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The Finite Hankel Transform Operator: Some Explicit and Local Estimates of the Eigenfunctions and Eigenvalues Decay Rates

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Abstract For fixed real numbers $c > 0$, $\alpha > -\frac{1}{2}$, the finite Hankel transform operator, denoted by $\mathcal{H}_c^\alpha$ is given by the integral operator defined on $L^2(0,1)$ with kernel $K_\alpha(x,y) = \sqrt{cxy}\,J_\alpha(cxy)$. To the operator $\mathcal{H}_c^\alpha$, we associate a positive, self-adjoint compact integral operator $\mathcal{Q}_c^\alpha = c\,\mathcal{H}_c^\alpha\,\mathcal{H}_c^\alpha$. Note that the integral operators $\mathcal{H}_c^\alpha$ and $\mathcal{Q}_c^\alpha$ commute with a Sturm-Liouville differential operator $\mathcal{D}_c^\alpha$. In this paper, we first give some useful estimates and bounds of the eigenfunctions $\varphi_{n,c}^{(\alpha)}$ of $\mathcal{H}_c^\alpha$ or $\mathcal{Q}_c^\alpha$. These estimates and bounds are obtained by using some special techniques from the theory of Sturm-Liouville operators, that we apply to the differential operator $\mathcal{D}_c^\alpha$. If $(\mu_{n,\alpha}(c))_n$ and $\lambda_{n,\alpha}(c) = c\,|\mu_{n,\alpha}(c)|^2$ denote the infinite and countable sequence of the eigenvalues of the operators $\mathcal{H}_c^{(\alpha)}$ and $\mathcal{Q}_c^\alpha$, arranged in the decreasing order of their magnitude, then we show an unexpected result that for a given integer $n \geq 0$, $\lambda_{n,\alpha}(c)$ is decreasing with respect to the parameter α. As a consequence, we show that for $\alpha \geq \frac{1}{2}$, the $\lambda_{n,\alpha}(c)$ and the $\mu_{n,\alpha}(c)$ have a super-exponential decay rate. Also, we give a lower decay rate of these eigenvalues. As it will be seen, the previous results are essential tools for the analysis of a spectral approximation scheme based on the eigenfunctions of the finite Hankel transform operator. Some numerical examples will be provided to illustrate the results of this work.

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1 Introduction

We first recall that for a bandwidth $c > 0$ and a real number $\alpha > -1/2$, the circular prolate spheroidal wave functions (CPSWFs), denoted by $(\varphi^{(\alpha)}_{n,c})_{n\geq 0}$ are the different eigenfunctions of the following finite Hankel transform operator, see for example [15,24]

$$\mathcal{H}^\alpha_c(f)(x) = \int_0^1 \sqrt{ctx}\, J_\alpha(ctx) f(t)\, dt, \quad x \in [0,1], \tag{1}$$

that is

$$\mathcal{H}^\alpha_c \varphi^{(\alpha)}_{n,c}(x)\, dy = \int_0^1 \sqrt{cxy}\, J_\alpha(cxy)\varphi^{(\alpha)}_{n,c}(y)\, dy = \mu_{n,\alpha}(c)\varphi^{(\alpha)}_{n,c}(x),$$
$$n \geq 0, \quad x \in [0,1]. \tag{2}$$

Here, J_α is the Bessel function of the first type and order $\alpha > -1/2$. We recall that the Hankel transform is defined on $L^2(0,\infty)$ by

$$\mathcal{H}^\alpha(f)(x) = \int_0^{+\infty} \sqrt{xy}\, J_\alpha(xy) f(y)\, dy. \tag{3}$$

Moreover, for $c > 0$, the Hankel Paley-Wiener space is the space of functions from $L^2(0,\infty)$, having compactly supported Hankel transforms, that is

$$\mathcal{B}^\alpha_c = \{f \in L^2(0,\infty);\ \ \mathrm{Supp}\mathcal{H}^\alpha(f) \subset [0,c]\}. \tag{4}$$

Although, in the literature, there exist extensive works devoted to the numerical computation of the CPSWFs and their associated eigenvalues $\mu_{n,\alpha}(c)$, see for example [2,15,21,24], very few references have dealt so far with the subject of the explicit estimates and bounds of the $\varphi^{(\alpha)}_{n,c}$, as well as the decay rate of the eigenvalues $\mu_{n,\alpha}(c)$ or $\lambda_{n,\alpha}(c)$. In particular in [17], the author has shown that asymptotically, the magnitudes of the $\mu_{n,\alpha}(c)$ have an exponential decay rate. Our objective from this work is to provide the reader with some useful explicit local estimates and bounds of the CPSWFs, as well as some explicit and simple lower and upper bounds of the eigenvalues $\lambda_{n,\alpha}(c)$, with $\alpha \geq 1/2$. To prove the upper bound, we first prove a surprising result given by Theorem 5 and states that for fixed n and c, the $\lambda_{n,\alpha}(c)$ are monotonically decreasing in α, that is for any integer $n \geq 0$, we have

$$\lambda_{n,\alpha}(c) \leq \lambda_{n,\alpha'}(c), \quad \forall\, \alpha \geq \alpha' > -\frac{1}{2}. \tag{5}$$

Then, by using the known exponential decay rate of the eigenvalues of the conventional PSWFs corresponding to the case $\alpha = 1/2$, see [7], one gets a simple upper bound for the $\lambda_{n,\alpha}(c),\ \alpha \geq 1/2$.

We should mention that the interest from the study of the eigenfunctions of the finite Hankel transform (CPSWFs) and in general the prolate spheroidal wave functions, comes from the fact they are widely used in various scientific area, such as applied mathematics, physics, engineering, see [12] for some of these concrete applications. In the pioneer work [24], D. Slepian has shown that the compact integral operator $\mathcal{H}_c^\alpha$ commutes with the following differential operator $\mathcal{D}_c^\alpha$ defined on $C^2([0,1])$ by

$$\mathcal{D}_c^\alpha(\phi)(x) = \frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\phi(x)\right] + \left(\frac{\frac{1}{4}-\alpha^2}{x^2} - c^2x^2\right)\phi(x). \tag{6}$$

Hence, $\varphi_{n,c}^{(\alpha)}$ is the $n-$th order bounded eigenfunction of the operator $-\mathcal{D}_c^\alpha$, associated with the eigenvalue $\chi_{n,\alpha}(c)$, that is

$$\begin{aligned} &-\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\varphi_{n,c}^{(\alpha)}(x)\right] - \left(\frac{\frac{1}{4}-\alpha^2}{x^2} - c^2x^2\right)\varphi_{n,c}^{(\alpha)}(x) \\ &= \chi_{n,\alpha}(c)\varphi_{n,c}^{(\alpha)}(x), \quad x \in [0,1]. \end{aligned} \tag{7}$$

In this work, we take advantage from the commutativity property of the operators $\mathcal{D}_c^\alpha$ and $\mathcal{H}_c^\alpha$ and prove some useful local estimates and bounds of the $\varphi_{n,c}^{(\alpha)}(x),\ x \in I$. Note that some estimates and bounds of the classical prolate spheroidal wave functions and their generalized versions, were already given in [5,16]. Nonetheless, in our present case of the CPSWFs, the techniques used in the previous references have to be modified and combined with new techniques based on the use of the Sturm-Liouville comparison theorem and Butlewski's theorem. These new techniques are needed in order to handle the extra difficulty caused by the singularity at $x = 0$, appearing in the differential operator $\mathcal{D}_c^\alpha$. Also, by using the characterization of the eigenvalues $\lambda_{n,\alpha}(c)$ in terms of an energy maximization problem, combined with Griffith's theorem which is a Paley-Wiener theorem for the Hankel transform, we prove an interesting result that the $\lambda_{n,\alpha}(c)$ are decreasing with respect to the parameter α. As a consequence, and by using the sharp decay rate of the eigenvalues of the finite Fourier transform, given in [7], we give a super-exponential decay rate of the $\lambda_{n,\alpha}(c)$, for $\alpha \geq \frac{1}{2}$.

This work is organised as follows. In Sect. 2, we give some mathematical preliminaries related to the properties and computation of the CPSWFs. In Sect. 3, we first provide a local estimate for the $\varphi_{n,c}^{(\alpha)}(x)$. Then, we give a bound of $|\varphi_{n,c}^{(\alpha)}(x)|$ for $x \in [0,1]$. The previous results are obtained under the condition that $\chi_{n,\alpha}(c) > c^2 + \alpha^2 - \frac{1}{4}$, where $\chi_{n,\alpha}(c)$ is the $n-$th eigenvalues $\chi_{n,\alpha}(c)$ of the differential operator $\mathcal{Q}_c^\alpha$. By using the classical Strurm-Liouville comparison theorem, we prove that $\chi_{n,\alpha}(c)$ passes through $c^2 + \alpha^2 - \frac{1}{4}$ when n is around $\frac{c}{\pi}$. In Sect. 4, we give an upper and a lower bound of the super-exponential decay rate of the eigenvalues $\lambda_{n,\alpha}(c)$. Finally, in Sect. 5, we provide the reader with some numerical examples that illustrate the different results of this work. Moreover, in this section, we also show that the $\varphi_{n,c}^{(\alpha)}$ are well adapted for the approximation of Hankel band-limited and almost band-limited functions.

2 Mathematical Preliminaries

In this section, we first give a brief description of the computation and the decay rate of the series expansion coefficients d_k^n of the eigenfunctions $\varphi_{n,c}^{(\alpha)}$ in an appropriate basis of $L^2([0,1])$. This basis is given by the orthogonal functions $\widetilde{T}_{k,\alpha}(x) = x^{\alpha+1/2}(-1)^k\sqrt{2(2k+\alpha+1)}P_k^{(\alpha,0)}(1-2x^2)$, $k \geq 0$. Here, $P_k^{(\alpha,\beta)}$ is the Jacobi polynomial of degree k and parameters $\alpha, \beta > -1$, normalized so that $P_k^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}$. Then, we relate the eigenvalues of the compact and positive operator

$$\mathcal{Q}_c^\alpha = c\mathcal{H}_c^\alpha\left(\mathcal{H}_c^\alpha\right)^* = c\mathcal{H}_c^\alpha\left(\mathcal{H}_c^\alpha\right) \tag{8}$$

to the solutions of a classical energy maximization problem over the Paley-Wiener space $\mathcal{B}_c^\alpha$, given by (4).

