J Fourier Anal Appl (2018) 24:759–779
https://doi.org/10.1007/s00041-016-9519-1

Commutators of Bilinear Pseudodifferential Operators and Lipschitz Functions

Árpád Bényi[1] · **Virginia Naibo**[2]

Received: 8 August 2016 / Published online: 7 December 2016
© Springer Science+Business Media New York 2016

Abstract Commutators of bilinear pseudodifferential operators and the operation of multiplication by a Lipschitz function are studied. The bilinear symbols of the pseudodifferential operators considered belong to classes that are shown to properly contain certain bilinear Hörmander classes of symbols of order one. The corresponding commutators are proved to be bilinear Calderón–Zygmund operators.

Keywords Commutators · Bilinear pseudodifferential operators · Bilinear Calderón–Zygmund theory

Mathematics Subject Classification Primary: 35S05 · 47G30 · 42B20; Secondary: 42B15

Communicated by Rodolfo H. Torres.

The first author is partially supported by a grant from the Simons Foundation (No. 246024). The second author is supported by NSF under Grant DMS 1500381.

✉ Virginia Naibo
vnaibo@math.ksu.edu

Árpád Bényi
arpad.benyi@wwu.edu

1 Department of Mathematics, Western Washington University, 516 High St, Bellingham, WA 98225, USA

2 Department of Mathematics, Kansas State University, 138 Cardwell Hall, 1228 N. 17th Street, Manhattan, KS 66506, USA

Birkhäuser

1 Introduction and Main Results

Let T be a linear pseudodifferential operator defined for functions on $\mathbb{R}^n$, let a be a function on $\mathbb{R}^n$, and consider the commutator $[T, a]$ defined by

$$[T, a](f) := T(af) - aT(f).$$

Given $1 < p < \infty$, estimates of the form

$$\|[T, a](f)\|_{L^p} \lesssim \|a\|_{\mathrm{Lip}^1} \|f\|_{L^p} \qquad \forall f \in L^p(\mathbb{R}^n), \tag{1.1}$$

have been extensively studied. In particular, Calderón proved in [9] that (1.1) holds when T is a pseudodifferential operator whose kernel is homogeneous of degree $-n-1$; in [11,12], Coifman and Meyer showed (1.1) when $T = T_\sigma$ and σ is a symbol in the Hörmander class $S^1_{1,0}$; this latter result was later extended by Auscher and Taylor in [1] to operators $T = T_\sigma$ with $\sigma \in \mathcal{B}S^1_{1,1}$, where the class $\mathcal{B}S^1_{1,1}$, which contains $S^1_{1,0}$ modulo symbols associated to smoothing operators, consists of symbols whose Fourier transforms in the first n-dimensional variable are appropriately compactly supported.

The result from [11,12] mentioned above was obtained by proving that, for each Lipschitz continuous function a on $\mathbb{R}^n$, the operator $f \mapsto [T, a](f)$ is a Calderón–Zygmund singular integral whose kernel constants are controlled by $\|a\|_{\mathrm{Lip}^1}$. The size and regularity estimates for the kernel are easily obtained after integration by parts while the L^2-boundedness follows from a local $L^4 - L^2$ estimate (see [11, pp. 113–114 and Proposition 6 on p. 105]).

On the other hand, Auscher and Taylor proved (1.1) in two different ways: one method is based on the use of paraproducts (see the proof of [1, Theorem 4.1]) while the other is based on the Calderón–Zygmund singular integral approach that relies on the $T(1)$ theorem (see the proof of [1, Theorem 4.4]). Theorem 4.4 in [1] actually extends [1, Theorem 4.1] to the larger class $\tilde{S}^1_{1,1}$ consisting of all symbols $\sigma \in S^1_{1,1}$ such that the symbol of T^*_σ, the transpose of T_σ, belongs to $S^1_{1,1}$ as well. For a host of related commutator estimates, we refer the reader to the articles [24,25] by Taylor and references therein.

Given a bilinear operator T defined for functions on $\mathbb{R}^n$ and a function a on $\mathbb{R}^n$, the following commutator operators are considered:

$$[T, a]_1(f, g) := T(af, g) - aT(f, g),$$
$$[T, a]_2(f, g) := T(f, ag) - aT(f, g).$$

Recently, Bényi and Oh [3] extended the results from [11,12] previously mentioned to this bilinear setting. More precisely, given a bilinear pseudodifferential operator T_σ with σ in the bilinear Hörmander class $BS^1_{1,0}$ and a Lipschitz function a on $\mathbb{R}^n$, it was proved in [3, Theorem 1] that $[T, a]_1$ and $[T, a]_2$ are bilinear Calderón–Zygmund operators; as a consequence these commutators enjoy boundedness properties of the form

$$\left\|[T,a]_j(f,g)\right\|_Z \lesssim \|a\|_{\mathrm{Lip}^1}\,\|f\|_X\,\|g\|_Y\,,\quad j=1,2,$$

for a variety of functional spaces X, Y, Z. For instance, one can take $X = L^{p_1}(\mathbb{R}^n)$, $Y = L^{p_2}(\mathbb{R}^n)$ and $Z = L^p(\mathbb{R}^n)$ where $1 < p_1, p_2 < \infty$, $\frac{1}{2} < p < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$; $X = Y = L^\infty(\mathbb{R}^n)$ and $Z = BMO$; or $X = Y = L^1(\mathbb{R}^n)$ and $Z = L^{\frac{1}{2},\infty}(\mathbb{R}^n)$, among other possibilities. It is only natural to wonder whether the bilinear setting admits counterparts to the results from [1] mentioned above.

In this article, we first introduce the bilinear versions of the linear classes $\mathcal{B}S^1_{1,1}$ and $\widetilde{S}^1_{1,1}$ and show that, modulo smoothing operators, they strictly contain the bilinear Hörmander classes $BS^1_{1,\delta}$ for every $0 \le \delta < 1$ (see Lemmas 2.1 and 2.2 in Sect. 2). The main theorem (Theorem 1.1) represents a bilinear counterpart of results in the spirit of [1, Theorems 4.1 and 4.4], which, in view of Lemmas 2.1 and 2.2, improves [3, Theorem 1] by enlarging the classes of symbols used from $BS^1_{1,\delta}$, $0 \le \delta < 1$, to $\mathcal{B}BS^1_{1,1}$ (notice that the proof of [3, Theorem 1], which is stated for $BS^1_{1,0}$, works without changes for the classes $BS^1_{1,\delta}$ when $0 \le \delta < 1$).

Theorem 1.1 *If $\sigma \in \mathcal{B}BS^1_{1,1}$ and a is a Lipschitz function in $\mathbb{R}^n$, then the commutators $[T_\sigma, a]_j$, $j = 1, 2$, are bilinear Calderón–Zygmund operators. In particular, $[T_\sigma, a]_j$, $j = 1, 2$, are bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $1 < p_1, p_2 < \infty$ and verify appropriate end-point boundedness properties. Moreover, the corresponding norms of the operators are controlled by $\|a\|_{Lip^1}$.*

In Sect. 2 we present some definitions, the statements of Lemmas 2.1 and 2.2, and some remarks that emphasize the importance of the estimates obtained in Lemma 2.2 for the proof of Theorem 1.1. These lemmas are then proved in Sect. 3. The proof of Theorem 1.1 is presented in Sect. 4.

