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Phase Retrievable Projective Representation Frames for Finite Abelian Groups

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Abstract We consider the problem of characterizing projective representations that admit frame vectors with the maximal span property, a property that allows for an algebraic recovering for the phase-retrieval problem. For a given multiplier μ of a finite abelian group G, we show that the representation dimension of any irreducible μ-projective representation of G is exactly the rank of the symmetric multiplier matrix associated with μ. With the help of this result we are able to prove that every irreducible μ-projective representation of a finite abelian group G admits a frame vector with the maximal span property, and obtain a complete characterization for all such frame vectors. Consequently the complement of the set of all the maximal span frame vectors for any projective unitary representation of any finite abelian group is Zariski-closed.

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These generalize some of the recent results about phase-retrieval with Gabor (or STFT) measurements.

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1 Introduction

The main purpose of this short note is to prove the existence of phase-retrievable frame vectors for every irreducible projective representation of finite abelian groups. Recall that a finite sequence $F = \{f_i\}_{i=1}^N$ of vectors in a finite dimensional Hilbert space H is called a *frame* for H if there are two constants $0 < C_1 \le C_2$ such that

$$C_1\|f\|^2 \le \sum_{i=1}^{N} |\langle f, f_i\rangle|^2 \le C_2\|f\|^2$$

holds for every $f \in H$. Equivalently, a finite sequence is a frame for H if and only if it is a spanning set of H. A frame $F = \{f_i\}_{i=1}^N$ is called a C-tight frame if $C_1 = C_2 = C$ and a Parseval frame if $C_1 = C_2 = 1$.

Like bases, frames are used in applications for signal decomposition and reconstructions through their dual frames. Define $\Theta_F: H \to \mathbb{F}^N$ ($\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$) by

$$\Theta_F(f) = \sum_{i=1}^{N} \langle f, f_i\rangle e_i, \quad \text{for all } f \in H,$$

where $\{e_i\}_{i=1}^N$ is the standard orthonormal basis for $\mathbb{F}^N$. Then Θ_F is the analysis operator of F and its synthesis operator is given by $\Theta_F^*(e_i) = f_i$. The frame operator is $S = \Theta_F^*\Theta_F$. Clearly, S is positive and invertible on H and satisfies the condition:

$$Sf = \sum_{i=1}^{N} \langle f, f_i\rangle f_i, \quad \text{for all } f \in H.$$

Replacing f by $S^{-1}f$ or applying S^{-1} to both sides or the above identity we get the reconstruction formula:

$$f = \sum_{i=1}^{N} \langle f, S^{-1}f_i\rangle f_i = \sum_{i=1}^{N} \langle f, f_i\rangle S^{-1}f_i, \quad \text{for all } f \in H.$$

The sequence $\{S^{-1}f_i\}_{i=1}^N$ is called the *standard or canonical dual frame* of F. In addition to the standard dual, when $n > N$ there exist infinitely many frames $\tilde{F} = \{\tilde{f}_i\}_{i=1}^N$ that also give us a reconstruction formula:

$$f = \sum_{i=1}^{N} \langle f, \tilde{f}_i \rangle f_i = \sum_{i=1}^{N} \langle f, f_i \rangle \tilde{f}_i, \qquad \text{i.e., } \Theta_{\tilde{F}}^{*} \Theta_F = I.$$

Any frame $\tilde{F} = \{\tilde{f}_i\}_{i=1}^N$ yielding the above reconstruction formula is called an *alternate dual frame* or just a *dual frame* for F. The mixed Gramian matrix for two finite sequences $\{x_i\}_{i=1}^N$ and $\{y_j\}_{j=1}^M$ is the $N \times M$ matrix $\Theta_Y \Theta_X^* = [\langle x_i\ ,\ y_j \rangle]$.

The phase retrieval problem considers recovering a signal of interest from magnitudes of its linear or nonlinear measurements. It arises in various fields of science and engineering applications, such as X-ray crystallography, coherent diffractive imaging, optics and more. Balan et al. are among the pioneers who initiated the investigation of the phase retrieval problem by using linear measurements against a frame (cf. [6–10]). For linear measurements with a frame $\{f_i\}_{i=1}^N$, it asks to reconstruct f from its intensity measurements $\{|\langle f, f_i\rangle|\}_{i=1}^N$. Clearly the intensity measurements are the same for both f and λf for every unimodular scalar λ. Therefore the phase retrieval problem asks to recover f up to an unimodular scalar.

Definition 1.1 A frame $\{f_i\}_{i=1}^N$ for H is called *phase retrievable* if the induced quotient map $\mathcal{A}{:}H/\mathbb{T} \to \mathbb{F}^N$ defined by $\mathcal{A}(f/\mathbb{T}) = \{|\langle f, f_i\rangle|\}_{i=1}^N$ is injective, where $\mathbb{T} = \{\lambda \in \mathbb{F}{:}|\lambda| = 1\}$.

A solution to the phase retrieval problem is impossible in the absence of injectivity of the map. Balan et al. obtained the following important characterizations of phase retrievable frames [6–8].

Theorem 1.1 *Let $\{f_i\}_{i=1}^N$ be a frame for H. If $\{f_i\}_{i=1}^N$ is phase retrievable, then it satisfies the complement property, i.e., for every $\Omega \subseteq \{1, \ldots, N\}$, either $\{f_i\}_{i\in\Omega}$ or $\{f_i\}_{i\in\Omega^c}$ spans $\mathbb{F}^n$. The complementary property is also sufficient when $\mathbb{F} = \mathbb{R}$, but not sufficient in the complex case $\mathbb{F} = \mathbb{C}$.*

Theorem 1.2 *Every generic frame $\{f_1, \ldots, f_N\}$ for $\mathbb{F}^n$ is phase retrievable if $N \geq 2n - 1$ in the real case $\mathbb{F} = \mathbb{C}$ or $N \geq 4n - 2$ in the complex case $\mathbb{F} = \mathbb{C}$.*

For the complex Hilbert space case, Theorem 1.2 was recently improved by Conca et al. [22] in which they proved that $4n - 4$ generic intensity measurements are sufficient for phase retrieval in $\mathbb{C}^n$, and $4n - 4$ measurements are necessary for phase retrieval in $\mathbb{C}^n$ when $n = 2^k + 1$.

