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Analytic Smoothing Effect for the Nonlinear Schrödinger Equations Without Square Integrability

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Abstract In this study we consider the Cauchy problem for the nonlinear Schrödinger equations with data which belong to L^p, $1 < p < 2$. In particular, we discuss analytic smoothing effect with data which satisfy exponentially decaying condition at spatial infinity in L^p, $1 < p < 2$. We construct solutions in the function space of analytic vectors for the Galilei generator and the analytic Hardy space with the phase modulation operator based on L^p.

Keywords Analytic smoothing effect · Analytic Hardy space · Nonlinear Schrödinger equations

Mathematics Subject Classification 35Q55

1 Introduction

We consider the local Cauchy problem for the nonlinear Schrödinger equations in setting of the Lebesgue space L^p, $1 < p < 2$:

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$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = f(u), \ (t,x) \in I_T \times \mathbb{R}^n, \\ u(0) = \phi \in L^p \end{cases} \tag{1.1}$$

where $I_T = [0, T]$, $T > 0$, $i = \sqrt{-1}$, $u : I_T \times \mathbb{R}^n \to \mathbb{C}$, $\partial_t = \partial/\partial t$, $\Delta = \sum_{j=1}^{n} \partial_j^2/\partial x_j^2$, and $n \geq 1$. The nonlinearity f satisfies the gauge condition

$$e^{i\theta} f(u) = f(e^{i\theta}u), \ \theta \in \mathbb{R}.$$

To be specific, we assume the following two types of nonlinearity in this study

$$f(u) = \lambda |u|^2 u, \tag{1.2}$$

$$f(u) = \lambda \left(|x|^{-\gamma} * |u|^2\right) u \tag{1.3}$$

where $\lambda \in \mathbb{C}$, $0 < \gamma < n$ and $\varphi * \psi$ denotes the convolution

$$(\varphi * \psi)(x) = \int_{\mathbb{R}^n} \varphi(x-y)\psi(y)dy, \ x \in \mathbb{R}^n.$$

There is a large literature on the Cauchy problem for the nonlinear Schrödinger equations in the L^2-based framework (see for instance [3,4,6,21,27–29] and reference therein).

Analyticity and analytic smoothing effect for solutions to nonlinear evolution equations have been studied in many papers ([5,8–11,13–15,20,22–24]).
In particular, analytic smoothing effect for the nonlinear Schrödinger equations and the Hartree equations in the L^2 setting with data which satisfy exponentially decaying condition has been studied in [8–11,13–15,23–25] (see also reference therein).

On the other hand, as far as the authors know, there are no results on analytic smoothing effect for the nonlinear Schrödinger equations in the L^p-framework with $p \neq 2$. In the present paper we discuss the problem of analytic smoothing effect for solutions to (1.1) in the L^p-framework. We believe this problem is interesting since exponentially decaying L^p-functions for $1 < p < 2$ do not necessarily belong to L^2. In fact, if $1 < p < 2$, there exists

$$\phi \in L^p \backslash L^2$$

such that

$$\sup_{\delta \in D} \left\| e^{\delta \cdot x} \phi \right\|_{L^p} < \infty.$$

See Proposition 1 below for details. Furthermore, it is sufficient to consider the case $1 < p < 2$, since any exponentially decaying L^q-function with $2 < q < \infty$ is an L^2-function. To be more precise, let $D \subset \mathbb{R}^n$ be a domain with $0 \in D$ and $2 < q < \infty$. If $\phi \in L^q$ satisfying

$$\sup_{\delta \in D} \left\| e^{\delta \cdot x} \phi \right\|_{L^q} < \infty,$$

then

$$\sup_{\delta\in D'} \left\|e^{\delta\cdot x}\phi\right\|_{L^1} \le C \sup_{\delta\in D} \left\|e^{\delta\cdot x}\phi\right\|_{L^q}$$

for $D' \Subset D$ (see Appendix of [15] and Chapter III of [26]). We put

$$\widehat{L^q} = \left\{\phi \in \mathcal{S}';\ \mathcal{F}[\phi] \in L^{q'}\right\}.$$

If $\phi \in \widehat{L^q}$ satisfying

$$\sup_{\delta\in D} \left\|e^{\delta\cdot x}\phi\right\|_{\widehat{L^q}} < \infty,$$

then by the boundedness of the Fourier transform, we have

$$\sup_{\delta\in D} \left\|e^{\delta\cdot x}\phi\right\|_{L^q} \le C \sup_{\delta\in D} \left\|e^{\delta\cdot x}\phi\right\|_{\widehat{L^q}} < \infty.$$

Hence if data satisfy exponentially decaying condition in L^q or $\widehat{L^q}$, then data decay exponentially in $L^1 \cap L^q \subset L^2$.

Analytic smoothing effect for the cubic nonlinear Schrödinger equations in minimal regularity Sobolev setting based on L^2 has been studied in [23,24]. Analytic smoothing effect for the nonlinear Schrödinger equations with non-gauge invariant quadratic nonlinearity in terms of the generator of dilations $P = 2t\partial_t + x\cdot\nabla$ in the framework of negative exponent Sobolev space H^s, $s > -3/4$ has been studied in [20].

In the previous papers [7,12,17–19,30,31], the authors have attempted to construct solutions and proved the existence of solutions to the nonlinear Schrödinger equations in the framework of L^p and $\widehat{L^p}$. Especially, in the L^p-setting, in [31] the author proved the existence of local solutions to the 1D cubic nonlinear Schrödinger equations for data in L^p with $1 < p < 2$. Later, in [12] we exploited his approach to show similar local well-posedness results for the Hartree equation for data in the Bessel potential spaces $H^{s,p}$ under suitable conditions on s, p, n. In particular, for $0 < \gamma < \min(2, n)$, we obtained the local well-posedness for data in L^p with $\max(\frac{2n}{n-\gamma+2}, \frac{2n}{n+\gamma}) < p < 2$. In this paper, based on these local existence results, we investigate analytic smoothing effect for the 1D cubic NLS and the Hatree equation for data in L^p when p is in the range stated above. To be more precise, our main purpose of this study is to show analytic smoothing effect for the Cauchy problem (1.1) in the framework of L^p-based spaces of functions as analytic vector for the Galilei generator $J = e^{i\frac{t}{2}\Delta}xe^{-i\frac{t}{2}\Delta}$ and the L^p-based analytic Hardy space characterized by the operator $A_\delta = e^{i\frac{t}{2}\Delta}e^{\delta\cdot x}e^{-i\frac{t}{2}\Delta}$.

Finally, the condition $p > \frac{2n}{n+\gamma}$ for the local well-posedness for the Hartree equation may seem unusual, since it is stronger than the condition for the cubic NLS. However, it is conjectured that this condition is optimal. See Remark 3 below.

We use the following notation throughout this paper. $L^p = L^p(\mathbb{R}^n)$, $1 \le p \le \infty$ is the usual Lebesgue space. The Fourier transform $\mathcal{F} : \psi \mapsto \widehat{\psi}$ is defined by

$$\widehat{\psi}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\psi(x)dx,\ \xi \in \mathbb{R}^n,$$

where $x \cdot y = \sum_{j=1}^n x_j y_j$ is the usual scaler product in $\mathbb{R}^n$, and $\mathcal{F}^{-1}$ is the inverse Fourier transform. The Schrödinger propagator is defined by $U(t)\psi = (U\psi)(t) = e^{i\frac{t}{2}\Delta}\psi = \mathcal{F}^{-1}\left[e^{-i\frac{t}{2}|\xi|^2}\mathcal{F}[\psi]\right]$, $t \in \mathbb{R}$, we often use the notation $U^{-1}(t)\psi = \left(U^{-1}\psi\right)(t) = U(-t)\psi$, $t \in \mathbb{R}$. As well known, U and U^{-1} have the following factorization formula (see Chapter 2 of [3]):

$$U(t)\psi = \mathcal{M}(t)D(t)\mathcal{F}\mathcal{M}(t)\psi,\ U^{-1}(t)\psi = \mathcal{M}(-t)\mathcal{F}^{-1}D^{-1}(t)\mathcal{M}(-t)\psi$$

