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A Hardy Inequality for Ultraspherical Expansions with an Application to the Sphere

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Abstract We prove a Hardy inequality for ultraspherical expansions by using a proper ground state representation. From this result we deduce some uncertainty principles for this kind of expansions. Our result also implies a Hardy inequality on spheres with a potential having a double singularity.

Keywords Hardy inequalities · Uncertainty principles · Ultraspherical expansions

Mathematics Subject Classification Primary: 42C10

1 Introduction and Main Result

For $d \geq 3$, the classical Hardy inequality states that

$$\frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{u^2(x)}{|x|^2}\, dx \leq \int_{\mathbb{R}^d} |\nabla u(x)|^2\, dx. \tag{1}$$

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Due to its applicability, there is an extensive literature about the topic (see the references in [16]) covering many extensions of this estimate in several and different directions. We are interested in one involving the fractional powers of the Laplacian. We can rewrite (1) as

$$\frac{(d-2)^2}{4}\int_{\mathbb{R}^d}\frac{u^2(x)}{|x|^2}\,dx \le \int_{\mathbb{R}^d} u(x)(-\Delta u(x))\,dx$$

and, taking the fractional Laplacian $(-\Delta)^\sigma$ defined by $\widehat{(-\Delta)^\sigma u} = |\cdot|^{2\sigma}\widehat{u}$, a natural extension is the inequality

$$C_{\sigma,d}\int_{\mathbb{R}^d}\frac{u^2(x)}{|x|^{2\sigma}}\,dx \le \int_{\mathbb{R}^d} u(x)(-\Delta)^\sigma u(x)\,dx, \tag{2}$$

for which the sharp constant $C_{\sigma,d}$ is well known (see [3,20]).
From (2), we deduce the positivity (in a distributional sense) of the operator

$$(-\Delta)^\sigma - \frac{C_{\sigma,d}}{|\cdot|^{2\sigma}}.$$

Our target is to provide a Hardy inequality like (2) related to ultraspherical expansions and apply it to prove the positivity of certain operator on the sphere with a potential having singularities in both poles of the sphere.

Let $C_n^\lambda(x)$ be the ultraspherical polynomial of degree n and order $\lambda > -1/2$. We consider $c_n^\lambda(x) = d_n^{-1}C_n^\lambda(x)$ with

$$d_n^2 = \int_{-1}^{1}\left(C_n^\lambda(x)\right)^2\,d\mu_\lambda(x), \qquad d\mu_\lambda(x) = (1-x^2)^{\lambda-1/2}\,dx.$$

The sequence of polynomials $\{c_n^\lambda\}_{n\ge 0}$ forms an orthonormal basis of the space $L^2_\lambda := L^2((-1,1), d\mu_\lambda)$. For each c_n^λ, it holds that $\mathcal{L}_\lambda c_n^\lambda = -(n+\lambda)^2 c_n^\lambda$, where

$$\mathcal{L}_\lambda = (1-x^2)\frac{d^2}{dx^2} - (2\lambda+1)x\frac{d}{dx} - \lambda^2.$$

The ultraspherical expansion of each appropriate function f defined in $(-1,1)$ is given by

$$f \longmapsto \sum_{n=0}^{\infty} a_n^\lambda(f)c_n^\lambda,$$

where $a_n^\lambda(f)$ is the n-th Fourier coefficient of f respect to $\{c_n^\lambda\}_{n\ge0}$, i.e.,

$$a_n^\lambda(f) = \int_{-1}^{1} f(y)c_n^\lambda(y)\,d\mu_\lambda(y).$$

The fractional powers of the operator $\mathcal{L}_\lambda$ are defined by

$$(-\mathcal{L}_\lambda)^{\sigma/2} f = \sum_{n=0}^{\infty} (n+\lambda)^\sigma a_n^\lambda(f) c_n^\lambda, \qquad \sigma > 0.$$

This operator should be the natural candidate to prove a Hardy type inequality for the ultraspherical expansion but, however, it is not the most appropriate in this setting. We have to consider another one with an analogous behaviour to $(-\mathcal{L}_\lambda)^{\sigma/2}$, in order to deduce some results on the sphere. For each $\sigma > 0$ we define (spectrally) the operator

$$A_\sigma^\lambda = \frac{\Gamma(\sqrt{-\mathcal{L}_\lambda} + \frac{1+\sigma}{2})}{\Gamma(\sqrt{-\mathcal{L}_\lambda} + \frac{1-\sigma}{2})}.$$

Then for f defined on the interval $(-1, 1)$

$$A_\sigma^\lambda f(x) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda+\frac{1+\sigma}{2})}{\Gamma(n+\lambda+\frac{1-\sigma}{2})} a_n^\lambda(f) c_n^\lambda(x).$$

Note that

$$\frac{\Gamma(n+\lambda+\frac{1+\sigma}{2})}{\Gamma(n+\lambda+\frac{1-\sigma}{2})} \simeq (n+\lambda)^\sigma, \tag{3}$$

then the behaviour of $(-\mathcal{L}_\lambda)^{\sigma/2}$ and A_σ^λ is similar. The natural Sobolev space to analyse Hardy type inequalities is

$$H_\lambda^\sigma = \left\{ f \in L_\lambda^2 : \|f\|_{H_\lambda^\sigma} := \left(\sum_{n=0}^{\infty} (n+\lambda)^\sigma (a_n^\lambda(f))^2 \right)^{1/2} < \infty \right\}.$$

We have to note that H_λ^σ is equivalent to the space $\mathcal{L}_{\lambda,\sigma}^2$ introduced in [5].