With the above results, construction or design for phase retrievable frames seems not a difficult task. For example, if $N \geq 2n-1$, then every n-independent (or full spark) frame (i.e., every n-vectors in the frame are linearly independent) has this property, and consequently it is phase retrievable in the real case. The main issue to the phase retrieval problem lies in the recovering algorithms due to the nonlinearity of the map $\mathcal{A}$. We refer to [1,3–5,11,12,14,15,17–19,23,25] for some detailed accounts on some recent developments and various kinds of approaches for the phase retrieval problem. In some special cases a linear reconstruction is also possible. For example, phase retrieval can be formulated as rank-one matrix recovery (phase-lifting) problem if a phase retrievable frame has the *maximal span property,* i.e., the span of $\{f_i \otimes f_i\}_{i=1}^N$ contains all the rank-one Hermitian operators [9,10]. In this case, $\{f_i \otimes f_i\}_{i=1}^N$ is a frame for the Hilbert space $\mathbb{H}^n$ (the space of all $n \times n$ Hermitian matrices) equipped with the Hilbert-Schmidt inner product $\langle A, B\rangle = \operatorname{tr}(AB^*)$. Let $\{A_i\}_{i=1}^N$ be a dual frame for $\{f_i \otimes f_i\}_{i=1}^N$. Then for every $x \in H$ we have

$$x \otimes x = \sum_{i=1}^{N} \langle x \otimes x, f_i \otimes f_i\rangle A_i = \sum_{i=1}^{N} |\langle x, f_i\rangle|^2 A_i,$$

and so x can be reconstructed (up to a multiple of unimodular scalar) by factorizing the above right hand side rank-one matrix.

Let G be the Gramian of the sequence $\{f_i \otimes f_i\}_{i=1}^N$, i.e. $G_{ij} = \operatorname{tr}(f_i \otimes f_i \cdot f_j \otimes f_j) = |\langle f_i, f_j\rangle|^2$. An explicit formula for one of the choices of a dual $\{A_i\}_{i=1}^N$ is obtained by Balan, Bodmann, Casazza, Edidin in [10,15].

Proposition 1.3 *If $\{f_i\}_{i=1}^N$ is a $\frac{N}{n}$-tight frame with the maximal span property, then*

$$x \otimes x = \sum_{i=1}^{N} |\langle x, f_i\rangle|^2 R_i,$$

where $R_i = \sum_{j=1}^N Q_{ij}(f_j \otimes f_j)$ and Q is the pseudo-inverse of G (i.e. $GQG = G$).

Let $k = n(n+1)/2$ or n^2, depending on whether H is real or complex. Then $\dim \mathbb{H}^n = k$, and hence $N \geq k$ if a frame $\{f_i\}_{i=1}^N$ has the maximal span property. A key element for this reconstruction requires the existence and constructions of frames with the maximal span property. In this note we investigate a special class of frames, namely, projection representation frames, that have the maximal span property. Typical examples include the frames obtained by Gabor representations (or STFT measurements):

Example 1.1 Let $H = \mathbb{C}^n$ and $w = (w(0), \ldots, w(n-1)) \in H$ be a window vector. For $x = (x(0), x(1), \ldots, x(n-1)) \in H$, the Gabor or STFT measurement of x is given by:

$$X_w(m,k) = \sum_{j=0}^{n-1} x(j)\overline{w(j-m)}e^{-2\pi ikj/n}, \quad (m,k) \in \mathbb{Z}_n \times \mathbb{Z}_n.$$

Let $T, E\colon \mathbb{C}^n \to \mathbb{C}^n$ be the unitary operators defined by

$$(Tw)(j) = w(j-1); \ (Ew)(j) = e^{2\pi ij/n}w(j).$$

Then $X_w(m,k) = \langle x, E^kT^m w\rangle$. The involved mapping $\pi\colon \mathbb{Z}_n \times \mathbb{Z}_n \to M_{n\times n}(\mathbb{C})$ defined by

$$\pi(m,k) = E^kT^m$$

is the Gabor representation of the group $G = \mathbb{Z}_n \times \mathbb{Z}_n$ on H. So the phase-retrieval problem with Gabor measurements asks to recover x from $|\langle x\,,\ E^k T^m w\,\rangle|$. This example has led to extensive research activities on the phase-retrieval problem in recent years (cf. [13,16,24,26,28,29]). One of the very basic questions is to characterize the window vectors w such that $\{\pi(m,k)w\}_{(m,k)\in G}$ is phase-retrievable or even has the maximal span property. A simple characterization for these type of frames was recently obtained by Bojarovska and Flinth.

Theorem 1.4 [16] *Let π be the Gabor representation of $G = \mathbb{Z}_n \times Z_n$ on $\mathbb{C}^n$ and $w \in \mathbb{C}^n$. Then $\{\pi(m,k)w\}_{(m,k)\in G}$ has the maximal span property if and only if $\langle\, \pi(m,k)w\,,\ w\,\rangle \neq 0$ for every $(m,k) \in G$.*

Note that the Gabor representation is a projective representation of the abelian group G. Moreover it is an irreducible representation meaning that $span\{\pi(m,k){:}(m,k) \in G\} = M_{n\times n}(\mathbb{C})$. This led us to consider the problem of characterizing the phase-retrievable frames for irreducible projective representations of any finite groups.