for $t \neq 0$, where the phase modulation operator $\mathcal{M}(t) : \psi \mapsto e^{i\frac{|x|^2}{2t}}\psi$, dilations $D(t) : \psi \mapsto (it)^{-n/2}\psi\left(\frac{\cdot}{t}\right)$ and its inverse $D^{-1}(t) = i^n D\left(t^{-1}\right)$, $t \neq 0$. We put the linear Schrödinger operator by

$$\mathcal{L} = i\partial_t + \frac{1}{2}\Delta = Ui\partial_t U^{-1}.$$

The Duhamel integral operator is defined by $S[f](t) = \left(\int_0^{\cdot} U(\cdot - s)f(\tau)d\tau\right)(t) = \int_0^t U(t-\tau)f(\tau)d\tau$, $t \in \mathbb{R}$. The Galilei generator is defined by

$$J(t) = U(t)xU(-t) = x + it\nabla,\ t \in \mathbb{R}.$$

We introduce an operator which gives analytic continuation (see also [11,15,23])

$$A_\delta(t) = U(t)e^{\delta\cdot x}U(-t),\ \delta \in \mathbb{R}^n$$

for $t \in \mathbb{R}$. We see that A_δ has another representation such as

$$A_\delta(t) = \mathcal{M}(t)e^{it\delta\cdot\nabla}\mathcal{M}(-t),\ t \neq 0,$$

where $e^{it\delta\cdot\nabla}\psi = \mathcal{F}^{-1}\left[e^{-t\delta\cdot\xi}\widehat{\psi}\right]$, $\delta \in \mathbb{R}^n$. The following commutation relation

$$[J, \mathcal{L}] = U[x, i\partial_t]U^{-1} = 0,\ [A_\delta, \mathcal{L}] = U[e^{\delta\cdot x}, i\partial_t]U^{-1} = 0$$

holds. The analytic Hardy space is defined by (see Chapter III of [26]):

$$\mathcal{H}^p(\Omega) = \left\{\psi :\ \text{analytic on } \mathbb{R}^n + i\Omega;\ \|\psi\|_{\mathcal{H}^p(\Omega)} = \sup_{y\in\Omega}\|\psi(\cdot + iy)\|_{L^p} < \infty\right\}$$

with domain $\Omega \subset \mathbb{R}^n$.

2 Function Spaces

We introduce the following basic function spaces

$$\begin{aligned}&\mathscr{X}(I_T)\\&\quad=\left\{u: I_T\times\mathbb{R}^n\to\mathbb{C};\ U^{-1}u\in C(I_T;L^p),\ \|u\|_{\mathscr{X}(I_T)}=\left\|U^{-1}u\right\|_{L^\infty(I_T;L^p)}<\infty\right\},\\&X^p_{q,\theta}(I_T)\\&\quad=\left\{u:\ I_T\times\mathbb{R}^n\to\mathbb{C};\ \|u\|_{X^p_{q,\theta}(I_T)}=\left\{\int_{I_T}\tau^{\theta q}\left\|\left(U^{-1}\mathcal{L}u\right)(\tau)\right\|^q_{L^p}d\tau\right\}^{1/q}<\infty\right\}\end{aligned}$$

for $1\le p,q\le\infty$, $\theta>0$ and we put the function spaces associated with $X^p_{q,\theta}(I_T)$ by

$$\widetilde{X}^p_{q,\theta}(I_T)=\left\{u\in X^p_{q,\theta}(I_T);\ u(0)\in L^p,\ \|u\|_{\widetilde{X}^p_{q,\theta}(I_T)}=\|u(0)\|_{L^p}+\|u\|_{X^p_{q,\theta}(I_T)}<\infty\right\}.$$

The function spaces such as above are firstly introduced in [31].

We introduce the following weighted function spaces:

$$G^a_p=\left\{\phi\in L^p;\ \|\phi\|_{G^a_p}=\sum_{\alpha\ge 0}\frac{a^\alpha}{\alpha!}\left\|x^\alpha\phi\right\|_{L^p}<\infty\right\},$$

$$G^D_p=\left\{\phi\in L^p;\ \|\phi\|_{G^D_p}=\sup_{\delta\in D}\left\|e^{\delta\cdot x}\phi\right\|_{L^p}<\infty\right\}.$$

The function space of analytic vectors for J is defined by

$$G^a_{p,q,\theta}(I_T)=\left\{u\in\widetilde{X}^p_{q,\theta}(I_T);\ \|u\|_{G^a_{p,q,\theta}(I_T)}=\|u(0)\|_{G^a_p}+\sum_{\alpha\ge 0}\frac{a^\alpha}{\alpha!}\left\|J^\alpha u\right\|_{X^p_{q,\theta}(I_T)}<\infty\right\},$$

for $a\in(0,\infty)^n$ and the analytic Hardy space with respect to A_δ is defined by

$$G^D_{p,q,\theta}(I_T)=\left\{u\in\widetilde{X}^p_{q,\theta}(I_T);\ \|u\|_{G^D_{p,q,\theta}(I_T)}=\|u(0)\|_{G^D_p}+\sup_{\delta\in D}\|A_\delta u\|_{X^p_{q,\theta}(I_T)}<\infty\right\}$$

for domain $D\subset\mathbb{R}^n$.

Our motivation of this study is based on the following proposition:

Proposition 1 *Let* $1<p<2$ *and bounded domain* $D\subset\mathbb{R}^n$. *Then*

$$G^D_p\backslash L^2\ne\varnothing.$$

Proof The following function belongs to $G_p^D \backslash L^2$,

$$\phi(x) = \begin{cases} |x|^{-n/(2-\varepsilon)}, & |x| \le 1, \\ e^{-r|x|}, & |x| > 1 \end{cases}$$

where $0 < \varepsilon < 2 - p$ and sufficiently large $r > 0$. □

The analyticity of functions which belong to $G^a_{p,q,\theta}(I_T)$ or $G^D_{p,q,\theta}(I_T)$ is shown by the following proposition:

Proposition 2 *Let* $1 \le p < 2$ *and* $\frac{1}{q} + \frac{1}{\theta} > 1$.

(1) *Let* $u \in G^a_{p,q,\theta}(I_T)$. *Then* $\left(\mathcal{M}^{-1}u\right)(t) \in \mathcal{H}^{p'}\left(\prod_{j=1}^n(-a_jt, a_jt)\right)$ $t \neq 0$, *where* $a \in (0, \infty)^n$.
(2) *Let* $u \in G^D_{p,q,\theta}(I_T)$. *Then* $\left(\mathcal{M}^{-1}u\right)(t) \in \mathcal{H}^{p'}(tD)$, $t \neq 0$, *where* $D \subset \mathbb{R}^n$ *is a domain with* $0 \in D$.

Proof Because

$$\widetilde{X}^p_{q,\theta}(I_T) \subset \mathscr{X}(I_T)$$

with

$$\begin{aligned} \|u\|_{\mathscr{X}(I_T)} &\le \|u(0)\|_{L^p} + T^{1-\theta\left(1-\frac{1}{q}\right)} \|u\|_{X^p_{q,\theta}(I_T)} \\ &\le \max\left\{1, T^{1-\theta\left(1-\frac{1}{q}\right)}\right\} \|u\|_{\widetilde{X}^p_{q,\theta}(I_T)}, \end{aligned}$$

we see that

$$\sup_{\delta \in \prod_{j=1}^n(-a_j,a_j)} \left\| e^{\delta\cdot x} U^{-1} u \right\|_{L^\infty(I_T;L^p)} \le \sum_{\alpha \ge 0} \frac{a^\alpha}{\alpha!} \left\| x^\alpha U^{-1} u \right\|_{L^\infty(I_T;L^p)} < \infty.$$