Note that thanks to the important commutativity property of the differential and integral operators $\mathcal{D}_c^\alpha$ and $\mathcal{H}_c^\alpha$, D. Slepian has developed in [24], an efficient computational scheme for the $\varphi_{n,c}^{(\alpha)}(x)$, $x \geq 0$, as well as for their corresponding eigenvalues $\chi_{n,\alpha}(c)$ and $\mu_{n,\alpha}(c)$. The Slepian scheme for the computation of $\varphi_{n,c}^{(\alpha)}(x)$, is given by the following series expansion,

$$\varphi_{n,c}^{(\alpha)}(x) = \sum_{k=0}^{+\infty} d_k^n\ \widetilde{T}_{k,\alpha}(x),$$
$$\widetilde{T}_{k,\alpha}(x) = (-1)^k\sqrt{2(2k+\alpha+1)}x^{\alpha+\frac{1}{2}}P_k^{(\alpha,0)}(1-2x^2), \quad x \in [0,1]. \tag{9}$$

Here, d_k^n are the expansion coefficients, given as the eigenvectors a tri-diagonal infinite order matrix. Moreover, by combining the integral equation (2) and the previous expansion, D. Slepian has given the following analytic extension of the $\varphi_{n,c}^{(\alpha)}$, over the unbounded interval $[1, +\infty)$,

$$\varphi_{n,c}^{(\alpha)}(x) = \frac{1}{\mu_{n,\alpha}(c)}\sum_{k\geq 0}(-1)^k d_k^n\sqrt{2(2k+\alpha+1)}\frac{J_{2k+\alpha+1}(cx)}{\sqrt{cx}}, \quad x \geq 1. \tag{10}$$

By evaluating the two expansions (9) and (10) at $x = 1$, one gets the following expression of the eigenvalues $\mu_{n,\alpha}(c)$,

$$\mu_{n,\alpha}(c) = \frac{1}{\sqrt{c}}\frac{\sum_{k\geq 0}(-1)^k d_k^n\sqrt{2(2k+\alpha+1)}J_{2k+\alpha+1}(c)}{\sum_{k=0}^{+\infty} d_k^n\ \widetilde{T}_{k,\alpha}(1)}. \tag{11}$$

In this work, we check that for a fixed positive integer n, the sequence $(d_k^n)_{k\geq 0}$ has a super-exponential decay rate. Consequently, the previous formulae for computing the $\varphi_{n,c}^{(\alpha)}$ and their eigenvalues $\mu_{n,\alpha}(c)$ are practical and highly accurate. Note that by using a slightly modified techniques of those used in [24], one can easily check that the expansion coefficients can be computed by solving the following tri-diagonal system

$$\sum_{k\geq 0}\left[c^2 a_{k,\alpha} d_{k-1}^n + \left(\left(2k+\alpha+\frac{1}{2}\right)\left(2k+\alpha+\frac{3}{2}\right)+c^2 b_{k,\alpha}\right) d_k^n + c^2 a_{k+1,\alpha} d_{k+1}^n\right]\widetilde{T}_{k,\alpha}(x)$$
$$= \chi_{n,\alpha}(c)\sum_{k\geq 0} d_k^n\, \widetilde{T}_{k,\alpha}, \tag{12}$$

where

$$a_{k+1,\alpha} = \frac{(k+1)(\alpha+k+1)}{(\alpha+2k+2)\sqrt{\alpha+2k+1}\sqrt{\alpha+2k+3)}},$$
$$b_{k,\alpha} = \frac{1}{2}\left(\frac{\alpha^2}{(\alpha+2k+2)(\alpha+2k)}+1\right). \tag{13}$$

The previous system can be written in the following eigensystem

$$MD = \chi_n D, \quad M = [m_{i,j}]_{i,j\geq 0}, \quad D = [d_k^n]_{k\geq 0}^T \tag{14}$$

with $m_{k,j} = 0$, if $\mid k-j \mid \geq 2$ and

$$m_{k,k-1} = c^2 a_{k,\alpha}, \quad m_{k,k} = \left(\alpha+2k+\frac{1}{2}\right)\left(\alpha+2k+\frac{3}{2}\right)$$
$$+ c^2 b_{k,\alpha}, m_{k,k+1} = m_{k,k+1}. \tag{15}$$

Also, note that the expansion coefficients $(d_k^n)_k$ are related to $\varphi_{n,c}^{(\alpha)}$ by the following relation,

$$d_k^n = \frac{(-1)^k\sqrt{2(2k+\alpha+1)}}{\mu_{n,\alpha}(c)}\int_0^1 \varphi_{n,c}^{(\alpha)}(y)\frac{J_{2k+\alpha+1}(y)}{\sqrt{cy}}dy. \tag{16}$$

In fact, from (2), we have

$$d_k^n = \frac{1}{\mu_{n,\alpha}(c)}\int_0^1 \widetilde{T}_{k,\alpha}(x)\int_0^1 \sqrt{cxy}J_\alpha(cxy)\varphi_{n,c}^{(\alpha)}(y)dydx$$
$$= \frac{1}{\mu_{n,\alpha}(c)}\int_0^1 \varphi_{n,c}^{(\alpha)}(y)\int_0^1 \sqrt{cxy}J_\alpha(cxy)\widetilde{T_{k,\alpha}}(x)dxdy$$

Also, from [15,24], one has

$$\int_0^1 \sqrt{cxy}J_\alpha(cxy)\widetilde{T}_{k,\alpha}(x)dx = (-1)^k\sqrt{2(2k+\alpha+1)}\frac{J_{2k+\alpha+1}(y)}{\sqrt{cy}}. \tag{17}$$

By combining the previous two equalities, one gets (16).

It is well known that the eigenvalues $\chi_{n,\alpha}(c)$ of the differential operator $\mathcal{D}_c^\alpha$ satisfy the differential equation

$$\frac{\partial \chi_{n,\alpha}(c)}{\partial c} = 2c\int_0^1 x^2 \left(\varphi_{n,c}^{(\alpha)}(x)\right)^2\, dx. \tag{18}$$

Since $\chi_{n,\alpha}(0) = (2n+\alpha+\frac{1}{2})(2n+\alpha+\frac{3}{2})$ is the $n-$th eigenvalue of the differential operator $\mathcal{D}_0^\alpha$, and since $0 \le c^2x^2 \le c^2$ for all $x \in [0,1]$, then by using the Min-Max principle, the $n-$th eigenvalue $\chi_{n,\alpha}(c)$ of the differential operator $\mathcal{D}_c^\alpha$ satisfies the following bounds,

$$\left(2n+\alpha+\frac{1}{2}\right)\left(2n+\alpha+\frac{3}{2}\right) \le \chi_{n,\alpha}(c) \le \left(2n+\alpha+\frac{1}{2}\right)\left(2n+\alpha+\frac{3}{2}\right) + c^2. \tag{19}$$

Next, we briefly check a classical result that the eigenvalues $\lambda_{n,\alpha}(c)$ of the integral operator $\mathcal{Q}_c^\alpha$, given by (8) are characterized as the solutions of an energy maximization problem over the Hankel Paley-Wiener space $\mathcal{B}_c^\alpha$. In fact, from [26, p. 154], we have

$$\begin{aligned} G_\alpha(x,y) &= \int_0^1 \sqrt{xy}\, J_\alpha(xt)J_\alpha(yt)t\, dt \\ &= \frac{\sqrt{xy}}{x^2-y^2}\big(xJ_{\alpha+1}(x)J_\alpha(y) - yJ_{\alpha+1}(y)J_\alpha(x)\big). \end{aligned} \tag{20}$$

On the other hand, by using the previous identity and since $\mathcal{H}_c^\alpha$ is self-adjoint, then a straightforward computation, gives us

$$\mathcal{Q}_c^\alpha(f)(x) = c\,\mathcal{H}_c^\alpha\mathcal{H}_c^\alpha(f)(x) = c\int_0^1 G_\alpha(cx,cy)f(y)\, dy, \tag{21}$$

where the kernel $G_\alpha(x,y)$ is given by (20). On the other hand, since the Hankel transform operator is its own inverse and since by Plancherel formula, we have for $f \in \mathcal{B}_c^\alpha$,

$$\|f\|^2_{L^2(0,\infty)} = \|\mathcal{H}^\alpha f\|^2_{L^2(0,\infty)} = c\int_0^1 (\mathcal{H}^\alpha(f))^2(cx)\, dx,$$

then, for $f \in \mathcal{B}_c^\alpha$, we have

$$\begin{aligned} \frac{\|f\|^2_{L^2(0,1)}}{\|f\|^2_{L^2(0,\infty)}} &= \frac{\int_0^1 (f(t))^2\, dt}{\|f\|^2_{L^2(0,\infty)}} \\ &= \frac{\int_0^1 \left(\int_0^c \sqrt{Xt}\, J_\alpha(Xt)\mathcal{H}^\alpha(f)(X)\, dX\right)\cdot\left(\int_0^c \sqrt{Yt}\, J_\alpha(Yt)\mathcal{H}^\alpha(f)(Y)\, dY\right)\, dt}{c\int_0^1 (\mathcal{H}^\alpha(f))^2(cx)\, dx} \end{aligned}$$

$$= \frac{c^2 \int_0^1 \left(\int_0^1 G_\alpha(cx, cy) \mathcal{H}^\alpha(f)(cy)\, dy \right) \cdot \mathcal{H}^\alpha(f)(cx)\, dx}{c \int_0^1 (\mathcal{H}^\alpha(f))^2(cx)\, dx}$$

$$= \frac{\int_0^1 \mathcal{Q}_c^\alpha \mathcal{H}^\alpha(f)(cx) \cdot \mathcal{H}^\alpha(f)(cx)\, dx}{\int_0^1 (\mathcal{H}^\alpha(f))^2(cx)\, dx}.$$

A standard result about the maximization of a quadratic form tells us that the solution of the energy maximization problem

$$\text{find } f = \arg \max_{f \in \mathcal{B}_c^\alpha} \frac{\|f\|^2_{L^2(0,1)}}{\|f\|^2_{L^2(0,\infty)}} \tag{22}$$

is given by the first eigenfunction, with the largest eigenvalue $\lambda_{0,\alpha}(c)$ of the operator $\mathcal{Q}_c^\alpha$. Since $\mathcal{Q}_c^\alpha = c\,\mathcal{H}_c^\alpha \mathcal{H}_c^\alpha$, then the eigenfunctions of $\mathcal{Q}_c^\alpha$ are also the eigenfunctions $\varphi_{n,c}^{(\alpha)}$ of $\mathcal{H}_c^\alpha$ and the eigenvalues of $\mathcal{Q}_c^\alpha$ are related to the eigenvalues of $\mathcal{H}_c^\alpha$ by the following rule

$$\lambda_{n,\alpha}(c) = c\,|\mu_{n,\alpha}(c)|^2, \quad n \geq 0. \tag{23}$$

Finally, we should mention that throughout this work, the eigenfunctions $\varphi_{n,c}^{(\alpha)}$ are normalized by the following rule,

$$\int_0^1 \left(\varphi_{n,c}^{(\alpha)}(x) \right)^2 dx = 1, \quad \int_0^\infty \left(\varphi_{n,c}^{(\alpha)}(x) \right)^2 dx = \frac{1}{\lambda_{n,\alpha}(c)}. \tag{24}$$

3 Some Explicit Estimates and Bounds of the Eigenfunctions

In this paragraph, we give an explicit upper bound of $|\varphi_{n,c}^{(\alpha)}(x)|$ with $x \in I = [0, 1]$, and $\alpha > -1/2$. To this end, we first show that under some conditions on n, c, the maximum of $|\varphi_{n,c}^{(\alpha)}(x)|$ is attained at $x = 1$. This is given by the following lemma.