Recall that a *projective unitary representation* π for a finite group G is a mapping $g \mapsto \pi(g)$ from G into the group $U(H)$ of all the unitary operators on a finite dimensional Hilbert space H such that $\pi(g)\pi(h) = \mu(g,h)\pi(gh)$ for all $g, h \in G$, where $\mu(g,h)$ is a scalar-valued function on $G \times G$ taking values in the circle group $\mathbb{T}$. This function $\mu(g,h)$ is then called a *multiplier of* π. In this case we also say that π is a μ-projective unitary representation. It is clear from the definition that we have

(i) $\mu(g_1, g_2g_3)\mu(g_2, g_3) = \mu(g_1g_2, g_3)\mu(g_1, g_2)$ for all $g_1, g_2, g_3 \in G$,
(ii) $\mu(g,e) = \mu(e,g) = 1$ for all $g \in G$, where e denotes the group unit of G. Any function $\mu{:}G \times G \mapsto \mathbb{T}$ satisfying $(i)-(ii)$ above will be called a *multiplier* or 2-*cocycle* of G. It follows from (i) and (ii) that we also have,
(iii) $\mu(g, g^{-1}) = \mu(g^{-1}, g)$ holds for all $g \in G$.

A projective unitary representation π of G on H is called irreducible if $span\{\pi(g){:}g \in G\} = B(H)$, the space of all the linear operators on H. The set of all possible multipliers on G can be given an abelian group structure by defining the product of two multipliers as their pointwise product. The resulting group we denote by $Z^2(G,\mathbb{T})$. The set of all the multipliers α satisfying,

$$\alpha(g,h) = \beta(gh)\beta(g)^{-1}\beta(h)^{-1}$$

for an arbitrary function $\beta{:}G \mapsto \mathbb{T}$ such that $\beta(e) = 1$ forms a subgroup $B(G,\mathbb{T})$ of $Z^2(G,\mathbb{T})$, and the quotient group $H^2(G,\mathbb{T}) = Z^2(G,\mathbb{T})/B^2(G,\mathbb{T})$ is the second cohomology group of G.

Let π be a projective group representation. A vector ξ is called a *π-maximal spanning frame vector* if $\{\pi(g)\xi\}_{g\in G}$ has the maximal spanning property. We will use $\mathcal{M}_\pi$ to denote the set of all π-maximal spanning frame vectors. The following elementary argument shows that if $\mathcal{M}_\pi$ is non empty, then a generic vector is a π-maximal span frame vector.

Proposition 1.5 *If $\mathcal{M}_\pi$ is not empty, then it is an open dense subset of H. In fact $\mathcal{M}_\pi^c$ is Zariski-closed.*

Proof We can assume that $H = \mathbb{F}^n$, and let $\{B_1, \ldots, B_k\}$ be a basis for $\mathbb{H}^n$. For each $g \in G$ and $x = (x_1, \ldots, x_n) \in H$, write

$$\pi(g)x \otimes \pi(g)x = \sum_{i=1}^{k} c_{i,g}(x) B_i,$$

where the coefficients $c_{i,g}(x)$ can be obtained by solving a system of linear equations and so they are quadratic polynomials of x_j when $\mathbb{F} = \mathbb{R}$ and in the complex case they are polynomials of u_i, v_i with u_i, v_i being the real and imaginary part of x_i. Let $C(x) = [c_{i,g}]_{k\times|G|}$, and P_Λ be the determinant of the submatrix of $C(x)$ consisting of the g-th columns for $g \in \Lambda$, where Λ is any subset of G with cardinality k. Then again each P_Λ is a polynomial of x_j or polynomial of u_i and v_i in the complex case. Clearly $x \in \mathcal{M}_\pi$ if and only if $\mathrm{rank}(C(x)) = k$. This implies that

$$\mathcal{M}_\pi = H \setminus \cap_{\Lambda \subset G, |\Lambda| = k} Z(P_\Lambda),$$

where $Z(P) = \{x \in H, P(x) = 0\}$.

Since $\mathcal{M}_\pi$ is nonempty, there exist $x \in H$ and Λ such that $P_\Lambda(x) \neq 0$. Thus P_Λ is a nonzero polynomial, and therefore we get that $H \setminus \mathcal{M}_\pi$ is Zariski-closed and so $\mathcal{M}_\pi$ is open and dense in H. □

We are interested in establishing sufficient and/or necessary conditions on π such that $\mathcal{M}_\pi$ is not empty.

Problem A Let π be a projective unitary representation of a finite group G on $\mathbb{C}^n$. Under what condition does π admit a frame vector with the maximal span property? Characterize or construct all such frame vectors.

For irreducible projective unitary representations we make the following conjecture:

Conjecture *Every irreducible μ-projective representation π admits a maximal spanning frame vector.*

The main purpose of this paper is to confirm this conjecture for abelian groups, and consequently we get a generalization of the known results for Gabor representations. We achieve this by obtaining a new formula for the dimension of irreducible projective representations for finite abelian groups.

Let μ be a 2-cocycle (or multiplier) of a finite abelian group G, and let C_μ be its associated symmetric multiplier matrix defined by $C_\mu = [c_{g,h}]$ with $c_{g,h} = \mu(g,h)\overline{\mu(h,g)}$.

It is well known that all the μ-projective representation of a finite abelian group G have the same representation dimension. The following result provides us with an explicit method to compute the representation dimension, which seems to be new.

Theorem 1.6 *If π is a irreducible μ-projective representation of a finite abelian group G on an n-dimensional complex Hilbert space H, then $\mathrm{rank}(C_\mu) = n^2$.*

With the help of the above result we will prove the following main result:

Theorem 1.7 *Suppose that π is a μ-projective unitary representation for a finite abelian group G on an n-dimensional complex Hilbert space H. If π is irreducible, then π admits a frame vector with the maximal span property. Moreover, $\{\pi(g)\xi\}_{g\in G}$ has the maximal span property if and only if $\langle \pi(g)\xi\,,\, \xi\,\rangle \neq 0$ for any $g \in G$.*

Remark From the computation perspective, typical phase-retrieval applications expect to use $O(n)$ or $O(n \log n)$ number of measurements. Although the frames of Theorem 1.7 are not optimal from this point of view, they are nevertheless important for indicating the strong and deep connections between the phase-retrieval problem and the general group representation theory. Our hope is that this result will stimulate further investigation on phase-retrievable group representation frames (not necessarily frames with the maximal span property) for more general finite groups. In addition, we remark that although $O(n^2)$ number of measurements are required for frames with the maximal span property to do phase-retrieval, the algorithm for signal recovering is very easy to perform due to the algebraic formula given in Proposition 1.3. Therefore Theorem 1.7 also has some attractiveness for computational implementations. Finally, while Theorem 1.4 is a special case of Theorem 1.7, the proof of Theorem 1.7 is completely new and the techniques utilized in the the proof of Theorem 1.4 do not apply to the general projective representation frame case.