Hence, it is sufficient to show real analyticity of $u \in G^\Omega_{p,q,\theta}$ with $\Omega = \prod_{j=1}^n(-a_j, a_j)$ in the case (1) and with $\Omega = D$ in the case (2). We see that

$$\begin{aligned} \left\| e^{\delta\cdot x} U(-t)u(t) \right\|_{L^1} &= \left\| e^{\delta\cdot x} \mathcal{F}^{-1} D^{-1}(t) \mathcal{M}(-t)u(t) \right\|_{L^1} \\ &= \left\| e^{\delta\cdot x} D(t) \mathcal{F}^{-1} \mathcal{M}(-t)u(t) \right\|_{L^1} \\ &= \left\| D(t) e^{t\delta\cdot x} \mathcal{F}^{-1} \mathcal{M}(-t)u(t) \right\|_{L^1} \\ &= |t|^{n/2} \left\| e^{t\delta\cdot x} \mathcal{F}^{-1} \mathcal{M}(-t)u(t) \right\|_{L^1} \\ &= |t|^{n/2} \left\| e^{-t\delta\cdot\xi} \mathcal{F}\left[(\mathcal{M}^{-1}u)(t)\right] \right\|_{L^1} < \infty \end{aligned}$$

for all $t \neq 0,\ \delta \in \Omega$. Therefore, $(\mathcal{M}^{-1}u)(t)$ is real analytic and has an analytic continuation

$$\begin{aligned}\left(\mathcal{M}^{-1}A_\delta u\right)(t,x) &= e^{it\delta\cdot\nabla}\mathcal{M}^{-1}u(t,x)\\ &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(x+it\delta)\cdot\xi}\mathcal{F}\left[\left(\mathcal{M}^{-1}u\right)(t)\right](\xi)d\xi\end{aligned}$$

for all $x+it\delta \in \mathbb{R}^n + it\Omega$ (see Appendix of [15]). Also we have

$$\sup_{\delta\in D}\left\|\left(\mathcal{M}^{-1}A_\delta u\right)(t)\right\|_{L^{p'}} \le C|t|^{-n\left(\frac{1}{2}-\frac{1}{p'}\right)}\sup_{\delta\in D}\left\|e^{\delta\cdot x}\left(U^{-1}u\right)(t)\right\|_{L^p} < \infty$$

and

$$\begin{aligned}\sup_{\delta\in\prod_{j=1}^n(-a_j,a_j)}\left\|\left(\mathcal{M}^{-1}A_\delta u\right)(t)\right\|_{L^{p'}} &\le \sum_{\alpha\ge 0}\frac{a^\alpha}{\alpha!}\left\|\left(J^\alpha u\right)(t)\right\|_{L^{p'}}\\ &\le C|t|^{-n\left(\frac{1}{2}-\frac{1}{p'}\right)}\sum_{\alpha\ge 0}\frac{a^\alpha}{\alpha!}\left\|x^\alpha\left(U^{-1}u\right)(t)\right\|_{L^p}\\ &< \infty.\end{aligned}$$

□

By the relation $e^{\delta\cdot x}U^{-1}U\phi = e^{\delta\cdot x}\phi$, we immediately have the following fact for the free solutions:

Corollary 1 *Let* $1 \le p < 2$.

(1) *If* $\phi \in G^a_p$, *then* $U(t)\phi,\ t \in \mathbb{R}\backslash\{0\}$, *is real analytic and has an analytic continuation to* $\mathbb{R}^n + it\prod_{j=1}^n(-a_j, a_j)$.

(2) *If* $\phi \in G^D_p$, *then* $U(t)\phi,\ t \in \mathbb{R}\backslash\{0\}$, *is real analytic and has an analytic continuation to* $\mathbb{R}^n + itD$.

3 Main Results

We put the interval $I_T = [0, T]$.

Theorem 1 *Let* $n = 1,\ 1 < p < 2$ *and* $a \in (0,\infty)$. *Then for any* $\eta > 0$ *there exists* $T = T(\eta) > 0$ *such that; for any* $\phi \in G^a_p$, *satisfying* $\|\phi\|_{G^a_p} \le \eta$ *then the Cauchy problem* (1.1)–(1.2) *has a unique solution* $u \in G^a_{p,p',2\left(\frac{1}{p}-\frac{1}{2}\right)}(I_T)$. *Furthermore,* $\left(\mathcal{M}^{-1}u\right)(t) \in \mathcal{H}^{p'}((-at, at)),\ t \in I_T\backslash\{0\}$.

Theorem 2 *Let* $n = 1,\ 1 < p < 2$. *Let a domain* $D \subset \mathbb{R}$ *satisfying* $0 \in D$ *and* $-D = D$. *Then for any* $\eta > 0$ *there exists* $T = T(\eta) > 0$ *such that; for any* $\phi \in G^D_p$, *satisfying* $\|\phi\|_{G_p^D} \leq \eta$ *then the Cauchy problem* (1.1)–(1.2) has a unique solution $u \in G^D_{p,p',2\left(\frac{1}{p}-\frac{1}{2}\right)}(I_T)$. Furthermore, $\left(\mathcal{M}^{-1}u\right)(t) \in \mathcal{H}^{p'}(tD)$, $t \in I_T\backslash\{0\}$.

Theorem 3 *Let* $n \geq 1$, $0 < \gamma < \min(n, 2)$, $\max\left(\frac{2n}{n+\gamma}, \frac{2n}{n-\gamma+2}\right) < p < 2$ *and* $a \in (0,\infty)^n$. *Then for any* $\eta > 0$ *there exists* $T = T(\eta) > 0$ *such that; for any* $\phi \in G^a_p$, *satisfying* $\|\phi\|_{G^a_p} \leq \eta$ *then the Cauchy problem* (1.1)–(1.3) has a unique solution $u \in G^a_{p,q,2n\left(\frac{1}{p}-\frac{1}{2}\right)}(I_T)$, with $q = \frac{2p}{(n+\gamma)p-2n}$. Furthermore, $\left(\mathcal{M}^{-1}u\right)(t) \in \mathcal{H}^{p'}\left(\prod_{j=1}^n(-a_jt, a_jt)\right)$, $t \in I_T\backslash\{0\}$.

Theorem 4 *Let* $n \geq 1$, $0 < \gamma < \min(n, 2)$ *and* $\max\left(\frac{2n}{n+\gamma}, \frac{2n}{n-\gamma+2}\right) < p < 2$. *Let a domain* $D \subset \mathbb{R}^n$ *satisfying* $0 \in D$ *and* $-D = D$. *Then for any* $\eta > 0$ *there exists* $T = T(\eta) > 0$ *such that; for any* $\phi \in G^D_p$, *satisfying* $\|\phi\|_{G^D_p} \leq \eta$ *then the Cauchy problem* (1.1)–(1.3) has a unique solution $u \in G^D_{p,q,2n\left(\frac{1}{p}-\frac{1}{2}\right)}(I_T)$, with $q = \frac{2p}{(n+\gamma)p-2n}$. Furthermore, $\left(\mathcal{M}^{-1}u\right)(t) \in \mathcal{H}^{p'}(tD)$, $t \in I_T\backslash\{0\}$.

Remark 1 $G^a_p \subset G^\Omega_p \subset G^b_p$, with $\Omega = \prod_{j=1}^n(-a_j, a_j)$, and $0 < b_j < a_j$, $j = 1, 2, \cdots, n$ (see Theorem 2 in [15]).

Remark 2 Let $1 \leq p \leq \infty$. We see that

$$G^a_p \subset G^b_1,\ 0 < b_j < a_j,\ G^{D_1}_p \subset G^{D_2}_1,\ D_2 \Subset D_1,$$
$$\mathcal{H}^p(D_1) \subset \mathcal{H}^\infty(D_2),\ D_2 \Subset D_1$$

where D_1, $D_2 \subset \mathbb{R}^n$ are domain (see Appendix of [15] and Chapter III of [26]). Therefore, the Cauchy data $\phi \in G^\Omega_p$, satisfy $\phi \in G^{\Omega'}_1$ and solutions obtained in Theorems 1–4, $\left(\mathcal{M}^{-1}u\right)(t) \in \mathcal{H}^{p'}(t\Omega)$, $t \neq 0$, satisfy $\left(\mathcal{M}^{-1}u\right)(t) \in \mathcal{H}^\infty(t\Omega')$, $t \neq 0$, where $\Omega' \Subset \Omega$ with $\Omega = \prod_{j=1}^n(-a_j, a_j)$ or $\Omega = D$.