2 Definitions and Preliminaries

Throughout the paper, the notation $\lesssim$ means $\le C$, where C is a constant that may only depend on some of the parameters used and not on the functions or symbols involved. For f in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, its Fourier transform in $\mathbb{R}^n$ is defined as

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx \quad \forall \xi \in \mathbb{R}^n.$$

Given a Lipschitz continuous function a defined on $\mathbb{R}^n$, set

$$\|a\|_{\mathrm{Lip}^1} := \sup\left\{\frac{|a(x)-a(y)|}{|x-y|} : x, y \in \mathbb{R}^n\right\}.$$

Let $K(x, y, z)$ be defined in $\mathbb{R}^{3n} \setminus \Delta$, where $\Delta := \{(x, x, x) : x \in \mathbb{R}^n\}$; K is said to be a bilinear Calderón–Zygmund kernel if there is a constant C_K such that for all $(x, y, z) \in \mathbb{R}^{3n} \setminus \Delta$ and $\alpha \in \mathbb{N}_0$ with $|\alpha| \le 1$,

$$|\partial^\alpha K(x,y,z)| \le \frac{C_K}{(|x-y|+|x-z|+|y-z|)^{2n+|\alpha|}}.$$

A bilinear operator T defined on $\mathcal{S}(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$ is called a bilinear Calderón–Zygmund operator if it is associated to a distributional kernel that coincides with a Calderón–Zygmund kernel on $\mathbb{R}^{3n}\setminus\Delta$ and can be extended to a bounded operator from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. We refer the reader to [13] regarding a systematic treatment of the theory of multilinear Calderón–Zygmund operators. We recall that if T is a bilinear Calderón–Zygmund operator, T can be extended to a bounded operator form $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for all $1<p_1,p_2<\infty$ and $\frac{1}{2}<p<\infty$ such that $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, from $L^1(\mathbb{R}^n)\times L^{\frac{p}{1-p}}(\mathbb{R}^n)$ into $L^{p,\infty}(\mathbb{R}^n)$ and from $L^{\frac{p}{1-p}}(\mathbb{R}^n)\times L^1(\mathbb{R}^n)$ into $L^{p,\infty}(\mathbb{R}^n)$ for $\frac{1}{2}\le p<1$, and from $L^\infty(\mathbb{R})\times L^\infty(\mathbb{R}^n)$ into BMO. The corresponding norms of the operator are controlled by $C_K+\|T\|_{L^2\times L^2\to L^1}$.

Consider $\delta\ge 0$, $\rho>0$ and $m\in\mathbb{R}$. An infinitely differentiable function $\sigma:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$ belongs to the bilinear Hörmander class $BS^m_{\rho,\delta}$ if for all multi-indices $\alpha,\beta,\gamma\in\mathbb{N}_0^n$ there exists a positive constant $C_{\alpha,\beta,\gamma}$ such that

$$|\partial_x^\alpha\partial_\xi^\beta\partial_\eta^\gamma\sigma(x,\xi,\eta)| \le C_{\alpha,\beta,\gamma}(1+|\xi|+|\eta|)^{m+\delta|\alpha|-\rho(|\beta+\gamma|)} \quad \forall x,\xi,\eta\in\mathbb{R}^n. \tag{2.2}$$

Given $\sigma\in BS^m_{\rho,\delta}$, the bilinear pseudodifferential operator associated to σ is defined by

$$T_\sigma(f,g)(x) := \int_{\mathbb{R}^{2n}}\sigma(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta \quad \forall x\in\mathbb{R}^n,\ f,g\in\mathcal{S}(\mathbb{R}^n).$$

Boundedness properties in the setting of Lebesgue spaces, Hardy spaces and BMO of bilinear operators with symbols in the bilinear Hörmander classes have been extensively studied; we refer the reader to [5,7,15,18,19,21,23] and the references therein. Boundedness properties in the context of Triebel-Lizorkin and Besov spaces have been proved in [2,4,20].

In this article, we will focus on classes of symbols related to $BS^m_{1,1}$, i.e. $\delta=\rho=1$. For $\sigma\in BS^m_{1,1}$ and $N,M\in\mathbb{N}_0$ define

$$\|\sigma\|_{N,M} := \sup_{\substack{|\alpha|\le N\\ |\beta|,|\gamma|\le M}}\ \sup_{x,\xi,\eta\in\mathbb{R}^n}|\partial_x^\alpha\partial_\xi^\beta\partial_\eta^\gamma\sigma(x,\xi,\eta)|(1+|\xi|+|\eta|)^{-m-|\alpha|+|\beta+\gamma|}.$$

The family of norms $\{\|\sigma\|_{N,M}\}_{N,M\in\mathbb{N}_0}$ defines a topology on $BS^m_{1,1}$ that makes $BS^m_{1,1}$ into a Fréchet space. For each $\alpha\in\mathbb{N}_0^n$ and $M\in\mathbb{N}$ we also set

$$\|\sigma\|_{\alpha,M} := \sup_{|\beta|,|\gamma|\le M}\ \sup_{x,\xi,\eta\in\mathbb{R}^n}|\partial_x^\alpha\partial_\xi^\beta\partial_\eta^\gamma\sigma(x,\xi,\eta)|(1+|\xi|+|\eta|)^{-m-|\alpha|+|\beta+\gamma|}.$$

Given $m\in\mathbb{R}$ and $r>0$, an infinitely differentiable function $\sigma:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$ belongs to $\mathcal{B}_r BS^m_{1,1}$ if

$$\sigma \in BS^m_{1,1} \text{ and } \operatorname{supp}(\widehat{\sigma}^1) \subset \{(\tau, \xi, \eta) \in \mathbb{R}^{3n} : |\tau| \leq r(|\xi| + |\eta|)\},$$

where $\widehat{\sigma}^1$ denotes the Fourier transform of σ with respect to its first variable in $\mathbb{R}^n$, that is, $\widehat{\sigma}^1(\tau, \xi, \eta) = \widehat{\sigma(\cdot, \xi, \eta)}(\tau)$ for all $\tau, \xi, \eta \in \mathbb{R}^n$. The classes $\mathcal{B}_r BS^m_{1,\delta}$ are defined analogously; they are now subsets of $BS^m_{1,\delta}$ with the support of $\widehat{\sigma}^1$ as above. The linear counterparts to these classes were introduced in [17]. The class $\mathcal{B}BS^m_{1,1}$ is defined as

$$\mathcal{B}BS^m_{1,1} := \bigcup_{r \in (0, \frac{1}{7})} \mathcal{B}_r BS^m_{1,1}.$$

If T is a bilinear operator, the transposes T^{*1} and T^{*2} are defined by

$$\langle T(f, g), h\rangle = \langle T^{*1}(h, g), f\rangle = \langle T^{*2}(f, h), g\rangle \quad \forall f, g, h \in \mathcal{S}(\mathbb{R}^n),$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing. Given a bilinear pseudodifferential operator T_σ we denote by σ^{*1} and σ^{*2} the symbols of its first and second transposes, respectively, and we introduce the class

$$\widetilde{BS}^m_{1,1} := \{\sigma \in BS^m_{1,1} : \sigma^{*1}, \sigma^{*2} \in BS^m_{1,1}\}.$$

For later use, we recall that the class of symbols $BS^0_{1,1}$ give rise to operators with bilinear Calderón–Zygmund kernels and, by [13, Corollary 1], the operators with symbols in $\widetilde{BS}^0_{1,1}$ are bilinear Calderón–Zygmund operators. The linear counterparts to $\widetilde{BS}^m_{1,1}$ have been studied in [8,16].

We have the following inclusions between the bilinear classes of symbols defined above.

Lemma 2.1 *If* $0 \leq \delta < 1$ *and* $r > 0$ *then*

$$BS^m_{1,\delta} \subsetneq \mathcal{B}_r BS^m_{1,\delta} + BS^{-\infty}_{1,\delta} \subset BS^m_{1,1}, \tag{2.3}$$

where $BS^{-\infty}_{1,\delta} := \bigcap_{\nu \in \mathbb{R}} BS^\nu_{1,\delta}$.

Lemma 2.2 *If* $0 < r < \frac{1}{3}$ *and* $m \in \mathbb{R}$ *then*

$$\mathcal{B}_r BS^m_{1,1} \subset \widetilde{BS}^m_{1,1}. \tag{2.4}$$

Moreover, suppose that $\sigma \in \mathcal{B}_r BS^m_{1,1}$ *is in* $\mathcal{S}(\mathbb{R}^{3n})$ *or is supported in* $\{(x, \xi, \eta) \in \mathbb{R}^{3n} : |\xi| + |\eta| > c\}$ *for some* $0 < c < 1$. *Then* $\sigma^{*j} \in \mathcal{B}_{\frac{2r}{1-r}} BS^m_{1,1}$ *for* $j = 1, 2$ *and given* $\alpha, \beta, \gamma \in \mathbb{N}^n_0$, *there exists* $M \in \mathbb{N}_0$ *such that*

$$\left|\partial^\alpha_x \partial^\beta_\xi \partial^\gamma_\eta \sigma^{*j}(x, \xi, \eta)\right| \lesssim \|\sigma\|_{\alpha, M} (1+|\xi|+|\eta|)^{m+|\alpha|-|\beta+\gamma|} \quad \forall x, \xi, \eta \in \mathbb{R}^n, \ j = 1, 2, \tag{2.5}$$

where M *and the implicit constant are independent of* σ.

