

Approximation by Polynomials in Sobolev Spaces with Jacobi Weight

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Abstract Polynomial approximation is studied in the Sobolev space $W_p^r(w_{\alpha,\beta})$ that consists of functions whose r -th derivatives are in weighted L^p space with the Jacobi weight function $w_{\alpha,\beta}$. This requires simultaneous approximation of a function and its consecutive derivatives up to s -th order with $s \leq r$. We provide sharp error estimates given in terms of $E_n(f^{(r)})_{L^p(w_{\alpha,\beta})}$, the error of best approximation to $f^{(r)}$ by polynomials in $L^p(w_{\alpha,\beta})$, and an explicit construction of the polynomials that approximate simultaneously with the sharp error estimates.

Keywords Approximation · Simultaneous approximation · Sobolev space · Jacobi weight

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1 Introduction

Polynomial approximation on a finite interval is a classical problem at the center of approximation theory. The purpose of this paper is to consider simultaneous approximation of a function and its derivatives by polynomials on an interval in L^p norms defined with respect to a Jacobi weight function. Although this problem has been studied by several researchers, our results are new in several aspects.

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Let $w_{\alpha,\beta}$ be the Jacobi weight function defined by $w_{\alpha,\beta}(x) := (1 - x)^\alpha(1 + x)^\beta$ for $\alpha, \beta > -1$ and $x \in (-1, 1)$. For $1 \leq p < \infty$, define

$$\|f\|_{L^p(w_{\alpha,\beta})} := \left(\int_{-1}^1 |f(x)|^p w_{\alpha,\beta}(x) dx \right)^{1/p},$$

and, for $p = \infty$, define this norm as the usual uniform norm $\|f\|_\infty$. For $r \in \mathbb{N}$, let $C^r[-1, 1]$ denote the space of functions that have r -th continuous derivatives on $[-1, 1]$. For $1 \leq p < \infty$, let $W_p^r(w_{\alpha,\beta})$ be the Sobolev space

$$W_p^r(w_{\alpha,\beta}) := \{f \in C^{r-1}[-1, 1] : f^{(r)} \in L^p(w_{\alpha,\beta})\},$$

and, for $p = \infty$, define this space as $C^r[-1, 1]$. We define the norm of $W_p^r(w_{\alpha,\beta})$ by

$$\|f\|_{W_p^r(w_{\alpha,\beta})} := \left(\sum_{k=0}^r \|f^{(k)}\|_{L^p(w_{\alpha,\beta})}^p \right)^{1/p}.$$

For $n \in \mathbb{N}$, let Π_n denote the space of polynomials of degree at most n in one variable. The standard error of best approximation by polynomials in Π_n is defined by

$$E_n(f)_{L^p(w_{\alpha,\beta})} := \inf_{p \in \Pi_n} \|f - p\|_{L^p(w_{\alpha,\beta})}.$$

The characterization of this quantity via an appropriate modulus of smoothness lies in the center of Approximation Theory and is widely studied; see, for example, [5, 6]. For $p = 2$, the n -th partial sum $S_n^{\alpha,\beta} f$ of the Fourier–Jacobi series satisfies

$$E_n(f)_{L^2(w_{\alpha,\beta})} = \|f - S_n^{\alpha,\beta} f\|_{L^2(w_{\alpha,\beta})}. \tag{1.1}$$

However, using $S_n^{\alpha,\beta} f$ for approximation in $W_2^r(w_{\alpha,\beta})$ gives a much weaker result than optimal (cf. [2, 7]), which will be discussed in Sect. 2 below.

Throughout this paper, we denote by c a generic constant, independent of n , whose value may vary from line by line. We prove two types of results for approximation in the Sobolev space.

Theorem 1.1 *Let $\alpha, \beta > -1$. Assume $f \in W_p^s(w_{\alpha,\beta})$ if $1 \leq p < \infty$ or $f \in C^s[-1, 1]$ if $p = \infty$. Then there exists a polynomial $p_n \in \Pi_{2n}$ such that*

$$\|f - p_n\|_{W_p^s(w_{\alpha,\beta})} \leq c E_n(f^{(s)})_{L^p(w_{\alpha,\beta})}, \quad 1 \leq p \leq \infty. \tag{1.2}$$

Theorem 1.2 *Let $\alpha, \beta > -1$. Assume $f \in W_p^s(w_{\alpha,\beta})$ if $1 \leq p < \infty$ or $f \in C^s[-1, 1]$ if $p = \infty$. Then there exists a polynomial $p_n \in \Pi_{2n}$ such that*

$$\|f^{(k)} - p_n^{(k)}\|_{L^p(w_{\alpha,\beta})} \leq c n^{-s+k} E_n(f^{(s)})_{L^p(w_{\alpha,\beta})}, \quad 1 \leq p \leq \infty, \tag{1.3}$$

provided either $\alpha = 0$ or $\beta = 0$.

Evidently, the estimate (1.3) is stronger than the estimate (1.2) but it holds under more restrictive conditions. Moreover, (1.3) is sharp; in fact, the order of the estimate is sharp for each fixed k .

The estimate (1.2) provides a sharp estimate for the error of best polynomial approximation in the Sobolev norm. It is known [13] that, for $r \in \mathbb{N}$,

$$E_n(f)_{L^p(w_{\alpha,\beta})} \leq c n^{-r} \|\phi^r f^{(r)}\|_{L^p(w_{\alpha,\beta})}, \quad \phi(x) := \sqrt{1-x^2}, \quad (1.4)$$

so that the righthand side of (1.2) and (1.3) can be stated with $E_n(f^{(s)})_{L^p(w_{\alpha,\beta})}$ replaced by $n^{-r+s} \|f^{(r)}\|_{L^p(w_{\alpha,\beta})}$. In the case $p = 2$, estimates in the form

$$\|f - p_n\|_{W_2^s(w_{\alpha,\beta})} \leq c n^{-r+s} \|f^{(r)}\|_{L^2(w_{\alpha,\beta})}, \quad 0 \leq s \leq r, \quad (1.5)$$

have been established and used in the spectral method for numerical solution of differential equations in some cases; more precisely, such an estimate was first established in [2,3] for the case of Chebyshev and Legendre polynomials ($\alpha = \beta = -1/2$ or 0) when $s = 1$, and later established for general α, β and $s = 1$ by several researchers, see [8,15] and references therein. In latter works, the norm in the lefthand side of (1.5) is often replaced by

$$\|f\|_{W_2^s(w_{\alpha,\beta})}^* := \left(\sum_{k=0}^s \|f^{(k)}\|_{L^2(w_{\alpha+k,\beta+k})}^2 \right)^{1/2}$$

and the norm in the righthand side is replaced by $\|f^{(r)}\|_{L^2(w_{\alpha+r,\beta+r})}$, which we call $*$ -version. By (1.4), our estimates can be stated in terms of the norm of $f^{(r)}$ for $r \geq s$, so that (1.2) is stronger than (1.5) and offers an estimate somewhat different from (1.5) in $*$ -version.

The estimate (1.3) is what is known as simultaneous approximation in the Approximation Theory community, where it is a folklore that each increased derivative reduces the order of approximation by n^{-1} . However, such estimates are usually established with a modulus of smoothness of $f^{(r)}$ in place of $E_n(f^{(r)})_{L^p(w_{\alpha,\beta})}$ in (1.3) in the literature (cf. [9,10]).

The polynomial p_n in the theorems can be expressed in a simple explicit formula. For $p = 2$, it is the n -th partial sum operator of the Fourier series in the Sobolev orthogonal polynomials in $W_2^s(w_{\alpha,\beta})$, which are polynomials that are orthogonal with respect to an inner product that involves derivatives. It should be pointed out, however, that this fact does not follow from the usual Hilbert space argument, since the norm of $W_2^s(w_{\alpha,\beta})$ is not arising from the square root of the inner product that defines the Sobolev orthogonality. Sobolev orthogonal polynomials have been studied extensively in the special function community (cf. [12] and the references therein), but not their orthogonal series, and in spectral method community, often with zero boundary at the end of the intervals. These polynomials are usually given in terms of the Jacobi polynomials with negative integer parameters, which requires appropriate extensions

that could be rather delicate (cf. [1,4,8,11,18]). We shall give a more direct definition of the Sobolev orthogonal polynomials in $W_2^s(w_{\alpha,\beta})$ that does not require such extensions.

The paper is organized as follows. In the following section, we discuss the approximation behavior of the L^2 partial sum operator $S_n^{\alpha,\beta} f$ in the Sobolev space, which gives suboptimal result. In Sect. 3, we consider a Sobolev inner product in $W_2^r(w_{\alpha,b})$ and define a family of its orthogonal polynomials, for all $\alpha, \beta > -1$, in an elegant formula that is more suitable for studying orthogonal series in terms of them. Approximation by polynomials or orthogonal series are studied in the following two sections. In particular, more elaborate version of Theorems 1.2 and 1.3 will be established in Sects. 4 and 5, respectively.