Remark 3 In Theorems 3 and 4, we need

$$p > \frac{2n}{n-\gamma+2} \text{ and } p > \frac{2n}{n+\gamma}.$$

The exponent $\frac{2n}{n-\gamma+2}$ appearing in the first condition is called a scaling limit which is well known and is considered as one candidate of the thresholds for the local well-posedness of (1.1)–(1.3). Thus our local result can reach almost critical L^p spaces if $n \geq 2$ and $\gamma > 1$. The exponent in the second condition, on the other hand, seems unfamiliar and one may wonder if the local result still holds for p below this exponent. However, it is conjectured that the Cauchy problem is ill posed for $p < \frac{2n}{n+\gamma}$, because of the singularity at zero frequency. This is deduced from the recent works [2] and [16] which study the well-posendess of (1.1)–(1.3) in $\widehat{L^p}$. For details, see the introduction in [16]. Note that $\frac{2n}{n+\gamma} \to 1$ as $\gamma \to n$ and thus the limit coincides with the lower threshold of the local results for the cubic NLS (Theorems 1 and 2). Note also that $p = \frac{2n}{n+\gamma}$ is the exponent such that the trilinear operator

$$(u_1, u_2, u_3) \mapsto \mathcal{F}\left[\left(|x|^{-\gamma} * (u_1\overline{u_2})\right) u_3\right]$$

is defined and is continuous from $L^p \times L^p \times L^p$ to L^p.

4 Key Lemmas

We introduce the following two types of trilinear form $\mathcal{T}_0$ and $\mathcal{T}_\gamma$ by

$$\mathcal{T}_0(u_1, u_2, u_3) = u_1\overline{u_2}u_3$$

and

$$\mathcal{T}_\gamma(u_1, u_2, u_3) = \left(|x|^{-\gamma} * u_1\overline{u_2}\right) u_3,$$

respectively. Then, we see that

$$\begin{aligned}
&U(-t)\mathcal{T}_0(u_1, u_2, u_3)\\
&\quad = \mathcal{M}(-t)\mathcal{F}^{-1}_{\xi\to x} i^n D(t^{-1})(\mathcal{M}(-t)u_1\overline{\mathcal{M}(-t)u_2})\mathcal{M}(-t)u_3\\
&\quad = Ct^{-n}\mathcal{M}(-t)\mathcal{F}^{-1}_{\xi\to x}(D(t^{-1})\mathcal{M}(-t)u_1\overline{D(t^{-1})\mathcal{M}(-t)u_2})D(t^{-1})\mathcal{M}(-t)u_3\\
&\quad = Ct^{-n}\left(\mathcal{M}(t)U(-t)u_1 * \left(\overline{\mathcal{M}(t)U(-t)u_2}(-\cdot)\right)\right) * U(-t)u_3
\end{aligned}$$

and

$$\begin{aligned}
&U(-t)\mathcal{T}_\gamma(u_1, u_2, u_3)\\
&\quad = \mathcal{M}(-t)\mathcal{F}^{-1}_{\xi\to x} i^n D(t^{-1})(|\xi|^{-\gamma} * \mathcal{M}(-t)u_1\overline{\mathcal{M}(-t)u_2})\mathcal{M}(-t)u_3\\
&\quad = C|t|^{-\gamma}\mathcal{M}(-t)\mathcal{F}^{-1}_{\xi\to x}(|\xi|^{-\gamma} * D(t^{-1})\mathcal{M}(-t)u_1\overline{D(t^{-1})\mathcal{M}(-t)u_2})\\
&\qquad \times D(t^{-1})\mathcal{M}(-t)u_3\\
&\quad = C|t|^{-\gamma}\left(|x|^{-(n-\gamma)}\mathcal{M}(t)U(-t)u_1 * \left(\overline{\mathcal{M}(t)U(-t)u_2}(-\cdot)\right)\right) * U(-t)u_3,
\end{aligned}$$

for $t \neq 0$.

Lemma 1 ([31]) *Let $n = 1$. We have*

$$\left\|\left(U^{-1}\mathcal{T}_0(u_1, u_2, u_3)\right)(t)\right\|_{L^1} \leq C|t|^{-1}\prod_{j=1}^{3}\left\|\left(U^{-1}u_j\right)(t)\right\|_{L^1},$$

for $t \neq 0$,

$$\sup_{\tau\in I_T}\left(\tau\left\|\left(U^{-1}\mathcal{T}_0\left(u_1,u_2,u_3\right)\right)(\tau)\right\|_{L^1}\right)$$
$$\leq C\prod_{j=1}^{3}\left\{\left\|u_j(0)\right\|_{L^1}+\int_{I_T}\left\|\left(U^{-1}\mathcal{L}u_j\right)(\tau)\right\|_{L^1}d\tau\right\},$$

and

$$\left\{\int_{I_T}\left\|\left(U^{-1}\mathcal{T}_0(u_1,u_2,u_3)\right)(\tau)\right\|_{L^2}^2 d\tau\right\}^{1/2}$$
$$\leq C\prod_{j=1}^{3}\left\{\left\|u_j(0)\right\|_{L^2}+\int_{I_T}\left\|\left(U^{-1}\mathcal{L}u_j\right)(\tau)\right\|_{L^2}d\tau\right\}.$$

Lemma 2 *Let* $n \geq 1$ *and* $0<\gamma<n$. *We have*

$$\left\|\left(U^{-1}\mathcal{T}_\gamma(u_1,u_2,u_3)\right)(t)\right\|_{L^{\frac{2n}{n+\gamma}}}\leq C|t|^{-\gamma}\prod_{j=1}^{3}\left\|\left(U^{-1}u_j\right)(t)\right\|_{L^{\frac{2n}{n+\gamma}}}$$

for all $t\neq 0$,

$$\sup_{\tau\in I_T}\left(\tau^{\gamma}\left\|\left(U^{-1}\mathcal{T}_\gamma(u_1,u_2,u_3)\right)(\tau)\right\|_{L^{\frac{2n}{n+\gamma}}}\right)$$
$$\leq C\prod_{j=1}^{3}\left(\left\|u_j(0)\right\|_{L^{\frac{2n}{n+\gamma}}}+\int_{I_T}\left\|\left(U^{-1}\mathcal{L}u_j\right)(\tau)\right\|_{L^{\frac{2n}{n+\gamma}}}d\tau\right)$$

and if $0<\gamma<\min(n,2)$, *then*

$$\left\{\int_{I_T}\left\|\left(U^{-1}\mathcal{T}_\gamma(u_1,u_2,u_3)\right)(\tau)\right\|_{L^2}^{2/\gamma}\right\}^{\gamma/2}$$
$$\leq C\prod_{j=1}^{3}\left\{\left\|u_j(0)\right\|_{L^2}+\int_{I_T}\left\|\left(U^{-1}\mathcal{L}u_j\right)(\tau)\right\|_{L^2}d\tau\right\}.$$