Lemma 1 *Let $c > 0$, $\alpha > -1/2$ be two real numbers. If $c^2 > \alpha^2 - \frac{1}{4}$, and $\chi_{n,\alpha}(c) > c^2 + \alpha^2 - \frac{1}{4}$, then we have*

$$\sup_{x \in [a_\alpha, 1]} \left| \varphi_{n,c}^{(\alpha)}(x) \right| = \left| \varphi_{n,c}^{(\alpha)}(1) \right|, \quad a_\alpha = \begin{cases} 0 & \text{if } \alpha^2 \leq 1/4 \\ \left(\frac{\alpha^2 - 1/4}{c^2} \right)^{1/4} & \text{if } \alpha > 1/2. \end{cases} \tag{25}$$

Proof We recall that $\varphi_{n,c}^{(\alpha)}$ is a solution of the following differential equation

$$\frac{d}{dt} \left[p(t) \left(\varphi_{n,c}^{(\alpha)} \right)'(t) \right] + q_\alpha(t) \varphi_{n,c}^{(\alpha)}(t) = 0,$$

with

$$p(t) = (1 - t^2), \quad q_\alpha(t) = \chi_{n,\alpha}(c) - c^2 t^2 + \frac{\frac{1}{4} - \alpha^2}{t^2}. \tag{26}$$

Next, consider the auxiliary function Q_α, defined on $[a_\alpha, 1]$ by

$$Q_\alpha(t) = \left(\varphi_{n,c}^{(\alpha)}(t)\right)^2 + \frac{p(t)}{q_\alpha(t)}\left(\left(\varphi_{n,c}^{(\alpha)}\right)'(t)\right)^2. \tag{27}$$

By computing the derivative of Q_α and then using the identity

$$\left(\varphi_{n,c}^{(\alpha)}\right)''(t) = \frac{2t}{1-t^2}\left(\varphi_{n,c}^{(\alpha)}\right)'(t) - \frac{1}{t^2}\left(\chi_n - c^2t^2 + \frac{1/4-\alpha^2}{t^2}\right)\varphi_{n,c}^{(\alpha)}(t),$$

one gets

$$Q'_\alpha(t) = \frac{2t}{q_\alpha^2(t)}\left(\chi_{n,\alpha}(c) + c^2 - 2c^2t^2 - \frac{\alpha^2-1/4}{t^4}\right)\left(\left(\varphi_{n,c}^{(\alpha)}\right)'(t)\right)^2. \tag{28}$$

Note that

$$\begin{aligned} &\chi_{n,\alpha}(c) + c^2 - 2c^2t^2 - \frac{\alpha^2-1/4}{t^4} \\ &\quad\geq 2c^2 + (\alpha^2-1/4) - 2c^2t^2 - \frac{\alpha^2-1/4}{t^4} \\ &\quad\geq (1-t^2)\left(2c^2 - (\alpha^2-1/4)\frac{1+t^2}{t^4}\right) \geq 0, \quad t \in (a_\alpha, 1]. \end{aligned} \tag{29}$$

Hence, by combining (28) and (29), one concludes that Q_α is increasing on [0, 1] and consequently,

$$\left(\varphi_{n,c}^{(\alpha)}\right)^2(t) \leq Q_\alpha(t) \leq \left(\varphi_{n,c}^{(\alpha)}\right)^2(1), \quad t \in [a_\alpha, 1],$$

which concludes the proof of the lemma. □

The following lemma provides us with a useful local estimate of the eigenfunctions $\varphi_{n,c}^{(\alpha)}$.

Lemma 2 *Under the notation and conditions of the previous lemma, we have for* $\alpha > -1/2$,

$$\sup_{t\in[a_\alpha,1]} \sqrt{1-t^2}\left|\varphi_{n,c}^{(\alpha)}(t)\right| \leq \sqrt{2}. \tag{30}$$

Proof We first consider the auxiliary function $K_\alpha(\cdot)$, defined by $K_\alpha(t) = (1-t^2)Q_\alpha(t)$, where Q_α is given by (27). Straightforward computations give us

$$
\begin{aligned}
K'_\alpha(t) &= -2t Q_\alpha(t) + (1-t^2) Q'_\alpha(t) \\
&= -2t\left(\varphi_{n,c}^{(\alpha)}\right)^2(t) + \frac{2t(1-t^2)}{q_\alpha(t)} \\
&\quad \times \left(\frac{\chi_{n,\alpha}(c) + c^2 - 2c^2t^2 + (1/4-\alpha^2)/t^4}{q_\alpha(t)} - 1\right)\left(\left(\varphi_{n,c}^{(\alpha)}\right)'(t)\right)^2 \\
&= -2t\left(\varphi_{n,c}^{(\alpha)}\right)^2(t) + \frac{2t(1-t^2)^2}{q_\alpha^2(t)}\left(c^2 - \frac{\alpha^2-1/4}{t^4}\right)\left(\left(\varphi_{n,c}^{(\alpha)}\right)'(t)\right)^2 \\
&\geq -2t\left(\varphi_{n,c}^{(\alpha)}\right)^2(t), \quad t \in [a_\alpha, 1].
\end{aligned}
$$

Hence, we have

$$
K_\alpha(x) = K_\alpha(x) - K_\alpha(1) \leq \int_x^1 2t\left(\varphi_{n,c}^{(\alpha)}\right)^2(t)\,dt \leq 2\int_0^1 \left(\varphi_{n,c}^{(\alpha)}\right)^2(t)\,dt = 2,
$$

which concludes the proof of the lemma. □

As a consequence of Lemmas 1 and 2, we obtain a bound for the eigenfunctions $\varphi_{n,c}^{(\alpha)}$, given by the following proposition.

Proposition 1 *Let $c > 0$, $\alpha > -1/2$ be two real numbers. If $c^2 > \alpha^2 - \frac{1}{4}$, and $\chi_{n,\alpha}(c) > c^2 + \alpha^2 - \frac{1}{4}$, then we have*

$$
\sup_{x\in[a_\alpha,1]} \left|\varphi_{n,c}^{(\alpha)}(x)\right| \leq 3\sqrt{\frac{3}{2}}\left(\chi_{n,\alpha}(c)\right)^{1/2}. \tag{31}
$$

Here, a_α is as given by (31).

Proof Without loss of generality, we may assume that $\varphi_{n,c}^{(\alpha)}(1) > 0$. By integrating (7) over the interval $[x, 1]$, with $x \in J = [a_\alpha, 1)$, one gets

$$
\left(\varphi_{n,c}^{(\alpha)}\right)'(x) = \frac{\chi_{n,\alpha}(c)}{1-x^2}\int_x^1 \left(1 - qt^2 - \frac{\alpha^2-1/4}{\chi_{n,\alpha}(c)\,t^2}\right)\varphi_{n,c}^{(\alpha)}(t)\,dt, \quad q = c^2/\chi_{n,\alpha}(c). \tag{32}
$$

Let G_α be the function defined on J by

$$
G_\alpha(t) = 1 - qt^2 - \frac{\alpha^2-1/4}{\chi_{n,\alpha}(c)\,t^2}.
$$

It can be easily checked that if $c^2 > \alpha^2 - \frac{1}{4}$, then G_α is decreasing and positive in J. Hence, by using (31) and (32), one gets

$$
\left|\left(\varphi_{n,c}^{(\alpha)}\right)'(x)\right| \leq \frac{\chi_{n,\alpha}(c)}{1-x^2} G_\alpha(x)\varphi_{n,c}^{(\alpha)}(1)(1-x) = \frac{\chi_{n,\alpha}}{1+x} G_\alpha(x)\varphi_{n,c}^{(\alpha)}(1).
$$

Consequently, we have,

$$\left|\varphi_{n,c}^{(\alpha)}(1)-\varphi_{n,c}^{(\alpha)}(x)\right| \leq \frac{\chi_{n,\alpha}(c)}{1+x} G_\alpha(x)(1-x)\varphi_{n,c}^{(\alpha)}(1). \qquad (33)$$

In a similar manner as it is done in [5], let $x_n \in J$ with

$$G_\alpha(x_n)\frac{1-x_n}{1+x_n} = \frac{a}{\chi_{n,\alpha}(c)}, \qquad (34)$$

where $a > 0$ is a constant to be fixed later on. By combining (33) and (34) and by using the result of lemma 1, one gets

$$\varphi_{n,c}^{(\alpha)}(1) \leq \frac{1}{1-a}\left|\varphi_{n,c}^{(\alpha)}(x_n)\right| \leq \frac{1}{1-a}\frac{\sqrt{2}}{\sqrt{1-x_n^2}}. \qquad (35)$$

On the other hand, since for any $x \in J$, we have $\frac{G_\alpha(x)}{1+x} \leq 1$, then from (34), we have

$$\frac{1}{\sqrt{1-x_n^2}} \leq \frac{1}{\sqrt{1-x_n}} \leq \sqrt{\frac{\chi_{n,\alpha}}{a}}.$$

By combining the previous two inequalities, one gets

$$\varphi_{n,c}^{(\alpha)}(1) \leq \frac{1}{a^{1/2}(1-a)}\left(\chi_{n,\alpha}(c)\right)^{1/2}.$$

To conclude the proof, it suffices to note that the minimum of the quantity $\frac{1}{a^{1/2}(1-a)}$ is obtained for $a = 1/3$. □

To extend the previous result to the case where $\alpha > \frac{1}{2}$, and the interval $[a_\alpha, 1]$ is substituted with the whole interval $[0, 1]$, we first need to locate the first positive zero of $\varphi_{n,c}^{(\alpha)}$. For this purpose, we use the following Sturm-Liouville comparison theorem, that compares the zeros of the eigenfunctions of two second order differential operators, see for example [3, page 4]

Theorem 1 (Sturm Comparison Theorem) *Let $p_i, r_i, i = 1, 2$ be two real continuous functions on the interval $[a, b]$ and let*

$$\left(p_1(x)u'\right)' + r_1(x)u = 0, \quad \left(p_2(x)v'\right)' + r_2(x)v = 0,$$

be two ODE with $0 < p_2(x) \leq p_1(x)$ and $r_1(x) \leq r_2(x)$. Then between any two zeros of u, there exists a zero of v.