2 Jacobi Polynomials and Fourier Jacobi Series

For $\alpha, \beta > -1$, the Jacobi polynomials are defined by [16, (4.21.2)],

$$P_n^{(\alpha,\beta)}(t) = \frac{(\alpha + 1)_n}{n!} {}_2F_1 \left(\begin{matrix} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix}; \frac{1-t}{2} \right) \tag{2.1}$$

in terms of the hypergeometric function ${}_2F_1$. For convenience, we shall define

$$J_n^{\alpha,\beta}(t) = \frac{2^n}{(n + \alpha + \beta + 1)_n} P_n^{(\alpha,\beta)}(t).$$

One advantage of this normalization is the following identity, by [16, (4.5.5)],

$$\frac{d}{dt} J_n^{\alpha,\beta}(t) = J_{n-1}^{\alpha+1,\beta+1}(t). \tag{2.2}$$

These polynomials are orthogonal with respect to the inner product

$$\langle f, g \rangle_{\alpha,\beta} := \int_{-1}^1 f(x)g(x)w_{\alpha,\beta}(x)dx.$$

The Fourier orthogonal expansion of f in $L^2(w_{\alpha,\beta})$ is defined by

$$f = \sum_{n=0}^{\infty} \widehat{f}_n^{\alpha,\beta} J_n^{\alpha,\beta}, \quad \text{where} \quad \widehat{f}_n^{\alpha,\beta} = \frac{\langle f, J_n^{\alpha,\beta} \rangle_{\alpha,\beta}}{h_n^{\alpha,\beta}}$$

and $h_n^{\alpha,\beta} := \langle J_n^{\alpha,\beta}, J_n^{\alpha,\beta} \rangle_{\alpha,\beta}$ is given by [16, (4.3.3)]

$$h_n^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}}{2n + \alpha + \beta + 1} \frac{\Gamma(n + \alpha + 1) \Gamma(n + \beta + 1) \Gamma(n + \alpha + \beta + 1)}{n! \Gamma(2n + \alpha + \beta + 1)^2}. \tag{2.3}$$

The n -th partial sum of this expansion, defined by

$$S_n^{\alpha,\beta} f(x) = \sum_{k=0}^n \widehat{f}_k^{\alpha,\beta} J_k^{\alpha,\beta}(x),$$

is the least square polynomial of degree n , that is, (1.1) holds. The operator $S_n^{\alpha,\beta}$ can be written as a linear integral operator.

For approximation in $L^p(w_{\alpha,\beta})$, $p \neq 2$, we can define a near best approximation operator as follows. We call η an admissible function if η is a C^∞ function on \mathbb{R}_+ satisfying $\eta(t) = 1$ for $0 \leq t \leq 1$ and $\eta(t) = 0$ for $t \geq 2$. For an admissible η , define

$$V_n^{\alpha,\beta} f := \sum_{k=0}^{2n} \eta\left(\frac{k}{n}\right) \widehat{f}_k^{\alpha,\beta} J_k^{\alpha,\beta}.$$

It is well known that $V_n^{\alpha,\beta}$ defines a bounded linear operator in $L^p(w_{\alpha,\beta})$ for $1 \leq p \leq \infty$ and it preserves polynomials up to degree n , that is, $V_n^{\alpha,\beta} f = f$ for $f \in \Pi_n$; consequently, the following theorem holds (see, for example, [17]).

Theorem 2.1 *Let $\alpha, \beta > -1$. For $f \in L^p(w_{\alpha,\beta})$ if $1 \leq p < \infty$, or $f \in C[-1, 1]$ if $p = \infty$,*

$$\|f - V_n^{\alpha,\beta} f\|_{L^p(w_{\alpha,\beta})} \leq c E_n(f)_{L^p(w_{\alpha,\beta})}, \quad 1 \leq p \leq \infty. \tag{2.4}$$

We will also need the Jackson type estimate for the error of best approximation. Let $\phi(x) := \sqrt{1 - x^2}$. The following theorem was established in [13].

Theorem 2.2 *Let $\alpha, \beta > -1$. For $f \in W_p^r(w_{\alpha,\beta})$ if $1 \leq p < \infty$, or $f \in C^r[-1, 1]$ if $p = \infty$,*

$$E_n(f)_{L^p(w_{\alpha,\beta})} \leq c n^{-r} \|\phi^r f^{(r)}\|_{L^p(w_{\alpha,\beta})}, \quad 1 \leq p \leq \infty. \tag{2.5}$$

In the rest of this section, we consider the approximation behavior of $S_n^{\alpha,\beta} f$ in $L^2(w_{\alpha,\beta})$ and in $W_2^r(w_{\alpha,\beta})$. Some of the results below are no doubt known but they provide contrast to our latter development and our proof is simple. We start with a lemma that is suggestive for our later study. Let ∂ denote the differential operator.

Lemma 2.3 *Let $\alpha, \beta > -1$. For $n = 1, 2, \dots$,*

$$\partial S_n^{\alpha,\beta} f = S_{n-1}^{\alpha+1,\beta+1}(\partial f). \tag{2.6}$$

Proof It is well-known that the Jacobi polynomials are eigenfunctions of a second order differential operator [16, (4.21.1)]

$$[w_{\alpha,\beta}(t)]^{-1} \frac{d}{dt} \left[(1 - t^2) w_{\alpha,\beta}(t) \right] \frac{d}{dt} J_n^{\alpha,\beta} = -\lambda_n J_n^{\alpha,\beta},$$

where $\lambda_n = n(n + \alpha + \beta + 1)$. Integrating by parts and applying this identity, we obtain by (2.2) that

$$\begin{aligned} \langle f', J_n^{\alpha+1, \beta+1} \rangle_{\alpha+1, \beta+1} &= \int_{-1}^1 f'(t) \left[w_{\alpha+1, \beta+1}(t) \frac{d}{dt} J_{n+1}^{\alpha, \beta}(t) \right] dt \\ &= - \int_{-1}^1 f(t) \frac{d}{dt} \left[w_{\alpha+1, \beta+1}(t) \frac{d}{dt} J_{n+1}^{\alpha, \beta}(t) \right] dt \\ &= \lambda_{n+1} \int_{-1}^1 f(t) J_{n+1}^{\alpha, \beta}(t) w_{\alpha, \beta}(t) dt = \lambda_{n+1} \langle f, J_{n+1}^{\alpha, \beta} \rangle_{\alpha, \beta}. \end{aligned}$$

Setting $f = J_{n+1}^{\alpha, \beta}(t)$, it follows readily from the definition that

$$\widehat{\partial f}_n^{\alpha+1, \beta+1} = \widehat{f}_{n+1}^{\alpha, \beta}, \quad n = 0, 1, 2, \dots \tag{2.7}$$

Consequently, by (2.2) again, we see that

$$\partial S_n^{\alpha, \beta} f(t) = \sum_{k=1}^n \widehat{f}_k^{\alpha, \beta} J_{k-1}^{\alpha+1, \beta+1}(t) = \sum_{k=0}^{n-1} \widehat{\partial f}_k^{\alpha+1, \beta+1} J_k^{\alpha+1, \beta+1}(t) = S_{n-1}^{\alpha+1, \beta+1} f(t).$$

This completes the proof. □

Theorem 2.4 *Let $\alpha, \beta > -1$ and $r \in \mathbb{N}$. For $f \in L^2(w_{\alpha, \beta})$ such that $f^{(r)} \in L^2(w_{\alpha+r, \beta+r})$,*

$$E_n(f)_{L^2(w_{\alpha, \beta})} \leq n^{-r} E_{n-r}(f^{(r)})_{L^2(w_{\alpha+r, \beta+r})}. \tag{2.8}$$

Proof By the Parseval identity, (2.7) and the formula for $h_k^{\alpha, \beta}$,

$$\begin{aligned} E_n(f)_{L^2(w_{\alpha, \beta})}^2 &= \|f - S_n^{\alpha, \beta} f\|_{L^2(w_{\alpha, \beta})}^2 = \sum_{k=n+1}^{\infty} \left| \widehat{f}_k^{\alpha, \beta} \right|^2 h_k^{\alpha, \beta} = \sum_{k=n}^{\infty} \left| \widehat{\partial f}_k^{\alpha+1, \beta+1} \right|^2 h_{k+1}^{\alpha, \beta} \\ &= \sum_{k=n}^{\infty} \left| \widehat{\partial f}_k^{\alpha+1, \beta+1} \right|^2 \frac{h_k^{\alpha+1, \beta+1}}{(k+1)(k+\alpha+\beta+2)} \\ &\leq n^{-2} E_{n-1}(f')_{L^2(w_{\alpha+1, \beta+1})}^2, \end{aligned}$$

where we have used the Parseval identity again at the last step. Iterating this inequality proves the stated result. □

The identity (2.6) allows us to derive error estimates for simultaneous approximation by $S_n^{\alpha, \beta} f$. For comparison with our later results, we formulate the following corollary.

Corollary 2.5 *Let $\alpha, \beta > -1$ and $r, s \in \mathbb{N}$. For $f \in L^2(w_{\alpha+r, \beta+r})$ such that $f^{(r)} \in L^2(w_{\alpha, \beta})$ and $n \geq r \geq s$,*

$$\left\| \phi^k \left(\partial^k f - \partial^k S_n^{\alpha, \beta} \right) \right\|_{L^2(w_{\alpha, \beta})} \leq cn^{-r+k} E_{n-r}(f^{(r)})_{L^2(w_{\alpha, \beta})}, \quad 0 \leq k \leq s. \tag{2.9}$$

Proof By (2.6), the left hand side of (2.9) can be rewritten as $E_{n-k}(f^{(k)})_{L^2(w_{\alpha+k,\beta+k})}$, which is bounded by, by (2.8), $n^{-r+k}E_{n-r}(f^{(r)})_{L^2(w_{\alpha+r-k,\beta+r-k})}$, in which we can remove $r - k$ since $w_{\alpha,\beta}(t) \leq 1$ if $\alpha, \beta \geq 0$ and $r - k \geq 0$. \square

It is possible to remove ϕ^k in the left hand side of (2.9) with the penalty of a higher power of n in the righthand side. This was first done in [2], see the proof in [3], for the Chebyshev and the Legendre cases with $r = 1$ and later extended to the Gegenbauer weight in [7], but with $\|f^{(r)}\|_{L^2(w_{\alpha,\beta})}$ in place of $E_{n-r}(f^{(r)})_{L^2(w_{\alpha,\beta})}$ in (2.10) below, which is weaker than (2.10) by (2.5). We give a complete proof for the Jacobi weight.

