dots n i_k\dots i_1} &= \sigma(n,\dots n, i_k,\dots, i_1)\\ &\quad\left(\frac{\partial v_{n\dots n i_k\dots i_2}}{\partial x^{i_1}} - \sum_{l=2}^{k}\Gamma^p_{i_l i_1} v_{n\dots n i_k\dots \hat{i}_l\dots i_2 p} - (m-k)\Gamma^p_{n i_1} v_{n\dots n i_k\dots i_2 p}\right)\\ &= J - J_k^1 - (m-k)J_k^2\end{aligned}$$

where

$$\begin{aligned} J &= \sigma(n,\dots n, i_k,\dots, i_1)\left(\frac{\partial v_{n\dots n i_k\dots i_2}}{\partial x^{i_1}}\right),\\ J_k^1 &= \sigma(n,\dots n, i_k,\dots, i_1)\left(\sum_{l=2}^{k}\Gamma^p_{i_l i_1} v_{n\dots n i_k\dots \hat{i}_l\dots i_2 p}\right),\\ J_k^2 &= \sigma(n,\dots n, i_k,\dots, i_1)\left(\Gamma^p_{n i_1} v_{n\dots n i_k\dots i_2 p}\right).\end{aligned}$$

Now, we will simplyfy each of the above terms one by one. Consider

$$
\begin{aligned}
J &= \sigma(n,\dots n, i_k,\dots,i_1)\left(\frac{\partial v_{n\dots ni_k\dots i_2}}{\partial x^{i_1}}\right)\\
&= \frac{\sigma(n,\dots n, i_k,\dots,i_2)}{m}\left(\frac{\partial v_{n\dots ni_k\dots i_2}}{\partial x^{i_1}} + (m-1)\frac{\partial v_{n\dots ni_k\dots i_1}}{\partial x^{n}}\right)\\
&= \frac{1}{m}\frac{\partial v_{n\dots ni_k\dots i_2}}{\partial x^{i_1}} + \frac{\sigma(n,\dots n, i_k,\dots,i_3)}{m}\left(\frac{\partial v_{n\dots ni_k\dots i_3 i_1}}{\partial x^{i_2}} + (m-2)\frac{\partial v_{n\dots ni_k\dots i_1}}{\partial x^{n}}\right)\\
&= \frac{1}{m}\sum_{l=1}^{k}\frac{\partial v_{n\dots ni_k\dots i_{l-1}i_{l+1}\dots i_1}}{\partial x^{i_l}} + \frac{m-k}{m}\frac{\partial v_{n\dots ni_k\dots i_1}}{\partial x^{n}}, \qquad \text{(repeating similar arguments).}\\
J_k^2 &= \sigma(n,\dots n, i_k,\dots,i_1)\left(\Gamma^p_{ni_1} v_{n\dots ni_k\dots i_2 p}\right)\\
&= \frac{\sigma(n,\dots n, i_k,\dots,i_2)}{m}\left(2\Gamma^p_{ni_1} v_{n\dots ni_k\dots i_2 p} + (m-2)\Gamma^p_{ni_2} v_{n\dots ni_k\dots i_3 i_1 p}\right)\\
&= \frac{2\sigma(n,\dots n, i_k,\dots,i_3)}{m(m-1)}\left(\Gamma^p_{i_2 i_1} v_{n\dots ni_k\dots i_3 p} + (m-2)\Gamma^p_{ni_1} v_{n\dots ni_k\dots i_2 p}\right)\\
&\quad + \frac{(m-2)\sigma(n,\dots n, i_k,\dots,i_3)}{m(m-1)}\left(2\Gamma^p_{ni_2} v_{n\dots ni_k\dots i_3 i_1 p} + (m-3)\Gamma^p_{ni_3} v_{n\dots ni_k\dots i_4 i_2 i_1 p}\right)\\