The following proposition gives a location of the first zero of $\varphi_{n,c}^{(\alpha)}$, where $\alpha > 1/2$.

Proposition 2 *Let $c > 0, \alpha > 1/2$ be two real numbers. Let $x_{1,n}$ be the first positive zero of $\varphi_{n,c}^{(\alpha)}$. Then for any integer n satisfying $\chi_{n,\alpha}(c) \geq 2c\sqrt{\alpha^2 - 1/4}$, we have*

$$a_\alpha^2 = \sqrt{\frac{\alpha^2 - \frac{1}{4}}{\chi_{n,\alpha}(c)}} \leq x_{1,n} \leq \frac{\pi + \frac{\pi}{2}\alpha - \frac{3}{4}}{\sqrt{\chi_{n,\alpha}(c) - (\alpha^2 - 1/4)}} = b_\alpha. \tag{36}$$

Proof To prove the previous lower bound, we first note that $(\varphi_{n,c}^{(\alpha)})'(0) = 0$ whenever $\alpha > 1/2$. Moreover, from the equality,

$$\frac{d}{dx}\left[(1-x^2)\left(\varphi_{n,c}^{(\alpha)}\right)'(x)\right] = \left(-\chi_{n,\alpha}(c) + c^2x^2 + \frac{\alpha^2 - 1/4}{x^2}\right)\varphi_{n,c}^{(\alpha)}(x), \tag{37}$$

one concludes that $\varphi_{n,c}^{(\alpha)}(x)$ and $(\varphi_{n,c}^{(\alpha)})'(x)$ have the same positive sign around $x = 0$, as long as the quantity $-\chi_{n,\alpha}(c) + c^2x^2 + \dfrac{\alpha^2 - 1/4}{x^2} \geq 0$. Straightforward computations show that this is the case when $0 < x \leq r_1$, with $r_1^2 = \dfrac{\chi_{n,\alpha}(c)}{2c^2}\left(1 - \sqrt{1 - 4c^2(\alpha^2 - 1/4)/(\chi_{n,\alpha}(c))^2}\right)$. Consequently, we have

$$x_{1,n} \geq r_1 \geq \sqrt{\frac{\alpha^2 - \frac{1}{4}}{\chi_{n,\alpha}(c)}}.$$

To prove the upper bound in (36), we use the change of function

$$U = (1-x^2)^{1/2}\varphi_{n,c}^{(\alpha)} \tag{38}$$

that transforms the differential equation (7) to the following equation for U, which has the same zeros as φ on (0, 1),

$$U'' + \left((1-x^2)^{-2} + \frac{\chi_{n,\alpha}(c) - c^2x^2}{1-x^2} + \frac{\frac{1}{4} - \alpha^2}{x^2(1-x^2)}\right)U = 0, \quad x \in (0,1). \tag{39}$$

Since $\chi_{n,\alpha}(c) \geq c^2 + \alpha^2 - \frac{1}{4}$, then we have $-c^2x^2 \geq -\chi_{n,\alpha}(c)x^2 + (\alpha^2 - 1/4)x^2$. Consequently, we have

$$\begin{aligned}(1-x^2)^{-2} + \frac{\chi_{n,\alpha}(c) - c^2x^2}{1-x^2} + \frac{\frac{1}{4} - \alpha^2}{x^2(1-x^2)} &\geq \frac{\chi_{n,\alpha}(c) - c^2x^2}{1-x^2} + \frac{\frac{1}{4} - \alpha^2}{x^2(1-x^2)}\\ &\geq \chi_{n,\alpha}(c) + \frac{\alpha^2 - 1/4}{1-x^2}\frac{x^4 - 1}{x^2}\\ &\geq \chi_{n,\alpha}(c) - (\alpha^2 - 1/4) - \frac{\alpha^2 - 1/4}{x^2}.\end{aligned}$$

Then, we use Sturm Comparison theorem to conclude that the first positive zero of U or of $\varphi_{n,c}^{(\alpha)}$ lies before the second zero of the bounded solution of the differential equation,

$$V'' + \left(\left(\chi_{n,\alpha}(c) - (\alpha^2 - 1/4)\right) - \frac{\alpha^2 - 1/4}{x^2}\right) V = 0, \quad x \in (0, 1). \tag{40}$$

It is well known that the bounded solution of the previous differential equation is given by

$$V(x) = \sqrt{x}\, J_\alpha \left(\sqrt{\chi_{n,\alpha}(c) - (\alpha^2 - 1/4)}x\right). \tag{41}$$

Note that since $x = 0$ is a first zero of V and since from [9], see also [10], an upper bound of $j_{\alpha,k}$, the $k-$th positive zero of the Bessel function $J_\alpha(\cdot)$ is given by

$$j_{\alpha,k} < k\pi + \frac{\pi}{2}\alpha - \frac{0.965}{4}\pi < k\pi + \frac{\pi}{2}\alpha - \frac{3}{4}. \tag{42}$$

Consequently, by using the Sturm comparison theorem applied to the equations (39) and (40), one concludes that the first positive zero of U or of $\varphi_{n,c}^{(\alpha)}$ lies before

$$\frac{j_{\alpha,1}}{\sqrt{\chi_{n,\alpha}(c) - (\alpha^2 - 1/4)}} \leq \frac{\pi + \frac{\pi}{2}\alpha - \frac{3}{4}}{\sqrt{\chi_{n,\alpha}(c) - (\alpha^2 - 1/4)}},$$

which concludes the proof of the proposition.

By using the results of proposition 1 and proposition 2, we get the following theorem that provides us with a bound for $|\varphi_{n,c}^{(\alpha)}(x)|$, $x \in [0, 1]$ which is valid for any $\alpha > -1/2$.

Theorem 2 *Let $c > 0$ and $\alpha > -1/2$, be such that $c^2 > \alpha^2 - 1/4$. Then, for any positive integer n with $\chi_{n,\alpha}(c) \geq c^2 + \alpha^2 - 1/4$ and $\frac{\sqrt{\chi_{n,\alpha}(c)}}{1 - b_\alpha} b_\alpha^{3/2} \leq 3\sqrt{3/2}$, where b_α is given by (36), we have*

$$\sup_{x \in [0,1]} \left|\varphi_{n,c}^{(\alpha)}(x)\right| \leq 3\sqrt{\frac{3}{2}} \left(\chi_{n,\alpha}(c)\right)^{1/2}. \tag{43}$$

Proof We first recall that if $\alpha^2 \leq \frac{1}{4}$, then $a_\alpha = 0$ and the inequality (43) follows from proposition 1. Hence, it suffices to consider the case where $\alpha > 1/2$. For this purpose, we use Butlewski's theorem, regarding the behaviour of the local extrema of the solution of a second order differential equation, see for example [4, p. 238]. More precisely, if ϕ is a solution of the differential equation

$$\frac{d}{dt}\left(p(t)y'(t)\right) + q(t)y(t) = 0, \quad t \in (a, b), \tag{44}$$

where $p(t)$ and $q(t)$ are two positive functions belonging to $C'(a, b)$, then the local maxima of $|\ \phi\ |$ is increasing or decreasing, according to the condition that $p(t)q(t)$ is decreasing or increasing. In our case, we have

$$p(t) = (1-t^2), \quad q(t) = q_n(t) = \chi_{n,\alpha}(c) - c^2t^2 - \frac{\alpha^2 - 1/4}{t^2}, \quad t \in (0,1).$$

Since

$$\frac{d}{dt}(p(t)q_n(t)) = \big(8c^2t^6 - 4\big(\chi_{n,\alpha}(c) + c^2\big)t^4 + 4\alpha^2 - 1\big)/(4t^3),$$

and since $\chi_{n,\alpha}(c) \geq 2\alpha^2 - 1/2$, then one can easily check that there exists a unique real number $t_{\alpha,n} \in \left[\sqrt{\frac{\alpha^2-1/4}{\chi_{n,\alpha}(c)}}, \frac{1}{2}\right]$, so that the function $p(t)q_n(t)$ is increasing in $(0, t_{\alpha,n})$ and decreasing in $(t_{\alpha,n}, 1)$. Hence, from Butlewski's theorem, the local maxima of $|\varphi_{n,c}^{(\alpha)}|$ are decreasing in $(0, t_{\alpha,n})$ and increasing in $(t_{\alpha,n}, 1)$. From the Proof of Proposition 2, we know that the first zero of $(\varphi_{n,c}^{(\alpha)})'$, denoted by $x'_{1,n}$ is located in $I_\alpha = [a_\alpha^2, b_\alpha]$, where a_α, b_α are given by (36). Hence, by integrating (37) over the interval $[x, x'_{1,n}]$, where $x \in I_\alpha$ and then using Hölder's inequality, one gets

$$\big(\varphi_{n,c}^{(\alpha)}\big)'(x) = \frac{-1}{1-x^2}\int_x^{x'_{1,n}} \left(-\chi_{n,\alpha}(c) + c^2t^2 + \frac{\alpha^2 - 1/4}{t^2}\right)\varphi_{n,c}^{(\alpha)}(t)\,dt, \quad a_\alpha^2 \leq x \leq b_\alpha.$$