Theorem 2.6 *Let $\alpha, \beta > -1, r = 1, 2, \dots$, and $f \in W_2^r(w_{\alpha,\beta})$. Then*

$$\left\| \partial^k f - \partial^k S_n^{\alpha,\beta} \right\|_{L^2(w_{\alpha,\beta})} \leq c_{\alpha,\beta} n^{-r+2k-1/2} E_{n-r}(f^{(r)})_{L^2(w_{\alpha,\beta})}, \quad 0 \leq k \leq r, \tag{2.10}$$

where $c_{\alpha,\beta}$ is proportional to $1/\sqrt{\min\{\alpha, \beta\} + 1}$ when $k = 1$. Moreover, the estimate (2.10) is sharp.

Comparing with (1.3), the order of n in (2.10) is much weaker, which shows that the least polynomials for $L^2(w_{\alpha,\beta})$ is not suitable for simultaneous approximation.

The proof of this theorem depends on two lemmas. The first one is an identity on the Jacobi polynomials.

Lemma 2.7 *For $\alpha, \beta > -1$ and $n \in \mathbb{N}$,*

$$J_n^{\alpha+1,\beta+1}(t) = \sum_{j=0}^n C_{n,j}^{\alpha,\beta} J_j^{\alpha,\beta}(t), \quad C_{n,j}^{\alpha,\beta} := (-1)^{j+n} A_j^{\alpha,\beta} B_n^{\alpha,\beta} + A_j^{\beta,\alpha} B_n^{\beta,\alpha}, \tag{2.11}$$

where

$$A_j^{\alpha,\beta} := \frac{(\alpha + \beta + 2)2j}{(\alpha + 1)_j} \quad \text{and} \quad B_n^{\alpha,\beta} := \frac{(\alpha + 1)_{n+1}}{(\alpha + \beta + 2)_{2n+1}}.$$

Proof The following relations on the Jacobi polynomials are stated in [18],

$$\begin{aligned} J_n^{\alpha,\beta}(t) &= J_n^{\alpha+1,\beta}(t) - \tau_n^{\alpha,\beta} J_{n-1}^{\alpha+1,\beta}(t), \\ J_n^{\alpha,\beta}(t) &= J_n^{\alpha,\beta+1}(t) + \tau_n^{\beta,\alpha} J_{n-1}^{\alpha,\beta+1}(t), \end{aligned} \tag{2.12}$$

where $\tau_n^{\alpha,\beta} := (n + \beta)/(2n + \alpha + \beta)(2n + \alpha + \beta + 1)$. Iterating these identities, it is easy to see that

$$\begin{aligned} J_n^{\alpha+1,\beta}(t) &= \frac{(\beta + 1)_n}{(\alpha + \beta + 2)_{2n}} \sum_{k=0}^n \frac{(\alpha + \beta + 2)_{2k}}{(\beta + 1)_k} J_k^{\alpha,\beta}, \\ J_n^{\alpha,\beta+1}(t) &= \frac{(\alpha + 1)_n}{(\alpha + \beta + 2)_{2n}} \sum_{k=0}^n (-1)^{n-k} \frac{(\alpha + \beta + 2)_{2k}}{(\alpha + 1)_k} J_k^{\alpha,\beta}. \end{aligned}$$

Together, these two identities imply that

$$\begin{aligned}
 J_n^{\alpha+1,\beta+1} &= \frac{(\beta + 2)_n}{(\alpha + \beta + 3)_{2n}} \sum_{k=0}^n \frac{(\alpha + \beta + 3)_{2k}}{(\beta + 2)_k} J_k^{\alpha,\beta+1} \\
 &= \frac{(\beta + 2)_n}{(\alpha + \beta + 3)_{2n}} \sum_{j=0}^n (-1)^j \frac{(\alpha + \beta + 2)_{2j}}{(\alpha + 1)_j} J_j^{\alpha,\beta} \\
 &\quad \sum_{k=j}^n (-1)^k \frac{2k + \alpha + \beta}{\alpha + \beta + 2} \frac{(\alpha + 1)_k}{(\beta + 2)_k},
 \end{aligned}$$

where we have interchanged the order of summations. By induction on n , we can establish that

$$\begin{aligned}
 \sum_{k=j}^n (-1)^k \frac{2k + \alpha + \beta}{\alpha + \beta + 2} \frac{(\alpha + 1)_k}{(\beta + 2)_k} &= (-1)^n \frac{\alpha + 1}{\alpha + \beta + 2} \frac{(\alpha + 2)_n}{(\beta + 2)_n} \\
 &\quad + (-1)^j \frac{\beta + 1}{\alpha + \beta + 2} \frac{(\alpha + 1)_j}{(\beta + 1)_j}
 \end{aligned}$$

from which the stated result follows from a quick simplification. □

Remark 2.8 The connection coefficients that appear when writing $J_n^{\alpha,\beta}$ in terms of $J_n^{\gamma,\delta}$ are non-negative if $\alpha = \gamma$ and $\beta > \delta > -1$ or $\beta = \delta$ and $\alpha > \gamma > -1$, or $\alpha = \beta > \gamma = \delta > -1$. It is interesting to observe that the coefficients in (2.11) may not be all positive when $\alpha \neq \beta$. For example, it is easy to see that the coefficient for $j = 1$ and $n = 4$ is negative if $\alpha > \beta$.

Our main effort lies in establishing the identity (2.13) in the following lemma.

Lemma 2.9 *Let $\alpha, \beta > -1$. If $f \in W_2^1(w_{\alpha,\beta})$, then*

$$\begin{aligned}
 S_{n-1}^{\alpha,\beta}(f') - \partial S_n^{\alpha,\beta}(f) &= \widehat{\partial} f_n^{\alpha,\beta} \sum_{j=0}^{n-1} \left((-1)^{n+j} A_j^{\alpha,\beta} B_n^{\alpha,\beta} + A_j^{\beta,\alpha} B_n^{\beta,\alpha} \right) J_j^{\alpha,\beta} \\
 &\quad + \widehat{\partial} f_{n+1}^{\alpha,\beta} \sum_{j=0}^{n-1} \left((-1)^{n+j} A_j^{\alpha,\beta} B_n^{\alpha,\beta} D_n^{\alpha,b} - A_j^{\beta,\alpha} B_n^{\beta,\alpha} D_n^{\beta,\alpha} \right) J_j^{\alpha,\beta}, \tag{2.13}
 \end{aligned}$$

where $A_j^{\alpha,\beta}$ and $B_n^{\alpha,\beta}$ are defined in (2.11) and

$$D_j^{\alpha,\beta} = \frac{(j + \beta + 1)}{(2j + \alpha + \beta + 2)(2j + \alpha + \beta + 3)}.$$

Proof First we assume that $f \in W_2^r(w_{\alpha,\beta})$ for r sufficiently large. Since $f' \in L^2(w_{\alpha,\beta})$, its Fourier orthogonal expansion is

$$f' = \sum_{j=0}^{\infty} \widehat{\partial f}_j^{\alpha,\beta} J_j^{\alpha,\beta}.$$

Moreover, since $L^2(w_{\alpha,\beta}) \subset L^2(w_{\alpha+1,\beta+1})$, we can also write, by (2.11), that

$$f' = \sum_{n=0}^{\infty} \widehat{\partial f}_n^{\alpha+1,\beta+1} J_n^{\alpha+1,\beta+1} = \sum_{n=0}^{\infty} \widehat{f}_{n+1}^{\alpha,\beta} \sum_{j=0}^n C_{n,j}^{\alpha,\beta} J_j^{\alpha,\beta} = \sum_{j=0}^{\infty} \left(\sum_{n=j}^{\infty} C_{n,j}^{\alpha,\beta} \widehat{f}_{n+1}^{\alpha,\beta} \right) J_j^{\alpha,\beta}.$$

Comparing the two expansions of f' , we conclude, by (2.11), that

$$\widehat{\partial f}_j^{\alpha,\beta} = \sum_{n=j}^{\infty} \widehat{f}_{n+1}^{\alpha,\beta} C_{n,j}^{\alpha,\beta} = (-1)^j A_j^{\alpha,\beta} \Sigma_{1,j} + A_j^{\beta,\alpha} \Sigma_{2,j}, \tag{2.14}$$

where

$$\Sigma_{1,j} := \sum_{k=j}^{\infty} (-1)^k \widehat{f}_{k+1}^{\alpha,\beta} B_k^{\alpha,\beta} \quad \text{and} \quad \Sigma_{2,j} := \sum_{k=j}^{\infty} \widehat{f}_{k+1}^{\alpha,\beta} B_k^{\beta,\alpha}.$$

The last two series are absolutely convergent, since $|\widehat{\partial f}_{k+1}^{\alpha,\beta}| (h_{k+1}^{\alpha,\beta})^{\frac{1}{2}} \leq E_k(f)_{\alpha,\beta}$, which decays fast by (2.8), and $B_k^{\alpha,\beta} / (h_{k+1}^{\alpha,\beta})^{\frac{1}{2}} \leq c/k^{2\beta}$ and $B_k^{\alpha,\beta} / (h_{k+1}^{\alpha,\beta})^{\frac{1}{2}} \leq c/k^{2\alpha}$. Since it is easy to check that $A_{j+1}^{\alpha,\beta} B_j^{\beta,\alpha} = A_{j+1}^{\beta,\alpha} B_j^{\alpha,\beta}$, we also have

$$\widehat{\partial f}_{j+1}^{\alpha,\beta} = \sum_{k=j+1}^{\infty} \widehat{f}_{k+1}^{\alpha,\beta} C_{j+1,k}^{\alpha,\beta} = (-1)^{j+1} A_{j+1}^{\alpha,\beta} \Sigma_{1,j} + A_{j+1}^{\beta,\alpha} \Sigma_{2,j}, \tag{2.15}$$

where we have used $\Sigma_{1,j+1} = \Sigma_{1,j} - (-1)^j \widehat{f}_{j+1}^{\alpha,\beta} B_j^{\alpha,\beta}$ and $\Sigma_{2,j+1} = \Sigma_{2,j} - \widehat{f}_{j+1}^{\alpha,\beta} B_j^{\beta,\alpha}$. Solving (2.14) and (2.15) and simplifying, we obtain

$$\begin{aligned} \Sigma_{1,j} &= (-1)^j B_j^{\alpha,\beta} \left(\widehat{\partial f}_j^{\alpha,\beta} - D_j^{\alpha,\beta} \widehat{\partial f}_{j+1}^{\alpha,\beta} \right), \\ \Sigma_{2,j} &= B_j^{\beta,\alpha} \left(\widehat{\partial f}_j^{\alpha,\beta} + D_j^{\beta,\alpha} \widehat{\partial f}_{j+1}^{\alpha,\beta} \right), \end{aligned} \tag{2.16}$$