With the previous notation our Hardy inequality for ultraspherical expansions is given in the following result.

Theorem 1 *Let* $\lambda > 0$ *and* $0 < \sigma < 1$. *Then for* $u \in H_\lambda^\sigma$

$$Q_{\sigma,\lambda} \int_{-1}^{1} \frac{u^2(x)}{(1-x^2)^{\sigma/2}} \, d\mu_\lambda(x) \le \int_{-1}^{1} u(x) A_\sigma^\lambda u(x) \, d\mu_\lambda(x), \tag{4}$$

where

$$Q_{\sigma,\lambda} = 2^\sigma \frac{\Gamma(\frac{\lambda}{2} + \frac{1+\sigma}{4})^2}{\Gamma(\frac{\lambda}{2} + \frac{1-\sigma}{4})^2}. \tag{5}$$

Inequality (4) can be rewritten in terms of the Fourier coefficients

$$Q_{\sigma,\lambda} \int_{-1}^{1} \frac{u^2(x)}{(1-x^2)^{\sigma/2}} \, d\mu_\lambda(x) \le \sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda+\frac{1+\sigma}{2})}{\Gamma(n+\lambda+\frac{1-\sigma}{2})} (a_n^\lambda(u))^2, \tag{6}$$

which is a kind of Pitt inequality for the ultraspherical expansions (for other Pitt inequalities see [4,11]). Note that for the right hand side of (4) we have, by (3),

$$\int_{-1}^{1} u(x) A_\sigma^\lambda u(x)\, d\mu_\lambda(x) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda+\frac{1+\sigma}{2})}{\Gamma(n+\lambda+\frac{1-\sigma}{2})} (a_n^\lambda(u))^2 \simeq \|u\|^2_{H_\lambda^\sigma},$$

so the space H_λ^σ is the adequated one.

The proof of Theorem 1 will be a consequence of a proper ground state representation in our setting, analogous to the given one in the Euclidean case in [9]. Following the ideas in that paper, we can see that the constant $Q_{\sigma,\lambda}$ is sharp but not achieved. Similar ideas have been recently exploited in [7,16].

From (4), by using Cauchy–Schwarz inequality, we can obtain a Heisenberg type uncertainty principle as it was done for the sublaplacian of the Heisenberg group in [10], and for the fractional powers of the same sublaplacian in [16].

Corollary 2 *Let* $\lambda > 0$ *and* $0 < \sigma < 1$. *Then for* $u \in H_\lambda^\sigma$

$$Q_{\sigma,\lambda} \left(\int_{-1}^{1} u^2(x)\, d\mu_\lambda(x) \right)^2 \le \int_{-1}^{1} u^2(x)(1-x^2)^{\sigma/2}\, d\mu_\lambda(x) \times \int_{-1}^{1} u(x) A_\sigma^\lambda u(x)\, d\mu_\lambda(x),$$

where $Q_{\sigma,\lambda}$ *is the constant given in* (5).

Pitt inequality (6) allows us to prove a logarithmic uncertainty principle for the ultraspherical expansions. The main idea comes from [3]. By an elementary argument, for a derivable function such that $\phi(0) = 0$ and $\phi(\sigma) > 0$ for $\sigma \in (0, \varepsilon)$, with $\varepsilon > 0$, it is verified that $\phi'(0_+) \ge 0$. Then, taking the function

$$\phi(\sigma) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda+\frac{1+\sigma}{2})}{\Gamma(n+\lambda+\frac{1-\sigma}{2})} (a_n^\lambda(u))^2 - Q_{\sigma,\lambda} \int_{-1}^{1} \frac{u^2(x)}{(1-x^2)^{\sigma/2}}\, d\mu_\lambda(x),$$

we have $\phi(0) = 0$ (this is Parseval identity) and, by (6), $\phi(\sigma) > 0$ for $\sigma \in (0, 1)$, then $\phi'(0_+) \ge 0$ and this inequality gives the logarithmic uncertainty principle, which is written as

$$\left(\log 2 + \psi\left(\frac{\lambda}{2} + \frac{1}{4} \right) \right) \int_{-1}^{1} u^2(x)\, d\mu_\lambda(x) \le \sum_{n=0}^{\infty} \psi\left(n+\lambda+\frac{1}{2} \right) (a_n(u))^2 + \int_{-1}^{1} \log(\sqrt{1-x^2}) u^2(x)\, d\mu_\lambda(x),$$

where $\psi(a) = \frac{\Gamma'(a)}{\Gamma(a)}$.

In next section we will show an application of Theorem 1 to obtain a Hardy inequality on the sphere. The results in Sect. 3 are the main ingredients in the proof of Theorem 1 which is given in last section of the paper.