Remark 2.1 Here we comment on a point that will be crucial in the proof of Theorem 1.1. It concerns the symbols of the transposes of operators with symbols in $\mathcal{B}_r BS^m_{1,1}$ that are a priori dilated in the spatial direction. Let $t > 0$, $\sigma \in \mathcal{B}_r BS^m_{1,1}$ for some $r > 0$, $m \in \mathbb{R}$ and set $\sigma^t(x, \xi, \eta) := \sigma(tx, \xi, \eta)$ for all $x, \xi, \eta \in \mathbb{R}^n$. It easily follows that $\sigma^t \in \mathcal{B}_{tr} BS^m_{1,1}$; moreover, if $\alpha \in \mathbb{N}_0^n$ and $M \in \mathbb{N}$ then $\|\sigma^t\|_{\alpha,M} \le t^{|\alpha|} \|\sigma\|_{\alpha,M}$. In the case $0 < r < \frac{1}{3}$, and for σ supported in $\{(x, \xi, \eta) \in \mathbb{R}^{3n} : |\xi| + |\eta| > 1\}$, the latter and (2.5) imply that given $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ there exists $M \in \mathbb{N}_0$ such that, for $j = 1, 2$,

$$\begin{aligned} &\left|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma (\sigma^t)^{*j}(x, \xi, \eta)\right| \\ &\qquad \lesssim t^{|\alpha|} \|\sigma\|_{\alpha,M} (1 + |\xi| + |\eta|)^{m+|\alpha|-|\beta+\gamma|} \quad \forall x, \xi, \eta \in \mathbb{R}^n, t \in (0, 1]. \end{aligned} \tag{2.6}$$

Remark 2.2 It is natural to consider what happens when transposition is applied to symbols that are dilated in the frequency variables. Setting $\sigma^{(t)}(x, \xi, \eta) := \sigma(x, t\xi, t\eta)$ for $t \in (0, 1]$, the inequality $t\big(1 + t(|\xi| + |\eta|)\big)^{-1} \lesssim (1 + |\xi| + |\eta|)^{-1}$ yields right away that $\sigma^{(t)} \in \mathcal{B}_r BS^m_{1,1}$ (with uniform in t derivatives estimates) as long as $\sigma \in \mathcal{B}_r BS^m_{1,1}$ for some $r > 0$ and $m \ge 0$. A straightforward calculation also shows that

$$(\sigma^t)^{*1}(x, \xi, \eta) = (\sigma^{(t)})^{*1}(tx, t^{-1}\xi, t^{-1}\eta).$$

While these considerations are of interest in their own right, they will play no role in the remainder of the paper.

3 Proofs of Lemma 2.1 and Lemma 2.2

Proof of Lemma 2.1. The second inclusion in (2.3) is straightforward. Given $m \in \mathbb{R}$, $0 \le \delta < 1$ and $r > 0$, we next prove that

$$BS^m_{1,\delta} \subset \mathcal{B}_r BS^m_{1,\delta} + BS^{-\infty}_{1,\delta} \tag{3.7}$$

and then provide an example which shows that such inclusion is proper.

The proof of (3.7) is modeled after arguments from [16, Proposition 3]. Let φ and ψ be infinitely differentiable functions defined on $\mathbb{R}^n$ and such that

$$\operatorname{supp}(\varphi) \subset \{\tau \in \mathbb{R}^n : |\tau| \le r\}, \quad \operatorname{supp}(\psi) \subset \{\tau \in \mathbb{R}^n : \tfrac{r}{4} \le |\tau| \le r\}$$

and

$$\varphi(\tau) + \sum_{k=1}^{\infty} \psi(2^{-k}\tau) = 1 \quad \forall \tau \in \mathbb{R}^n.$$

Fix $\sigma \in BS^m_{1,\delta}$; if $\chi(\xi, \eta)$ is a smooth function supported in $\{(\xi, \eta) \in \mathbb{R}^{2n} : |(\xi, \eta)| \le 2\}$ and equal to 1 in $\{(\xi, \eta) \in \mathbb{R}^{2n} : |(\xi, \eta)| \le 1\}$ then it easily follows that

$\chi\sigma \in BS^{-\infty}_{1,\delta}$. Therefore, without loss of generality we can assume that $\mathrm{supp}(\sigma) \subset \{(x,\xi,\eta) \in \mathbb{R}^{3n} : |(\xi,\eta)| > 1\}$. We have that

$$\sigma(x,\xi,\eta) = \sigma_1(x,\xi,\eta) + \sigma_2(x,\xi,\eta),$$

where

$$\widehat{\sigma_1}^1(\tau,\xi,\eta) = \varphi\left(\tfrac{\tau}{|(\xi,\eta)|}\right)\widehat{\sigma}^1(\tau,\xi,\eta), \quad \widehat{\sigma_2}^1(\tau,\xi,\eta) = \left(1-\varphi\left(\tfrac{\tau}{|(\xi,\eta)|}\right)\right)\widehat{\sigma}^1(\tau,\xi,\eta).$$

We will prove that $\sigma_1 \in \mathcal{B}_r BS^m_{1,\delta}$ and that $\sigma_2 \in BS^{-\infty}_{1,\delta}$. In order to prepare for this, we first make some observations. Let $k \in \mathbb{N}_0$, $t := 2^k\,|(\xi,\eta)|$ for $\xi,\eta \in \mathbb{R}^n$ and H be a smooth function defined on $\mathbb{R}$. Note that

$$\partial_{\xi_j}(H(t)) = 2^k \tfrac{\xi_j}{|(\xi,\eta)|}\,\partial_t H(t) = \tfrac{\xi_j}{|(\xi,\eta)|^2}\,t\partial_t H(t)$$

where ξ_j denotes the jth component of ξ; there is an analogous formula if a derivative with respect to a component of η is taken. Moreover,

$$\partial_\xi^\beta \partial_\eta^\gamma(H(t)) = |(\xi,\eta)|^{-|\beta+\gamma|}\sum_{\ell=1}^{|\beta+\gamma|} H_{\ell,\beta,\gamma}(\xi,\eta)\,(t\partial_t)^\ell H(t), \tag{3.8}$$

where $H_{\ell,\beta,\gamma}$ are continuous homogeneous functions of degree zero defined on $\mathbb{R}^{2n} \setminus \{(0,0)\}$. In the particular case when $H(t) = t^n h(ty)$, for a smooth function h defined on $\mathbb{R}^n$ and fixed $y \in \mathbb{R}^n$, we have that there exists a smooth function h_ℓ defined on $\mathbb{R}^n$, independent of k, (ξ,η) and y, such that

$$(t\partial_t)^\ell(t^n h(ty)) = t^n h_\ell(ty). \tag{3.9}$$

Indeed, note that the Fourier transform with respect to y of $(t\partial_t)^\ell(t^n h(ty))$, evaluated at τ, is $(t\partial_t)^\ell(\widehat{h}(\frac{\tau}{t}))$. For $\ell = 1$, $(t\partial_t)(\widehat{h}(\frac{\tau}{t})) = -\nabla\widehat{h}(\frac{\tau}{t})\cdot\frac{\tau}{t}$. Define h_1 so that $\widehat{h_1}(z) = -\nabla\widehat{h}(z)\cdot z$; then $(t\partial_t)(\widehat{h}(\frac{\tau}{t})) = \widehat{h_1}(\frac{\tau}{t})$ and taking the inverse Fourier transform with respect to τ and evaluating at y it follows that $t\partial_t(t^n h(ty)) = t^n h_1(ty)$. In general, define $\widehat{h_\ell}(z) := -\nabla\widehat{h_{\ell-1}}(z)\cdot z$ for $\ell \geq 2$.