Proof

$$\begin{aligned}
&|t|^{\gamma}\left\|U(-t)\mathcal{T}_\gamma(u_1,u_2,u_3)\right\|_{L^{p_0}}\\
&\quad= C\left\||x|^{-(n-\gamma)}\mathcal{M}(t)U(-t)u_1*\left(\overline{\mathcal{M}(t)U(-t)u_2}(-\cdot)\right)*U(-t)u_3\right\|_{L^{p_0}}\\
&\quad\leq C\left\||x|^{-(n-\gamma)}\mathcal{M}(t)U(-t)u_1*\left(\overline{\mathcal{M}(t)U(-t)u_2}(-\cdot)\right)\right\|_{L^{\rho_1}}\|U(-t)u_3\|_{L^{p_3}}\\
&\quad= C\left\|\mathcal{F}^{-1}\left[|\xi|^{-\gamma}*\widehat{\mathcal{M}(t)U(-t)}u_1\widehat{\left(\overline{\mathcal{M}(t)U(-t)u_2}(-\cdot)\right)}\right]\right\|_{L^{\rho_1}}\|U(-t)u_3\|_{L^{p_3}}
\end{aligned}$$

$$\leq C \left\| |\xi|^{-\gamma} * \widehat{\mathcal{M}(t)U(-t)u_1} \left(\overline{\widehat{\mathcal{M}(t)U(-t)u_2}}(-\cdot) \right) \right\|_{L^{\rho_1'}} \|U(-t)u_3\|_{L^{p_3}}$$

for $\frac{1}{p_0} = \frac{1}{\rho_1} + \frac{1}{p_3} - 1$, with $2 \leq \rho_1 \leq \infty$. By the Hardy–Littlewood–Sobolev inequality, with $\frac{1}{\rho_1'} = \frac{1}{\rho_2} + \frac{\gamma}{n} - 1$, $2 \leq \rho_2 \leq \infty$ and $n - \gamma < \frac{n}{\rho_2}$, we have

$$\begin{aligned}
&|t|^{\gamma} \left\| U(-t)\mathcal{T}_{\gamma}(u_1, u_2, u_3) \right\|_{L^{p_0}} \\
&\quad \leq C \left\| |\xi|^{-\gamma} * \widehat{\mathcal{M}(t)U(-t)u_1} \left(\overline{\widehat{\mathcal{M}(t)U(-t)u_2}}(-\cdot) \right) \right\|_{L^{\rho_1'}} \|U(-t)u_3\|_{L^{p_3}} \\
&\quad \leq C \left\| \mathcal{F}\left[\mathcal{M}(t)U(-t)u_1 * \left(\overline{\mathcal{M}(t)U(-t)u_2} \right)(-\cdot) \right] \right\|_{L^{\rho_2}} \|U(-t)u_3\|_{L^{p_3}} \\
&\quad \leq C \left\| \mathcal{M}(t)U(-t)u_1 * \left(\overline{\mathcal{M}(t)U(-t)u_2} \right)(-\cdot) \right\|_{L^{\rho_2'}} \|U(-t)u_3\|_{L^{p_3}} \\
&\quad \leq C \prod_{j=1}^{3} \left\| U(-t)u_j \right\|_{L^{p_j}}
\end{aligned}$$

for $\frac{1}{\rho_2'} = \frac{1}{p_1} + \frac{1}{p_2} - 1$ and $\frac{1}{p_0} = \sum_{j=1}^{3} \frac{1}{p_j} + \frac{\gamma}{n} - 1$. In particular, $p_j = \frac{2n}{n+\gamma}$, $j = 0, 1, 2, 3$, satisfies these conditions. By

$$u_j = Uu_j(0) - iS\left[\mathcal{L}u_j\right],\ U^{-1}u_j = u_j(0) - iU^{-1}S\left[\mathcal{L}u_j\right],$$

we obtain the first and second inequalities. Finally, by the Hardy–Littlewood–Sobolev inequality with $\frac{\gamma}{3n} = \frac{3n-2\gamma}{3n} + \frac{\gamma}{n} - 1$ and the Hölder inequality with $\frac{1}{2} = \frac{\gamma}{3n} + \frac{3n-2\gamma}{6n}$, we have

$$\left\{ \int_{I_T} \left\| \mathcal{T}_{\gamma}(u_1, u_2, u_3) \right\|_{L^2}^{2/\gamma} \right\}^{\gamma/2} \leq C \prod_{j=1}^{3} \left\| u_j \right\|_{L^{6/\gamma}\left(I_T; L^{\frac{6n}{3n-2\gamma}} \right)},$$

where $\left(\frac{6}{\gamma}, \frac{6n}{3n-2\gamma} \right)$ is an admissible pair and by the Strichartz estimate

$$\begin{aligned}
&\left\{ \int_{I_T} \left\| \mathcal{T}_{\gamma}(u_1, u_2, u_3)(\tau) \right\|_{L^2}^{2/\gamma} d\tau \right\}^{\gamma/2} \\
&\quad \leq C \prod_{j=1}^{3} \left\{ \left\| u_j(0) \right\|_{L^2} + \int_{I_T} \left\| \left(U^{-1}\mathcal{L}u_j \right)(\tau) \right\|_{L^2} d\tau \right\}.
\end{aligned}$$

This completes the proof. □

We obtain the following two inequalities by the multi-linear interpolation between $\tau^{-\alpha}L^{\infty}\left(\tau^{-2}d\tau, I_T; L^1\right)$ and $\tau^{-\alpha}L^q(\tau^{-2}d\tau, I_T; L^2)$, for $\alpha = 1, \gamma$ respectively (see

Chapter 4 of [1] and [31]), where

$$\|u\|_{\tau^{\alpha}L^{r}(\tau^{-2}d\tau,I_T;L^p)} = \left\{\int_{I_T} \tau^{\alpha r}\,\|u(\tau,\cdot)\|_{L^p}^{r}\,\tau^{-2}d\tau\right\}^{1/r},\ 1\le r<\infty,\ 1\le p\le\infty$$

and

$$\|u\|_{\tau^{\alpha}L^{\infty}(\tau^{-2}d\tau,I_T;L^p)} = \sup_{\tau\in I_T}\tau^{\alpha}\,\|u(\tau,\cdot)\|_{L^p},\ 1\le p\le\infty.$$

Lemma 3 ([12,31]) *We have*

$$\left\{\int_{I_T}\tau^{2\left(\frac{1}{p}-\frac{1}{2}\right)q}\left\|\left(U^{-1}\mathcal{T}_0(u_1,u_2,u_3)\right)(\tau)\right\|_{L^p}^{q}d\tau\right\}^{1/q} \le C\prod_{j=1}^{3}\left\{\left\|u_j(0)\right\|_{L^p}+\int_{I_T}\left\|\left(U^{-1}\mathcal{L}u_j\right)(\tau)\right\|_{L^p}d\tau\right\},$$

for $n=1,\ 1<p<2,\ q=p'$ *and*

$$\left\{\int_{I_T}\tau^{2n\left(\frac{1}{p}-\frac{1}{2}\right)q}\left\|\left(U^{-1}\mathcal{T}_{\gamma}(u_1,u_2,u_3)\right)(\tau)\right\|_{L^p}^{q}d\tau\right\}^{1/q} \le C\prod_{j=1}^{3}\left\{\left\|u_j(0)\right\|_{L^p}+\int_{I_T}\left\|\left(U^{-1}\mathcal{L}u_j\right)(\tau)\right\|_{L^p}d\tau\right\},$$

for $0<\gamma<\min(n,2),\ \frac{2n}{n+\gamma}<p<2,\ q=\frac{2p}{(n+\gamma)p-2n}$.

Lemma 4

(1) *Let* $1\le p<2,\ 1\le q\le\infty,\ \theta>0$ *and let* $u\in G^{D}_{p,q,\theta}(I_T)$. *Then*

$$A_{\delta}(|u|^2u) = A_{\delta}u\overline{A_{-\delta}u}A_{\delta}u$$

for all $\delta\in D$.

(2) *Let* $\frac{2n}{n+\gamma}\le p<2,\ 1\le q\le\infty,\ \theta>0$ *and let* $u\in G^{D}_{p,q,\theta}(I_T)$. *Then*

$$A_{\delta}\left((|x|^{-\gamma}*|u|^2)u\right) = \left(|x|^{-\gamma}*(A_{\delta}u\overline{A_{-\delta}u})\right)A_{\delta}u$$

for all $\delta\in D$.