2 Proofs of the Main Results

Let π be a projective unitary representation of G on an n-dimensional complex Hilbert space H. For each $\xi \in H$, consider the matrix

$$A(\xi) = [a_{g,h}(\xi)]_{G\times G}$$

with $a_{g,h}(\xi) = \langle \pi(h)\pi(g)\xi\,,\, \pi(g)\xi\,\rangle$. We first establish the following sufficient condition:

Lemma 2.1 *If there exists $\xi \in H$ such that $A(\xi)$ has rank n^2 (where $n = \dim H$), then π is irreducible and $\{\pi(g)\xi\}_{g\in G}$ has the maximal span property.*

Proof Let $X = \{\pi(g)\}_{g\in G}$ and $Y = \{\pi(g)\xi \otimes \pi(g)\xi\}_{g\in G}$ be two sequences in $B(H)$ equipped with the trace inner product. Note that the mixed Gramian matrix $\Theta_Y \Theta_X^*$ is exactly the matrix $A(\xi)$ which is assumed to have rank n^2. Thus $\mathrm{rank}(\Theta_Y) \geq n^2$ and $\mathrm{rank}(\Theta_X) \geq n^2$. Since we also have $\mathrm{rank}(\Theta_Y) \leq n^2$ and $\mathrm{rank}(\Theta_X) \leq n^2$, we get that $\mathrm{rank}(\Theta_Y) = n^2 = \mathrm{rank}(\Theta_X)$, which implies that π is irreducible and $\{\pi(g)\xi\}_{g\in G}$ has the maximal span property. □

Lemma 2.2 *Suppose that π is a μ-projective unitary representation for a finite group G on an n-dimensional complex Hilbert space H. Then there exists $\xi \in H$ such $\langle \pi(g)\xi\,,\, \xi\,\rangle \neq 0$ for all $g \in G$. Moreover, the set of all such vectors ξ is open and dense in H.*

Proof We can assume that $H = \mathbb{C}^n$. By the Baire-Category theorem it suffices to prove that for each $g \in G$, the set $\{\xi \in \mathbb{C}^n : \langle \pi(g)\xi\,,\, \xi\,\rangle \neq 0\}$ is open and dense in

$\mathbb{C}^n$. Since $\langle \pi(g)\xi , \xi \rangle$ is a quadratic polynomial of ξ, we only need to point out that this is a nonzero polynomial. Indeed, if $\langle \pi(g)\xi , \xi \rangle = 0$ for all $\xi \in \mathbb{C}^n$, then we have $\pi(g) = 0$, which is a contradiction. □

Lemma 2.3 *Suppose that π is a μ-projective unitary representation for an abelian group G. If there exists $\xi \in H$ such that $\{\pi(g)\xi\}_{g\in G}$ has the maximal spanning property, then $\langle \pi(g)\xi , \xi \rangle \neq 0$ for every $g \in G$.*

Proof Since $\{\pi(g)\xi\}_{g\in G}$ has the maximal spanning property we have that $span\{\pi(g)\xi \otimes \pi(g)\xi : g \in G\} = B(H)$. So if $\langle \pi(h)\xi, \xi\rangle = 0$ for some $h \in G$, then for every $g \in G$ we get

$$\begin{aligned} |\langle \pi(h)\pi(g)\xi , \pi(g)\xi \rangle| &= |\langle \pi(g^{-1})\pi(h)\pi(g)\xi , \xi \rangle| = |\langle \pi(g^{-1}hg)\xi , \xi \rangle| \\ &= |\langle \pi(h)\xi , \xi \rangle| = 0. \end{aligned}$$

Thus $\mathrm{tr}(\pi(h)(\pi(g)\xi \otimes \pi(g)\xi)) = 0$, and so $\pi(h) = 0$ which leads to a contradiction. □

Lemma 2.4 [2] *Let μ be a multiplier for an abelian group G. Then all the irreducible μ-projective representations have the same representation dimension.*

Let μ be a multiplier for a finite group G. While all the irreducible μ-projective representations have the same representation dimension, in general it is not easy to find an explicit formula for the representation dimension. The following theorem tells us that the representation dimension of the irreducible μ-projective representations is uniquely determined by the rank of its symmetric multiplier matrix. This seems to be new and it does provide us a very easy way to compute the representation dimension for any given multiplier μ.

Let μ be a multiplier for an abelian group G. Recall that the *symmetric multiplier matrix* is defined by $C_\mu = [c_{g,h}]$ with $c_{g,h} = \mu(g, h)\overline{\mu(h, g)}$.

Theorem 2.5 *Suppose that π is a μ-projective unitary representation for an abelian group G on an n-dimensional Hilbert space $H = \mathbb{C}^n$. Then rank$(C_\mu) \leq n^2$. Moreover, π is an irreducible μ-representation if and only if rank$(C_\mu) = n^2$.*

Proof By Lemma 2.2, there exists $\eta \in \mathbb{C}^n$ such that $\langle \pi(g)\eta , \eta\rangle \neq 0$ for any $g \in G$. Let $\Theta_1: M_{n\times n}(\mathbb{C}) \mapsto \ell^2(G)$ be the analysis operator for $\{\pi(g)\}_{g\in G}$, and $\Theta_2: M_{n\times n}(\mathbb{C}) \mapsto \ell^2(G)$ be the analysis operator for $\{\pi(g)\eta \otimes \pi(g)\eta\}_{g\in G}$. Then we have

$$\Theta_2\Theta_1^* = [\langle \pi(g)\pi(h)\eta , \pi(h)\eta \rangle]_{G\times G}.$$

Note that

$$\langle \pi(g)\pi(h)\eta , \pi(h)\eta \rangle = c_{g,h}\langle \pi(g)\eta , \eta \rangle.$$

and $\langle \pi(g)\eta , \eta \rangle \neq 0$ for every $g \in G$. So we get that

$$\mathrm{rank}(C_\mu) = \mathrm{rank}(\Theta_2\Theta_1^*) \leq \mathrm{rank}(\Theta_1) = \dim\, (span\{\pi(g): g \in G\}) \leq n^2.$$

Now assume that $\operatorname{rank}(C_\mu) = n^2$. Then the above inequality implies that $\dim(span\{\pi(g): g \in G\}) = n^2$, and thus π is irreducible. Conversely, let us assume that π is irreducible. We will prove that $\operatorname{rank}(C_\mu) = n^2$.