2 An Application to the Sphere

It is well known that $L^2(\mathbb{S}^d) = \oplus_{n=0}^{\infty}\mathcal{H}_n(\mathbb{S}^d)$, where $\mathcal{H}_n(\mathbb{S}^d)$ is the set of spherical harmonics of degree n in $d+1$ variables. If we consider the shifted Laplacian on the sphere

$$-\Delta_{\mathbb{S}^d} = -\tilde{\Delta}_{\mathbb{S}^d} + \left(\frac{d-1}{2}\right)^2,$$

where $-\tilde{\Delta}_{\mathbb{S}^d}$ is the Laplace-Beltrami operator on $\mathbb{S}^d$, it is verified that

$$-\Delta_{\mathbb{S}^d}\mathcal{H}_n(\mathbb{S}^d) = \left(n + \frac{d-1}{2}\right)^2 \mathcal{H}_n(\mathbb{S}^d).$$

In this way, the analogous of the operator A_σ^λ on $\mathbb{S}^d$ is defined by

$$\begin{aligned}\mathbf{A}_\sigma f &= \frac{\Gamma\left(\sqrt{-\Delta_{\mathbb{S}^d}} + \frac{1+\sigma}{2}\right)}{\Gamma\left(\sqrt{-\Delta_{\mathbb{S}^d}} + \frac{1-\sigma}{2}\right)} f \\ &= \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{d-1}{2} + \frac{1+\sigma}{2}\right)}{\Gamma\left(n + \frac{d-1}{2} + \frac{1-\sigma}{2}\right)} \operatorname{proj}_{\mathcal{H}_n(\mathbb{S}^d)} f,\end{aligned}$$

where $\operatorname{proj}_{\mathcal{H}_n(\mathbb{S}^d)} f$ denotes the projection of f onto the eigenspace $\mathcal{H}_n(\mathbb{S}^d)$.

The operator $\mathbf{A}_\sigma$ becomes the fractional powers of the Laplacian in the Euclidean space through conformal transforms as was observed by Branson in [6]. So $\mathbf{A}_\sigma$ is the natural operator to prove a Hardy type inequality on the sphere. In our proof, we will write $\mathbf{A}_\sigma$ in terms of A_σ^λ and this is the main reason to consider A_σ^λ in the case of the ultraspherical expansions. An analogous of the Hardy-Littlewood-Sobolev inequality for $\mathbf{A}_\sigma$ and some other inequalities for it were given by Beckner in [2]. The operators $\mathbf{A}_\sigma$ also appear in [18, p. 151] and [17, p. 525].

Each point $x \in \mathbb{S}^d$ can be written as

$$x = (t, \sqrt{1-t^2}x_1', \ldots, \sqrt{1-t^2}x_d'),$$

for $t \in (-1, 1)$ and $x' := (x_1', \ldots, x_d') \in \mathbb{S}^{d-1}$, and so

$$\int_{\mathbb{S}^d} f(x)\,dx = \int_{-1}^{1}\int_{\mathbb{S}^{d-1}} f(t, \sqrt{1-t^2}x')(1-t^2)^{(d-2)/2}\,dx'\,dt.$$

With these coordinates, see [19, Sect. 3], we have that an orthonormal basis for each $\mathcal{H}_n(\mathbb{S}^d)$ is given by

$$\phi_{n,j,k}(x) = \psi_{n,j}(t) Y^d_{j,k}(x'), \qquad j = 0, \ldots, n,$$

with

$$\psi_{n,j}(t) = (1-t^2)^{j/2} c_{n-j}^{j+(d-1)/2}(t)$$

and $\{Y^d_{j,k}\}_{k=1,\ldots,d(j)}$ an orthonormal basis of spherical harmonics on $\mathbb{S}^{d-1}$ of degree j. The value $d(j)$ indicates the dimension of $\mathcal{H}_j(\mathbb{S}^{d-1})$; i.e.,

$$d(j) = (2j+d-2)\frac{(j+d-3)!}{j!(d-2)!}.$$

Then, the orthogonal projection of f onto the eigenspace $\mathcal{H}_n(\mathbb{S}^d)$ can be written as

$$\mathrm{proj}_{\mathcal{H}_n(\mathbb{S}^d)} f = \sum_{j=0}^{n} \sum_{k=1}^{d(j)} f_{n,j,k} \phi_{n,j,k},$$

with

$$f_{n,j,k} = \int_{-1}^{1} G_{j,k}(t) c_{n-j}^{j+(d-1)/2}(t)(1-t^2)^{j+(d-2)/2}\, dt,$$

$$G_{j,k}(t) = (1-t^2)^{-j/2} F_{j,k}(t) \quad \text{and} \quad F_{j,k}(t) = \int_{\mathbb{S}^{d-1}} f(t, \sqrt{1-t^2}x') Y^d_{j,k}(x')\, dx'.$$

It is easy to observe that

$$f(x) = \sum_{j=0}^{\infty} \sum_{k=1}^{d(j)} F_{j,k}(t) Y^d_{j,k}(x') = \sum_{j=0}^{\infty} \sum_{k=1}^{d(j)} (1-t^2)^{j/2} G_{j,k}(t) Y^d_{j,k}(x').$$

Moreover, from the definition of $\mathbf{A}_\sigma$, we have

$$\mathbf{A}_\sigma f(x) = \sum_{j=0}^{\infty} \sum_{k=1}^{d(j)} (1-t^2)^{j/2} A_\sigma^{j+(d-1)/2} G_{j,k}(t) Y^d_{j,k}(x').$$

Now, considering the Sobolev space

$$\mathbf{H}^\sigma = \left\{ f \in L^2(\mathbb{S}^d) : \|f\|_{\mathbf{H}^\sigma} := \left(\sum_{n=0}^{\infty} \left(n + \frac{d-1}{2}\right)^\sigma \| \mathrm{proj}_{\mathcal{H}_n(\mathbb{S}^d)} f \|^2_{L^2(\mathbb{S}^d)} \right)^{1/2} < \infty \right\},$$

we have the following Hardy inequality on the sphere.