Proof Let $t\neq 0$. It is sufficient to show

$$e^{\delta\cdot x}U(-t)\left[(|u|^2u)(t)\right]\in L^1$$

for all $\delta \in D$, by Proposition 1 above. Indeed, we have

$$\begin{aligned} &e^{\delta\cdot x}U(-t)\left[(|u|^2u)(t)\right] \\ &\quad = Ct^{-n}\left(e^{\delta\cdot x}\mathcal{M}(t)U(-t)u(t) * \left(e^{\delta\cdot x}\overline{\mathcal{M}(t)U(-t)u(t)}(-\cdot)\right)\right) \\ &\qquad * e^{\delta\cdot x}U(-t)u(t) \in L^1 \end{aligned}$$

for all $\delta \in D$, because $e^{\delta\cdot x}U(-t)u(t) \in L^1 \cap L^p$ for all $\delta \in D$. Hence $(\mathcal{M}^{-1}|u|^2u)(t)$ is analytic on $\mathbb{R}^n + itD$ and its analytic continuation is represented as

$$(\mathcal{M}^{-1}|u|^2u)(t, x + it\delta) = e^{it\delta\cdot\nabla}(\mathcal{M}^{-1}|u|^2u)(t, x),\ \ x + it\delta \in \mathbb{R}^n + itD,$$

and

$$A_\delta(|u|^2u)(t) = \mathcal{M}(t)e^{it\delta\cdot\nabla}\mathcal{M}(-t)(|u|^2u)(t) = (A_\delta u)\,(t)\overline{(A_{-\delta}u)\,(t)}\,(A_\delta u)\,(t)$$

for all $\delta \in D$. Similarly, we have

$$\begin{aligned} &e^{\delta\cdot x}U(-t)[((|x|^{-\gamma} * |u|^2)u)(t)] \\ &\quad = C|t|^{-\gamma}\left(|x|^{-(n-\gamma)}e^{\delta\cdot x}\mathcal{M}(t)U(-t)u(t) * \left(e^{\delta\cdot x}\overline{\mathcal{M}(t)U(-t)u(t)}(-\cdot)\right)\right) \\ &\qquad * e^{\delta\cdot x}U(-t)u(t) \in L^{\frac{2n}{n+\gamma}} \end{aligned}$$

for all $\delta \in D$ by Lemma 2 above and hence

$$\begin{aligned} &e^{\delta\cdot x}U(-t)[((|x|^{-\gamma} * |u|^2)u)(t)] \\ &\quad = C|t|^{-\gamma}\left(|x|^{-(n-\gamma)}e^{\delta\cdot x}\mathcal{M}(t)U(-t)u(t) * \left(e^{\delta\cdot x}\overline{\mathcal{M}(t)U(-t)u(t)}(-\cdot)\right)\right) \\ &\qquad * e^{\delta\cdot x}U(-t)u(t) \in L^1 \end{aligned}$$

for all $\delta \in D$. □

5 Proof of Theorem 1

We define a complete metric space $\left(B_T^a(R), d\right)$ by

$$B_T^a(R) = \left\{u \in G^a_{p,p',2\left(\frac{1}{p}-\frac{1}{2}\right)}(I_T);\ u(0) = \phi,\ \sum_{\alpha\geq 0}\frac{a^\alpha}{\alpha!}\left\|J^\alpha u\right\|_{X^p_{p',2\left(\frac{1}{p}-\frac{1}{2}\right)}(I_T)} \leq R\right\},$$

$$d(u, v) = \sum_{\alpha\geq 0}\frac{a^\alpha}{\alpha!}\left\|J^\alpha(u - v)\right\|_{X^p_{p',2\left(\frac{1}{p}-\frac{1}{2}\right)}(I_T)}.$$

We show the map $\Phi : u \mapsto \Phi u$, $\Phi u = U\phi - i\lambda S[|u|^2 u]$, is a contraction mapping in $(B_T^a(R), d)$. We have

$$J^\alpha \Phi u = U x^\alpha \phi - i\lambda \sum_{\beta+\gamma+\delta=\alpha} \frac{\alpha!(-1)^{|\gamma|}}{\beta!\gamma!\delta!} S\left[J^\beta u \overline{J^\gamma u} J^\delta u\right]$$

and

$$\begin{aligned} U^{-1}\mathcal{L} J^\alpha \Phi u &= \lambda U^{-1} \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \frac{\alpha!(-1)^{|\alpha_2|}}{\alpha_1!\alpha_2!\alpha_3!} J^{\alpha_1} u \overline{J^{\alpha_2} u} J^{\alpha_3} u \\ &= \lambda \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \frac{\alpha!(-1)^{|\alpha_2|}}{\alpha_1!\alpha_2!\alpha_3!} U^{-1} \mathcal{T}_0\left(J^{\alpha_1} u, J^{\alpha_2} u, J^{\alpha_3} u\right). \end{aligned}$$

Therefore,

$$\begin{aligned} &\left\{\int_{I_T} \tau^{2\left(\frac{1}{p}-\frac{1}{2}\right)p'} \left\|\left(U^{-1}\mathcal{L} J^\alpha \Phi u\right)(\tau)\right\|_{L^p}^{p'} d\tau\right\}^{1/p'} \\ &\le C \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!} \prod_{j=1}^3 \left\{\left\|x^{\alpha_j}\phi\right\|_{L^p} + \int_{I_T} \left\|\left(U^{-1}\mathcal{L} J^{\alpha_j} u\right)(\tau)\right\|_{L^p} d\tau\right\} \\ &= C \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \alpha! \prod_{j=1}^3 \left\{\frac{1}{\alpha_j!} \left\|x^{\alpha_j}\phi\right\|_{L^p} + \frac{1}{\alpha_j!} \int_{I_T} \left\|\left(U^{-1}\mathcal{L} J^{\alpha_j} u\right)(\tau)\right\|_{L^p} d\tau\right\} \end{aligned}$$

and

$$\begin{aligned} &\int_{I_T} \left\|\left(U^{-1}\mathcal{L} J^{\alpha_j} u\right)(\tau)\right\|_{L^p} d\tau \\ &= \int_{I_T} \tau^{-2\left(\frac{1}{p}-\frac{1}{2}\right)} \tau^{2\left(\frac{1}{p}-\frac{1}{2}\right)} \left\|\left(U^{-1}\mathcal{L} J^{\alpha_j} u\right)(\tau)\right\|_{L^p} d\tau \\ &\le \left\{\int_{I_T} \tau^{p-2} d\tau\right\}^{1/p} \left\{\int_{I_T} \tau^{2\left(\frac{1}{p}-\frac{1}{2}\right)p'} \left\|\left(U^{-1}\mathcal{L} J^{\alpha_j} u\right)(\tau)\right\|_{L^p}^{p'} d\tau\right\}^{1/p'} \\ &= T^{\frac{1}{p'}} \left\{\int_{I_T} \tau^{2\left(\frac{1}{p}-\frac{1}{2}\right)p'} \left\|\left(U^{-1}\mathcal{L} J^{\alpha_j} u\right)(\tau)\right\|_{L^p}^{p'} d\tau\right\}^{1/p'}. \end{aligned}$$

Also we have the difference term

$$\begin{aligned} &U^{-1}\mathcal{L}\left(\Phi J^\alpha u - \Phi J^\alpha v\right) \\ &= \lambda \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \frac{\alpha!(-1)^{|\alpha_2|}}{\alpha_1!\alpha_2!\alpha_3!} U^{-1}\Big[\mathcal{T}_0\left(J^{\alpha_1} u, J^{\alpha_2} u, J^{\alpha_3}(u-v)\right) \\ &\quad + \mathcal{T}_0\left(J^{\alpha_1} v, J^{\alpha_2} v, J^{\alpha_3}(u-v)\right) + \mathcal{T}_0\left(J^{\alpha_1} u, J^{\alpha_2}(u-v), J^{\alpha_3} v\right)\Big]. \end{aligned}$$

Therefore,

$$\sum_{\alpha\geq 0}\frac{a^{\alpha}}{\alpha!}\left\|J^{\alpha}\Phi u\right\|_{X^{p}_{p',2\left(\frac{1}{p}-\frac{1}{2}\right)}(I_T)}\leq C\left(\eta+T^{\frac{1}{p'}}R\right)^{3},$$

$$d(\Phi u,\Phi v)\leq CT^{\frac{1}{p'}}\left(\eta+T^{\frac{1}{p'}}R\right)^{2}d(u,v)$$

and Φ is a contraction mapping with $R>0$ and $T>0$ satisfying

$$\begin{cases} T^{\frac{1}{p'}}<\min\left(\frac{1}{2^{2/3}C\eta^2},\frac{2^{1/3}-1}{2C\eta^2}\right),\\ R=2C\eta^3. \end{cases}$$