&= \frac{2}{m(m-1)}\Gamma^p_{i_2 i_1} v_{n\dots ni_k\dots i_3 p} + \frac{2(m-2)\sigma(n,\dots n, i_k,\dots,i_3)}{m(m-1)}\sum_{q=1}^{2}\Gamma^p_{ni_q} v_{n\dots ni_k\dots i_3\hat{i_q}i_1 p}\\
&\quad + \frac{(m-3)(m-2)\sigma(n,\dots n, i_k,\dots,i_3)}{m(m-1)}\Gamma^p_{ni_3} v_{n\dots ni_k\dots i_2 i_1 p}\\
&= \frac{2}{m(m-1)}\sum_{q,r=1,q\neq r}^{3}\Gamma^p_{i_q i_r} v_{n\dots ni_k\dots i_4\hat{i_q}\hat{i_r}i_1 p}\\
&\quad + \frac{2(m-3)\sigma(n,\dots n, i_k,\dots,i_4)}{m(m-1)}\sum_{q=1}^{3}\Gamma^p_{ni_q} v_{n\dots ni_k\dots i_4\hat{i_q}i_1 p}\\
&\quad + \frac{(m-4)(m-3)\sigma(n,\dots n, i_k,\dots,i_4)}{m(m-1)}\Gamma^p_{ni_4} v_{n\dots ni_k\dots i_5 i_3 i_2 i_1 p},\\
&= \frac{2(k-1)}{m(m-1)}\sum_{q,r=1,q\neq r}^{k}\Gamma^p_{i_q i_r} v_{n\dots ni_k\dots\hat{i_q}\dots\hat{i_r}\dots i_1 p} + \frac{2(m-k)}{m(m-1)}\sum_{q=1}^{k}\Gamma^p_{ni_q} v_{n\dots ni_k\dots\hat{i_q}\dots i_1 p}\\
&\qquad\qquad\qquad\qquad \text{(repeating similar calculation } (k-3)\text{-times).}\\
J_k^1 &= \sigma(n,\dots n, i_k,\dots,i_1)\left(\sum_{l=2}^{k}\Gamma^p_{i_l i_1} v_{n\dots ni_k\dots\hat{i_l}\dots i_2 p}\right)\\
&= \frac{\sigma(n,\dots n, i_k,\dots,i_2)}{m}\left(2\sum_{l=2}^{k}\Gamma^p_{i_l i_1} v_{n\dots ni_k\dots\hat{i_l}\dots i_2 p} + (m-2)\sum_{l=3}^{k}\Gamma^p_{i_l i_2} v_{n\dots ni_k\dots\hat{i_l}\dots i_3 i_1 p}\right.\\
&\quad \left. + (m-2)\Gamma^p_{i_3 i_2} v_{n\dots ni_k\dots i_4 i_1 p}\right)\\
&= \frac{\sigma(n,\dots n, i_k,\dots,i_3)}{m(m-1)}\left\{2(k-1)\Gamma^p_{i_2 i_1} v_{n\dots ni_k\dots i_3 p} + (m-2)\left(2\sum_{l=3}^{k}\Gamma^p_{i_l i_1} v_{n\dots ni_k\dots\hat{i_l}\dots i_2 p}\right.\right.
\end{aligned}
$$

$$+2\Gamma^p_{i_3 i_1} v_{n\ldots n i_k \ldots i_4 i_2 p} + 2\sum_{l=3}^{k} \Gamma^p_{i_l i_2} v_{n\ldots n i_k \ldots \hat{i}_l \ldots i_3 i_1 p} + (m-3)\sum_{l=4}^{k} \Gamma^p_{i_l i_3} v_{n\ldots n i_k \ldots \hat{i}_l \ldots i_4 i_2 i_1 p}$$

$$+(m-3)\Gamma^p_{i_3 i_4} v_{n\ldots n i_k \ldots i_5 i_2 i_1 p} + 2\Gamma^p_{i_3 i_2} v_{n\ldots n i_k \ldots i_4 i_1 p} + (m-3)\Gamma^p_{i_3 i_4} v_{n\ldots n i_k \ldots i_5 i_2 i_1 p}\Big)\Big\}$$