Theorem 3 *Let $d \geq 2$, $0 < \sigma < 1$, and e_d be the north pole of the sphere $\mathbb{S}^d$. Then for $f \in \mathbf{H}^\sigma$*

$$2^\sigma Q_{\sigma,(d-1)/2} \int_{\mathbb{S}^d} \frac{f^2(x)}{(|x-e_d||x+e_d|)^\sigma}\, dx \leq \int_{\mathbb{S}^d} f(x)\mathbf{A}_\sigma f(x)\, dx, \tag{7}$$

where $Q_{\sigma,(d-1)/2}$ is the constant given in (5).

Proof By the orthogonality of the spherical harmonics, it is elementary to show that

$$\int_{\mathbb{S}^d} f(x)\mathbf{A}_\sigma f(x)\, dx = \sum_{j=0}^{\infty}\sum_{k=1}^{d(j)} \int_{-1}^{1} G_{j,k}(t) A_\sigma^{j+(d-1)/2} G_{j,k}(t)\, d\mu_{j+(d-1)/2}.$$

Now, applying Theorem 1, we deduce that

$$\int_{\mathbb{S}^d} f(x)\mathbf{A}_\sigma f(x)\, dx \geq \sum_{j=0}^{\infty}\sum_{k=1}^{d(j)} Q_{\sigma,j+(d-1)/2} \int_{-1}^{1} \frac{F_{j,k}^2(t)}{(1-t^2)^{\sigma/2}}\, d\mu_{(d-1)/2}.$$

It is known (see [20]) that for $0 < x \leq y$ and $j \geq 0$ we have that $\frac{\Gamma(j+y)}{\Gamma(j+x)} \geq \frac{\Gamma(y)}{\Gamma(x)}$. So, $Q_{\sigma,j+(d-1)/2} \geq Q_{\sigma,(d-1)/2}$ and

$$\int_{\mathbb{S}^d} f(x)\mathbf{A}_\sigma f(x)\, dx \geq Q_{\sigma,(d-1)/2} \sum_{j=0}^{\infty}\sum_{k=1}^{d(j)} \int_{-1}^{1} \frac{F_{j,k}^2(t)}{(1-t^2)^{\sigma/2}}\, d\mu_{(d-1)/2}.$$

The proof of (7) is finished by using the identity

$$\sum_{j=0}^{\infty}\sum_{k=1}^{d(j)} \int_{-1}^{1} \frac{F_{j,k}^2(t)}{(1-t^2)^{\sigma/2}}\, d\mu_{(d-1)/2} = 2^\sigma \int_{\mathbb{S}^d} \frac{f^2(x)}{(|x-e_d||x+e_d|)^\sigma}\, dx.$$

□

The analogous role on the sphere of radially symmetric functions is played by functions which are invariant under the action of $SO(d-1)$. By $SO(d-1)$-invariance we mean that f is invariant under the action of the group $SO(d-1)$ on $\mathbb{S}^{d-1}$ whenever $SO(d-1)$ is embedded into $SO(d)$ in a suitable way. Each function f of this kind can be written as $f(x) = g(\langle x, e_d\rangle)$, for a certain function g defined in $(-1, 1)$. Then for this kind of functions Theorem 3 reduces to Theorem 1 with $\lambda = (d-1)/2$, in this way we can deduce that the constant $2^\sigma Q_{\sigma,(d-1)/2}$ in (7) is sharp.

As in the classic case, from Theorem 3 we deduce that in a distributional sense

$$\mathbf{A}_\sigma - \frac{2^\sigma Q_{\sigma,(d-1)/2}}{(|x-e_d||x+e_d|)^\sigma} \geq 0.$$

Note that in this case we are perturbing the operator $\mathbf{A}_\sigma$ adding a potential with singularities in both poles of the sphere.

3 Auxiliary Results

The following lemmas give the tools to prove Theorem 1. To be more precise, Lemma 1 provides a nonlocal representation of the operator A_σ^λ with a kernel having nice properties for our target. Lemma 2 shows the action of the operator A_σ^λ on the family of weights $(1-x^2)^{-(\lambda/2+(1-\sigma)/4)}$.

For $f, g \in L_\lambda^2$ we are going to set up the notation

$$\langle f, g\rangle_\lambda = \int_{-1}^{1} f(x)g(x)\,d\mu_\lambda(x)$$

to simplify the writing.