We first introduce a couple of notations: let $\hat{G}$ be the dual group of G, and $\bar{\pi}: g \mapsto \overline{\pi(g)}$, the complex conjugation of $\pi(g)$. Then $\bar{\pi}$ is a projective representation with multiplier $\bar{\mu}$. Consider the group representation $\pi \otimes \bar{\pi}: g \mapsto \pi(g) \otimes \overline{\pi(g)}$. Then it is a projective representation with multiplier $\mu\bar{\mu} = 1$, and so it is a group representation. Hence $\pi \otimes \bar{\pi}$ can be decomposed as the direct sum of one-dimensional group representations of G. Moreover, each one dimensional representation of G appears at most once in the direct sum decomposition of $\pi \otimes \bar{\pi}$. Let $T_\mu = \{\chi \in \hat{G}: \chi \subset \pi \otimes \bar{\pi}\}$. Then T_μ is a subgroup of $\hat{G}$. Define

$$G_\mu = T_\mu^\perp = \{g \in G: \chi(g) = 1, \forall \chi \in T_\mu\}.$$

Note that $|T_\mu| = \dim\ H \times \dim\ H = n^2$. Thus $|G_\mu| = [G:G_\mu] = |T_\mu| = n^2$.

Since G is abelian, it is easy to verify that $c: G \times G \mapsto \mathbb{T}$ defined by $c(g,h) = c_{gh} = \mu(g,h)\overline{\mu(h,g)}$ is a bi-homomorphism, i.e., $c(gg',h) = c(g,h)c(g'h)$ and $c(g,hh') = c(g,h)c(g,h')$ for all $g, g', h, h' \in G$. This induces a homomorphism $\lambda_\mu: G \mapsto \hat{G}$. By Proposition 2.4 in [21] we know that

$$G_\mu = Ker(\lambda_\mu) = \{g \in G: \lambda_\mu(g) = 1\}.$$

Therefore we get

$$|\lambda_\mu(G)| = [G:Ker(\lambda_\mu)] = n^2.$$

Recall that the characters of G are linearly independent. Since each row of the symmetric multiplier matrix C_μ defines a character of G by $h \mapsto c(g,h)$, we get that the rank of C_μ is equal to the number of different characters appeared in the rows of C_μ. By the definition of λ_μ, we know that this number is exactly the cardinality of the image of λ_μ. This implies that $\operatorname{rank}(C_\mu) = |\lambda_\mu(G)| = n^2$ as claimed. □

Corollary 2.6 *Let μ be a multiplier of an abelian group G and $n^2 = rank(C_\mu)$. Then every n-dimensional μ-projective representation π of G is irreducible.*

Proof Let σ be an irreducible subrepresentation of π on a d-dimensional π-invariant subspace. Then, by Theorem 2.5, the representation dimension of σ is equal to $\operatorname{rank}(C_\mu) = d^2$. This implies that $d = n$ and thus $\sigma = \pi$. Therefore π is irreducible. □

Proof of Theorem 1.7 Assume that π is an irreducible μ-projective representation of G on $H = \mathbb{C}^n$. By Lemma 2.3 we know that if $\{\pi(g)\xi\}_{g\in G}$ has the maximal span property, then $\langle \pi(g)\xi\ ,\ \xi \rangle \neq 0$ for every $g \in G$. Therefore, to complete the proof, it suffices to show that $\{\pi(g)\xi\}_{g\in G}$ has the maximal span property when $\langle \pi(g)\xi\ ,\ \xi \rangle \neq 0$ for every $g \in G$.

Let Θ_1 and $\Theta_2: M_{n\times n}(\mathbb{C}) \mapsto \ell^2(G)$ be the analysis operators defined in the proof of Theorem 2.5. Then we know that $\operatorname{rank}(\Theta_2\Theta_1^*) = \operatorname{rank}(C_\mu)$. Since π is irreducible, we get that $\mathrm{rank}(\Theta_1^*) = n^2$ and by Theorem 2.5 that $\mathrm{rank}(C_\mu) = n^2$. This implies that $\mathrm{rank}(\Theta_2\Theta_1^*) = n^2$ which implies that $\mathrm{rank}(\Theta_2) = n^2$ since we also have $\mathrm{rank}(\Theta_2) \leq n^2$. Therefore $\{\pi(g)\xi \otimes \pi(g)\xi : g \in G\}$ spans $M_{n\times n}(\mathbb{C})$, i.e., $\{\pi(g)\xi\}_{g\in G}$ has the maximal span property. □

3 Examples and Discussions

Since the Gabor representation is a very special irreducible projective unitary representation for the group $\mathbb{Z}_n \times \mathbb{Z}_n$ for a special multiplier μ, it implies that Theorem 1.4 is a very special case of Theorem 1.7. The following example presents us all the possible irreducible projective unitary representations for the group $\mathbb{Z}_n \times \mathbb{Z}_n$.