Now, by (2.2) and (2.11),

$$\partial S_n^{\alpha,\beta}(f) = \sum_{k=0}^{n-1} \widehat{f}_{k+1}^{\alpha,\beta} J_k^{\alpha+1,\beta+1} = \sum_{j=0}^{n-1} \sum_{k=j}^{n-1} \widehat{f}_{k+1}^{\alpha,\beta} C_{k,j}^{\alpha,\beta} J_j^{\alpha,\beta},$$

so that, by (2.14), we conclude that

$$\begin{aligned}
 S_{n-1}^{\alpha,\beta}(f') - \partial S_n^{\alpha,\beta}(f) &= \sum_{j=0}^{n-1} \left(\widehat{f}_j^{\alpha,\beta} - \sum_{k=j}^{n-1} \widehat{f}_{k+1}^{\alpha,\beta} C_{k,j}^{\alpha,\beta} \right) J_j^{\alpha,\beta} = \sum_{j=0}^{n-1} \sum_{k=n}^{\infty} \widehat{f}_{k+1}^{\alpha,\beta} C_{k,j}^{\alpha,\beta} J_j^{\alpha,\beta} \\
 &= \sum_{j=0}^{n-1} \left[(-1)^j A_j^{\alpha,\beta} \Sigma_{1,n} + A_j^{\beta,\alpha} \Sigma_{2,n}^{\beta,\alpha} \right] J_j^{\alpha,\beta}. \tag{2.17}
 \end{aligned}$$

Inserting the expressions for $\Sigma_{1,n}$ and $\Sigma_{2,n}$ in (2.15) completes the proof for smooth f . Since both sides of (2.13) is bounded for $f \in W_2^1(w_{\alpha,\beta})$, as shown in the proof of Theorem 2.6, the identity holds in $W_2^1(w_{\alpha,\beta})$ by the usual density argument. \square

Proof of Theorem 2.6 Assuming (2.13), we proceed with the proof. First we consider the case $k = 1$. Let $f \in W_2^r(w_{\alpha,\beta})$. By the triangle inequality,

$$\begin{aligned}
 &\| \partial f - \partial S_n^{\alpha,\beta} f \|_{L^2(w_{\alpha,\beta})} \leq \\
 &\| f' - S_{n-1}^{\alpha,\beta}(f') \|_{L^2(w_{\alpha,\beta})} + \| S_{n-1}^{\alpha,\beta}(f') - \partial S_n^{\alpha,\beta}(f) \|_{L^2(w_{\alpha,\beta})}.
 \end{aligned}$$

The first term in the right hand side is bounded by $E_{n-1}(f')_{\alpha,\beta}$, which is small than the desired bound. We now bound the second term. By (2.13)

$$\begin{aligned}
 &\| S_{n-1}^{\alpha,\beta}(f') - \partial S_n^{\alpha,\beta}(f) \|_{L^2(w_{\alpha,\beta})}^2 \\
 &= \left| \widehat{\partial f}_n^{\alpha,\beta} \right|^2 \sum_{j=0}^{n-1} \left| (-1)^{n+j} A_j^{\alpha,\beta} B_n^{\alpha,\beta} + A_j^{\beta,\alpha} B_n^{\beta,\alpha} \right|^2 h_j^{\alpha,\beta} \\
 &\quad + \left| \widehat{\partial f}_{n+1}^{\alpha,\beta} \right|^2 \sum_{j=0}^{n-1} \left| (-1)^{n+j} A_j^{\alpha,\beta} B_n^{\alpha,\beta} D_n^{\alpha,\beta} - A_j^{\beta,\alpha} B_n^{\beta,\alpha} D_n^{\beta,\alpha} \right|^2 h_j^{\alpha,\beta}.
 \end{aligned}$$

By the expression of $A_j^{\alpha,\beta}$ and $B_n^{\alpha,\beta}$ in (2.11), it is not difficult to verify that

$$\begin{aligned}
 &\frac{|B_n^{\alpha,\beta}|^2}{h_n^{\alpha,\beta}} \sum_{j=0}^{n-1} |A_j^{\alpha,\beta}|^2 h_j^{\alpha,\beta} = \frac{n(n+\alpha)(n+\alpha+1)^2}{(\beta+1)(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2} \sim \frac{n}{\beta+1} \\
 &\frac{|B_n^{\beta,\alpha}|^2}{h_n^{\alpha,\beta}} \sum_{j=0}^{n-1} |A_j^{\beta,\alpha}|^2 h_j^{\alpha,\beta} = \frac{n(n+\beta)(n+\beta+1)^2}{(\alpha+1)(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2} \sim \frac{n}{\alpha+1},
 \end{aligned}$$

and $h_n^{\alpha,\beta} |D_n^{\alpha,\beta}|^2 / h_{n+1}^{\alpha,\beta} \sim 1$. Consequently, we deduce that

$$\| S_{n-1}^{\alpha,\beta}(f') - \partial S_n^{\alpha,\beta}(f) \|_{L^2(w_{\alpha,\beta})}^2 \leq c_{\alpha,\beta} n \left(\left| \widehat{\partial f}_n^{\alpha,\beta} \right|^2 h_n^{\alpha,\beta} + \left| \widehat{\partial f}_{n+1}^{\alpha,\beta} \right|^2 h_{n+1}^{\alpha,\beta} \right)$$

$$\leq c_{\alpha,\beta} n [E_{n-1}(f')_{\alpha,\beta}]^2,$$

where the last step follows from the Parseval identity. This proves (2.10) for $k = 1$ and $r = 1$, which implies the case $k = 1$ and $r \geq 1$ by (2.8).

The case $k > 1$ follows inductively. Our main effort lies in proving the inequality

$$\left\| \partial^m \left(S_{n-1}^{\alpha,\beta}(f') - \partial S_n^{\alpha,\beta} f \right) \right\|_{L^2(w_{\alpha,\beta})} \leq c n^{2m+1} E_{n-1}(f')_{\alpha,\beta} \tag{2.18}$$

for $m = 1, 2, \dots$. Using (2.13) and $\partial^m J_j^{\alpha,\beta} = J_{j-m}^{\alpha+m,\beta+m}$, we see that the main ingredient is the estimate the sum

$$\begin{aligned} \left\| \sum_{j=m}^{n-1} A_j^{\alpha,\beta} \left| J_{j-m}^{\alpha+m,\alpha+m} \right| \right\|_{L^2(w_{\alpha,\beta})}^2 &\leq n \sum_{j=m}^{n-1} \left| A_j^{\alpha,\beta} \right|^2 h_{j-m}^{\alpha+m,\beta+m} \\ &\leq c n \sum_{j=m}^{n-1} \left| A_j^{\alpha,\beta} \right|^2 h_{j-m}^{\alpha,\beta} j^{2m-1} \leq c n^{4m+2\beta+2}. \end{aligned}$$

where the first inequality follows from the Cauchy–Schwartz inequality and the second one follows from

$$h_n^{\alpha+m,\beta+m} / h_n^{\alpha,\beta} \sim n^{2m-1}, \tag{2.19}$$

which can be easily verified by (2.3) and the asymptotic of the Gamma function, and the third one follows from a straightforward estimate. Consequently, it follows readily that

$$\frac{|B_n^{\alpha,\beta}|^2}{h_n^{\alpha,\beta}} \left\| \sum_{j=m}^{n-1} A_j^{\alpha,\beta} \left| J_{j-m}^{\alpha+m,\alpha+m} \right| \right\|_{L^2(w_{\alpha,\beta})}^2 \leq c n^{2m+1}$$

and the similar estimate holds when α and β are exchanged in $A_j^{\alpha,\beta}$ and $B_n^{\alpha,\beta}$. These estimates allow us to estimate, by (2.13), that

$$\left\| \partial^m \left(S_{n-1}^{\alpha,\beta}(f') - \partial S_n^{\alpha,\beta} f \right) \right\|_{L^2(w_{\alpha,\beta})} \leq c n^{2m+1} \left(\left| \widehat{\partial f}_n^{\alpha,\beta} \right|^2 h_n^{\alpha,\beta} + \left| \widehat{\partial f}_{n+1}^{\alpha,\beta} \right|^2 h_{n+1}^{\alpha,\beta} \right),$$

from which (2.18) follows readily.

Assume now (2.10) has been established for a fixed k , we prove that it also holds for $k + 1$. By the triangle inequality,

$$\begin{aligned} &\left\| \partial^{k+1} f - \partial^{k+1} S_n^{\alpha,\beta} f \right\|_{L^2(w_{\alpha,\beta})} \\ &\leq \left\| \partial^k f' - \partial^k S_{n-1}^{\alpha,\beta}(f') \right\|_{L^2(w_{\alpha,\beta})} + \left\| \partial^k \left[S_{n-1}^{\alpha,\beta}(f') - \partial S_n^{\alpha,\beta}(f) \right] \right\|_{L^2(w_{\alpha,\beta})}. \end{aligned}$$

The second term is the right hand side can be bounded, by applying (2.18) with $m = k$ and (2.8), by $c n^{2k+1/2} E_{n-1}(f')_{\alpha,\beta} \leq c n^{2k+1/2} n^{-r+1} E_{n-r}(f^{(r)})_{\alpha,\beta}$, in which the power of n can be written as $-r + 2(k + 1) - 1/2$, which agrees with that in (2.10) for $k + 1$, whereas the first term in the right hand side can be bounded, by induction hypothesis with r replaced by $r - 1$, by a bound that is less than the above bound. This completes the proof of (2.10) for $k = 1$ and the proof.