$$= \frac{(m-2)\sigma(n,\ldots,n,i_k,\ldots,i_3)}{m(m-1)} \left\{ 2\sum_{q=1}^{2} \left(\sum_{l=3}^{k} \Gamma^p_{i_l i_q} v_{n\ldots n i_k \ldots \hat{i}_l \ldots \hat{i}_q i_1 p} + \Gamma^p_{i_3 i_q} v_{n\ldots n i_k \ldots i_4 \hat{i}_q i_1 p} \right) \right.$$

$$\left. + (m-3)\left(\sum_{l=4}^{k} \Gamma^p_{i_l i_3} v_{n\ldots n i_k \ldots \hat{i}_l \ldots i_4 i_2 i_1 p} + 2\Gamma^p_{i_3 i_4} v_{n\ldots n i_k \ldots i_5 i_2 i_1 p} \right) \right\}$$

$$+ \frac{2(k-1)}{m(m-1)} \Gamma^p_{i_2 i_1} v_{n\ldots n i_k \ldots i_3 p}$$

$$= \frac{(m-3)\sigma(n,\ldots,n,i_k,\ldots,i_4)}{m(m-1)} \left\{ 2\sum_{q=1}^{3} \left(\sum_{l=4}^{k} \Gamma^p_{i_l i_q} v_{n\ldots n i_k \ldots \hat{i}_l \ldots \hat{i}_q i_1 p} \right) \right.$$

$$+4\sum_{q=1}^{3} \left(\Gamma^p_{i_4 i_q} v_{n\ldots n i_k \ldots \hat{i}_q i_1 p} \right)$$

$$\left. + (m-4)\left(\sum_{l=5}^{k} \Gamma^p_{i_l i_4} v_{n\ldots n i_k \ldots \hat{i}_l \ldots i_1 p} + 3\Gamma^p_{i_5 i_4} v_{n\ldots n i_k \ldots i_1 p} \right) \right\}$$

$$+ \frac{2(k-1)}{m(m-1)} \sum_{q,r=1,q\neq r}^{3} \Gamma^p_{i_q i_r} v_{n\ldots n i_k \ldots \hat{i}_q \hat{i}_r i_1 p}$$

Repeating this expansion for (k-2) times more to get

$$J^1_k = \frac{(m-k+1)\sigma(n,\ldots,n,i_k)}{m(m-1)} \left\{ 2\sum_{q=1}^{k-1} \Gamma^p_{i_k i_q} v_{n\ldots n i_{k-1} \ldots \hat{i}_l \ldots i_1 p} + 2(k-2)\sum_{q=1}^{k-1} \Gamma^p_{i_k i_q} v_{n\ldots n i_{k-1} \ldots \hat{i}_l \ldots i_1 p} \right.$$

$$\left. +(m-k)(k-1)\Gamma^p_{n i_k} v_{n\ldots n i_{k-1} \ldots i_1 p} \right\} + \frac{2(k-1)}{m(m-1)} \sum_{q,r=1,q\neq r}^{k-1} \Gamma^p_{i_q i_r} v_{n\ldots n i_k \ldots \hat{i}_q \hat{i}_r i_1 p}$$

$$= \frac{(m-k+1)\sigma(n,\ldots,n,i_k)}{m(m-1)} \left\{ 2(k-1)\sum_{q=1}^{k-1} \Gamma^p_{i_k i_q} v_{n\ldots n i_{k-1} \ldots \hat{i}_q \ldots i_1 p} \right.$$

$$\left. +(m-k)(k-1)\Gamma^p_{n i_k} v_{n\ldots n i_{k-1} \ldots i_1 p} \right\} + \frac{2(k-1)}{m(m-1)} \sum_{q,r=1,q\neq r}^{k-1} \Gamma^p_{i_q i_r} v_{n\ldots n i_k \ldots \hat{i}_q \hat{i}_r i_1 p}$$

$$= \frac{2(k-1)}{m(m-1)} \sum_{q,r=1,q\neq r}^{k} \Gamma^p_{i_q i_r} v_{n\ldots n i_k \ldots \hat{i}_q \ldots \hat{i}_r \ldots i_1 p} + \frac{2(k-1)(m-k)}{m(m-1)} \sum_{q=1}^{k} \Gamma^p_{n i_q} v_{n\ldots n i_k \ldots \hat{i}_q \ldots i_1 p}$$