On the other hand, from the expression of $(\varphi_{n,c}^{(\alpha)}(x))''$, one can easily check that this later is positive whenever $0 < x \leq a_\alpha^2$. Consequently, we have

$$\left|\big(\varphi_{n,c}^{(\alpha)}\big)'(x)\right| \leq \frac{\chi_{n,\alpha}(c)}{1-x^2}\int_x^{x'_{1,n}} \varphi_{n,c}^{(\alpha)}(t)\,dt, \quad 0 < x < b_\alpha.$$

By using the expression of b_α as well as the conditions on $\chi_{n,\alpha}(c)$, together with Hölder's inequality applied to the above integral, one gets

$$\left|\big(\varphi_{n,c}^{(\alpha)}\big)'(x)\right| \leq \frac{\sqrt{b_\alpha}}{1-b_\alpha}\chi_{n,\alpha}(c).$$

Finally, since $\varphi_{n,c}^{(\alpha)}(0) = 0$ and since $x'_{1,n} < b_\alpha$, then we have

$$\left|\varphi_{n,c}^{(\alpha)}\big(x'_{1,n}\big)\right| < \frac{b_\alpha^{3/2}}{1-b_\alpha}\chi_{n,\alpha}(c) \leq 3\sqrt{3/2}\sqrt{\chi_{n,\alpha}(c)}.$$

Finally, from the previous analysis, we have

$$\sup_{x\in[0,1]}\left|\varphi_{n,c}^{(\alpha)}(x)\right| \leq \max\left(\left|\varphi_{n,c}^{(\alpha)}\big(x'_{1,n}\big)\right|, \left|\varphi_{n,c}^{(\alpha)}(1)\right|\right) \leq 3\sqrt{3/2}\sqrt{\chi_{n,\alpha}(c)},$$

which concludes the proof of the theorem. □

The following theorem tells us that $\chi_{n,\alpha}(c)$ passes through $c^2+\alpha^2-\frac{1}{4}$ for n around $\frac{\sqrt{c^2+\alpha^2-1/4}}{\pi}$. A similar result if the special case where $\alpha = 1/2$ has been already given by Theorem 3.1 in [22, p.35].

Theorem 3 *Consider two real numbers $c > 0$, $\alpha > -\frac{1}{2}$, with $c^2 \geq \frac{1}{4} - \alpha^2$, then,*

(*a*) *For any positive integer $n < \frac{c}{\pi} - \frac{\alpha}{2}$, we have $\chi_{n,\alpha}(c) < c^2 + \alpha^2 - \frac{1}{4}$.*

(*b*) *For any integer $n > \frac{\sqrt{c^2+\alpha^2-1/4}}{\pi} + \frac{5}{3}$, we have $\chi_{n,\alpha}(c) > c^2 + \alpha^2 - \frac{1}{4}$.*

Proof To alleviate notation, we let φ, χ denote the eigenfunction $\varphi^{(\alpha)}_{n,c}$ and its associated eigenvalue $\chi_{n,\alpha}(c)$, respectively. We want to prove that solutions on $(0, 1)$ of the differential equation

$$((1-x^2)\varphi')' + \left(\chi - c^2x^2 + \frac{\frac{1}{4}-\alpha^2}{x^2}\right)\varphi = (P_1(x)\varphi')' + r_1(x)\varphi = 0, \tag{45}$$

have at least $\frac{\sqrt{\chi-(\alpha^2-1/4)}}{\pi} - \frac{\alpha}{2}$ zeros. As it is done in the proof of proposition 2, the change of function $U = (1-x^2)^{1/2}\varphi$ leads to the equation for U, given by (39). Since $\chi \geq c^2 + \alpha^2 - \frac{1}{4}$, then from the Sturm comparison theorem, the number of zeros of $\varphi^{(\alpha)}_{n,c}$ is bounded below by the number of zeros of the function $V(\cdot)$, given by (41). Since a bound of the $k-$th zero of the Bessel function $J_\alpha(\cdot)$ is given by (42), then n, the number of zeros of $\varphi^{(\alpha)}_{n,c}$ is bounded below by $\left[\frac{\sqrt{\chi_{n,\alpha}(c)-(\alpha^2-1/4)}}{\pi} - \frac{\alpha}{2}\right]$. Finally, to conclude the proof of (a), it suffices to note that $\chi \geq c^2 + \alpha^2 - \frac{1}{4}$ and use the previous bound below of the number of zeros n.

Next to prove (b), we divide the interval $(0, 1)$ into the two subintervals $(0, 1-\eta)$, $[1-\eta, 1)$ with $\eta \in (0, \frac{1}{2})$ to be fixed later on. We first bound the number n_η of zeros of φ or of U, in the interval $(0, 1-\eta)$. Since for $0 < x < 1-\eta$, we have $(1-x^2)^{-2} \leq \eta^{-2}$, then by using the Sturm-Liouville comparison theorem applied to (45) and the differential equation

$$V''(x) + (\chi + \eta^{-2})V(x) = 0, \quad x \in (0, 1-\eta), \tag{46}$$

one concludes that

$$n_\eta \leq \frac{(1-\eta)\sqrt{\chi+\eta^{-2}}}{\pi} + 1. \tag{47}$$

It remains to find a bound for $n'_\eta = n - n_\eta$. We now compare the Eq. (45) with an appropriate second order differential equation on the interval $[1-\eta, 1)$. We may assume that $\eta \leq 1-\sqrt{\frac{5}{6}}$. Since in this last interval, we have $P_1(x) = (1+x)(1-x) \geq (2-\eta)(1-x)$ and since $\chi \leq c^2 + \alpha^2 - \frac{1}{4}$ and $\frac{1+x^2}{x^2} \leq \frac{11}{5}$, then we have

$$r_1(x) = \chi - c^2x^2 + \frac{\frac{1}{4}-\alpha^2}{x^2} \leq \chi(1-x^2) + \left(\frac{1}{4}-\alpha^2\right)\left(\frac{1}{x^2}-x^2\right)$$

$$\le \chi(1-x^2) + (1/4-\alpha^2)(1-x^2)\frac{1+x^2}{x^2}$$
$$\le (\chi + 11/20)(1-x^2) \le 2(\chi+11/20)(1-x) = r_2(x).$$

Hence, we use the Sturm-Liouville comparison theorem applied to the Eq. (45) and the following equation

$$(2-\eta)((1-x)U')' + 2(\chi+11/20)(1-x)U = 0, \quad x \in (1-\eta, 1). \tag{48}$$

The previous equation is rewritten as

$$U'' - \frac{U'}{1-x} + \frac{2(\chi+11/20)}{2-\eta}U = 0, \quad x \in (1-\eta, 1).$$

If we let $v(x) = U(1-x)$ and take $t = 1-x$ as a new variable, then the previous equation is reduced to the Bessel equation with solution $v(t) = J_0(bt)$ on $(0,\eta)$, with $b^2 = \frac{2\chi+11/10}{2-\eta}$. Moreover, since from [26, p. 489], the $m-$th zeros of $J_0(x)$ lies in the interval $\left((m+\frac{3}{4})\pi, (m+\frac{7}{8})\pi\right)$, then $v(t)$ has at most $\left[\sqrt{\frac{2(\chi+11/20)}{2-\eta}}\frac{\eta}{\pi} - \frac{3}{4}\right]$ zeros in $(0,\eta)$. By using Sturm comparison theorem, one concludes that n'_η, the number of zeros of φ in $(1-\eta, 1)$ is bounded as follows,

$$n'_\eta \le \left[\sqrt{\frac{2(\chi+11/20)}{2-\eta}}\frac{\eta}{\pi} - \frac{3}{4}\right] + 1 \le \frac{1}{\pi}\eta\sqrt{\frac{2(\chi+11/20)}{2-\eta}} + \frac{1}{4}.$$

Straightforward manipulations show that

$$n = n_\eta + n'_\eta \le \frac{1-\eta}{\pi}\sqrt{\chi}\left(1+\frac{1}{2}\eta^{-2}\chi^{-1}\right) + 1 + \frac{\eta}{\pi}\sqrt{\frac{2}{2-\eta}(\chi+11/20)+1/4}$$
$$\le \frac{\sqrt{\chi}}{\pi} + \frac{1}{2\pi}\eta^{-2}\chi^{-1/2} + 5/4 + \frac{\eta}{\pi}\left(\frac{11/10+\eta\chi}{2\sqrt{\chi}}\right)$$

since $n \ge 1$, then from (19), we have $\chi \ge 6$. Moreover, by choosing $\eta = \chi^{-\frac{1}{4}}$, one gets

$$n = n_\eta + n'_\eta \le \frac{\sqrt{\chi}}{\pi} + \frac{1+\frac{11}{20}6^{-3/4}}{\pi} + \frac{5}{4} \le \frac{\sqrt{\chi}}{\pi} + \frac{5}{3},$$

that is $\sqrt{\chi_{n,\alpha}(c)} \ge \pi(n-5/3)$, which allows us to conclude for (b).

4 Eigenvalues Behaviour and Decay of the Finite Hankel Transform Operator

In this paragraph, we prove an important property of the eigenvalues $\lambda_{n,\alpha}(c)$, that is for fixed integer $n \geq 0$ and real numbers $c > 0$, $\alpha > \alpha' > -1/2$, we have $\lambda_{n,\alpha'}(c) < \lambda_{n,\alpha}(c)$. To prove this result, we need the following Paley-Wiener theorem for the Hankel transform, given by J. L. Griffith in [11].

Theorem 4 [11] *Let* $\alpha > -\dfrac{1}{2}$ *and* $p, q > 0$ *with* $\dfrac{1}{p} + \dfrac{1}{q} = 1$. *Let* f *be an even function of exponential type 1. If* $1 < p \leq 2$ *and* $t^{\alpha+1/2} f(t) \in L^p(0, +\infty)$, *then* f *can be represented by*

$$f(z) = \int_0^1 (xz)^{-\alpha} J_\alpha(xz)\phi(x)\, dx, \quad z \in \mathbb{C},$$

with $x^{-\alpha-1/2}\phi(x) \in L^q(0, 1)$. *Conversely, if* f *has this representation and* $x^{-\alpha-1/2}\phi(x) \in L^p(0, 1)$, $1 < p \leq 2$, *then* f *is an even entire function of exponential type* 1 *such that* $t^{\alpha+1/2} f(t) \in L^q(0, \infty)$.