To show that the order is sharp, we consider $g(t) = \widehat{J}_{n+1}^{\alpha-k,\beta-k}(t)$, which is well defined for n large even if $\alpha < 0$ or $\beta < 0$. Then $g^{(k)}(t) = \widehat{J}_{n+1-k}^{\alpha,\beta}(t)$. Since the orthogonal expansion of $\widehat{J}_{n+1-k}^{\alpha,\beta}$ is itself, $E_{n-k}(g^{(k)})_{\alpha,\beta} = \|\widehat{J}_{n+1-k}^{\alpha,\beta}\|_{\alpha,\beta}$. Furthermore, by (2.12), $J_{n+1-k}^{\alpha,\beta} - J_{n+1-k}^{\alpha+k,\beta+k}$ is a polynomial of degree $n - k$, so that

$$\partial^k S_n^{\alpha,\beta} g(t) = S_{n-k}^{\alpha+k,\beta+k} (g^{(k)}) = S_{n-k}^{\alpha+k,\beta+k} (\widehat{J}_{n+1-k}^{\alpha,\beta}) = \widehat{J}_{n+1-k}^{\alpha,\beta} - \widehat{J}_{n+1-k}^{\alpha+k,\beta+k}.$$

Consequently, $\partial^k g - \partial^k S_n^{\alpha,\beta} g = \widehat{J}_{n+1-k}^{\alpha+k,\beta+k}$. It then follows from (2.19) that (2.10) is sharp for $k = r$. □

3 Sobolev Orthogonal Polynomials and Orthogonal Expansions

As mentioned in the Sect. 1, for approximation in the Sobolev space $W_p^s(w_{\alpha,\beta})$, we need to work with the Jacobi polynomials with parameters α, β being negative integers. Setting α, β as negative integers in (2.1) leads to a reduction of polynomial degrees, which causes problems when one considers orthogonal expansions. There have been several ways of remedying the definition of the Jacobi polynomials in the literature; see, for example, [1, 4, 8, 11] and the references therein. Motivating by the study in [18], which will be explained in the end of this section, we give another definition that can be regarded as either avoiding delicate extensions of the Jacobi polynomials to negative integers or as an alternative definition that holds for all negative indices.

Definition 3.1 Let $\alpha, \beta > -1$ and $s \in \mathbb{N}$. For $\theta \in [-1, 1]$ and $n \in \mathbb{N}_0$, define

$$J_n^{\alpha-s,\beta-s}(x) = J_{n,\theta}^{\alpha-s,\beta-s}(x) := \begin{cases} \frac{(x - \theta)^n}{n!}, & 0 \leq n \leq s - 1, \\ \int_{\theta}^x \frac{(x - t)^{s-1}}{(s - 1)!} J_{n-s}^{\alpha,\beta}(t) dt, & n \geq s. \end{cases} \tag{3.1}$$

It is evident that $J_n^{\alpha-s,\beta-s}$ is a polynomial of degree n . Furthermore, these polynomials evidently satisfy the following properties:

$$\partial^s J_n^{\alpha-s,\beta-s}(x) = J_{n-s}^{\alpha,\beta}(x), \quad n \geq s; \tag{3.2}$$

$$\partial^k J_n^{\alpha-s,\beta-s}(\theta) = \begin{cases} \delta_{k,n}, & n \leq s - 1, \\ 0, & n \geq s, \end{cases} \quad 0 \leq k \leq s - 1, \tag{3.3}$$

where ∂^k denotes the k -th derivative. Comparing with (2.2), the identity (3.2) suggests that these polynomials can be regarded as an extension of the Jacobi polynomials with negative parameters when $\alpha - s \leq -1$ and/or $\beta - s \leq -1$. If $\alpha - s > -1$ and $\beta - s > -1$, then both $J_n^{\alpha-s, \beta-s}$ and $\mathcal{J}_n^{\alpha-s, \beta-s}$ satisfy (3.2), but $J_n^{\alpha-s, \beta-s}$ does not satisfy (3.3). These polynomials are orthogonal with respect to the inner product

$$\langle f, g \rangle_{\alpha, \beta}^{-s} := \int_{-1}^1 f^{(s)}(t)g^{(s)}(t)w_{\alpha, \beta}(t)dt + \sum_{k=0}^{s-1} \lambda_k f^{(k)}(\theta)g^{(k)}(\theta),$$

where λ_k are positive constants.

Theorem 3.2 For $\alpha, \beta > -1$ and $s \in \mathbb{N}$. The polynomial $\mathcal{J}_{n, \theta}^{\alpha-s, \beta-s}$ is orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_{\alpha, \beta}^{-s}$ and its norm square, $\mathfrak{h}_n^{\alpha-s, \beta-s} := \langle \mathcal{J}_n^{\alpha-s, \beta-s}, \mathcal{J}_n^{\alpha-s, \beta-s} \rangle_{\alpha, \beta}^{-s}$, satisfies

$$\mathfrak{h}_n^{\alpha-s, \beta-s} = \lambda_n, \quad 0 \leq n \leq s - 1, \quad \text{and} \quad \mathfrak{h}_n^{\alpha-s, \beta-s} = h_{n-s}^{\alpha, \beta}, \quad n \geq s.$$

Proof Let $m \leq n$. We consider the orthogonality of $\mathcal{J}_n^{\alpha-s, \beta-s}$ and $\mathcal{J}_m^{\alpha-s, \beta-s}$. If $n \leq s - 1$ then, by (3.2) and (3.3),

$$\langle \mathcal{J}_n^{\alpha-s, \beta-s}, \mathcal{J}_m^{\alpha-s, \beta-s} \rangle_{\alpha, \beta}^{-s} = \sum_{k=0}^{s-1} \lambda_k \partial^k \mathcal{J}_n^{\alpha-s, \beta-s}(\theta) \partial^k \mathcal{J}_m^{\alpha-s, \beta-s}(\theta) = \lambda_n \delta_{n,m}.$$

Whereas if $n \geq s$, then, by (3.2) and (3.3),

$$\langle \mathcal{J}_n^{\alpha-s, \beta-s}, \mathcal{J}_m^{\alpha-s, \beta-s} \rangle_{\alpha, \beta}^{-s} = \int_{-1}^1 J_{n-s}^{\alpha, \beta}(x) J_{m-s}^{\alpha, \beta}(x) w_{\alpha, \beta}(x) dx = \delta_{n,m} h_{n-s}^{\alpha, \beta}$$

by the orthogonality of the Jacobi polynomials. □

As we mentioned before, the inner product $\langle \cdot, \cdot \rangle_{\alpha, \beta}$ and its associated orthogonal polynomials have been studied in the literature (see, [12] and its references). Instead of starting with an extension of the Jacobi polynomials to parameters being negative integers and constructing orthogonal polynomials accordingly, our construction is more direct with a strikingly, in comparison, simple proof and works for all real parameters.

For $f \in \mathcal{W}_p^s(w_{\alpha, \beta})$, we can study the Fourier orthogonal expansion of f with respect to the orthogonal system $\mathcal{J}_{n, \theta}^{\alpha-s, \beta-s}$,

$$f = \sum_{n=0}^{\infty} \widehat{\mathfrak{h}}_n^{\alpha-s, \beta-s} \mathcal{J}_{n, \theta}^{\alpha-s, \beta-s} \quad \text{with} \quad \widehat{\mathfrak{h}}_n^{\alpha-s, \beta-s} = \widehat{\mathfrak{h}}_{n, \theta}^{\alpha-s, \beta-s} := \frac{\langle f, \mathcal{J}_{n, \theta}^{\alpha-s, \beta-s} \rangle_{\alpha, \beta}^{-s}}{\mathfrak{h}_n^{\alpha-s, \beta-s}}.$$

The n -th partial sum of this expansion is defined by

$$S_n^{\alpha-s, \beta-s} f = S_{n, \theta}^{\alpha-s, \beta-s} f := \sum_{k=0}^n \widehat{J}_k^{\alpha-s, \beta-s} \mathcal{J}_{k, \theta}^{\alpha-s, \beta-s}.$$

For $s = 0$, the operator $S_n^{\alpha, \beta} f = S_n^{\alpha, \beta} f$ is the partial sum of the usual Jacobi expansion. This operator satisfies several simple properties and can be written, in particular, in terms of the partial sum $S_{n-s}^{\alpha, \beta} f$, where we define $S_n^{\alpha, \beta} f = 0$ if $n < 0$.