We next prove that $\sigma_1 \in \mathcal{B}_r BS^m_{1,\delta}$. In view of the support of φ, $\mathrm{supp}(\widehat{\sigma_1}^1) \subset \{(\tau,\xi,\eta) \in \mathbb{R}^{3n} : |\tau| \leq r\,|(\xi,\eta)|\} \subset \{(\tau,\xi,\eta) \in \mathbb{R}^{3n} : |\tau| \leq r(|\xi|+|\eta|)\}$. Set $t := |(\xi,\eta)|$ (i.e. $k = 0$ above) and note that

$$\sigma_1(x,\xi,\eta) = \int_{\mathbb{R}^n} t^n \check{\varphi}(ty)\sigma(x-y,\xi,\eta)\,dy.$$

Applying (3.8) with $H(t)=t^n\check{\varphi}(ty)$ and (3.9) with $h=\check{\varphi}$, it follows that

$$\begin{aligned}\partial_x^\alpha\partial_\xi^\beta\partial_\eta^\gamma\sigma_1(x,\xi,\eta) &= \sum_{\substack{\beta_1+\beta_2=\beta\\ \gamma_1+\gamma_2=\gamma}}\int_{\mathbb{R}^n}\partial_\xi^{\beta_1}\partial_\eta^{\gamma_1}(t^n\check{\varphi}(ty))\partial_x^\alpha\partial_\xi^{\beta_2}\partial_\eta^{\gamma_2}\sigma(x-y,\xi,\eta)\,dy\\ &= \sum_{\substack{\beta_1+\beta_2=\beta\\ \gamma_1+\gamma_2=\gamma}}|(\xi,\eta)|^{-|\beta_1+\gamma_1|}\sum_{\ell=1}^{|\beta_1+\gamma_1|}H_{\ell,\beta_1,\gamma_1}(\xi,\eta)\\ &\quad\times\int_{\mathbb{R}^n}t^nh_\ell(ty)\partial_x^\alpha\partial_\xi^{\beta_2}\partial_\eta^{\gamma_2}\sigma(x-y,\xi,\eta)\,dy. \end{aligned}\tag{3.10}$$

Using that $|(\xi,\eta)|\sim 1+|\xi|+|\eta|$ for (ξ,η) in the support of σ, that $H_{\ell,\beta,\gamma}$ are bounded functions (since they are homogeneous of degree zero and continuous), that $\sigma\in BS^m_{1,\delta}$, and that $\left|\int_{\mathbb{R}^n}t^nh_\ell(ty)\,dy\right|\le\|h_\ell\|_{L^1}$, (3.10) gives

$$\left|\partial_x^\alpha\partial_\xi^\beta\partial_\eta^\gamma\sigma_1(x,\xi,\eta)\right|\lesssim(1+|\xi|+|\eta|)^{m+\delta|\alpha|-|\beta+\gamma|},$$

as desired.

We now proceed to show that $\sigma_2\in BS^{-\infty}_{1,\delta}$. We have

$$\widehat{\sigma_2}^1(\tau,\xi,\eta)=\left(1-\varphi\left(\tfrac{\tau}{|(\xi,\eta)|}\right)\right)\widehat{\sigma}^1(\tau,\xi,\eta)=\sum_{k=1}^\infty\psi\left(\tfrac{\tau}{2^k|(\xi,\eta)|}\right)\widehat{\sigma}^1(\tau,\xi,\eta).$$

For $k\in\mathbb{N}$ define $\sigma^{(k)}(x,\xi,\eta)$ so that $\widehat{\sigma^{(k)}}^1(\tau,\xi,\eta)=\psi\left(\tfrac{\tau}{2^k|(\xi,\eta)|}\right)\widehat{\sigma}^1(\tau,\xi,\eta)$. Setting $t=2^k|(\xi,\eta)|$, we have

$$\sigma^{(k)}(x,\xi,\eta)=\int_{\mathbb{R}^n}t^n\check{\psi}(ty)\sigma(x-y,\xi,\eta)\,dy;$$

using (3.8) with $H(t)=t^n\check{\psi}(ty)$ and (3.9) with $h=\check{\psi}$, we obtain

$$\begin{aligned}\partial_x^\alpha\partial_\xi^\beta\partial_\eta^\gamma\sigma_2^{(k)}(x,\xi,\eta) &= \sum_{\substack{\beta_1+\beta_2=\beta\\ \gamma_1+\gamma_2=\gamma}}|(\xi,\eta)|^{-|\beta_1+\gamma_1|}\sum_{\ell=1}^{|\beta_1+\gamma_1|}H_{\ell,\beta_1,\gamma_1}(\xi,\eta)\\ &\quad\times\int_{\mathbb{R}^n}t^nh_\ell(ty)\partial_x^\alpha\partial_\xi^{\beta_2}\partial_\eta^{\gamma_2}\sigma(x-y,\xi,\eta)\,dy.\end{aligned}$$

We will show that given $\nu\in\mathbb{R}$ there exists $L\in\mathbb{N}$ such that

$$\left|\int_{\mathbb{R}^n}t^nh_\ell(ty)\partial_x^\alpha\partial_\xi^{\beta_2}\partial_\eta^{\gamma_2}\sigma(x-y,\xi,\eta)\,dy\right|\lesssim 2^{-kL}(1+|\xi|+|\eta|)^{\nu+\delta|\alpha|-|\beta_2+\gamma_2|}\tag{3.11}$$

for all $k \in \mathbb{N}$ and $x, \xi, \eta \in \mathbb{R}^n$ and where the implicit constants may depend on $\nu, \alpha, \beta, \gamma, \delta$ and σ. Since $|(\xi, \eta)| \sim 1 + |\xi| + |\eta|$ in the support of σ and $H_{\ell,\beta_1,\gamma_1}$ are bounded functions, (3.11) gives that $\left|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma_2^{(k)}(x, \xi, \eta)\right| \lesssim 2^{-kL}(1 + |\xi| + |\eta|)^{\nu+\delta|\alpha|-|\beta+\gamma|}$ for all $k \in \mathbb{N}$, $x, \xi, \eta \in \mathbb{R}^n$. Adding over all $k \in \mathbb{N}$ it follows that $\sigma_2 \in BS^{\nu}_{1,\delta}$ and, because ν is arbitrary, one then concludes that $\sigma_2 \in BS^{-\infty}_{1,\delta}$.

In order to prove (3.11) and for ease of notation, set $\theta := \partial_x^\alpha \partial_\xi^{\beta_2} \partial_\eta^{\gamma_2} \sigma$. Let $L \in \mathbb{N}$ be such that $m - (1-\delta)L \le \nu$ (recall that $0 \le \delta < 1$) and write

$$\theta(x-y, \xi, \eta) = \sum_{|\mu| \le L-1} \frac{(-1)^{|\mu|}}{\mu!} \partial_x^\mu \theta(x, \xi, \eta) y^\mu + \sum_{|\mu|=L} \frac{(-1)^{|\mu|}}{\mu!} \partial_z^\mu \theta(z, \xi, \eta)|_{z=z_{x,y}} y^\mu,$$

for some appropriate $z_{x,y} \in \mathbb{R}^n$. From the definition of h_ℓ we have that $\text{supp}(\widehat{h_\ell}) \subset \text{supp}(\widehat{h}) = \text{supp}(\psi)$. Taking into account that ψ is supported in an annulus, we have that

$$\sum_{|\mu| \le L-1} \frac{(-1)^{|\mu|}}{\mu!} \partial_x^\mu \theta(x, \xi, \eta) \int_{\mathbb{R}^n} t^n h_\ell(ty) y^\mu \, dy = 0. \tag{3.12}$$

In addition, using the definition of θ, that $\sigma \in BS^m_{1,\delta}$, that $t = 2^k |(\xi, \eta)|$ and that $|(\xi, \eta)| > 1$ in the support of σ, it follows that

$$\begin{aligned} &\left| \sum_{|\mu|=L} \frac{(-1)^{|\mu|}}{\mu!} \int_{\mathbb{R}^n} t^n h_\ell(ty) \partial_z^\mu \theta(z, \xi, \eta)|_{z=z_{x,y}} y^\mu \, dy \right| \\ &\qquad \lesssim \sum_{|\mu|=L} \frac{1}{\mu!} (1 + |\xi| + |\eta|)^{m+\delta|\mu+\alpha|-|\beta_2+\gamma_2|} t^{-|\mu|} \\ &\qquad\quad \times \left\| h_\ell(\cdot) \, |\cdot|^{|\mu|} \right\|_{L^1} \\ &\qquad \sim (1 + |\xi| + |\eta|)^{m-(1-\delta)L+\delta|\alpha|-|\beta_2+\gamma_2|} 2^{-kL} \\ &\qquad \lesssim 2^{-kL} (1 + |\xi| + |\eta|)^{\nu+\delta|\alpha|-|\beta_2+\gamma_2|}, \end{aligned} \tag{3.13}$$

where in the last inequality it was used that $m - (1-\delta)L \le \nu$. The estimates (3.12) and (3.13) imply (3.11).