Example 3.1 Let $G = \mathbb{Z}_n \times \mathbb{Z}_n$ and $H^2(G, \mathbb{T})$ be the second cohomology group. Then $H^2(G, \mathbb{T}) \cong \mathbb{Z}_n$. Let $\xi = e^{2\pi i/n}$. Let $\alpha \in Z^2(G, \mathbb{T})$ be given by

$$\alpha((m, k), (m', k')) = \xi^{-mk'}.$$

Then $[\alpha] \in H^2(G, \mathbb{T})$ is a generator. To understand all the projective representations of G, it suffices to understand the α^a-projective representations of G for each $a \in \{0, 1, \dots, n-1\}$. Denote by IrrRep_G^a the set of isomorphic classes of irreducible α^a-projective representations of G.

Fix $a \in \{0, 1, \dots, n-1\}$. Let $a' = n/\tilde{a}$, where $\tilde{a}$ denotes the greatest common factor of n and a. Let $H = \langle a' \rangle \times \mathbb{Z}_n \lhd G$. Then

$$\alpha^a(x, y) = \alpha^a(y, x) = 1 \text{ for any } x,\ y \in H.$$

In particular, H is α^a-symmetric. It is also easy to check that H is maximal α^a-symmetric. Hence the objects in IrrRep_G^a all have dimension $[G{:}H] = a'$ and there are $\tilde{a}^2$ objects in IrrRep_G^a (cf. [20, Sect. 2.3]).

The irreducible projective representations of G are induced from one-dimensional linear representations of H (cf. [20, Proposition 2.14]). Let $u \in \mathbb{Z}_{\tilde{a}}$ and $v \in \mathbb{Z}_n$. Let $\chi_{u,v}{:}H \to \mathbb{C}^\times$ be the one-dimensional linear representation given by $(m, k) \mapsto \xi^{mu+kv}$. Let $\pi_{u,v} = \alpha^a \mathrm{Ind}_H^G \chi_{u,v}$. Here the induction is with respect to α^a (cf. [20, Sect. 2.2]). Then $\pi_{u,v} \in \mathrm{IrrRep}_G^a$. One may check that $\pi_{u,v} \cong \pi_{u',v'}$ if and only if $v \equiv v' \pmod{\tilde{a}}$.

In matrix form, we may describe $\pi_{u,v}$ as follows. Let V be a $\mathbb{C}$-vector space with dimension a'. Fix $\{e_0, e_1, \dots, e_{a'-1}\}$ a basis of V. Define $\pi_{u,v}((1, 0))$ and $\pi_{u,v}((0, 1))$ by

$$\pi_{u,v}((1, 0))e_i = \xi^u e_{i+1}, \quad \pi_{u,v}((0, k))e_i = \chi_{u,v}^{(i,0)}((0, k))e_i \text{ for } 0 \leq i \leq a' - 1.$$

Here the subscript i is considered modulo a', $\chi_{u,v}^{(i,0)}$ is the α^a-twist of $\chi_{u,v}$ by $(i, 0)$ (cf. [20, Proposition 2.10]). More precisely, as $a' \times a'$ matrices,

$$\pi((0, k)) = \tilde{E}^k, \quad \pi((m, 0)) = \xi^{mu}\tilde{T}^m,$$

where

$$\tilde{E} = \mathrm{diag}(\xi^v, \xi^{v+a}, \ldots, \xi^{v+(a'-1)a}),$$
$$\tilde{T} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{a' \times a'} . \tag{3.1}$$

If we take $u = v = 0$, $a = 1$, then it is the representation in Example 1.1. With this description, one may find all $w \in V$ with the maximal span property by applying Theorem 1.7.

This example can be further generalized by considering the tensor product of group representations.

Example 3.2 Let G_1 and G_2 be two finite groups. Let $\alpha_i \in Z^2(G_i, \mathbb{T})$ be a 2-cocycle of G_i $(i = 1, 2)$. Define a map $\alpha_1 \times \alpha_2 : (G_1 \times G_2) \times (G_1 \times G_2) \to \mathbb{T}$ by

$$(\alpha_1 \times \alpha_2)((g_1, g_2), (h_1, h_2)) = \alpha_1(g_1, h_1)\alpha_2(g_2, h_2) \text{ for all } g_1, h_1 \in G_1,\ g_2, h_2 \in G_2.$$

It is easy to check that $\alpha_1 \times \alpha_2 \in Z^2(G_1 \times G_2, \mathbb{T})$. Let (π_1, V_1, α_1) and (π_2, V_2, α_2) be projective unitary representations of G_1 and G_2, respectively, on finite dimensional Hilbert spaces V_1 and V_2. Define a map $\pi_1 \times \pi_2 : G_1 \times G_2 \to B(V_1 \otimes V_2)$ by

$$\pi_1 \times \pi_2((g_1, g_2)) = \pi_1(g_1) \otimes \pi_2(g_2) \text{ for all } g_1 \in G_1,\ g_2 \in G_2.$$

Then $(\pi_1 \times \pi_2, V_1 \otimes V_2, \alpha_1 \times \alpha_2)$ is a projective representation of $G_1 \times G_2$. If moreover π_1 and π_2 are unitary projective representations, then so is $\pi_1 \times \pi_2$. In this situation, we have

(1) if π_i is irreducible $(i = 1, 2)$, then $\pi_1 \times \pi_2$ is an irreducible projective representation of $G_1 \times G_2$;
(2) each irreducible projective representation of $G_1 \times G_2$ with multiplier $\alpha = \alpha_1 \times \alpha_2$ is isomorphic to a representation $\pi_1 \times \pi_2$, where π_i is an irreducible projective representation of G_i with multiplier α_i $(i = 1, 2)$.

It is easy to prove:

Proposition 3.1 *If $\xi_i \in V_i$ is a π_i-maximal spanning frame vector* $(i = 1, 2)$*, then $\xi_1 \otimes \xi_2$ is a $(\pi_1 \times \pi_2)$-maximal spanning frame vector.*

If $G = \prod_{n_i} (\mathbb{Z}_{n_i} \times \mathbb{Z}_{n_i})$ with n_i pairwise coprime, then $H^2(G, \mathbb{T}) = \prod_i H^2(\mathbb{Z}_{n_i} \times \mathbb{Z}_{n_i}, \mathbb{T})$. Hence every element of $H^2(G, \mathbb{T})$ is represented by a cocycle in $\prod_i Z^2(\mathbb{Z}_{n_i} \times \mathbb{Z}_{n_i}, \mathbb{T})$. Then for projective representations of G, we obtain all the vectors with maximal span property by the last example.