Lemma 1 *Let $\lambda > 0$ and $0 < \sigma < 1$. If f is a finite linear combination of ultraspherical polynomials, then*

$$A_\sigma^\lambda f(x) = \int_{-1}^{1} (f(x) - f(y))\, K_\sigma^\lambda(x, y)\, d\mu_\lambda(y) + E_{\sigma,\lambda} f(x), \qquad x \in (-1, 1), \quad (8)$$

where the kernel is given by

$$K_\sigma^\lambda(x, y) = D_{\sigma,\lambda} \int_{-1}^{1} \frac{d\mu_{\lambda-1/2}(t)}{(1 - xy - \sqrt{1-x^2}\sqrt{1-y^2}\,t)^{\lambda+(1+\sigma)/2}},$$

with

$$D_{\sigma,\lambda} = \frac{c_\lambda^2}{2^{\lambda+(1+\sigma)/2}} \frac{\Gamma(\frac{1-\sigma}{2})\Gamma(\lambda+\frac{1+\sigma}{2})}{|\Gamma(-\sigma)|\Gamma(1+\lambda)}, \qquad c_\lambda = \frac{\Gamma(2\lambda+1)}{2^{2\lambda}(\Gamma(\lambda+1/2))^2},$$

and

$$E_{\sigma,\lambda} = \frac{\Gamma(\lambda+\frac{1+\sigma}{2})}{\Gamma(\lambda+\frac{1-\sigma}{2})}.$$

Moreover, for $f \in H_\lambda^\sigma$ we have

$$\langle A_\sigma^\lambda f, f\rangle_\lambda = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} (f(x) - f(y))^2 K_\sigma^\lambda(x, y)\, d\mu_\lambda(y)\, d\mu_\lambda(x) + E_{\sigma,\lambda}\langle f, f\rangle_\lambda \quad (9)$$

Proof We start with the identity

$$\int_0^\infty \left(e^{-(n+\lambda)t} - e^{-(\sigma-1)t/2}\right)(\sinh t/2)^{-\sigma-1}\, dt = 2^{1+\sigma}\Gamma(-\sigma)\frac{\Gamma(n+\lambda+\frac{1+\sigma}{2})}{\Gamma(n+\lambda+\frac{1-\sigma}{2})} \tag{10}$$

for $\lambda > 0$ (actually it is also true for values $\lambda > -1/2$) and $0 < \sigma < 1$. To deduce the previous identity it is enough to apply integration by parts with $u = e^{-(n+\lambda+(1-\sigma)/2)t} - 1$ and $v = -2e^{-\sigma t/2}(\sinh t/2)^{-\sigma}/\sigma$, and use [14, Eq. 8, p. 367]

$$\int_0^\infty e^{-\rho t}\,(\cosh(ct)-1)^\nu\, dt = \frac{\Gamma(\frac{\rho}{c}-\nu)\Gamma(2\nu+1)}{2^\nu c\Gamma(\frac{\rho}{c}+\nu+1)}$$

for $c > 0$, $2\nu > -1$, and $\rho > c\nu$.

Now, we consider the Poisson operator for ultraspherical expansions. It is given by

$$e^{-t\sqrt{-\mathcal{L}_\lambda}} f(x) = \sum_{n=0}^\infty e^{-(n+\lambda)t} a_n^\lambda(f) c_n^\lambda(x) = \int_{-1}^1 f(y) P_t^\lambda(x,y)\, d\mu_\lambda(y),$$

with

$$P_t^\lambda(x,y) = \sum_{n=0}^\infty e^{-(n+\lambda)t} c_n^\lambda(x) c_n^\lambda(y).$$

By the product formula for ultraspherical polynomials [8, Eq. B.2.9, p. 419]

$$\frac{C_n^\lambda(x)C_n^\lambda(y)}{C_n^\lambda(1)} = c_\lambda \int_{-1}^1 C_n^\lambda(xy + \sqrt{1-x^2}\sqrt{1-y^2}t)\, d\mu_{\lambda-1/2}(t), \qquad \lambda > 0,$$

the identity [8, Eq. B.2.8. p. 419]

$$\sum_{n=0}^\infty \frac{n+\lambda}{\lambda} C_n^\lambda(x) r^n = \frac{1-r^2}{(1-2xr+r^2)^{\lambda+1}}, \qquad 0 \le r < 1,$$

and the relation $d_n^2 = \frac{\lambda}{c_\lambda(n+\lambda)} C_n^\lambda(1)$, we deduce the expression

$$P_t^\lambda(x,y) = \frac{c_\lambda^2}{2^\lambda} \int_{-1}^1 \frac{\sinh t}{(\cosh t - w(s))^{\lambda+1}}\, d\mu_{\lambda-1/2}(s),$$

with $w(s) = xy + \sqrt{1-x^2}\sqrt{1-y^2}s$. The previous identity for P_t^λ is not new, it appears as formula (2.12) in [12].