6 Proof of Theorem 2

We define a complete metric space $\left(B_T^D(R),d\right)$ by

$$B_T^D(R)=\left\{u\in G^D_{p,p',2\left(\frac{1}{p}-\frac{1}{2}\right)}(I_T);\ u(0)=\phi,\ \sup_{\delta\in D}\|A_\delta u\|_{X^{p}_{p',2\left(\frac{1}{p}-\frac{1}{2}\right)}(I_T)}\leq R\right\},$$

$$d(u,v)=\sup_{\delta\in D}\|A_\delta(u-v)\|_{X^{p}_{p',2\left(\frac{1}{p}-\frac{1}{2}\right)}(I_T)}.$$

We show the map $\Phi:u\mapsto\Phi u$, $\Phi u=U\phi-i\lambda S[|u|^2u]$, is a contraction mapping in $(B_T^D(R),d)$. We have

$$A_\delta\Phi u=Ue^{\delta\cdot x}\phi-i\lambda S\left[A_\delta u\overline{A_{-\delta}u}A_\delta u\right],$$

and

$$U^{-1}\mathcal{L}A_\delta\Phi u=\lambda U^{-1}A_\delta u\overline{A_{-\delta}u}A_\delta u=\lambda U^{-1}\mathcal{T}_0(A_\delta u,A_{-\delta}u,A_\delta u).$$

Therefore,

$$\left\{\int_{I_T}\tau^{2\left(\frac{1}{p}-\frac{1}{2}\right)p'}\left\|\left(U^{-1}\mathcal{L}A_\delta\Phi u\right)(\tau)\right\|_{L^p}^{q}d\tau\right\}^{1/q}$$
$$\leq C\prod_{j=1}^{3}\left\{\left\|e^{(-1)^{j+1}\delta\cdot x}\phi\right\|_{L^p}+\int_{I_T}\left\|\left(U^{-1}\mathcal{L}A_{(-1)^{j+1}\delta}u\right)(\tau)\right\|_{L^p}d\tau\right\}$$

and

$$\int_{I_T}\left\|\left(U^{-1}\mathcal{L}A_{(-1)^{j+1}\delta}u\right)(\tau)\right\|_{L^p}d\tau$$

$$
\begin{aligned}
&= \int_{I_T} \tau^{-2\left(\frac{1}{p}-\frac{1}{2}\right)} \tau^{2\left(\frac{1}{p}-\frac{1}{2}\right)} \left\| \left(U^{-1}\mathcal{L}A_{(-1)^{j+1}\delta}u\right)(\tau)\right\|_{L^p} d\tau \\
&\le \left\{\int_{I_T} \tau^{p-2} d\tau\right\}^{1/p} \left\{\int_{I_T} \tau^{2\left(\frac{1}{p}-\frac{1}{2}\right)p'} \left\| \left(U^{-1}\mathcal{L}A_{(-1)^{j+1}\delta}u\right)(\tau)\right\|_{L^p}^{p'} d\tau\right\}^{1/p'} \\
&= T^{\frac{1}{p'}} \left\{\int_{I_T} \tau^{2\left(\frac{1}{p}-\frac{1}{2}\right)p'} \left\| \left(U^{-1}\mathcal{L}A_{(-1)^{j+1}\delta}u\right)(\tau)\right\|_{L^p}^{p'} d\tau\right\}^{1/p'} .
\end{aligned}
$$

Also we have the difference term

$$
\begin{aligned}
&U^{-1}\mathcal{L}(\Phi u - \Phi v) \\
&\quad = \lambda U^{-1}\left[\mathcal{T}_0\left(A_\delta u, A_{-\delta}u, A_\delta(u-v)\right) + \mathcal{T}_0\left(A_\delta v, A_{-\delta}v, A_\delta(u-v)\right)\right. \\
&\qquad \left. + \mathcal{T}_0\left(A_\delta u, A_{-\delta}(u-v), A_\delta v\right)\right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sup_{\delta\in D} \|A_\delta \Phi u\|_{X^p_{p',2\left(\frac{1}{p}-\frac{1}{2}\right)}(I_T)} &\le C\left(\eta + T^{\frac{1}{p'}}R\right)^3, \\
d(\Phi u, \Phi v) &\le C T^{\frac{1}{p'}} \left(\eta + T^{\frac{1}{p'}}R\right)^2 d(u,v)
\end{aligned}
$$

and Φ is a contraction mapping with $R > 0$ and $T > 0$ satisfying

$$
\begin{cases}
T^{\frac{1}{p'}} < \min\left(\frac{1}{2^{2/3}C\eta^2}, \frac{2^{1/3}-1}{2C\eta^2}\right), \\
R = 2C\eta^3.
\end{cases}
$$

7 Proof of Theorem 3

We define a complete metric space $\left(B^a_T(R), d\right)$ by

$$
\begin{aligned}
&B^a_T(R) = \left\{ u \in G^a_{p,q,2n\left(\frac{1}{p}-\frac{1}{2}\right)}(I_T);\ u(0) = \phi,\ \sum_{\alpha\ge 0} \frac{a^\alpha}{\alpha!} \left\|J^\alpha u\right\|_{X^p_{q,2n\left(\frac{1}{p}-\frac{1}{2}\right)}(I_T)} \le R \right\}, \\
&d(u,v) = \sum_{\alpha\ge 0} \frac{a^\alpha}{\alpha!} \left\|J^\alpha(u-v)\right\|_{X^p_{q,2n\left(\frac{1}{p}-\frac{1}{2}\right)}(I_T)} .
\end{aligned}
$$

We show the map $\Phi : u \mapsto \Phi u$, $\Phi u = U\phi - i\lambda S[|u|^2 u]$, is a contraction mapping in $(B^a_T(R), d)$. We have

$$
J^\alpha \Phi u = U x^\alpha \phi - i\lambda \sum_{\beta+\gamma+\delta=\alpha} \frac{\alpha!(-1)^{|\gamma|}}{\beta!\gamma!\delta!} S\left[J^\beta u \overline{J^\gamma u} J^\delta u\right]
$$

and

$$U^{-1}\mathcal{L}J^{\alpha}\Phi u = \lambda U^{-1}\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\frac{\alpha!(-1)^{|\alpha_2|}}{\alpha_1!\alpha_2!\alpha_3!}\left(|x|^{-\gamma}*J^{\alpha_1}u\overline{J^{\alpha_2}u}\right)J^{\alpha_3}u$$
$$=\lambda\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\frac{\alpha!(-1)^{|\alpha_2|}}{\alpha_1!\alpha_2!\alpha_3!}U^{-1}\mathcal{T}_{\gamma}\left(J^{\alpha_1}u,\,J^{\alpha_2}u,\,J^{\alpha_3}u\right).$$

Therefore,

$$\left\{\int_{I_T}\tau^{2n\left(\frac{1}{p}-\frac{1}{2}\right)q}\left\|\left(U^{-1}\mathcal{L}J^{\alpha}\Phi u\right)(\tau)\right\|_{L^p}^q d\tau\right\}^{1/q}$$
$$\leq C\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}\prod_{j=1}^{3}\left\{\left\|x^{\alpha_j}\phi\right\|_{L^p}+\int_{I_T}\left\|\left(U^{-1}\mathcal{L}J^{\alpha_j}u\right)(\tau)\right\|_{L^p}d\tau\right\}$$
$$=C\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\alpha!\prod_{j=1}^{3}\left\{\frac{1}{\alpha_j!}\left\|x^{\alpha_j}\phi\right\|_{L^p}+\frac{1}{\alpha_j!}\int_{I_T}\left\|\left(U^{-1}\mathcal{L}J^{\alpha_j}u\right)(\tau)\right\|_{L^p}d\tau\right\}$$

and

$$\int_{I_T}\left\|\left(U^{-1}\mathcal{L}J^{\alpha_j}u\right)(\tau)\right\|_{L^p}d\tau$$
$$=\int_{I_T}\tau^{-2n\left(\frac{1}{p}-\frac{1}{2}\right)}\tau^{2n\left(\frac{1}{p}-\frac{1}{2}\right)}\left\|\left(\partial_\tau U^{-1}J^{\alpha_j}u\right)(\tau)\right\|_{L^p}d\tau$$
$$\leq\left\{\int_{I_T}\tau^{-2n\left(\frac{1}{p}-\frac{1}{2}\right)q'}d\tau\right\}^{1/q'}\left\{\int_{I_T}\tau^{2n\left(\frac{1}{p}-\frac{1}{2}\right)q}\left\|\left(U^{-1}\mathcal{L}J^{\alpha_j}u\right)(\tau)\right\|_{L^p}^q d\tau\right\}^{1/q}$$
$$=T^{\frac{(2+n-\gamma)p-2n}{(2-n-\gamma)p+2n}}\left\{\int_{I_T}\tau^{2n\left(\frac{1}{p}-\frac{1}{2}\right)q}\left\|\left(U^{-1}\mathcal{L}J^{\alpha_j}u\right)(\tau)\right\|_{L^p}^q d\tau\right\}^{1/q}.$$