For an abelian group, all the irreducible μ-projective unitary representations have the same representations dimension n, and we have proved that $\mathrm{rank}(C_\mu) = n^2$ which is a key ingredient in the proof of our main theorem. However, this is not the case for non-abelian groups since irreducible representations with respect to the same multiplier could have different representation dimensions. This leads to the following question:

Problem B Let μ be a multiplier for a finite non-abelian group G such that all the irreducible μ-projective unitary representations have the same representation dimension n. Is it still true that $\mathrm{rank}(C_\mu) = n^2$?

The following is an example which tells us that this indeed is true for some special groups.

Example 3.3 Consider the metacyclic groups of type $G = \mathbb{Z}_m \ltimes \mathbb{Z}_p$ with p a prime. Fix a presentation of G

$$G = \langle a, b \mid a^m = 1, b^p = 1, bab^{-1} = a^r \rangle,$$

where $r \in \mathbb{Z}_{\geq 0}$ and $r^p \equiv 1 \pmod{m}$. By [27, Theorem 2.11.3],

$$H^2(G, \mathbb{C}^\times) = \begin{cases} 0 & \text{if } p \nmid (m, r-1), \\ \mathbb{Z}_p & \text{if } p \mid (m, r-1). \end{cases}$$

In the following, we assume that $p \mid (m, r-1)$. Fix ζ a primitive l-th root of unity, where $l = (m, 1 + r + \cdots + r^{p-1})$. Define $\alpha : G \times G \to \mathbb{T}$ by

$$\alpha(a^i b^j, a^{i'} b^{j'}) = \begin{cases} 1 & \text{if } j = 0, \\ \zeta^{i'(1+r+\cdots+r^{j-1})} & \text{otherwise.} \end{cases}$$

By [27, Lemmas 2.11.1, 2.11.3 and Theorem 2.11.3], this α is a well-defined element in $Z^2(G, \mathbb{T})$ and it represents a generator of $H^2(G, \mathbb{T})$. If we arrange the elements of G in the order as $1,\ a,\ a^2,\ \ldots,\ a^{m-1},\ b,\ ab,\ \ldots,\ a^{m-1}b,\ \ldots,\ b^{p-1},\ ab^{p-1},\ \ldots,\ a^{m-1}b^{p-1}$, by writing down C_α explicitly, one sees that C_α is given by

$$\begin{pmatrix} A & X_1^{-1}A & \cdots & X_{p-1}^{-1}A \\ AX_1 & X_1^{-1}AX_1 & \cdots & X_{p-1}^{-1}AX_1 \\ \cdots & \cdots & \cdots & \cdots \\ AX_{p-1} & X_1^{-1}AX_{p-1} & \cdots & X_{p-1}^{-1}AX_{p-1} \end{pmatrix}_{p \times p},$$

where A is the $m \times m$ matrix with all entries equal to 1, and

$$X_i = \mathrm{diag}(1, \zeta^{1+r+\cdots+r^{i-1}}, \ldots, \zeta^{(m-1)(1+r+\cdots+r^{i-1})}).$$

Note that the rank of

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & \zeta & \cdots & \zeta^{m-2} & \zeta^{m-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \zeta^{1+r+\cdots+r^{p-2}} & \cdots & \zeta^{(m-2)(1+r+\cdots+r^{p-2})} & \zeta^{(m-1)(1+r+\cdots+r^{p-2})} \end{pmatrix}_{p \times m},$$

is p by Vandermonde determinant. The rank of C_α is p^2. On the other hand, by [20, Corollary 3.11], every irreducible α-projective representation of G has dimension p. Hence Question 4.1 has an affirmative answer in this case.

Moreover, all the irreducible α-projective representations of G admit maximal span vectors. Indeed, let $\pi : G \to \mathrm{GL}(V)$ be an irreducible α-projective representation. By Lemma 2.1, it suffices to show that there exists an element $\xi \in V$ with $\mathrm{rank}((A(\xi)) = p^2$.

Let $m' = m/p$. Then $m \mid m'(r-1)$. The subgroup $K = \langle a^{m'}, b\rangle$ of G is abelian and $K \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Furthermore, the restriction $\alpha|_{K\times K} \in Z^2(K, \mathbb{T})$ represents a generator of $H^2(K, \mathbb{T}) \cong \mathbb{Z}_p$. Hence $\pi|_K : K \to \mathrm{GL}(V)$ is an irreducible projective representation of K. For any $\xi \in V$, we have

$$\mathrm{rank}((\langle \pi(g)\pi(h)\xi, \pi(h)\xi\rangle)_{g,h\in G} \geq \mathrm{rank}((\langle \pi(g)\pi(h)\xi, \pi(h)\xi\rangle)_{g,h\in K}.$$

The claim follows from the abelian case.

One of our ultimate goals is to confirm the conjecture that every irreducible μ-representation admits a maximal spanning frame vector. By Lemma 2.1 we know that π will have a maximal spanning frame vector ξ if $\mathrm{rank}((A(\xi)) = n^2$, where n is the representation dimension and $A(\xi)$ is the mixed Gramian of $\{\pi(g)\xi \otimes \pi(g)\xi\}_{g\in G}$ and $\{\pi(g)\}_{g\in G}$. So we make the following conjecture: let π be an irreducible μ-projective unitary representation of a finite group G on an n-dimensional Hilbert space H. Then there exists a vector $\xi \in H$ such that $\mathrm{rank}(A(\xi)) = n^2$. We point out that in the abelian group case, we have $\mathrm{rank}(A(\xi)) = \mathrm{rank}(C_\mu)$ for every $\xi \in H$ that satisfies the requirement $\langle \pi(g)\xi, \xi\rangle \neq 0$ for every $g \in G$. Thus this is true in the abelian group case.