By using the previous theorem, we prove the following lemma that compares two Paley-Wiener spaces for Hankel band-limited functions. We should mention that the previous theorem is still valid if the interval $(0, 1)$ is substituted with the interval $(0, c)$.

Lemma 3 *Let* $\alpha \geq \alpha' > -\frac{1}{2}$ *be two real numbers, then the Hankel Paley-Wiener spaces* B_c^α *and* $B_c^{\alpha'}$ *satisfy the following inclusion relation,*

$$\mathcal{B}_c^\alpha \subset x^{\alpha-\alpha'} \cdot \mathcal{B}_c^{\alpha'}, \quad \alpha > \alpha'. \tag{49}$$

Here, $\mathcal{B}_c^\alpha$ *is as given by* (4).

Proof Since $f \in \mathcal{B}_c^\alpha$, then for $x \geq 0$, we have

$$f(x) = \int_0^c \sqrt{xy} J_\alpha(xy)\mathcal{H}^\alpha(f)(y)dy = x^{\alpha+\frac{1}{2}} \int_0^c (xy)^{-\alpha} J_\alpha(xy) y^{\alpha+\frac{1}{2}}\mathcal{H}^\alpha(f)(y)dy.$$

It follows that

$$x^{-\alpha-\frac{1}{2}} f(x) = \int_0^c (xy)^{-\alpha} J_\alpha(xy) y^{\alpha+\frac{1}{2}}\mathcal{H}^\alpha(f)(y)dy.$$

Let $\phi(y) = y^{\alpha+\frac{1}{2}}\mathcal{H}^\alpha(f)(y)$, then $y^{-\alpha-\frac{1}{2}}\phi(y) \in L^2[0, c]$. By using the previous Griffith's theorem with $p = q = 2$, one concludes that the function $g = x^{-\alpha-\frac{1}{2}} f$ is an even entire function of exponential type 1. Moreover, since $f = x^{\alpha+\frac{1}{2}} g \in L^2(0, +\infty)$ and since $\alpha > \alpha' > -\frac{1}{2}$, then we have

$$x^{\alpha'+\frac{1}{2}} g \in L^2(0, +\infty). \tag{50}$$

Again by using Griffith's theorem, one concludes that there exists a function φ such that $x^{-\alpha'-\frac{1}{2}}\varphi \in L^2[0,c]$ and

$$g(x) = \int_0^c (xy)^{-\alpha'} J_{\alpha'}(xy)\varphi(y)dy.$$

Hence

$$x^{\alpha'+1/2}g(x) = \int_0^c \sqrt{xy}J_{\alpha'}(xy)y^{-\alpha'-\frac{1}{2}}\varphi(y)dy.$$

It follows from (50) that $x^{\alpha'+1/2}g = x^{\alpha'-\alpha}f \in L^2(0,+\infty)$ and $\mathcal{H}^\alpha(x^{\alpha'-\alpha}f) = x^{-\alpha'-\frac{1}{2}}\varphi 1_{[0,c]}$. That is $x^{\alpha'-\alpha}f \in \mathcal{B}_c^{\alpha'}$ and $f \in x^{\alpha-\alpha'}\mathcal{B}_c^{\alpha'}$.

By using the previous lemma, we show that for a fixed integer $n \geq 0$, the eigenvalues $\lambda_{n,\alpha}(c)$ is decreasing with respect to the parameter $\alpha > -\frac{1}{2}$. This unexpected result is one of the main results of this work and it is given by the following theorem.

Theorem 5 *Let $\left(\lambda_{n,\alpha}(c)\right)_{n\geq 0}$ be the sequence of the eigenvalues of the operator $\mathcal{Q}_c^\alpha = c\,\mathcal{H}_c^\alpha\mathcal{H}_c^\alpha$, then for any integer $n \geq 0$, we have*

$$\lambda_{n,\alpha}(c) \leq \lambda_{n,\alpha'}(c), \quad \forall \alpha \geq \alpha' > -\frac{1}{2}. \tag{51}$$

Proof We first recall that if A is a self-adjoint compact operator on a Hilbert space H, with positive eigenvalues $(\lambda_n)_n$ arranged in decreasing order, then by Min-Max theorem, we have

$$\lambda_k = \max_{S_k} \min_{x\in S_k} \frac{< Ax, x >}{\|x\|^2},$$

where S_k is a subspace of H of dimension k. In the special case where $H = \mathcal{B}_c^\alpha$, $A = \mathcal{Q}_c^\alpha$ and by using the discussion given in Sect. 2, that relates the energy maximization problem to the eigenvalues $\lambda_{n,\alpha}(c)$, one concludes that

$$\lambda_{n,\alpha}(c) = \begin{cases} \max\limits_{f\in\mathcal{B}_c^\alpha} \dfrac{||f||^2_{L^2[0,1]}}{||f||^2_{L^2(0,+\infty)}}, & \text{if } n = 0 \\ \max\limits_{S_n\subset\mathcal{B}_c^\alpha} \min\limits_{f\in S_n} \dfrac{||f||^2_{L^2[0,1]}}{||f||^2_{L^2(0,+\infty)}}, & \text{if } n \geq 1, \end{cases}$$

where the S_n are subspaces of $\mathcal{B}_c^\alpha$ of dimensions n. Next, let $\alpha > \alpha' > -\frac{1}{2}$, then by using Lemma 3, we get

$$\lambda_{0,\alpha}(c) \leq \max_{f\in x^{\alpha-\alpha'}\mathcal{B}_c^{\alpha'}} \frac{||f||^2_{L^2[0,1]}}{||f||^2_{L^2(0,+\infty)}} = \max_{f\in\mathcal{B}_c^{\alpha'}} \frac{||x^{\alpha-\alpha'}f||^2_{L^2[0,1]}}{||x^{\alpha-\alpha'}f||^2_{L^2(0,+\infty)}}.$$

On the other hand, for $f \in \mathcal{B}_c^{\alpha'}$, we have

$$\frac{||x^{\alpha-\alpha'} f||^2_{L^2(0,+\infty)}}{||x^{\alpha-\alpha'} f||^2_{L^2[0,1]}} = 1 + \frac{||x^{\alpha-\alpha'} f||^2_{L^2[1,+\infty)}}{||x^{\alpha-\alpha'} f||^2_{L^2[0,1]}} \geq 1 + \frac{||f||^2_{L^2[1,+\infty)}}{||f||^2_{L^2[0,1]}} = \frac{||f||^2_{L^2(0,+\infty)}}{||f||^2_{L^2[0,1]}},$$

which implies that

$$\frac{||x^{\alpha-\alpha'} f||^2_{L^2[0,1]}}{||x^{\alpha-\alpha'} f||^2_{L^2[0,+\infty)}} \leq \max_{f \in \mathcal{B}_c^{\alpha'}} \frac{||f||^2_{L^2[0,1]}}{||f||^2_{L^2[0,+\infty)}}. \tag{52}$$

That is

$$\lambda_{0,\alpha}(c) \leq \lambda_{0,\alpha'}(c).$$

Similarly, for $n \geq 1$, and by using Lemma 3, we get

$$\begin{aligned}
\lambda_{n,\alpha}(c) &\leq \max_{S_n \subset x^{\alpha-\alpha'} \mathcal{B}_c^{\alpha'}} \min_{f \in S_n} \frac{||f||^2_{L^2[0,1]}}{||f||^2_{L^2[0,+\infty)}} \\
&\leq \max_{x^{\alpha'-\alpha} S_n \subset \mathcal{B}_c^{\alpha'}} \min_{f \in S_n} \frac{||f||^2_{L^2[0,1]}}{||f||^2_{L^2[0,+\infty)}} \\
&\leq \max_{x^{\alpha'-\alpha} S_n \subset \mathcal{B}_c^{\alpha'}} \min_{g \in x^{\alpha'-\alpha} S_n} \frac{||x^{\alpha-\alpha'} g||^2_{L^2[0,1]}}{||x^{\alpha-\alpha'} g||^2_{L^2[0,+\infty)}} \\
&\leq \max_{H_n \subset \mathcal{B}_c^{\alpha'}} \min_{g \in H_n} \frac{||x^{\alpha-\alpha'} g||^2_{L^2[0,1]}}{||x^{\alpha-\alpha'} g||^2_{L^2[0,+\infty)}}
\end{aligned}$$

Hence, by (52) we get $\lambda_{n,\alpha}(c) \leq \max_{H_n \subset H B_c^{\alpha'}} \min_{g \in H_n} \frac{||g||^2_{L^2[0,1]}}{||g||^2_{L^2[0,+\infty[}} = \lambda_{n,\alpha'}(c)$, which completes the proof of the theorem.

Note that in the special case where $\alpha = \frac{1}{2}$, we have $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin(x)$ and the $\varphi_{n,c}^{1/2}$ are the solutions of the eigen-problem

$$\sqrt{\frac{2}{\pi}} \int_0^1 \sin(cxy) \varphi_{n,c}^{1/2}(y)\, dy = \mu_{n,1/2}(c) \varphi_{n,c}^{1/2}(x), \quad x \in [0, 1]. \tag{53}$$

Moreover, it is well known that the solutions of the previous eigen-problem are given by the classical prolate spheroidal wave functions of odd orders $\psi_{2n+1,c}$. These PSWFs are solutions of the integral equations,

$$2i\int_0^1 \sin(cxy)\psi_{2n+1,c}(y)\,dy = \mu_{2n+1}(c)\psi_{2n+1,c}(x), \tag{54}$$

$$\int_{-1}^1 \frac{\sin c(x-y)}{\pi(x-y)}\psi_{2n+1,c}(y)\,dy = \lambda_{2n+1}(c)\psi_{2n+1,c}(x) \quad x\in[0,1]. \tag{55}$$

From the previous three equalities, one gets the following identity relating the eigenvalues of $\mathcal{Q}_c^{1/2}$ to the eigenvalues associated to the classical PSWFs of odd orders,

$$\lambda_{n,1/2}(c) = c\big|\mu_{n,1/2}(c)\big|^2 = \frac{c}{2\pi}\big|\mu_n(c)\big|^2 = \lambda_{2n+1}(c), \quad n\geq 0. \tag{56}$$

Note that unlike the eigenvalues $\lambda_{n,\alpha}(c)$, the behaviour and the sharp decay rate of eigenvalues $\lambda_n(c)$ associated with the classical PSWFs, are well known in the literature, see for example [7,13,20,25]. In particular, it has been shown in [7] that the sharp asymptotic decay rate of the $(\lambda_n(c))$ is given by $e^{-2n\log\left(\frac{4n}{ec}\right)}$. More precisely, for any real $0<a<\frac{4}{e}$, there exists a constant M_a such that $\lambda_{n,c}\leq e^{-2n\log(\frac{an}{c})}$, for $n\geq cM_a$. Moreover, for any real $b>\frac{4}{e}$, there exists a constant M_b such that $\lambda_{n,c}\geq e^{-2n\log\left(\frac{bn}{c}\right)}$, for $n\geq cM_b$. By combining the monotonicity of the $\lambda_{n,\alpha}(c)$ with respect to the parameter α, the identity (56) and the previous decay rate of the classical eigenvalues $\lambda_n(c)$, one gets the following corollary that provides us with a super-exponential decay rate of the $\lambda_{n,\alpha}(c)$.