Lemma 3.3 *Let $\alpha, \beta > -1$ and $s \in \mathbb{N}$. For $f \in W_p^s(w_{\alpha, \beta})$,*

- (1) $\widehat{J}_n^{\alpha-s, \beta-s} = f^{(n)}(\theta)$ if $0 \leq n \leq s - 1$, and $\widehat{J}_n^{\alpha-s, \beta-s} = \widehat{\partial^s f}_{n-s}^{\alpha, \beta}$ if $n \geq s$;
- (2) $\partial^s S_n^{\alpha-s, \beta-s} f = S_{n-s}^{\alpha, \beta} f^{(s)}$ if $n \geq s$;
- (3) For $n = 0, 1, \dots$,

$$S_n^{\alpha-s, \beta-s} f(x) = \sum_{k=0}^{\min\{n, s-1\}} f^{(k)}(\theta) \frac{(x - \theta)^k}{k!} + \int_{\theta}^x \frac{(x - t)^{s-1}}{(s - 1)!} S_{n-s}^{\alpha, \beta} f^{(s)}(t) dt.$$

Proof As in the proof of the Theorem 3.2, it is easy to see that, if $n \leq s - 1$, then $\langle f, \mathcal{J}_n^{\alpha-s, \beta-s} \rangle_{\alpha, \beta}^{-s} = \lambda_n f^{(n)}(\theta)$ and $h_n^{\alpha-s, \beta-s} = \lambda_n$, whereas if $n \geq s$ then $\langle f^{(s)}, \mathcal{J}_{n-s}^{\alpha, \beta} \rangle_{\alpha, \beta}$ and $h_n^{\alpha-s, \beta-s} = h_{n-s}^{\alpha, \beta}$ for $n \geq s$, from which (1) for $n \geq s$ follows readily. The case for $n \leq s - 1$ follows similarly. By the definition of the partial sum operator, we then obtain, for $n \geq s$,

$$\begin{aligned} S_n^{\alpha-s, \beta-s} f(x) &= \sum_{k=0}^{s-1} f^{(k)}(\theta) \mathcal{J}_k^{\alpha-s, \beta-s}(x) + \sum_{k=s}^n \widehat{\partial^s f}_k^{\alpha, \beta} \mathcal{J}_k^{\alpha-s, \beta-s}(x) \\ &= \sum_{k=0}^{s-1} f^{(k)}(\theta) \frac{(x - \theta)^k}{k!} + \sum_{k=s}^n \widehat{\partial^s f}_k^{\alpha, \beta} \int_{\theta}^x \frac{(x - t)^{s-1}}{(s - 1)!} J_{k-s}^{\alpha, \beta}(t) dt, \end{aligned}$$

which is, after changing the order of the sum and the integral, exactly the right hand side of (3). Finally, taking s -th derivative of the identity in (3) proves (2). □

In [11], we extended the Jacobi polynomials to allow the parameters to be negative integers. The extended polynomial, again denoted by $J_n^{\alpha, \beta}$, satisfies (2.2) for all $\alpha, \beta \in \mathbb{R}$, $n \in \mathbb{N}$, and its leading coefficient is $x^n/n!$, that is, $J_n^{\alpha, \beta}(x) = x^n/n! + \dots$ Based on this extension of the Jacobi polynomials, we proved in [18] that the polynomials

$$\widehat{J}_n^{\alpha-s, \beta-s}(t) = \begin{cases} J_n^{\alpha-s, \beta-s}(t), & n \geq s, \\ J_n^{\alpha-s, \beta-s}(t) - \sum_{k=0}^{s-1} J_{n-k}^{\alpha-s+k, \beta-s+k}(t) \frac{(t - 1)^k}{k!}, & n < s \end{cases}$$

are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_{0,\beta}^{-s}$ with $\theta = 1$. Below we prove that $\widehat{J}_n^{-s,\beta-s}(x) \equiv \mathcal{J}_n^{-s,\beta-s}(x)$ for $n \geq 0$. Indeed, if $n < s$, we use (2.2) and the leading coefficient of $J_n^{\alpha,\beta}$ to conclude, by the Taylor expansion, that

$$\widehat{J}_n^{-s,\beta-s}(x) = J_n^{-s,\beta-s}(x) - \sum_{k=0}^{n-1} \partial^k J_n^{-s,\beta-s}(1) \frac{(x-1)^k}{k!} = \frac{(x-1)^n}{n!} = \mathcal{J}_n^{-s,\beta-s}(x),$$

whereas if $n \geq s$, the equivalence follows from the identity [18, (2.9)]

$$J_n^{-s,\beta-s}(x) = \frac{(-1)^s (n-s)!}{n!} (1-x)^s J_{n-s}^{0,\beta}(x), \quad n \geq s,$$

and the integral form of the Taylor reminder formula. In fact, it is this equivalence that motivated our definition of $\mathcal{J}_n^{\alpha-s,\beta-s}$ for all $\alpha, \beta \in \mathbb{R}$.

4 Polynomial Approximation in Sobolev Spaces

We shall show that $\mathcal{S}_{n,\theta}^{\alpha-s,\beta-s} f$ approximates f with the least error, up to a multiple constant, in the $W_2^s(w_{\alpha,\beta})$ norm. We also define a near best approximation polynomial that will play the role of $\mathcal{S}_{n,\theta}^{\alpha,\beta} f$ when $p \neq 2$. Fix θ , we define $\mathcal{V}_n^{\alpha-s,\beta-s} = \mathcal{V}_{n,\theta}^{\alpha-s,\beta-s}$ for $s = 1, 2, \dots$ by

$$\mathcal{V}_{n,\theta}^{\alpha-s,\beta-s} f(x) := \sum_{k=0}^{s-1} f^{(k)}(\theta) \frac{(x-\theta)^k}{k!} + \int_{\theta}^x \frac{(x-t)^{s-1}}{(s-1)!} V_n^{\alpha,\beta} f^{(s)}(t) dt. \quad (4.1)$$

It is easy to see that $\mathcal{V}_{n,\theta}^{\alpha-s,\beta-s} f$ is a polynomial of degree $2n + s$ and it preserves polynomials of degree $\leq n$. Furthermore, it satisfies

$$\partial^s \mathcal{V}_{n,\theta}^{\alpha-s,\beta-s} f(x) = V_n^{\alpha,\beta} f^{(s)}(x). \quad (4.2)$$

To study the approximation property of $\mathcal{S}_{n,\theta}^{\alpha,\beta} f$ or $\mathcal{V}_{n,\theta}^{\alpha-s,\beta-s} f$, we will need the weighted Hardy inequality with respect to the Jacobi weight:

Lemma 4.1 For $\alpha, \beta > -1$ and $f \in L^p(w_{\alpha,\beta})$, $1 < p < \infty$,

$$\left(\int_{-1}^1 \left(\int_{-1}^x |f(t)| dt \right)^p w_{\alpha,\beta}(x) dx \right)^{1/p} \leq c \left(\int_{-1}^1 |f(x)|^p w_{\alpha,\beta}(x) dx \right)^{1/p}$$

if and only if $\beta < p - 1$.

Proof With $w_{\alpha,\beta}$ replaced by a general weight function w , it is known (see, for example, [14]) that the inequality holds if and only if

$$\sup_{-1 < x < 1} \left(\int_x^1 w(t) dt \right)^{1/p} \left(\int_{-1}^x w(t)^{1-q} dt \right)^{1/q} < \infty,$$

where $q = p/(p - 1)$. If $-1 < x \leq 0$, then the first integral is finite and the second integral is finite if and only if, since $1 - t \sim 1, \beta(1 - q) > -1$, or $\beta < 1/(q - 1) = p - 1$. If $0 \leq x < 1$, then the first integral is bounded by $c(1 - x)^{(\alpha+1)/p}$, whereas the second integral is bounded by, since $1 + t \sim 1, (1 - x)^{(\alpha(1-q)+1)/q}$, it is easy to verify that their product is finite. \square

Theorem 4.2 *Let $\alpha, \beta > -1$ and $s \in \mathbb{N}$. For $f \in W_p^s(w_{\alpha,\beta})$ if $1 \leq p < \infty$, or $f \in C^s[-1, 1]$ if $p = \infty$, the estimate*

$$\|f - \mathcal{V}_{n,\theta}^{\alpha-s,\beta-s} f\|_{W_p^s(w_{\alpha,\beta})} \leq c E_n(f^{(s)})_{L^p(w_{\alpha,\beta})} \tag{4.3}$$

holds under either one of the following assumptions:

- (1) $\theta \in (-1, 1)$;
- (2) $\theta = -1, \beta < p - 1$ if $1 < p \leq \infty$ and $\beta \leq 0$ if $p = 1$;
- (3) $\theta = 1, \alpha < p - 1$ if $1 < p \leq \infty$ and $\alpha \leq 0$ if $p = 1$.

Proof By the definition of the $\|\cdot\|_{W_p^s(w_{\alpha,\beta})}$ norm, we need to show that

$$\|\partial^k f - \partial^k \mathcal{V}_{n,\theta}^{\alpha-s,\beta-s} f\|_{L^p(w_{\alpha,\beta})} \leq c E_n(f^{(s)})_{L^p(w_{\alpha,\beta})}, \quad 0 \leq k \leq s. \tag{4.4}$$

Since $\partial^s f - \partial^s \mathcal{V}_{n,\theta}^{\alpha-s,\beta-s} f = f^{(s)} - V_n^{\alpha,\beta} f^{(s)}$, the estimate (4.4) when $k = s$ follows immediately. By the Taylor reminder formula, we can write

$$f(x) = \sum_{k=0}^{s-1} f^{(k)}(\theta) \frac{(x - \theta)^k}{k!} + \int_{\theta}^x \frac{(x - t)^{s-1}}{(s - 1)!} f^{(s)}(t) dt,$$

so that

$$f(x) - \mathcal{V}_{n,\theta}^{\alpha-s,\beta-s} f(x) = \int_{\theta}^x \frac{(t - \theta)^{s-1}}{(s - 1)!} \left[f^{(s)}(t) - V_n^{\alpha,\beta} f^{(s)}(t) \right] dt$$

Taking k -th derivatives, $1 \leq k \leq s - 1$, only reduces the power of $(t - \theta)^{s-1}$ by k . Since we need to consider $k = 0, 1, \dots, s - 1$, we can ignore the factor $(t - \theta)^{s-k-1}$. For $p = \infty$, the estimate (4.4) follows trivially from the above identity.