Also we have the difference term

$$U^{-1}\mathcal{L}\left(\Phi J^{\alpha}u-\Phi J^{\alpha}v\right)$$
$$=\lambda\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\frac{\alpha!(-1)^{|\alpha_2|}}{\alpha_1!\alpha_2!\alpha_3!}U^{-1}\Big[\mathcal{T}_{\gamma}\left(J^{\alpha_1}(u-v),\,J^{\alpha_2}u,\,J^{\alpha_3}u\right)$$
$$+\mathcal{T}_{\gamma}\left(J^{\alpha_1}v,\,J^{\alpha_2}v,\,J^{\alpha_3}(u-v)\right)+\mathcal{T}_{\gamma}\left(J^{\alpha_1}v,\,J^{\alpha_2}(u-v),\,J^{\alpha_3}u\right)\Big].$$

Therefore,

$$\sum_{\alpha\geq 0}\frac{a^{\alpha}}{\alpha!}\left\|J^{\alpha}\Phi u\right\|_{X^p_{q,2n\left(\frac{1}{p}-\frac{1}{2}\right)}(I_T)}\leq C\left(\eta+T^{\frac{(2+n-\gamma)p-2n}{(2-n-\gamma)p+2n}\frac{1}{q'}}R\right)^3,$$

$$d(\Phi u, \Phi v)d \leq CT^{\frac{(2+n-\gamma)p-2n}{(2-n-\gamma)p+2n}\frac{1}{q'}}\left(\eta + T^{\frac{(2+n-\gamma)p-2n}{(2-n-\gamma)p+2n}\frac{1}{q'}}R\right)^2 d(u,v)$$

and Φ is a contraction mapping with $R > 0$ and $T > 0$ satisfying

$$\begin{cases} T^{\frac{(2+n-\gamma)p-2n}{(2-n-\gamma)p+2n}\frac{1}{q'}} < \min\left(\frac{1}{2^{2/3}C\eta^2}, \frac{2^{1/3}-1}{2C\eta^2}\right), \\ R = 2C\eta^3. \end{cases}$$

8 Proof of Theorem 4

We define a complete metric space $\left(B_T^D(R), d\right)$ by

$$B_T^D(R) = \left\{u \in G^D_{p,q,2n\left(\frac{1}{p}-\frac{1}{2}\right)}(I_T);\ u(0)=\phi,\ \sup_{\delta\in D}\|A_\delta u\|_{X^p_{q,2n\left(\frac{1}{p}-\frac{1}{2}\right)}(I_T)} \leq R\right\},$$
$$d(u,v) = \sup_{\delta\in D}\|A_\delta(u-v)\|_{X^p_{q,2n\left(\frac{1}{p}-\frac{1}{2}\right)}(I_T)}.$$

We show the map $\Phi : u \mapsto \Phi u$, $\Phi u = U\phi - i\lambda S[|u|^2u]$, is a contraction mapping in $(B_T^D(R), d)$. We have

$$A_\delta\Phi u = Ue^{\delta\cdot x}\phi - i\lambda S\left[A_\delta u\overline{A_{-\delta}u}A_\delta u\right],$$

and

$$U^{-1}\mathcal{L}A_\delta\Phi u = \lambda U^{-1}\left(|x|^{-\gamma} * A_\delta u\overline{A_{-\delta}u}\right)A_\delta u = \lambda U^{-1}\mathcal{T}_\gamma(A_\delta u, A_{-\delta}u, A_\delta u).$$

Therefore,

$$\left\{\int_{I_T}\tau^{2n\left(\frac{1}{p}-\frac{1}{2}\right)q}\left\|\left(U^{-1}\mathcal{L}A_\delta\Phi u\right)(\tau)\right\|^q_{L^p}d\tau\right\}^{1/q}$$
$$\leq C\prod_{j=1}^{3}\left\{\left\|e^{(-1)^{j+1}\delta\cdot x}\phi\right\|_{L^p} + \int_{I_T}\left\|\left(U^{-1}\mathcal{L}A_{(-1)^{j+1}\delta}u\right)(\tau)\right\|_{L^p}d\tau\right\}$$

and

$$\int_{I_T}\left\|\left(U^{-1}\mathcal{L}A_{(-1)^{j+1}\delta}u\right)(\tau)\right\|_{L^p}d\tau$$
$$= \int_{I_T}\tau^{-2n\left(\frac{1}{p}-\frac{1}{2}\right)}\tau^{2n\left(\frac{1}{p}-\frac{1}{2}\right)}\left\|\left(U^{-1}\mathcal{L}A_{(-1)^{j+1}\delta}u\right)(\tau)\right\|_{L^p}d\tau$$
$$\leq \left\{\int_{I_T}\tau^{-2n\left(\frac{1}{p}-\frac{1}{2}\right)q'}d\tau\right\}^{1/q'}\left\{\int_{I_T}\tau^{2n\left(\frac{1}{p}-\frac{1}{2}\right)q}\left\|\left(U^{-1}\mathcal{L}A_{(-1)^{j+1}\delta}u\right)(\tau)\right\|^q_{L^p}d\tau\right\}^{1/q}$$

$$= T^{\frac{(2+n-\gamma)p-2n}{(2-n-\gamma)p+2n}\frac{1}{q'}} \left\{ \int_{I_T} \tau^{2n\left(\frac{1}{p}-\frac{1}{2}\right)q} \left\| \left(U^{-1}\mathcal{L}A_{(-1)^{j+1}\delta}u \right)(\tau) \right\|_{L^p}^q d\tau \right\}^{1/q}.$$

Also we have the difference term

$$\begin{aligned} &U^{-1}\mathcal{L}(\Phi u - \Phi v) \\ &\quad = \lambda U^{-1}\left[\mathcal{T}_\gamma\left(A_\delta(u-v), A_{-\delta}u, A_\delta u\right) + \mathcal{T}_\gamma\left(A_\delta v, A_{-\delta}v, A_\delta(u-v)\right)\right. \\ &\qquad \left. + \mathcal{T}_\gamma\left(A_\delta v, A_{-\delta}(u-v), A_\delta u\right)\right]. \end{aligned}$$

Therefore,

$$\sup_{\delta\in D} \|A_\delta \Phi u\|_{X^p_{p',2n\left(\frac{1}{p}-\frac{1}{2}\right)}(I_T)} \le C\left(\eta + T^{\frac{(2+n-\gamma)p-2n}{(2-n-\gamma)p+2n}\frac{1}{q'}} R\right)^3,$$

$$d(\Phi u, \Phi v) \le C T^{\frac{(2+n-\gamma)p-2n}{(2-n-\gamma)p+2n}\frac{1}{q'}} \left(\eta + T^{\frac{(2+n-\gamma)p-2n}{(2-n-\gamma)p+2n}\frac{1}{q'}} R\right)^2 d(u,v)$$

and Φ is a contraction mapping with $R > 0$ and $T > 0$ satisfying

$$\begin{cases} T^{\frac{(2+n-\gamma)p-2n}{(2-n-\gamma)p+2n}\frac{1}{q'}} < \min\left(\frac{1}{2^{2/3}C\eta^2}, \frac{2^{1/3}-1}{2C\eta^2}\right), \\ R = 2C\eta^3. \end{cases}$$

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