Corollary 1 *Let $c>0$ and $\alpha\geq\frac{1}{2}$ be two positive real numbers. Then for any $0<a<\frac{8}{e}$, there exits a constant M_a such that*

$$\lambda_{n,\alpha}(c)\leq e^{-4n\log(\frac{an}{c})}, \quad n\geq cM_a. \tag{57}$$

Unfortunately and unlike the classical case, we don't have a precise asymptotic lower decay rate of the $\lambda_{n,\alpha}(c)$. Nonetheless, the following proposition gives us a bound below for the asymptotic decay rate of the $\lambda_{n,\alpha}(c)$, with a similar type of the super-exponential decay of the bound above.

Proposition 3 *Let $c>0$ be a positive real number, then there exists a constant δ_0 and a positive integer k_0 such that for any integer $n\geq\max\left(\frac{c}{2},\frac{c}{\pi}+k_0\right)$ and $\chi_{n,\alpha}(c)>\max\left(2\alpha^2-1/2, c^2(4\alpha^2-1)\right), c^2/\chi_{n,\alpha}(c)$, we have*

$$|\lambda_{n,\alpha}(c)|\geq\delta_0 e^{-A(2n+\alpha+1)\log\left(\frac{\pi}{c}(n+k_0)\right)}, \tag{58}$$

for some positive constant A.

Proof It is well known, see [24] that $\mu_{n,\alpha}(c)$ satisfies the differential equation,

$$\frac{\partial\mu_{n,\alpha}(c)}{\partial c} = \frac{\mu_{n,\alpha}(c)}{2c}\left(\left(\varphi_{n,c}^{(\alpha)}(1)\right)^2-1\right).$$

Here, we recall that $\varphi_{n,c}$ is normalized so that $\|\varphi_{n,c}\|_{L^2(0,1)} = 1$. It can be easily checked that in this case, $\lambda_{n,\alpha}(c)$ satisfies

$$\frac{\partial}{\partial c}\left(\log(\lambda_{n,\alpha}(c))\right) = \frac{(\varphi^{(\alpha)}_{n,c}(1))^2}{c}. \tag{59}$$

On the other hand, from [1], there exists a positive integer k_0 and a positive real number δ_0 such that

$$\lambda_{[\frac{c}{\pi}]-k_0,\alpha}(c) \geq \delta_0.$$

Since the $(\lambda_{n,\alpha}(c))_n$ are arranged in the decreasing order, then the previous inequality implies that $\lambda_{n,\alpha}(c) \geq \delta_0$ for any integer $n \leq \left[\frac{c}{\pi}\right] - k_0$, or any $c \geq c_n = \pi(n+k_0)$. Also, by using (59), one gets

$$\lambda_{n,\alpha}(c) = \lambda_{n,\alpha}(c_n)\exp\left(-\int_c^{c_n}\frac{(\varphi^{(\alpha)}_{n,\tau}(1))^2}{\tau}\right)d\tau \geq \delta_0\exp\left(-\int_c^{c_n}\frac{(\varphi^{(\alpha)}_{n,\tau}(1))^2}{\tau}\right)d\tau. \tag{60}$$

On the other hand, it has been shown in [14] that for $\alpha \geq \frac{1}{2}$ and $\chi_{n,\alpha}(c) > \max\left(2\alpha^2 - 1/2, c^2(4\alpha^2-1)\right)$, the WKB uniform approximation of the $\varphi^{(\alpha)}_{n,c}$ is given by

$$\sup_{x\in[\gamma_n,1]}\left|\varphi^{(\alpha)}_{n,c}(x) - \frac{A_n\chi_n^{1/4}\sqrt{S_n(x)}J_0(\sqrt{\chi_n}S_n(x))}{(1-x^2)^{1/4}r_n(x)^{1/4}}\right| \leq \frac{C_{q_n}}{\sqrt{\chi_n}}, \tag{61}$$

where A_n is a normalization constant, C_{q_n} is a constant depending only on $q_n = c^2/\chi_{n,\alpha}(c) < 1$, $\gamma_n = \sqrt{\frac{2\alpha^2-1/2}{\chi_{n,\alpha}(c)}}$, and

$$r_n(t) = 1 - q_nt^2 - \frac{\alpha^2-1/4}{t^2\chi_{n,\alpha}(c)}, \quad S_n(x) = \int_x^1\sqrt{\frac{r_n(t)}{1-t^2}}\,dt.$$

Also, from [14], we know that in the neighbourhood of $x = 1$, the quantity $\frac{\sqrt{S_n(x)}J_0(\sqrt{\chi_n}S_n(x))}{(1-x^2)^{1/4}r_n(x)^{1/4}}$ is bounded uniformly in n. Moreover, by using the same techniques as those used in [6] for the approximation of the normalization constant appearing in the WKB approximation of the classical PSWFs, one concludes that our normalization constants A_n are also bounded uniformly in n as soon as $q_n = c^2/\chi_{n,\alpha}(c) \leq \widetilde{q} < 1$. Consequently, if we also assume that $n \geq c/2$, then using the previous analysis together with the upper bound of $\chi_{n,\alpha}(c)$, given by (19), one concludes that there exists a constant B such that

$$\left(\varphi^{(\alpha)}_{n,c}(1)\right)^2 \leq B\sqrt{\chi_{n,\alpha}(c)} \leq \sqrt{2}B\cdot(2n+\alpha+1). \tag{62}$$

It is easy to see that the previous inequality is still valid for any $c_n = \pi(n+k_0) \leq \tau \leq c$. Finally, by substituting c with τ in (62) and using (60), one gets the desired result (58).

As a consequence of the previous proposition, we have the following corollary showing the super-exponential decay rate of the $|\mu_{n,\alpha}|$ does not invalidate an exponential decay of the expansion coefficients $(d_k^n)_k$, given by (16).

Corollary 2 *Under the hypotheses on the integer n, given by the previous proposition, there exist two positive constants A, M such that for any integer $k \geq n$, we have*

$$|d_k^n| \leq \frac{M}{\sqrt{c\pi(2k+\alpha+/2)}} \exp\left(-(2k+\alpha+1)\log\left(\frac{4k+2\alpha+2}{ec}\right) + An\log\left(\frac{\pi n}{c}\right)\right). \tag{63}$$

Proof We first note that the Bessel function satisfies the following bound, see for example [4], $|J_\alpha(x)| \leq \frac{|x|^\alpha}{2^\alpha\Gamma(\alpha+1)}$ $\forall \alpha > \frac{-1}{2}$. Here, Γ denotes the Gamma function. By combining (16) and the previous inequality, one obtains

$$\begin{aligned}
|d_k^n| &\leq \frac{\sqrt{2(2k+\alpha+1)}}{|\mu_{n,\alpha}(c)|}\int_0^1 \left|\varphi_{n,c}^{(\alpha)}(y)\right| \frac{|cy|^{2k+\alpha+1}}{2^{2k+\alpha+1}\Gamma(2k+\alpha+2)\sqrt{cy}}dy \\
&\leq \frac{\sqrt{2(2k+\alpha+1)}c^{2k+\alpha+\frac{1}{2}}}{|\mu_{n,\alpha}(c)|2^{2k+\alpha+1}\Gamma(2k+\alpha+2)}\int_0^1 \left|\varphi_{n,c}^{(\alpha)}(y)\right| y^{2k+\alpha+\frac{1}{2}}dy \\
&\leq \frac{\sqrt{2(2k+\alpha+1)}c^{2k+\alpha+\frac{1}{2}}}{|\mu_{n,\alpha}(c)|2^{2k+\alpha+1}\Gamma(2k+\alpha+2)\left(2k+\alpha+\frac{3}{2}\right)}.
\end{aligned}$$

The last inequality follows from the Hölder's inequality applied to the integral $\int_0^1 |\varphi_{n,c}^{(\alpha)}(y)| y^{2k+\alpha+\frac{1}{2}}dy$. On the other hand, it is well known that $\Gamma(s+1) \geq \sqrt{2\pi}s^{s+\frac{1}{2}}\exp(-s)$. Consequently, we have

$$|d_k^n| \leq \frac{\sqrt{2}}{|\mu_{n,\alpha}|}\frac{1}{\sqrt{c\pi(2k+\alpha+3/2)}}e^{-(2k+\alpha+1)\log\left(\frac{4k+2\alpha+2}{ec}\right)}. \tag{64}$$

Finally, by combining the previous inequality and (58) and taking into account that $\lambda_{n,\alpha}(c) = c|\mu_{n,\alpha}(c)|^2$, one gets the desired inequality (63).

5 Numerical Results

In this paragraph, we give some numerical examples that illustrate the various results of the previous sections. Moreover, we show that the eigenfunctions of the finite Hankel transform operator are well adapted for the approximation of Hankel- and almost Hankel Band-limited functions.