After putting the values of J, J^1_k and J^2_k in dv, we get

$$(\mathrm{d}v)_{n\ldots n i_k \ldots i_1} = \frac{(m-k)}{m} \frac{\partial v_{n\ldots n i_k \ldots i_1}}{\partial x^n} - \frac{2(m-k)}{m} \sum_{l=1}^{k} \Gamma^p_{i_l n} v_{n\ldots n i_k \ldots \hat{i}_l \ldots i_1 p}$$

$$+\frac{1}{m}\sum_{l=1}^{k}\frac{\partial v_{n\ldots ni_k\ldots\hat{i}_l\ldots i_1}}{\partial x^{i_l}}-\frac{2}{m}\sum_{l,q=1,l\neq q}^{k}\Gamma^{p}_{i_l i_q}v_{n\ldots ni_k\ldots\hat{i}_l\ldots\hat{i}_q\ldots i_1 p}.$$

□

Proof of Estimate (11) Let ${}^tL=\frac{\Phi_\xi\partial_\xi}{i\lambda|\Phi_\xi|^2}$. Then as already noted

$${}^tL^N(e^{i\lambda\Phi(x,\xi)})=e^{i\lambda\Phi(x,\xi)}.$$

Consider,

$$\begin{aligned}&\left|\iint_{|x-y|>\delta/C_0}({}^tL^N(e^{i\lambda\Phi(y,x,\xi,\eta)}))\tilde{\tilde{a}}_N(x,\xi)f_{i_1\ldots i_m}(z)\tilde{b}^{i_1}(x,\xi)\ldots\tilde{b}^{i_m}(x,\xi)dxd\xi\right|\\&\leq\left|\iint_{|x-y|>\delta/C_0}e^{i\lambda\Phi(y,x,\xi,\eta)}L^N(\tilde{\tilde{a}}_N(x,\xi)f_{i_1\ldots.i_m}(z)\tilde{b}^{i_1}(x,\xi)\ldots\tilde{b}^{i_m}(x,\xi))dxd\xi\right|\\&\quad+N\int_{|x-y|>\delta/C_0}e^{-\lambda\delta^2/2}\left|f_{i_1\ldots i_m}(x)B^{i_1\ldots i_m}(x,\xi_{bdry})\right|dx.\end{aligned}$$

Using the fact that, f is compactly supported and using (7), we get (11). □

Proof of the Estimate (13) Consider

$$\left|\iint_{|x-y|<\delta/C_0}\left(e^{i\lambda\Phi(y,x,\xi,\eta)}\right)\tilde{\tilde{a}}_N(x,\xi)f_{i_1\ldots i_m}(z)\tilde{b}^{i_1}(x,\xi)\ldots\tilde{b}^{i_m}(x,\xi)dxd\xi\right|.$$

Rewrite the above as :

$$\begin{aligned}&\left|\int_{|x-y|<\delta/C_0}(e^{i\lambda\Phi(y,x,\xi_c,\eta)})(e^{-i\lambda\Phi(y,x,\xi_c,\eta)})\right.\\&\quad\left.\int_{|\xi-\eta|<\delta/C_0}(e^{i\lambda\Phi(y,x,\xi,\eta)})\tilde{\tilde{a}}_N(x,\xi)f_{i_1\ldots i_m}(z)\tilde{b}^{i_1}(x,\xi)\ldots\tilde{b}^{i_m}(x,\xi)dxd\xi\right|\end{aligned}$$

Using [18, Remark 2.10], we get

$$\begin{aligned}&\left|\int_{|x-y|<\delta/C_0}(e^{i\lambda\Phi(y,x,\xi_c,\eta)})f_{i_1\ldots i_m}(x)\right.\\&\quad\sum_{0\leq k\leq\lambda/C}C_n\frac{1}{k!}\lambda^{-n/2-k}\left(\frac{\Delta}{2}\right)^k\left(\tilde{\tilde{a}}_N(x,\xi_c)\tilde{b}^{i_1}(x,\xi_c)\ldots\tilde{b}^{i_m}(x,\xi_c)\right)\\&\quad\left.+R(x,y,\eta,\lambda)dx\right|\end{aligned}$$

Lemma 8

$$\sum_{0\le k\le\lambda/C} C_n \frac{1}{k!}\lambda^{-n/2-k}\left(\frac{\Delta}{2}\right)^k \left(\tilde{\tilde{a}}_N(x,\xi_c)\tilde{b}^{i_1}(x,\xi_c)\cdots\tilde{b}^{i_m}(x,\xi_c)\right)$$

is a formal analytic symbol.