Combining (10) and the definition of the Poisson operator, it is clear that

$$A_\sigma^\lambda f(x) = \frac{1}{2^{1+\sigma}\Gamma(-\sigma)} \int_0^\infty \left(e^{-t\sqrt{-\mathcal{L}_\lambda}} f(x) - f(x)e^{-(\sigma-1)t/2}\right)(\sinh t/2)^{-\sigma-1}\, dt,$$

which can be splitted in

$$
\begin{aligned}
&A_\sigma^\lambda f(x)\\
&= \frac{1}{2^{1+\sigma}\Gamma(-\sigma)} \int_0^\infty \left(e^{-t\sqrt{-\mathcal{L}_\lambda}} f(x) - f(x) e^{-t\sqrt{-\mathcal{L}_\lambda}} 1(x)\right) (\sinh t/2)^{-\sigma-1}\, dt\\
&\quad + \frac{f(x)}{2^{1+\sigma}\Gamma(-\sigma)} \int_0^\infty \left(e^{-t\sqrt{-\mathcal{L}_\lambda}} 1(x) - e^{-(\sigma-1)t/2}\right) (\sinh t/2)^{-\sigma-1}\, dt. \qquad (11)
\end{aligned}
$$

From the obvious identity

$$
e^{-t\sqrt{-\mathcal{L}_\lambda}} 1(x) = \int_{-1}^1 P_t^\lambda(x,y)\, d\mu_\lambda(y) = e^{-\lambda t},
$$

for the second term in (11) we have

$$
\begin{aligned}
&\frac{f(x)}{2^{1+\sigma}\Gamma(-\sigma)} \int_0^\infty \left(e^{-t\sqrt{-\mathcal{L}_\lambda}} 1(x) - e^{-(\sigma-1)t/2}\right) (\sinh t/2)^{-\sigma-1}\, dt\\
&= \frac{f(x)}{2^{1+\sigma}\Gamma(-\sigma)} \int_0^\infty \left(e^{-\lambda t} - e^{-(\sigma-1)t/2}\right) (\sinh t/2)^{-\sigma-1}\, dt\\
&= E_{\sigma,\lambda} f(x),
\end{aligned}
$$

where we have used (10) with $n = 0$.

The first integral in (11) verifies

$$
\begin{aligned}
&\frac{1}{2^{1+\sigma}\Gamma(-\sigma)} \int_0^\infty \left(e^{-t\sqrt{-\mathcal{L}_\lambda}} f(x) - f(x) e^{-t\sqrt{-\mathcal{L}_\lambda}} 1(x)\right) (\sinh t/2)^{-\sigma-1}\, dt\\
&= \frac{1}{2^{1+\sigma}|\Gamma(-\sigma)|} \int_0^\infty \int_{-1}^1 P_t^\lambda(x,y)(f(x) - f(y))\, d\mu_\lambda(y)\, (\sinh t/2)^{-\sigma-1}\, dt\\
&= \frac{1}{2^{1+\sigma}|\Gamma(-\sigma)|} \int_{-1}^1 (f(x) - f(y)) \int_0^\infty P_t^\lambda(x,y)\, (\sinh t/2)^{-\sigma-1}\, dt\, d\mu_\lambda(y)\\
&= \int_{-1}^1 (f(x) - f(y))\, K_\sigma^\lambda(x,y)\, d\mu_\lambda(y),
\end{aligned}
$$

with

$$
K_\sigma^\lambda(x,y) = \frac{1}{2^{1+\sigma}|\Gamma(-\sigma)|} \int_0^\infty P_t^\lambda(x,y)\, (\sinh t/2)^{-\sigma-1}\, dt.
$$

In last computation we have used Fubini theorem. This is justified for finite combinations of ultraspherical polynomials by using the estimate

$$
P_t^\lambda(x,y) \le \frac{C \sinh t}{(1-x^2)^{\lambda/2}(1-y^2)^{\lambda/2}(\cosh t - xy - \sqrt{1-x^2}\sqrt{1-y^2})},
$$

which follows from the elementary inequality

$$\int_{-1}^{1} \frac{(1-s^2)^{\lambda-1}}{(A-Bs)^{\lambda+1}}\,ds \le \frac{C}{B^{\lambda}(A-B)}, \qquad A > B > 0, \quad \lambda > 0,$$

and the mean value theorem.

Indeed, taking $C_f = \max\{|f'(x)| : x \in [-1,1]\}$ and using the inequality $1 - xy - \sqrt{1-x^2}\sqrt{1-y^2} \ge C|x-y|^2$, we have

$$\begin{aligned}
&\int_0^{\infty} \int_{-1}^{1} P_t^{\lambda}(x,y)|f(x)-f(y)|\,d\mu_{\lambda}(y)\,(\sinh t/2)^{-\sigma-1}\,dt \\
&\le \frac{C_f}{(1-x^2)^{\lambda/2}} \left(C_1 \int_0^1 \int_{-1}^{1} \frac{t^{-\sigma}|x-y|}{t^2+|x-y|^2}(1-y^2)^{\lambda/2-1/2}\,dy\,dt \right. \\
&\quad \left. + C_2 \int_1^{\infty} \int_{-1}^{1} e^{-(\sigma+1)t/2}|x-y|(1-y^2)^{\lambda/2-1/2}\,dy\,dt \right) =: \frac{C_f}{(1-x^2)^{\lambda/2}}(I_1+I_2).
\end{aligned}$$

Obviously, I_2 is a finite integral. For I_1 the change of variable $t = |x-y|s$ gives

$$I_1 \le C_1 \int_0^{\infty} \frac{s^{-\sigma}}{s^2+1}\,ds \int_{-1}^{1} |x-y|^{-\sigma}(1-y^2)^{\lambda/2-1/2}\,dy < \infty.$$