Given $r > 0$ and $m \in \mathbb{R}$, we next present an example of symbol σ such that

$$\sigma \in \mathcal{B}_r BS^m_{1,1} \setminus \bigcup_{\delta \in [0,1)} BS^m_{1,\delta}.$$

Let $\Psi \in \mathcal{S}(\mathbb{R}^{2n})$ be such that $\text{supp}(\Psi) \subset \{(\xi, \eta) \in \mathbb{R}^{2n} : 1 < |\xi| + |\eta| < 2\}$ and $\{g_k\}_{k\in\mathbb{N}_0}$ be a family of infinitely differentiable functions defined in $\mathbb{R}^n$ that satisfy

$$\left|\partial^\alpha g_k(x)\right| \lesssim 2^{k(m+|\alpha|)} \quad \forall x \in \mathbb{R}^n, k \in \mathbb{N}_0.$$

Consider $\Phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\text{supp}(\widehat{\Phi}) \subset \{\tau \in \mathbb{R}^n : |\tau| < r\}$ and set $\Phi_{2^{-k}}(x) := 2^{nk}\Phi(2^k x)$ for $x \in \mathbb{R}^n$. It easily follows that the symbol σ defined by

$$\sigma(x, \xi, \eta) := \sum_{k\in\mathbb{N}_0} (g_k * \Phi_{2^{-k}})(x)\, \Psi(2^{-k}\xi, 2^{-k}\eta), \quad x, \xi, \eta \in \mathbb{R}^n,$$

belongs to $\mathcal{B}_r BS^m_{1,1}$. Indeed, notice first that if $2^k \leq |\xi|+|\eta| < 2^{k+1}$ for some $k \in \mathbb{N}_0$,

$$\text{supp}(\widehat{\sigma}^1(\cdot, \xi, \eta)) \subset \text{supp}(\widehat{\Phi}(2^{-k}\cdot)) \subset \{\tau \in \mathbb{R}^n : |\tau| < 2^k r\}.$$

Since $|\xi| + |\eta| \geq 2^k$, we get the support condition for $\widehat{\sigma}^1(\cdot, \xi, \eta)$. Secondly, we have for $2^k \leq |\xi| + |\eta| < 2^{k+1}$ that $\left|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)\right| \lesssim 2^{k(m+|\alpha|)} 2^{-k(|\beta+\gamma|)} \lesssim (1 + |\xi| + |\eta|)^{m+|\alpha|-|\beta+\gamma|}$. If $|\xi| + |\eta| < 1$ then $\sigma(\cdot, \xi, \eta) = 0$ and therefore the support condition and the estimate follow trivially.

Next we choose $\{g_k\}_{k\in\mathbb{N}_0}$ so that $\sigma \notin BS^m_{1,\delta}$ for any $0 \leq \delta < 1$. For instance, consider $g_k(x) = 2^{km} e^{2\pi i 2^k x_0 \cdot x}$ for a fixed point $x_0 \in \mathbb{R}^n$ with all its component different from zero and $\widehat{\Phi}(x_0) \neq 0$. If $\xi, \eta \in \mathbb{R}^n$ are such that $2^k < |\xi| + |\eta| < 2^{k+1}$ then $\sigma(x, \xi, \eta) = (g_k * \Phi_{2^{-k}})(x)\, \Psi(2^{-k}\xi, 2^{-k}\eta)$ and

$$\left|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)\right| = 2^{k(m+|\alpha|-|\beta|-|\gamma|)} \left|(2\pi x_0)^\alpha\, \widehat{\Phi}(x_0)\right| \left|(\partial_\xi^\beta \partial_\eta^\gamma \Psi)(2^{-k}\xi, 2^{-k}\eta)\right|.$$

If $\sigma \in BS^m_{1,\delta}$ then $\left|(\partial_\xi^\beta \partial_\eta^\gamma \Psi)(2^{-k}\xi, 2^{-k}\eta)\right| \leq C_{\alpha,\beta,\gamma} 2^{-k(1-\delta)|\alpha|}$ for all $\xi, \eta \in \mathbb{R}^n$ and all $k \in \mathbb{N}_0$, which is impossible for $0 \leq \delta < 1$. □

Proof of Lemma 2.2. We first assume that the symbol σ is in the Schwartz class so that the corresponding calculations are properly justified. We then use an approximation argument to obtain the results for symbols that are not rapidly decreasing and are supported away from the origin in the frequency variables. Finally, we show the inclusion (2.4).

Given $\sigma \in \mathcal{B}_r BS^m_{1,1} \cap \mathcal{S}(\mathbb{R}^{3n})$ for some $0 < r < \frac{1}{3}$ and $m \in \mathbb{R}$, we have to prove that σ^{*1} and σ^{*2} belong to $\mathcal{B}_{\frac{2r}{1-r}} BS^m_{1,1}$ and satisfy (2.5). We will work with σ^{*1}; an analogous reasoning is valid for the symbol σ^{*2}.

The ideas below, which exploit the fact that $\widehat{\sigma}^1$ is supported in $\{(\tau, \xi, \eta) \in \mathbb{R}^{3n} : |\tau| \leq r(|\xi| + |\eta|)\}$, are inspired by those in the proof of [6, Theorem 2.1] about the fact that $BS^\nu_{\rho,\delta}$ is closed under transposition for $\nu \in \mathbb{R}$, $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$.

It holds that

$$T_{\sigma^{*1}}(f, g)(x) = \int_{\mathbb{R}^{3n}} \bar{\sigma}(y, \xi, \eta) f(y) \widehat{g}(\eta) e^{-2\pi i (y-x)\cdot\xi} e^{2\pi i x\cdot\eta} d\xi d\eta dy,$$

where $\bar{\sigma}(y,\xi,\eta) = \sigma(y,-\xi-\eta,\eta)$. The fact that $\sigma \in BS^m_{1,1}$ easily implies that $\bar{\sigma} \in BS^m_{1,1}$ and that if $\alpha \in \mathbb{N}^n_0$ and $M \in \mathbb{N}_0$ then $\|\bar{\sigma}\|_{\alpha,M} \lesssim \|\sigma\|_{\alpha,2M}$. Indeed, this follows from the formula

$$\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \bar{\sigma}(x,\xi,\eta) = \sum_{\gamma_1+\gamma_2=\gamma} (-1)^{|\beta+\gamma_1|} (\partial_x^\alpha \partial_\xi^{\beta+\gamma_1} \partial_\eta^{\gamma_2}\sigma)(x,-\xi-\eta,\eta).$$

Moreover,

$$\sigma^{*1}(x,\xi,\eta) = \int_{\mathbb{R}^{2n}} \bar{\sigma}(x+y,z+\xi,\eta)e^{-2\pi i z\cdot y}\,dydz.$$

We will first proceed to prove (2.5) for the case $\alpha=\beta=\gamma=0$. We have

$$\begin{aligned}\sigma^{*1}(x,\xi,\eta) &= \int_{\mathbb{R}^{2n}} \bar{\sigma}(x+y,z+\xi,\eta)e^{-2\pi i z\cdot y}\,dydz\\ &= \int_{\mathbb{R}^{2n}} \sigma(x+y,-z-\xi-\eta,\eta)e^{-2\pi i z\cdot y}\,dydz\\ &= \int_{\mathbb{R}^{n}} \widehat{\sigma}^1(z,-z-\xi-\eta,\eta)e^{2\pi i z\cdot x}\,dz\\ &= \int_{|z|\le r(|z+\xi+\eta|+|\eta|)} \widehat{\sigma}^1(z,-z-\xi-\eta,\eta)e^{2\pi i z\cdot x}\,dz,\end{aligned} \tag{3.14}$$

where we have used that $\sigma \in \mathcal{B}_r BS^m_{1,1}$. The condition $|z| \le r(|z+\xi+\eta|+|\eta|)$ implies that $|z| \le \frac{2r}{1-r}A$, where $A := 1+|\xi|+|\eta|$. Therefore, if $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is such that $\psi \equiv 1$ in $\{z : |z| \le \frac{2r}{1-r}\}$, we have

$$\begin{aligned}\sigma^{*1}(x,\xi,\eta) &= \int_{\mathbb{R}^{n}} \psi(A^{-1}z)\widehat{\sigma}^1(z,-z-\xi-\eta,\eta)e^{2\pi i z\cdot x}\,dz\\ &= \int_{\mathbb{R}^{2n}} \psi(A^{-1}z)\bar{\sigma}(x+y,z+\xi,\eta)e^{-2\pi i z\cdot y}\,dydz.\end{aligned}$$

For $L \in \mathbb{N}$ with $2L > n$, write

$$e^{-2\pi i z\cdot y} = (1+(2\pi A)^2\,|y|^2)^{-L}(1+A^2(-\Delta_z))^L e^{-2\pi i z\cdot y}.$$