So far we have only considered irreducible projective unitary representations with maximal spanning frame vectors. We still don't know the answer to the following problem.

Problem C Let π be an irreducible projective unitary representation of a finite group G. Is it possible that there exists a phase-retrievable frame $\{\pi(g)\xi\}$ which does not have the maximal span property?

For arbitrary projective unitary representations we post the following general problem:

Problem D Let π be a projective unitary representation of a finite group G on an n-dimensional Hilbert space H. Under what condition does π admit a phase-retrievable frame vector ξ? (i.e., $\{\pi(g)\xi\}$ is a phase-retrievable frame). If π admits a maximal spanning frame vector, must π be irreducible?

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References

1. Alexeev, B., Bandeira, A.S., Fickus, M., Mixon, D.G.: Phase retrieval with polarization. SIAM J. Imaging Sci. **7**, 35–66 (2014)
2. Backhouse, N.B., Bradley, C.J.: Projective representations of Abelian groups. Proc. Am. Math. Soc. **36**, 260–266 (1972)
3. Balan, R.: Stability of phase retrievable frames. In: Proceedings of SPIE, Wavelets and Sparsity XV, 88580H (2013)
4. Balan, R., Wang, Y.: Invertibility and robustness of phaseless reconstruction. Appl. Comput. Harmon. Anal. **38**, 469–488 (2015)
5. Balan, R., Zou, D.: On Lipschitz analysis and Lipschitz synthesis for the phase retrieval problem. Linear Algebra Appl. **496**, 152–181 (2016)
6. Balan, R., Casazza, P.G., Edidin, D.: On signal reconstruction from the absolute value of the frame coefficients. In: Proceedings of SPIE, vol. 5914, pp. 1–8 (2005)
7. Balan, R., Casazza, P., Edidin, D.: On signal reconstruction without phase. Appl. Comput. Harmon. Anal. **20**, 345–356 (2006)
8. Balan, R., Casazza, P.G., Edidin, D.: Equivalence of reconstruction from the absolute value of the frame coefficients to a sparse representation problem. IEEE Signal Process. Lett. **14**(5), 341–345 (2007)
9. Balan, R., Bodmann, B.G., Casazza, P.G., Edidin, D.: Fast algorithms for signal reconstruction without phase. In: Proceedings of SPIE-Wavelets XII, San Diego, vol. 6701, pp. 670111920–670111932 (2007)
10. Balan, R., Bodmann, B.G., Casazza, P.G., Edidin, D.: Painless reconstruction from magnitudes of frame vectors. J. Fourier Anal. Appl. **15**, 488–501 (2009)
11. Bandeira, A.S., Cahill, J., Mixon, D.G., Nelson, A.A.: Saving phase: injectivity and stability for phase retrieval. Appl. Comput. Harmon. Anal. **37**, 106–125 (2014)
12. Bandeira, A.S., Chen, Y., Mixon, D.G.: Phase retrieval from power spectra of masked signals. Inf. Inference **3**, 83–102 (2014)
13. Bendory, T., Eldar, Y.C.: A least squares approach for stable phase retrieval from short-time Fourier transform magnitude, preprint arXiv:1510.00920 (2015)
14. Bodmann, B.G., Hammen, N.: Stable phase retrieval with low-redundancy frames. Adv. Comput. Math. **41**, 317–33 (2015)
15. Bodmann, B.G., Casazza, P.G., Edidin, D., Balan, R.: Frames for Linear Reconstruction Without Phase. CISS Meeting, Princeton, NJ (2008)
16. Bojarovska, I., Flinth, A.: Phase retrieval from Gabor measurements. J. Fourier Anal. Appl. **22**, 542–567 (2016)
17. Candès, E.J., Li, X.: Solving quadratic equations via PhaseLift when there are about as many equations as unknowns. Found. Comput. Math. **14**, 1017–1026 (2014)
18. Candès, E.J., Eldar, Y.C., Strohmer, T., Voroninski, V.: Phase retrieval via matrix completion. SIAM J. Imaging Sci. **6**, 199–225 (2013)
19. Candès, E.J., Strohmer, T., Voroninski, V.: PhaseLift: exact and stable signal recovery from magnitude measurements via convex programming. Commun. Pure Appl. Math. **66**, 1241–1274 (2013)
20. Cheng, C.: A character theory for projective representations of finite groups. Linear Algebra Appl. **469**, 230–242 (2015)
21. Cheng, C., Fu, J.: On the rings of projective characters of Abelian groups and Dihedral groups, preprint (2016)
22. Conca, A., Edidin, D., Hering, M., Vinzant, C.: An algebraic characterization of injectivity in phase retrieval. Appl. Comput. Harmon. Anal. **38**, 346–356 (2015)
23. Eldar, Y.C., Sidorenko, P., Mixon, D.G., Barel, S., Cohen, O.: Sparse phase retrieval from short-time Fourier measurements. IEEE Signal Process. Lett. **22**, 638–642 (2015)
24. Eldar, Y.C., Hammen, N., Mixon, D.: Recent advances in phase retrieval. In: IEEE Signal Processing Magazine, pp. 158–162 (September 2016)
25. Fickus, M., Mixon, D.G., Nelson, A.A., Wang, Y.: Phase retrieval from very few measurements. Linear Algebra Appl. **449**, 475–499 (2014)
26. Jaganathan, K., Eldar, Y.C., Hassibi, B.: Phase retrieval: an overview of recent developments. arXiv:1510.07713
27. Karpilovsky, G.: The Schur Multiplier. London Mathematical Society Monographs. Clarendon Press, Oxford (1987)

28. Li, L., Cheng, C., Han, D., Sun, Q., Shi, G.: Phase retrieval from multiple-window short-time Fourier measurements. IEEE Signal Process. Lett. **24**, 372–376 (2017)
29. Nawab, S.H., Quatieri, T.F., Lim, J.S.: Signal reconstruction from short-time Fourier transform magnitude. IEEE Trans. Acoust. Speech Signal Process. **31**, 986–998 (1983)