Example 1 In this example, we illustrate one of the main results of this work, which is given by Theorem 5. That is the eigenvalues $\lambda_{n,\alpha}(c)$ are decreasing with respect to the parameter α. For this purpose, we have considered the values of $c = 10\pi$ and the four values of $\alpha = 0, 1, 2, 3$. Then, we have used formula (11) and computed highly accurate approximation of the eigenvalues $\lambda_{n,\alpha}(c)$ with $0 \leq n \leq 40$. In Fig. 1a, we have plot the graphs of the significant eigenvalues $\lambda_{n,\alpha}(c)$ with the various values of n and α. In order to check that the decay with respect to the parameter α holds also for the very small eigenvalues, we have plot in Fig. 1b the graphs of the $\log(\lambda_{n,\alpha}(c))$. Note that the results given by the previous figures indicate what was expected by Theorem 5, that is the $\lambda_{n,\alpha}(c)$ are decreasing with respect to the parameter α.

Example 2 In this example, we give some numerical tests that illustrate the super-exponential decay rate of the eigenvalues $\lambda_{n,\alpha}(c)$, given by Corollary 1. For this purpose, we have considered the value of $\alpha = 1$ and the three different values of $c = 5\pi, 10\pi, 15\pi$, and computed highly accurate values of the eigenvalues $\lambda_{n,\alpha}(c)$, for $n \geq \frac{\sqrt{c^2+\alpha^2-1/4}}{\pi} + \frac{5}{3}$. By Theorem 2, these values of n correspond to the case where $\chi_{n,\alpha}(c) \geq c^2 + \alpha^2 - \frac{1}{4}$. As in the classical case, the critical value of $n = n_c = \frac{\sqrt{c^2+\alpha^2-1/4}}{\pi}$ corresponds to the beginning of the plunge region of the eigenvalues $\lambda_{n,\alpha}(c)$. In Fig. 2, we plot the graphs of the highly accurate values of the $\log(\lambda_{n,\alpha}(c))$, as well as the graphs of $-4n \log\left(\frac{8n}{ec}\right)$, the logarithm of the optimal theoretical super-exponential decay rate, as given by Corollary 1. Note that for the different values of c, the theoretical asymptotic decay rate given by corollary 1 is very close to the actual decay rate.

Example 3 In this last example, we illustrate the quality of the spectral approximation of the Hankel band-limited and almost Hankel band-limited functions, by the orthogonal projection over $\text{Span}\{\varphi_{n,c}^{(\alpha)},\ 0 \leq n \leq N\}$. Note that the concept of almost band-limited functions has been introduced in the framework of the classical Fourier

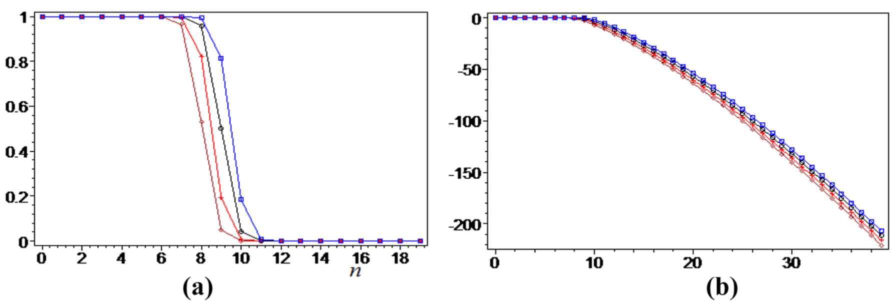

Fig. 1 **a** Graph of the $\lambda_{n,\alpha}(c)$ for $c = 10\pi$, $\alpha = 0$ (blue), 1 (black), 2 (red), 3 (brown), **b** same as **a** with the graphs of $\log\left(\lambda_{n,\alpha}(c)\right)$

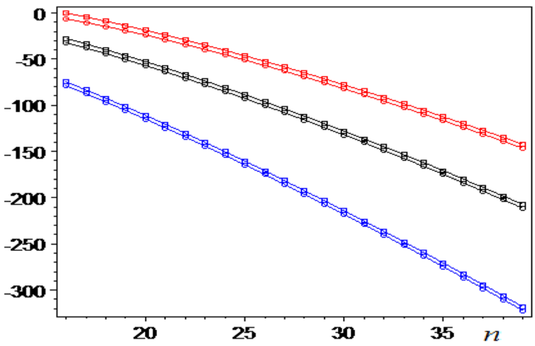

Fig. 2 Graph of the $\log(\lambda_{n,\alpha}(c))$ for $\alpha = 1$, $c = 5\pi$ (blue circles), $c = 10\pi$ (black circles), $c = 15\pi$ (red circles), versus the corresponding graphs of $-4n \log\left(\frac{8n}{ec}\right)$ (boxes) (Color figure online)

transform by Landau, see [19]. In a similar manner, the almost Hankel band-limited functions are defined as follows.

Definition 1 Let Ω be a measurable set of $\mathbb{R}_+$ and let $\epsilon_\Omega > 0$ be a positive real number. A function $f \in L^2(0, +\infty)$ is said to be $\epsilon_\Omega-$almost band-limited to Ω if

$$\left\| \mathcal{H}^\alpha f - \chi_\Omega \mathcal{H}^\alpha f \right\|_{L^2(0,+\infty)} \leq \epsilon_\Omega. \tag{65}$$

Here χ_Ω denotes the characteristic function of Ω.

Note that the $\varphi_{n,c}^{(\alpha)}$ are the radial parts of the $2D$ PSWFs that are concentrated on the unit disc. In [23], the author has developed a $2D$ PSWFs based quadrature scheme for $2D-$bandlimited functions. The proposed quadrature scheme is restricted to the unit disk and it is used to derive an approximation scheme of $2D-$bandlimited functions over the unit disk. Recently in [18], a similar scheme is developed for the approximation of functions that are almost bandlimited and space-concentrated on a disk. In this last example, we should restrict ourselves to the $1D-$case. Since the $\varphi_{n,c}^{(\alpha)}$ are Hankel $c-$band-limited functions, then they are $0-$almost band-limited to $\Omega = [0, c]$. Next, for an integer $N \geq 1$, let $S_N^\alpha(f)$ be the N-th partial sum of the expansion of f, in the basis $\{\varphi_{n,c}^{(\alpha)},\ n \geq 0\}$, that is

$$S_N^\alpha f(x) = \sum_{n=0}^{N} \left\langle f, \varphi_{n,c}^{(\alpha)} \right\rangle_{L^2(0,1)} \varphi_{n,c}^{(\alpha)}(x), \tag{66}$$

where $< \cdot, \cdot >$ denotes the usual inner product of $L^2(0, 1)$. The quality of approximation of the classical Fourier almost $c-$band-limited functions, by the classical PSWFs has been given in [8]. Moreover, in [13], this quality of approximation has been extended to the expansion with respect to some families of classical orthogonal polynomials. By straightforward modifications of the techniques used in [13], one gets the following proposition that provides us with the quality of approximation of almost Hankel $c-$band-limited function by the $\varphi_{n,c}^{(\alpha)}$.

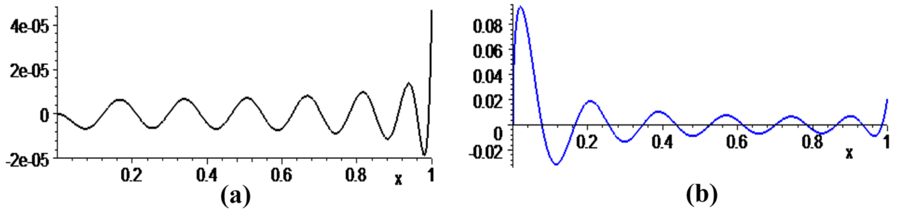

Fig. 3 **a** Graph of the approximation error $f_1(x) - S_N^{\alpha_1} f_1$ for $\alpha_1 = 3/2,\ c = 10\pi,\ N = 11$, **b** Graph of the approximation error $f_2(x) - S_N^{\alpha_2} f_2$ for $\alpha_2 = 1,\ c = 10\pi,\ N = 11$

Proposition 4 *If f is an $L^2(0, +\infty)$ function that is $\epsilon_\Omega-$ almost Hankel band-limited in $\Omega = [0, c]$, then for any positive integer N, we have*

$$\left(\int_0^{+1} |f(t) - S_N^\alpha f(t)|^2 dt\right)^{1/2} \leq \left(\epsilon_\Omega + \sqrt{\lambda_{N,\alpha}(c)}\right) \|f\|_{L^2(0,+\infty)}. \tag{67}$$

To illustrate the previous spectral approximation result, we have considered the following Hankel and almost Hankel band-limited functions, given by

$$f_1(x) = \frac{J_{\alpha_1+1}(ax)}{\sqrt{x}},\ \alpha_1 = \frac{3}{2},\ a = 20; \qquad f_2(x) = x^{\alpha_2 - 1/2} \exp(-x),\ \ \alpha_2 = 1,$$

respectively. Note that the Hankel transforms of f_1 and f_2 are given by

$$\mathcal{H}^{\alpha_1}(f_1)(s) = a^{-\alpha_1-1} s^{\alpha_1+1/2} \mathbf{1}_{[0,a]}(s),$$
$$\mathcal{H}^{\alpha_2}(f_2)(s) = \frac{2^{\alpha_2}\Gamma(\alpha_2 + 1/2)}{\sqrt{\pi}} \frac{s^{\alpha_2+1/2}}{(1+s^2)^{\alpha_2+1/2}}.$$

Hence, $f_1 \in B_c^{\alpha_1}$, for any $c \geq a$. Moreover, straightforward computations show that f_2 is $\epsilon_\Omega-$concentrated on $\Omega = [0, c]$, with

$$\epsilon_\Omega = \frac{2^{\alpha_2}\Gamma(\alpha_2 + 1/2)}{\sqrt{\pi}} \frac{\sqrt{1+2c^2}}{2(1+c^2)^2}.$$

In the special case where $\alpha_2 = 1$ and $c = 10\pi$, we have $\epsilon_\Omega \approx 0.0225$. For this last value of $c = 10\pi$, we have computed the N-th partial sum $S_N^{\alpha_1} f_1$ and $S_N^{\alpha_2} f_2$, with $N = 11$. The approximation errors $f_1(x) - S_N^{\alpha_1} f_1$ and $f_2(x) - S_N^{\alpha_2} f_2$ are given in Fig. 3a, b, respectively. Note that as predicted by the previous proposition, the first approximation error is proportional to $\sqrt{\lambda_{N,\alpha_1}(c)}$, and the second one is proportional to ϵ_Ω.

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