Integration by parts gives

$$\sigma^{*1}(x,\xi,\eta) = \int_{\mathbb{R}^{2n}} q(x,y,z,\xi,\eta)e^{-2\pi i z\cdot y}\,dydz, \tag{3.15}$$

where

$$q(x,y,z,\xi,\eta) = \frac{(1+A^2(-\Delta_z))^L\left(\psi(A^{-1}z)\bar{\sigma}(x+y,z+\xi,\eta)\right)}{(1+(2\pi A)^2\,|y|^2)^L}.$$

We next estimate q. If $P_L = \{\gamma = (\gamma_1, \cdots, \gamma_n) : \gamma_i$ even and $|\gamma| = 2j,\ j = 0, \cdots, L\}$, then

$$\begin{aligned}(1 + A^2(-\Delta_z))^L &\left(\psi(A^{-1}z)\bar{\sigma}(x + y, z + \xi, \eta)\right)\\ &= \sum_{\gamma \in P_L} C_\gamma A^{|\gamma|}\partial_z^\gamma \left(\psi(A^{-1}z)\bar{\sigma}(x + y, z + \xi, \eta)\right)\\ &= \sum_{\gamma \in P_L} A^{|\gamma|} \sum_{\gamma_1+\gamma_2=\gamma} C_{\gamma_1,\gamma_2}(\partial_z^{\gamma_1}\psi)(A^{-1}z)A^{-|\gamma_1|}\\ &\quad \times\ \partial_z^{\gamma_2}(\bar{\sigma}(x + y, z + \xi, \eta)).\end{aligned}$$

Using the fact that $\bar{\sigma} \in BS^m_{1,1}$, we get

$$\begin{aligned}&\left|(1 + A^2(-\Delta_z))^L \left(\psi(A^{-1}z)\bar{\sigma}(x + y, z + \xi, \eta)\right)\right|\\ &\quad\lesssim \|\bar{\sigma}\|_{0,2L} \sum_{\substack{\gamma \in P_L\\ \gamma_1+\gamma_2=\gamma}} A^{|\gamma_2|}\left|(\partial_z^{\gamma_1}\psi)(A^{-1}z)\right| (1 + |z + \xi| + |\eta|)^{m-|\gamma_2|}.\end{aligned} \tag{3.16}$$

Recall that the support of ψ is compact and that $\psi \equiv 1$ in $\{z : |z| \le \frac{2r}{1-r}\}$. By choosing ψ such that its support is contained in $\{z : |z| \le \frac{2r}{1-r} + \varepsilon\}$ for some $\varepsilon = \varepsilon_r > 0$ for which $\frac{2r}{1-r} + \varepsilon < 1$ (this is possible for $0 < r < \frac{1}{3}$), we have that

$$1 + |z + \xi| + |\eta| \sim A,\ \text{ for } z \text{ such that } A^{-1}z \in \operatorname{supp}(\psi).$$

This observation and (3.16) give that

$$|q(x, y, z, \xi, \eta)| \lesssim \|\bar{\sigma}\|_{0,2L} \frac{A^m}{(1 + A^2 |y|^2)^L}.$$

Since $2L > n$,

$$\begin{aligned}\left|\sigma^{*1}(x, \xi, \eta)\right| &\lesssim \|\bar{\sigma}\|_{0,2L}\, A^m \int_{|z|\lesssim A} \int_y \frac{1}{(1 + A^2 |y|^2)^L}\, dydz \sim \|\bar{\sigma}\|_{0,2L}\, A^m\\ &\lesssim \|\sigma\|_{0,4L}\, A^m,\end{aligned}$$

obtaining the desired result with $M = 4L$ in (2.5).

The estimate (2.5) for any $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ follows from the previous case once a few observations are made. The formulas

$$\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma^{*1}(x, \xi, \eta) = \int_{\mathbb{R}^{2n}} (\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \bar{\sigma})(x + y, z + \xi, \eta)e^{-2\pi i z\cdot y}\, dydz$$

and

$$\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \bar{\sigma}(x,\xi,\eta) = \sum_{\gamma_1+\gamma_2=\gamma} (-1)^{|\beta+\gamma_1|} (\partial_x^\alpha \partial_\xi^{\beta+\gamma_1} \partial_\eta^{\gamma_2} \sigma)(x, -\xi-\eta, \eta)$$

imply that

$$\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma^{*1}(x,\xi,\eta) = \sum_{\gamma_1+\gamma_2=\gamma} (-1)^{|\beta+\gamma_1|} (\partial_x^\alpha \partial_\xi^{\beta+\gamma_1} \partial_\eta^{\gamma_2} \sigma)^{*1}(x,\xi,\eta).$$

Moreover, for γ_1 and γ_2 such that $\gamma_1+\gamma_2=\gamma$, $\partial_x^\alpha \partial_\xi^{\beta+\gamma_1} \partial_\eta^{\gamma_2} \sigma \in \mathcal{B}_r BS_{1,1}^{m+|\alpha|-|\beta+\gamma|}$. Then (2.5) follows after applying the previous case to each of these terms and observing that for each $K \in \mathbb{N}_0$ there exists $M \in \mathbb{N}_0$ independent of σ such that the norm in $BS_{1,1}^{m+|\alpha|-|\beta+\gamma|}$ given by $\left\| \partial_x^\alpha \partial_\xi^{\beta+\gamma_1} \partial_\eta^{\gamma_2} \sigma \right\|_{0,K}$ is controlled by the norm in $BS_{1,1}^m$ given by $\|\sigma\|_{\alpha,M}$.

Finally, note that σ^{*1} belongs to $\mathcal{B}_{\frac{2r}{1-r}} BS_{1,1}^m$. Indeed, we have just proved that it belongs to $BS_{1,1}^m$; moreover, (3.14) shows that

$$\widehat{\sigma^{*1}}^1(\tau,\xi,\eta) = \widehat{\sigma}^1(\tau, -\tau-\xi-\eta, \eta),$$

whose support is contained in $\{(\tau,\xi,\eta) \in \mathbb{R}^{3n} : |\tau| \le \frac{2r}{1-r}(|\xi|+|\eta|)\}$.

We next prove the results for a symbol $\sigma \in \mathcal{B}_r BS_{1,1}^m$ supported in $\{(x,\xi,\eta) \in \mathbb{R}^{3n} : |\xi|+|\eta| > c\}$ for some $0 < c < 1$. Again, we will just work with σ^{*1} and show that it satisfies (2.5) and it belongs to $\mathcal{B}_{\frac{2r}{1-r}} BS_{1,1}^m$. Let $\theta \in \mathcal{S}(\mathbb{R}^n)$ be such that $\theta(0)=1$ and $\mathrm{supp}(\widehat{\theta}) \subset \{\tau \in \mathbb{R}^n : |\tau| < 1\}$ and consider $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$ satisfying $\varphi(0,0)=1$. Set $\Psi(x,\xi,\eta) := \theta(x)\varphi(\xi,\eta)$ and $\sigma_\varepsilon(x,\xi,\eta) := \sigma(x,\xi,\eta)\Psi(\varepsilon c x, \varepsilon\xi, \varepsilon\eta)$ for $\varepsilon > 0$. Note that $\sigma_\varepsilon \in \mathcal{S}(\mathbb{R}^{3n})$ and due to the supports of σ, $\widehat{\sigma}^1$ and $\widehat{\theta}$ it follows that

$$\mathrm{supp}(\widehat{\sigma_\varepsilon}^1) \subset \{(\tau,\xi,\eta) \in \mathbb{R}^{3n} : |\tau| \le (r+\varepsilon)(|\xi|+|\eta|)\}.$$

As a consequence, $\sigma_\varepsilon \in \mathcal{B}_{r+\varepsilon} BS_{1,1}^m$ and, if $r+\varepsilon < \frac{1}{3}$, the previous case applies giving that $\sigma_\varepsilon^{*1} \in \mathcal{B}_{\frac{2(r+\varepsilon)}{1-r-\varepsilon}}$ and that (2.5) holds for σ_ε^{*1}.