To obtain the expression of K_{σ}^{λ} we observe that

$$\begin{aligned}
&K_{\sigma}^{\lambda}(x,y) \\
&= \frac{c_{\lambda}^2}{2^{\lambda+1+\sigma}|\Gamma(-\sigma)|} \int_0^{\infty} \int_{-1}^{1} \frac{\sinh t}{(\cosh t - w(s))^{\lambda+1}}\,d\mu_{\lambda-1/2}(s)\,(\sinh t/2)^{-\sigma-1}\,dt \\
&= \frac{c_{\lambda}^2}{2^{\lambda+(1+\sigma)/2}} \frac{\Gamma(\frac{1-\sigma}{2})\Gamma(\lambda+\frac{1+\sigma}{2})}{|\Gamma(-\sigma)|\Gamma(\lambda+1)} \int_{-1}^{1} \frac{d\mu_{\lambda-1/2}(s)}{(1-w(s))^{\lambda+(1+\sigma)/2}},
\end{aligned}$$

where we have applied Fubini theorem and the change of variable $2(\sinh t/2)^2 = z(1-w(s))$ in last equality. With the last identity we have concluded the proof of (8).

To prove (9) we follow the argument in [16, Lemma 5.1]. First, we observe that the kernel $K_{\sigma}^{\lambda}(x,y)$ is positive and symmetric in the sense that $K_{\sigma}^{\lambda}(x,y) = K_{\sigma}^{\lambda}(y,x)$. Then, (9) is clear when f is a finite linear combination of ultraspherical polynomials. For $f \in H_{\lambda}^{\sigma}$ we consider a sequence of finite linear combinations of ultraspherical polynomials $\{p_k\}_{k\ge 0}$ such that p_k converges to f in H_{λ}^{σ}. Then, by using the definition of A_{σ}^{λ}, it is clear that $\langle A_{\sigma}^{\lambda} p_k, p_k\rangle_{\lambda}$ converges to $\langle A_{\sigma}^{\lambda} f, f\rangle_{\lambda}$. Moreover, the result for polynomial functions implies

$$\begin{aligned}
\langle A_{\sigma}^{\lambda} p_k, p_k\rangle_{\lambda} &= \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} (p_k(x)-p_k(y))^2 K_{\sigma}^{\lambda}(x,y)\,d\mu_{\lambda}(y)\,d\mu_{\lambda}(x) \\
&\quad + E_{\sigma,\lambda}\langle p_k, p_k\rangle_{\lambda} < \infty. \qquad (12)
\end{aligned}$$

Consequently, the functions $P_k(x, y) = p_k(x) - p_k(y)$ form a Cauchy sequence in $L^2((-1, 1) \times (-1, 1), d\omega)$ where $d\omega(x, y) = K_\sigma^\lambda(x, y)\, d\mu_\lambda(x)\, d\mu_\lambda(y)$ which converges to $f(x) - f(y)$ in this norm. Hence, passing to the limit in (12), we complete the proof of the lemma. □

Lemma 2 *Let* $\lambda > 0$ *and* $2\lambda + 1 > \sigma > 0$. *Then*

$$A_\sigma^\lambda\left(\frac{1}{(1-x^2)^{\lambda/2+(1-\sigma)/4}}\right) = \frac{Q_{\sigma,\lambda}}{(1-x^2)^{\lambda/2+(1+\sigma)/4}}, \tag{13}$$

where $Q_{\sigma,\lambda}$ *is the constant given in* (5).

Proof First of all, we have to realize that the ultraspherical polynomial $C_n^\lambda(x)$ is odd for $n = 2m + 1$, $m \in \mathbb{Z}^+$; therefore, for $\beta > 0$, the function $(1 - x^2)^{\beta-1} C_{2m+1}^\lambda(x)$ is an odd function and its integral over the interval $(-1, 1)$ is zero. For $n = 2m$ we use [15, Eq. 15, p. 519] to obtain

$$\begin{aligned}
&\int_{-1}^{1} (1-x^2)^{\beta-1} C_{2m}^\lambda(x)\, dx \\
&= \sqrt{\pi}\,\frac{(2\lambda)_{2m}}{(2m)!}\,\frac{\Gamma(\beta)}{\Gamma(\beta+1/2)}\,{}_3F_2(-2m, 2\lambda+2m, \beta; 2\beta, \lambda+1/2; 1) \\
&= \pi\,\frac{(2\lambda)_{2m}}{(2m)!}\,\frac{\Gamma(\beta)\Gamma(\lambda+1/2)\Gamma(\beta-\lambda+1/2)}{\Gamma(1/2-m)\Gamma(\lambda+m+1/2)\Gamma(\beta+m+1/2)\Gamma(\beta-\lambda-m+1/2)},
\end{aligned}$$

where in last identity we have evaluated the hypergeometric function with the so-called Watson formula [13, Eq. 16.4.6, p. 406]. Therefore, if we denote $\alpha = \lambda/2 + (1-\sigma)/4$, we obtain that

$$\int_{-1}^{1} (1-x^2)^{\alpha-1} C_{2m}^\lambda(x)\, dx = R_{\sigma,\lambda} \int_{-1}^{1} (1-x^2)^{\alpha+\sigma/2-1} C_{2m}^\lambda(x)\, dx, \tag{14}$$

with

$$\begin{aligned}
R_{\sigma,\lambda} = {} & \frac{\Gamma(\alpha)\Gamma(\alpha-\lambda+1/2)}{\Gamma(\alpha+\sigma/2)\Gamma(\alpha-\lambda+1/2+\sigma/2)} \\
& \times \frac{\Gamma(\alpha+m+1/2+\sigma/2)\Gamma(\alpha-\lambda-m+1/2+\sigma/2)}{\Gamma(\alpha+m+1/2)\Gamma(\alpha-\lambda-m+1/2)}.
\end{aligned}$$