For $p > 1$, applying the Hölder inequality, we obtain

$$\left\| f - \mathcal{V}_{n,\theta}^{\alpha-s,\beta-s} f \right\|_{L^p(w_{\alpha,\beta})}^p$$

$$\begin{aligned} &\leq \int_{-1}^1 \left| \int_{\theta}^x |f^{(s)}(t) - V_n^{\alpha,\beta} f^{(s)}(t)| dt \right|^p w_{\alpha,\beta}(x) dx \\ &\leq \int_{-1}^1 \left| \int_{\theta}^x |f(t) - V_n^{\alpha,\beta} f(t)|^p w_{\alpha,\beta}(t) dt \right| \cdot \left| \int_{\theta}^x w_{\alpha,\beta}(t)^{-q/p} dt \right|^{p/q} w_{\alpha,\beta}(x) dx \\ &\leq \|f^{(s)} - V_n^{\alpha,\beta} f^{(s)}\|_{L^p(w_{\alpha,\beta})}^p \int_{-1}^1 \left| \int_{\theta}^x w_{\alpha,\beta}(t)^{-q/p} dt \right|^{p/q} w_{\alpha,\beta}(x) dx. \end{aligned}$$

If $\theta \in (-1, 1)$, it is easy to see, since $w_{\alpha,\beta}$ has no singularity at θ , that the last integral is bounded by a constant for all $p > 1$. If $\theta = -1$, the inequality follows from Lemma 4.1 and the case $\theta = 1$ follows likewise from the dual Hardy inequality [14]. A direct proof can also be easily carried out by dividing the integral over $[-1, 1]$ into two integrals over $[-1, 0]$ and $[0, 1]$, respectively, so that

$$\begin{aligned} \int_{-1}^1 \left| \int_{-1}^x w_{\alpha,\beta}(t)^{-q/p} dt \right|^{p/q} w_{\alpha,\beta}(x) dx &\leq c \int_{-1}^0 \left| \int_{-1}^x (1+t)^{-\beta q/p} dt \right|^{p/q} w_{\alpha,\beta}(x) dx \\ &\quad + \int_0^1 \left| \int_{-1}^x w_{\alpha,\beta}(t)^{-q/p} dt \right|^{p/q} w_{\alpha,\beta}(x) dx; \end{aligned}$$

the first term in the right hand side is bounded if $-\beta q/p > -1$ or $\beta < p/q = p - 1$, whereas the second term can be seen to be bounded, after splitting the inner integral as a sum of two, one over $[-1, 0]$ and the other over $[0, x]$. The case $\theta = 1$ works similarly.

For $p = 1$, we divide the integral into two terms and exchanging the orders of the integrals in each term,

$$\begin{aligned} &\|f - \mathcal{V}_{n,\theta}^{\alpha-s,\beta-s} f\|_{L^1(w_{\alpha,\beta})} \\ &\leq \left(\int_{-1}^{\theta} \int_x^{\theta} + \int_{\theta}^1 \int_{\theta}^x \right) |f^{(s)}(t) - V_n^{\alpha,\beta} f^{(s)}(t)| dt w_{\alpha,\beta}(x) dx \\ &= \int_{-1}^{\theta} |f^{(s)}(t) - V_n^{\alpha,\beta} f^{(s)}(t)| \int_{-1}^t w_{\alpha,\beta}(x) dx dt \\ &\quad + \int_{\theta}^1 |f^{(s)}(t) - V_n^{\alpha,\beta} f^{(s)}(t)| \int_t^1 w_{\alpha,\beta}(x) dx dt. \end{aligned}$$

For $\theta \in (-1, 1)$, $w_{\alpha,\beta}(x) \sim (1-x)^\alpha$ in the first term in the right hand side, which implies that $\int_{-1}^t w_{\alpha,\beta}(x) dx \leq cw_{\alpha,\beta}(t)$, whereas $w_{\alpha,\beta}(x) \sim (1+x)^\beta$ in the second term, which implies that $\int_t^1 w_{\alpha,\beta}(x) dx \leq cw_{\alpha,\beta}(t)$. This proves the case for $\theta \in (-1, 1)$. For $\theta = -1$, there is only the second term, for which we use $(1+x)^\beta \leq (1+t)^\beta$ for $\beta \leq 0$ to conclude that $\int_t^1 w_{\alpha,\beta}(x) dx \leq cw_{\alpha,\beta}(t)$ for $\beta \leq 0$. The case $\theta = 1$ is proved similarly. The proof is completed. \square

Evidently, Theorem 1.1 follows from this theorem with any $\theta \in (0, 1)$. In the case of $p = 2$, we can replace $\mathcal{V}_{n,\theta}^{\alpha-s,\beta-s} f$ by the partial sum operator $\mathcal{S}_n^{\alpha-s,\beta-s} f$.

Corollary 4.3 *Let $\alpha, \beta > -1, s \in \mathbb{N}$. For $f \in W_2^s(w_{\alpha,\beta})$, the estimate*

$$\|f - \mathcal{S}_{n,\theta}^{\alpha-s,\beta-s} f\|_{W_2^s(w_{\alpha,\beta})} \leq c E_{n-s}(f^{(s)})_{L^2(w_{\alpha,\beta})} \tag{4.5}$$

holds if (a) $\theta \in (-1, 1)$, or (b) $\theta = -1$ and $\beta < 1$, or (c) $\theta = 1$ and $\alpha < 1$.

In particular, if $f \in W_2^r(w_{\alpha,\beta})$ for $r \geq s$, then we have

$$\|f - \mathcal{S}_n^{\alpha-s,\beta-s} f\|_{W_2^s(w_{\alpha,\beta})} \leq c n^{-r+s} E_{n-r}(f^{(r)})_{L^2(w_{\alpha+r-s,\beta+r-s})}$$

and the righthand is also bounded by $c n^{-r+s} \|\phi^{r-s} f^{(r)}\|_{L^p(w_{\alpha,\beta})}$ by Theorem 2.2.

5 Simultaneous Approximation

We now consider the simultaneous approximation by polynomials. Our goal is to establish the estimate of the type

$$\|\partial^k f - \partial^k \mathcal{V}_{n,\theta}^{\alpha-s,\beta-s} f\|_{L^p(w_{\alpha,\beta})} \leq c n^{-s+k} E_n(f^{(s)})_{L^p(w_{\alpha,\beta})}, \quad 0 \leq k \leq s.$$

Our result in the previous section shows that this estimate holds for $k = s$. We apply a technique called Aubin-Nitsche duality (see [3, p. 321] and [2]) to handle the case $0 \leq k \leq s - 1$. For this, we need to work with either $\theta = -1$ or $\theta = 1$. We mainly work with $\theta = -1$, since the other case is similar.

Let g be a measurable function on $[-1, 1]$. For $0 \leq k \leq s - 1$, we define a function

$$u_{g,k}(x) := \int_{-1}^x \frac{(x-t)^{s-1}}{(s-1)!} \frac{1}{w_{\alpha,\beta}(t)} \int_t^1 \frac{(y-t)^{s-k-1}}{(s-k-1)!} g(y) w_{\alpha,\beta}(y) dy dt. \tag{5.1}$$

It is straightforward to verify that $u_{g,k}$ is the solution of the following boundary value problem of a $(2s - k)$ -th order differential equation:

$$\begin{aligned} (-1)^{s-k} [w_{\alpha,\beta}(x)]^{-1} \frac{d^{s-k}}{dx^{s-k}} (w_{\alpha,\beta}(x) u^{(s)}(x)) &= g(x), \quad -1 \leq x \leq 1, \\ u^{(j)}(-1) &= 0, \quad 0 \leq j \leq s - 1, \\ \lim_{x \rightarrow 1} \frac{d^j}{dx^j} (w_{\alpha,\beta}(x) u^{(s)}(x)) &= 0, \quad 0 \leq j \leq s - k - 1. \end{aligned} \tag{5.2}$$

Lemma 5.1 *If $v \in W_p^s$ satisfies $v^{(j)}(-1) = 0$ for $0 \leq j \leq s - k - 1$, then*

$$\int_{-1}^1 u_{g,k}^{(s)}(x) v^{(s-k)}(x) w_{\alpha,\beta}(x) dx = \int_{-1}^1 g(x) v(x) w_{\alpha,\beta}(x) dx.$$

Proof By the differential equation in (5.2),

$$\int_{-1}^1 g(x) v(x) w_{\alpha,\beta}(x) dx = (-1)^{s-k} \int_{-1}^1 \frac{d^{s-k}}{dx^{s-k}} (w_{\alpha,\beta}(x) u_{g,k}^{(s)}(x)) v(x) dx.$$

With the help of the boundary conditions of $u_{g,k}^{(s)}$ at $x = 1$ and v at -1 , successive integration by parts proves the stated identity. \square

Throughout the rest of the paper, we let p and q be related by $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 5.2 *Let $\alpha > -1$ and $g \in L^q(w_{\alpha,\beta})$, $1 \leq q \leq \infty$. Then, for $1 \leq q \leq \infty$,*

$$\|\phi^{s-k} u_{g,k}^{(2s-k)}\|_{L^q(w_{\alpha,\beta})} \leq c \|g\|_{L^q(w_{\alpha,\beta})}, \quad 0 \leq k \leq s - 1, \tag{5.3}$$

provided (a) $\beta = 0$ or (b) $s = 1$, $\beta < p/2 - 1$ if $1 \leq q < \infty$ and $\beta \leq -1/2$ if $q = \infty$.