Proof Let,

$$a_k = \frac{1}{k!}\left(\frac{\Delta}{2}\right)^k \left(\tilde{\tilde{a}}_N(x,\xi_c)\tilde{b}^{i_1}(x,\xi_c)\cdots\tilde{b}^{i_m}(x,\xi_c)\right)$$

Then from the Cauchy integral formula [18, Exercise 2.4],

$$\begin{aligned}
|a_k| &\le C_n(k+1)^{n/2}(k-1)!2^k \sup_{B_1(\xi_c)} \left(\tilde{\tilde{a}}_N(x,\xi)\tilde{b}^{i_1}(x,\xi)\ldots\tilde{b}^{i_m}(x,\xi)\right)\\
&\le C1_n(k+1)^{n/2}(k-1)!2^k\\
&\le C2_n(k+1)^{n/2}e^{2-k}(k-1)^{k-1/2}2^k \text{ (Using Stirling's approximation)}\\
&\le C2_n\left(\frac{2}{e}\right)^{k+1}(k+1)^{n/2+k}\\
&\le \tilde{C_n}^{k+n/2}(k+n/2)^{n/2+k}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{0\le k\le\lambda/C} C_n \frac{1}{k!}\lambda^{-n/2-k}\left(\frac{\Delta}{2}\right)^k \left(\tilde{\tilde{a}}_N(x,\xi_c)\tilde{b}^{i_1}(x,\xi_c)\ldots\tilde{b}^{i_m}(x,\xi_c)\right)\\
&\quad = \sum_{0\le k\le\lambda/C} \lambda^{-n/2-k} a_{k+n/2}
\end{aligned}$$

is a formal analytic symbol $B^{i_1\ldots i_m}(x,y,\eta;\lambda)$ by [18, Exercise 1.1]. □

Hence,

$$\begin{aligned}
&\int_{|x-y|<\delta/C_0} (e^{i\lambda\Phi(y,x,\xi,\eta)})\tilde{\tilde{a}}_N(x,\xi) f_{i_1\ldots i_m}(z)\tilde{b}^{i_1}(x,\xi)\ldots\tilde{b}^{i_m}(x,\xi)dxd\xi\\
&\quad = \int_{|x-y|<\delta/C_0} (e^{i\lambda\Phi(y,x,\xi_c,\eta)}) f_{i_1\ldots i_m}(x)B^{i_1\ldots i_m}(x,y,\eta;\lambda)dx\\
&\qquad + \int_{|x-y|<\delta/C_0} (e^{i\lambda\Phi(y,x,\xi_c,\eta)}) f_{i_1\ldots i_m}(x)R(x,y,\eta;\lambda)dx.
\end{aligned}$$

But,

$$\left|\int_{|x-y|<\delta/C_0} (e^{i\lambda\Phi(y,x,\xi_c,\eta)}) f_{i_1\ldots i_m}(x)R(x,y,\eta;\lambda)dx\right| = \mathcal{O}(e^{-\lambda/c}),$$

since,

$$|R(x, y, \eta; \lambda)| \leq |\Omega|/Ce^{-\lambda/c},$$

see [18, Remark 2.10]. So, this along with (10) and (11), gives us:

$$\left| \int_{|x-y|<\delta/C_0} (e^{i\lambda\Phi(y,x,\xi_c,\eta)}) f_{i_1\ldots i_m}(x) B^{i_1\ldots i_m}(x, y, \eta; \lambda) dx \right| = \mathcal{O}(e^{-\lambda/c}).$$

□

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