We will estimate the norms of σ_ε and σ_ε^{*1} as elements of $BS_{1,1}^m$. For $\alpha,\beta,\gamma \in \mathbb{N}_0^n$, the Leibniz rule gives that $\left|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma_\varepsilon(x,\xi,\eta)\right|$ is pointwise bounded by a linear combination of terms of the form

$$\left|\partial_x^{\alpha_1} \partial_\xi^{\beta_1} \partial_\eta^{\gamma_1} \sigma(x,\xi,\eta) \partial_x^{\alpha_2} \partial_\xi^{\beta_2} \partial_\eta^{\gamma_2} \Psi(\varepsilon c x, \varepsilon\xi, \varepsilon\eta)\right| c^{|\alpha_2|} \varepsilon^{|\alpha_2+\beta_2+\gamma_2|}$$

with $\alpha_1+\alpha_2=\alpha$, $\beta_1+\beta_2=\beta$ and $\gamma_1+\gamma_2=\gamma$. Using that $\sigma \in BS_{1,1}^m$, $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$, θ has bounded derivatives, $\varepsilon(1+|\varepsilon\xi|+|\varepsilon\eta|)^{-1} \lesssim (1+|\xi|+|\eta|)^{-1}$ for $0 < \varepsilon < 1$, and $0 < c < 1$, the latter can be pointwise estimated by

$$\|\sigma\|_{\alpha_1, \max(|\beta|,|\gamma|)} \, \varepsilon^{|\alpha_2|} (1+|\xi|+|\eta|)^{m+|\alpha|-|\beta+\gamma|}.$$

That is, with $M_{\beta,\gamma} := \max(|\beta|, |\gamma|)$ and for all $x, \xi, \eta \in \mathbb{R}^n$, we have

$$\left|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma_\varepsilon(x,\xi,\eta)\right| \lesssim \Big(\|\sigma\|_{\alpha, M_{\beta,\gamma}} + \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_2\neq 0}} \|\sigma\|_{\alpha_1, M_{\beta,\gamma}}\, \varepsilon^{|\alpha_2|}\Big) \times (1+|\xi|+|\eta|)^{m+|\alpha|-|\beta+\gamma|}. \tag{3.17}$$

As a consequence,

$$\left|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma_\varepsilon(x,\xi,\eta)\right| \lesssim \|\sigma\|_{|\alpha|, M_{\beta,\gamma}}\, (1+|\xi|+|\eta|)^{m+|\alpha|-|\beta+\gamma|} \quad \text{for } 0<\varepsilon<\tfrac{1}{3}-r. \tag{3.18}$$

The estimate (2.5) for σ_ε^{*1} and (3.18) imply the existence of $M \in \mathbb{N}$ such that

$$\begin{aligned}\left|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma_\varepsilon^{*1}(x,\xi,\eta)\right| &\lesssim \|\sigma_\varepsilon\|_{\alpha, M}\, (1+|\xi|+|\eta|)^{m+|\alpha|-|\beta+\gamma|} \\ &\lesssim \|\sigma\|_{|\alpha|, M}\, (1+|\xi|+|\eta|)^{m+|\alpha|-|\beta+\gamma|}, \quad \text{for } 0<\varepsilon<\tfrac{1}{3}-r.\end{aligned} \tag{3.19}$$

By (3.19), it follows that $\{\sigma_\varepsilon^{*1}\}_{0<\varepsilon<\frac{1}{3}-r}$ is a bounded family in the topology of $BS^m_{1,1}$; therefore there exist a sequence $\{\sigma_{\varepsilon_k}^{*1}\}_{k\in\mathbb{N}}$, with $\varepsilon_k \to 0$ as $k\to\infty$, and a symbol $\Sigma \in BS^m_{1,1}$ such that $\sigma_{\varepsilon_k}^{*1} \to \Sigma$, as $k\to\infty$, in the $\mathcal{C}^\infty$ topology on compact sets of $\mathbb{R}^n$ (see, for instance, [22, pp. 245–246], where such fact is shown for the linear Hörmander classes).

Using (3.18) and (3.19) for $\alpha=\beta=\gamma=0$, the Dominated Convergence Theorem implies that for all $f, g, h \in \mathcal{S}(\mathbb{R}^n)$

$$\langle T_\sigma(f,g), h\rangle = \lim_{k\to\infty} \langle T_{\sigma_{\varepsilon_k}}(f,g), h\rangle = \lim_{k\to\infty} \langle T_{\sigma_{\varepsilon_k}^{*1}}(h,g), f\rangle = \langle T_\Sigma(h,g), f\rangle.$$

We then conclude that $T_\sigma^{*1} = T_\Sigma$ and therefore $\sigma^{*1} = \Sigma$. The estimate (3.17) and the first inequality in (3.19) imply that

$$\left|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma_{\varepsilon_k}^{*1}(x,\xi,\eta)\right| \lesssim \Big(\|\sigma\|_{\alpha, M} + \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_2\neq 0}} \|\sigma\|_{\alpha_1, M}\, \varepsilon_k^{|\alpha_2|}\Big)(1+|\xi|+|\eta|)^{m+|\alpha|-|\beta+\gamma|}.$$

As $\epsilon_k \to 0$, we obtain (2.5) for σ^{*1}.

Finally, let us prove that $\widehat{\sigma^{*1}}^1$ is supported in $\{(\tau,\xi,\eta)\in\mathbb{R}^{3n} : |\tau| \le \frac{2r}{1-r}(|\xi|+|\eta|)\}$. Fix $\xi, \eta \in \mathbb{R}^n$ and let ϕ be an infinitely differentiable function defined on $\mathbb{R}^n$ and with compact support contained in $\{\tau\in\mathbb{R}^n : |\tau| > \frac{2r}{1-r}(|\xi|+|\eta|)\}$. For ε_k small enough, we have that $\operatorname{supp}(\phi) \subset \{\tau\in\mathbb{R}^n : |\tau| > \frac{2(r+\varepsilon_k)}{1-r-\varepsilon_k}(|\xi|+|\eta|)\}$. The Dominated Convergence Theorem and the fact that $\widehat{\sigma_{\varepsilon_k}^{*1}}^1$ is supported in $\{(\tau,\xi,\eta)\in\mathbb{R}^{3n} : |\tau| \le \frac{2(r+\varepsilon_k)}{1-r-\varepsilon_k}(|\xi|+|\eta|)\}$ imply that

$$\begin{aligned}\langle \widehat{\sigma^{*1}}^1(\cdot,\xi,\eta),\phi\rangle &= \int_{\mathbb{R}^n} \sigma^{*1}(x,\xi,\eta)\widehat{\phi}(x)\,dx\\ &= \lim_{k\to\infty}\int_{\mathbb{R}^n} \sigma^{*1}_{\varepsilon_k}(x,\xi,\eta)\widehat{\phi}(x)\,dx = \lim_{k\to\infty}\langle \widehat{\sigma^{*1}_{\varepsilon_k}}^1(\cdot,\xi,\eta),\phi\rangle = 0.\end{aligned}$$

For the inclusion (2.4), let $\sigma \in \mathcal{B}_r BS^m_{1,1}$ for some $0<r<\frac{1}{3}$ and $m\in\mathbb{R}$. Consider an infinitely differentiable function χ defined on $\mathbb{R}^{2n}$, supported on $\{(\xi,\eta)\in\mathbb{R}^{2n} : |\xi|+|\eta|\leq 2\}$ and identically equal to one on $\{(\xi,\eta)\in\mathbb{R}^{2n} : |\xi|+|\eta|\leq 1\}$; then $\sigma(x,\xi,\eta)=\chi(\xi,\eta)\sigma(x,\xi,\eta)+(1-\chi(\xi,\eta))\sigma(x,\xi,\eta)$ for all $x,\xi,\eta\in\mathbb{R}^n$. The symbol $(1-\chi)\sigma$ belongs to $\mathcal{B}_r BS^m_{1,1}$ and is supported on the set where $|\xi|+|\eta|>1$. The above results then give in particular that $((1-\chi)\sigma)^{*j}\in BS^m_{1,1}$ for $j=1,2$. It easily follows that $\chi\sigma\in BS^m_{1,0}$; since this class is closed under transposition ([6, Theorem 2.1]), we have $(\chi\sigma)^{*j}\in BS^m_{1,0}\subset BS^m_{1,1}$ for $j=1,2$. As a consequence $\sigma^{*j}\in BS^m_{1,1}$ for $j=1,2$ and therefore $\sigma\in \widetilde{BS}^m_{1,1}$. □

4 Proof of Theorem 1.1

Bilinear versions of the $T(1)$ theorem were first studied in [10] and [13]. The proof of Theorem 1.1 makes use of a bilinear formulation of the $T(1)$ theorem given in [14, Theorem 1.1], which states that a bilinear operator T associated to a standard kernel is a Calderón–Zygmund operator if $T(1,1)$, $T^{*1}(1,1)$ and $T^{*2}(1,1)$ (which can be properly defined) are in BMO and T satisfies a certain weak boundedness property.