In this way, if we prove the identity

$$R_{\sigma,\lambda} = Q_{\sigma,\lambda}^{-1} \frac{\Gamma(2m+2\alpha+\sigma)}{\Gamma(2m+2\alpha)} \tag{15}$$

we will conclude the proof, because (14) implies

$$a_n^\lambda\left(\frac{1}{(1-x^2)^{\alpha+\sigma/2}}\right) = Q_{\sigma,\lambda}^{-1} \frac{\Gamma(n+2\alpha+\sigma)}{\Gamma(n+2\alpha)} a_n^\lambda\left(\frac{1}{(1-x^2)^{\alpha}}\right),$$

where we have had in mind that the n-th Fourier coefficient is null when $n = 2m + 1$.

Let us check that (15) actually holds. Using the reflection formula [1, Eq. 6.1.17, p. 256] twice we have

$$\begin{aligned}\frac{\Gamma(\alpha-\lambda-m+1/2+\sigma/2)}{\Gamma(\alpha-\lambda-m+1/2)} &= \frac{\Gamma(\alpha+m+\sigma/2)}{\Gamma(\alpha+m)}\frac{\sin(\pi(\alpha-\lambda-m+1/2))}{\sin(\pi(\alpha-\lambda-m+1/2+\sigma/2))} \\ &= \frac{\Gamma(\alpha+m+\sigma/2)}{\Gamma(\alpha+m)}\frac{\Gamma(\alpha)\Gamma(\alpha-\lambda+1/2+\sigma/2)}{\Gamma(\alpha+\sigma/2)\Gamma(\alpha-\lambda+1/2)},\end{aligned}$$

and then

$$\begin{aligned}R_{\sigma,\lambda} &= \frac{\Gamma(\alpha)^2}{\Gamma(\alpha+\sigma/2)^2}\frac{\Gamma(\alpha+m+\sigma/2)\Gamma(\alpha+m+\sigma/2+1/2)}{\Gamma(\alpha+m)\Gamma(\alpha+m+1/2)} \\ &= Q_{\sigma,\lambda}^{-1}\frac{\Gamma(2m+2\alpha+\sigma)}{\Gamma(2m+2\alpha)},\end{aligned}$$

by the duplication formula [1, Eq. 6.1.18, p. 256]. □

4 Proof of Theorem 1

Polarizing the identity (9) in Lemma 1 we obtain

$$\langle g, A_\sigma^\lambda f\rangle_\lambda = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} F(x,y)K_\sigma^\lambda(x,y)\,d\mu_\lambda(y)\,d\mu_\lambda(x) + E_{\sigma,\lambda}\langle g, f\rangle_\lambda, \tag{16}$$

with $F(x,y) = (g(x)-g(y))(f(x)-f(y))$.

Let us take $g(x) = (1-x^2)^{-\lambda/2-(1-\sigma)/4}$ and $f(x) = u^2(x)/g(x)$ for $u \in H_\lambda^\sigma$. Then

$$F(x,y) = (u(x)-u(y))^2 - g(x)g(y)\left(\frac{u(x)}{g(x)} - \frac{u(y)}{g(y)}\right)^2$$

and (16) becomes

$$\begin{aligned}&\langle g, A_\sigma^\lambda f\rangle_\lambda \\ &= \langle u, A_\sigma^\lambda u\rangle_\lambda - \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} g(x)g(y)\left(\frac{u(x)}{g(x)} - \frac{u(y)}{g(y)}\right)^2 K_\sigma^\lambda(x,y)\,d\mu_\lambda(y)\,d\mu_\lambda(x).\end{aligned}$$

Now, by (13), we have

$$\langle g, A_\sigma^\lambda f\rangle_\lambda = \langle A_\sigma^\lambda g, f\rangle_\lambda = Q_{\sigma,\lambda}\int_{-1}^{1}\frac{u^2(x)}{(1-x^2)^{\sigma/2}}\,d\mu_\lambda(x)$$

and then we can deduce the ground state representation

$$\langle u, A_\sigma^\lambda u\rangle_\lambda - Q_{\sigma,\lambda} \int_{-1}^{1} \frac{u^2(x)}{(1-x^2)^{\sigma/2}}\, d\mu_\lambda(x) = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} g(x)g(y)\left(\frac{u(x)}{g(x)} - \frac{u(y)}{g(y)}\right)^2 \times K_\sigma^\lambda(x,y)\, d\mu_\lambda(y)\, d\mu_\lambda(x). \qquad (17)$$

So, due to the positivity of the kernel K_σ^λ, we conclude the proof.

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