Proof By (5.1), taking s -th derivative gives

$$u_{g,k}^{(s)}(x) := \frac{1}{w_{\alpha,\beta}(x)} \int_x^1 \frac{(y-x)^{s-k-1}}{(s-k-1)!} g(y) w_{\alpha,\beta}(y) dy.$$

Taking another $(s - k)$ -th order derivatives by the product rule, we conclude that

$$u_{g,k}^{(2s-k)}(x) = -g(x) + \sum_{j=1}^{s-k} \binom{s-k}{j} \frac{d^j}{dx^j} \left(\frac{1}{w_{\alpha,\beta}(x)} \right) \int_x^1 \frac{(y-x)^{j-1}}{(j-1)!} g(y) w_{\alpha,\beta}(y) dy.$$

If $\beta = 0$, then $\left| \frac{d^j}{dx^j} \left(\frac{1}{w_{\alpha,0}(x)} \right) \right| (1-x)^{j-1} = |(-\alpha)_j| (1-x)^{-\alpha-1}$. Hence, by $(y-x)^{j-1} \leq (1-x)^{j-1}$ for $x \leq y \leq 1$, it follows that

$$\left| u_{g,k}^{(2s-k)}(x) \right| \leq |g(x)| + c \sum_{j=1}^{s-k} \frac{1}{(1-x)^{\alpha+1}} \int_x^1 |g(y)| w_{\alpha,0}(y) dy.$$

If $q = \infty$, it follows immediately that $\|u_{g,k}^{(2s-k)}\|_{\infty} \leq c \|g\|_{\infty}$. For $1 < q < \infty$, by the Hölder inequality and $\phi(x)^{s-k} \leq c(1-x)^{1/2}$ as $0 \leq k \leq s - 1$,

$$\begin{aligned} & \int_{-1}^1 \left| \frac{\phi(x)^{s-k}}{(1-x)^{\alpha+1}} \int_x^1 g(y) w_{\alpha,0}(y) dy \right|^q w_{\alpha,0}(x) dx \\ & \leq c \int_{-1}^1 \int_x^1 |g(y)|^q w_{\alpha,0}(y) dy \left| \int_x^1 w_{\alpha,0}(t) dt \right|^{q/p} \frac{w_{\alpha,0}(x)}{(1-x)^{(\alpha+1/2)q}} dx \\ & \leq c \int_{-1}^1 |g(y)|^q w_{\alpha,\beta}(y) dy \int_{-1}^1 (1-x)^{q/2-1} dx \leq c \|g\|_{L^q(w_{\alpha,0})}^q, \end{aligned}$$

whereas for $q = 1$, exchanging the order of integration shows that

$$\int_{-1}^1 \left| \frac{\phi(x)^{s-k}}{(1-x)^{\alpha+1}} \int_x^1 g(y) w_{\alpha,0}(y) dy \right| w_{\alpha,0}(x) dx$$

$$\leq c \int_{-1}^1 |g(y)|w_{\alpha,0}(y) \int_{-1}^y (1-x)^{\alpha+1/2}(1-x)^{-\alpha-1} dx dy \leq c \|g\|_{L^1(w_{\alpha,0})}.$$

Putting this together, we have completed proof of (5.3) under (a).

We now prove (5.3) under the condition in (b). Since $s = 1, k$ has to be zero. Since

$$u''_{g,0}(x) = \frac{d}{dx} (w_{\alpha,\beta}(x))^{-1} \int_x^1 g(y)w_{\alpha,\beta}(y) dy - g(x)$$

and $\frac{d}{dx} (w_{\alpha,\beta}(x))^{-1} = (\alpha(1-x)^{-1} - \beta(1+x)^{-1})(w_{\alpha,\beta}(x))^{-1}$, it is easy to see that our main task is to establish the inequality

$$\int_{-1}^1 \left| \frac{\phi(x)}{(1-x)w_{\alpha,\beta}(x)} \int_x^1 g(y)w_{\alpha,\beta}(y) dy \right|^q w_{\alpha,\beta}(x) dx \leq c \|g\|_{L^q(w_{\alpha,\beta})}^q.$$

For $q = \infty$, it is easy to see that we need $\beta + 1 - 1/2 \leq 0$ or $\beta \leq -1/2$. For $1 < q < \infty$, we can follow the proof in (a) and conclude that the inequality holds if $-\beta(q-1) > q/2 - 1$ or $\beta < p/2 - 1$. This also holds for $q = 1$ as can be seen by exchanging the order of the integrals. \square

It is worthwhile to mention that if $s > 1$ and $\beta \neq 0$, then we apply the product rule on $1/w_{\alpha,\beta}(x) = (1-x)^{-\alpha}(1+x)^{-\beta}$ to obtain

$$\left| \frac{d^j}{dx^j} \left(\frac{1}{w_{\alpha,\beta}(x)} \right) \right| (1-x)^{j-1} \leq c \sum_{i=0}^j (1-x)^{-\alpha+i-1} (1+x)^{-\beta-i},$$

so that, since the summand in the last expression is independent of j ,

$$\left| u_{g,k}^{(2s-k)}(x) \right| \leq |g(x)| + c \sum_{i=0}^{s-k} \frac{(1-x)^{i-1}}{(1-x)^\alpha(1+x)^{\beta+i}} \int_x^1 |g(y)|w_{\alpha,\beta}(y) dy.$$

Following the same proof as before, it is not difficult to see that we need $\beta < (1-s/2)p - 1$ for (5.3). Since $\beta > -1$, this makes sense only if $s = 1$ and $\beta < p/2 - 1$.

Theorem 5.3 *Let $\alpha, \beta > -1$ and $f \in W_p^s(w_{\alpha,\beta})$ for $1 \leq p < \infty$ and $f \in C^s[-1, 1]$ if $p = \infty$. Then*

$$\| \partial^k f - \partial^k \mathcal{V}_{n,\theta}^{\alpha-s,\beta-s} f \|_{L^p(w_{\alpha,\beta})} \leq c n^{-s+k} E_n(f^{(s)})_{L^p(w_{\alpha,\beta})}, \quad 0 \leq k \leq s, \quad (5.4)$$

holds under either one of the following sets of assumptions:

- (a) $\theta = -1$, with (i) $s \in \mathbb{N}, \beta = 0$ and $1 \leq p \leq \infty$, or (ii) $s = 1, \beta < p/2 - 1$ for $1 < p \leq \infty$ and $\beta \leq p/2 - 1$ if $p = 1$;
- (b) $\theta = 1$, with (i) $s \in \mathbb{N}, \alpha = 0$ and $1 \leq p \leq \infty$, or (ii) $s = 1, \alpha < p/2 - 1$ for $1 < p \leq \infty$ and $\alpha \leq p/2 - 1$ if $p = 1$.

Furthermore, for $p = 2$, we can replace (5.4) by

$$\|\partial^k f - \partial^k \mathcal{S}_{n,\theta}^{\alpha-s,\beta-s} f\|_{L^2(w_{\alpha,\beta})} \leq c n^{-s+k} E_{n-s}(f^{(s)})_{L^2(w_{\alpha,\beta})}, \quad 0 \leq k \leq s. \quad (5.5)$$

Proof Let us first consider the case $\theta = -1$ and write $\mathcal{V}_n^{\alpha-s,\beta-s} f = \mathcal{V}_{n,-1}^{\alpha-s,\beta-s} f$ for convenience. Applying Lemma 5.1 with $v = \partial^k f - \partial^k \mathcal{V}_n^{\alpha-s,\beta-s} f$ and using (4.2), we have

$$\begin{aligned} \langle \partial^k f - \partial^k \mathcal{V}_n^{\alpha-s,\beta-s} f, g \rangle_{\alpha,\beta} &= \langle f^{(s)} - V_n^{\alpha,\beta} f^{(s)}, u_{g,k}^{(s)} \rangle_{\alpha,\beta} \\ &= \left\langle f^{(s)} - V_n^{\alpha,\beta} f^{(s)}, u_{g,k}^{(s)} - V_{\lfloor \frac{n}{2} \rfloor}^{\alpha,\beta} u_{g,k}^{(s)} \right\rangle_{\alpha,\beta}, \end{aligned}$$

where the second equation follows from the orthogonality of $f^{(s)} - \mathcal{V}_n^{\alpha,\beta} f^{(s)}$ to any polynomials of degree at most n . By Theorem 2.2 and (5.3),

$$\begin{aligned} \|u_{g,k}^{(s)} - V_{\lfloor \frac{n}{2} \rfloor}^{\alpha,\beta} u_{g,k}^{(s)}\|_{L^q(w_{\alpha,\beta})} &\leq c E_{\lfloor \frac{n}{2} \rfloor} \left(u_{g,k}^{(s)} \right)_{L^q(w_{\alpha,\beta})} \\ &\leq c n^{-s+k} \|\phi^{s-k} u_{g,k}^{(2s-k)}\|_{L^q(w_{\alpha,\beta})} \leq c n^{-s+k} \|g\|_{L^q(w_{\alpha,\beta})}, \end{aligned}$$

so that, by the Hölder inequality, we conclude that

$$\begin{aligned} \left| \langle \partial^k f - \partial^k \mathcal{V}_n^{\alpha-s,\beta-s} f, g \rangle_{\alpha,\beta} \right| &\leq c \|f^{(s)} - \mathcal{V}_n^{\alpha,\beta} f^{(s)}\|_{L^p(w_{\alpha,\beta})} \|u_{g,k}^{(s)} - V_{\lfloor \frac{n}{2} \rfloor}^{\alpha,\beta} u_{g,k}^{(s)}\|_{L^q(w_{\alpha,\beta})} \\ &\leq c n^{-s+k} E_n(f)_{L^p(w_{\alpha,\beta})} \|g\|_{L^q(w_{\alpha,\beta})}. \end{aligned}$$

Applying the above inequality to the following expression of the L^p norm

$$\|\partial^k f - \partial^k \mathcal{V}_n^{\alpha-s,\beta-s} f\|_{L^p(w_{\alpha,\beta})} = \sup_{\|g\|_{L^q(w_{\alpha,\beta})} \neq 0} \frac{\langle \partial^k f - \partial^k \mathcal{V}_n^{\alpha-s,\beta-s} f, g \rangle_{\alpha,\beta}}{\|g\|_{L^q(w_{\alpha,\beta})}}$$

completes the proof (5.4) for $\theta = -1$.

The proof for $\theta = 1$ follows similarly with $u_{g,k}$ in (5.1) replaced by

$$u_{g,k}(x) := \int_x^1 \frac{(t-x)^{s-1}}{(s-1)!} \frac{1}{w_{\alpha,\beta}(t)} \int_{-1}^t \frac{(t-y)^{s-k-1}}{(s-k-1)!} g(y) w_{\alpha,\beta}(y) dy dt$$

for $0 \leq k \leq s - 1$. The proof follows along the same line and we omit the details. \square

Evidently, Theorem 1.2 follows from this theorem with either $\alpha = 0$ or $\beta = 0$.

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