Proof of Theorem 1.1. The proof is divided into several subsections that follow closely the argument proving [3, Theorem 1]. The relevant difference between the proofs has to do with the verification of the BMO condition and the weak boundedness property for the commutators. The argument requires checking that the symbols σ_j and $\widetilde{\sigma}_j$ defined in Sect. 4.2 and their transposes are in $BS^0_{1,\delta}$, $0\leq\delta<1$, in the case of [3, Theorem 1] and in the class $BS^0_{1,1}$ in the present case. The former is straightforward since the classes $BS^0_{1,\delta}$ are closed under transposition for all $0\leq\delta<1$ ([6, Theorem 2.1]), while the latter needs extra work because the class $BS^0_{1,1}$ is not closed under transposition ([4, Corollary 2]). It is here that Remark 2.1 will play an important role.

4.1 Kernels of Commutators

In this section, we check that the kernels of the bilinear commutators $[T_\sigma,a]_j$, $j=1,2$, are Calderón–Zygmund for any $\sigma\in BS^1_{1,1}$ and any Lipschitz continuous function a defined on $\mathbb{R}^n$. Let K_j be the kernel of $[T_\sigma,a]_j$, $j=1,2$. Then, we have

$$\begin{aligned}K_1(x,y,z) &= \big(a(y)-a(x)\big)K(x,y,z),\\ K_2(x,y,z) &= \big(a(z)-a(x)\big)K(x,y,z),\end{aligned}$$

where K is the kernel of T_σ. Since $\sigma \in BS^1_{1,1}$ we have that

$$|K(x,y,z)| \lesssim \big(|x-y|+|x-z|+|y-z|\big)^{-1-2n},$$
$$|\nabla K(x,y,z)| \lesssim \big(|x-y|+|x-z|+|y-z|\big)^{-2-2n}$$

on $\mathbb{R}^{3n} \setminus \Delta$ ([6, Theorem 5.1 (last item)], see also [3, Lemma 3]). These estimates combined with the fact that a is Lipschitz continuous in $\mathbb{R}^n$ give that

$$|K_j(x,y,z)| \lesssim \|a\|_{\mathrm{Lip}^1}\big(|x-y|+|x-z|+|y-z|\big)^{-2n},$$
$$|\nabla K_j(x,y,z)| \lesssim \|a\|_{\mathrm{Lip}^1}\big(|x-y|+|x-z|+|y-z|\big)^{-2n-1}$$

on $\mathbb{R}^{3n} \setminus \Delta$ and for $j=1,2$.

4.2 The BMO Condition

We now verify that the commutators $[T_\sigma, a]_1$ and $[T_\sigma, a]_2$ satisfy the BMO conditions of the bilinear $T(1)$ theorem for any $\sigma \in \mathcal{B}BS^1_{1,1}$ and any Lipschitz continuous function a defined on $\mathbb{R}^n$. We will assume without loss of generality that $\sigma(x,0,0)=0$; notice that $[T_\sigma, a]_j = [T_{\sigma_0}, a]_j$ where $\sigma_0 = \sigma - \sigma(\cdot,0,0)$ and, moreover, $\sigma_0 \in \mathcal{B}BS^1_{1,1}$ as well.

By the Fundamental Theorem of Calculus we have

$$\sigma(x,\xi,\eta) = \sum_{j=1}^n (\xi_j \sigma_j(x,\xi,\eta) + \eta_j \widetilde{\sigma}_j(x,\xi,\eta)),$$

where $\xi = (\xi_1, \cdots, \xi_n)$, $\eta = (\eta_1, \cdots, \eta_n)$,

$$\sigma_j(x,\xi,\eta) = \int_0^1 \partial_{\xi'_j}\sigma(x,\xi',t\eta)\Big|_{\xi'=t\xi} dt \quad \text{and}$$
$$\widetilde{\sigma}_j(x,\xi,\eta) = \int_0^1 \partial_{\eta'_j}\sigma(x,t\xi,\eta')\Big|_{\eta'=t\eta} dt. \tag{4.20}$$

That is,

$$T_\sigma(f,g) = \frac{1}{2\pi i}\sum_{j=1}^n \big[T_{\sigma_j}(\partial_j f, g) + T_{\widetilde{\sigma}_j}(f, \partial_j g)\big].$$

Lemma 4.1 below gives that $\sigma_j, \widetilde{\sigma}_j \in \widetilde{BS}^0_{1,1}$ for all j (we state and prove Lemma 4.1 at the end of this section in order to ease the flow of the proof). We then have

$$[T_\sigma, a]_1(1,1) = T_\sigma(a,1) - aT_\sigma(1,1) = \frac{1}{2\pi i}\sum_{j=1}^n \big[T_{\sigma_j}(\partial_j a, 1) + \underbrace{T_{\widetilde{\sigma}_j}(a, \partial_j 1)}_{=0}\big].$$

Recall that the symbols in the class $\widetilde{BS}^0_{1,1}$ give rise to bilinear Calderón–Zygmund operators ([13, Corollary 1, p. 155]); as a consequence, T_{σ_j} is bounded from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into BMO. Since $\partial_j a \in L^\infty$, we conclude that $T_{\sigma_j}(\partial_j a, 1) \in BMO$ and $\left\|T_{\sigma_j}(\partial_j a, 1)\right\|_{BMO} \lesssim \left\|\partial_j a\right\|_{L^\infty}$. Therefore, $[T_\sigma, a]_1(1,1) \in BMO$ and its norm is controlled by $\|a\|_{\mathrm{Lip}^1}$. An analogous proof shows the result for $[T_\sigma, a]_2(1,1)$.

We next prove that $\big([T_\sigma, a]_j\big)^{*k}(1,1) \in BMO$ for $j = 1, 2$ and $k = 1, 2$. The following identities were proved in [3, Lemma 6]:

$$\big([T_\sigma, a]_1\big)^{*1} = -[T_{\sigma^{*1}}, a]_1, \quad \big([T_\sigma, a]_1\big)^{*2} = [T_{\sigma^{*2}}, a]_1 - [T_{\sigma^{*2}}, a]_2,$$
$$\big([T_\sigma, a]_2\big)^{*1} = [T_{\sigma^{*1}}, a]_2 - [T_{\sigma^{*1}}, a]_1, \quad \big([T_\sigma, a]_2\big)^{*2} = -[T_{\sigma^{*2}}, a]_2.$$

Now, let χ be an infinitely differentiable function defined on $\mathbb{R}^{2n}$, supported on $\{(\xi, \eta) \in \mathbb{R}^{2n} : |\xi| + |\eta| \le 2\}$ and identically equal to one on $\{(\xi, \eta) \in \mathbb{R}^{2n} : |\xi| + |\eta| \le 1\}$; write $\sigma(x, \xi, \eta) = \chi(\xi, \eta)\sigma(x, \xi, \eta) + (1 - \chi(\xi, \eta))\sigma(x, \xi, \eta)$ for $x, \xi, \eta \in \mathbb{R}^n$. We have $\chi\sigma \in BS^1_{1,0}$ and $(1-\chi)\sigma \in \mathcal{B}_r BS^1_{1,1}$ for some $0 < r < \frac{1}{7}$. Since $BS^1_{1,0}$ is closed under transposition, we obtain $(\chi\sigma)^{*k} \in BS^1_{1,0}$ for $k = 1, 2$; moreover, by Lemma 2.2 $((1-\chi)\sigma)^{*k} \in \mathcal{B}_{\frac{2r}{1-r}} BS^1_{1,1}$ for $k = 1, 2$ (note that $\frac{2r}{1-r} < \frac{1}{3}$). Then Lemma 4.1 below can again be applied to σ^{*1} and σ^{*2}. As a consequence, the previous reasoning allows to conclude that $[T_{\sigma^{*k}}, a]_j(1,1) \in BMO$ with norm controlled by $\|a\|_{\mathrm{Lip}^1}$ for $j = 1, 2$ and $k = 1, 2$. In view of the formulas for $\big([T_\sigma, a]_j\big)^{*k}$ given above, the desired results follow.

4.3 The Weak Boundedness Property

For $x \in \mathbb{R}^n$ and $t > 0$, set $B_x(t) := \{x \in \mathbb{R}^n : |x| \le t\}$. Given $M \in \mathbb{N}_0$, an infinitely differentiable function ϕ defined on $\mathbb{R}^n$ is called a normalized bump function of order M if $\mathrm{supp}(\phi) \subset B_0(1)$ and $\|\partial^\alpha \phi\|_{L^\infty} \le 1$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le M$. For such function ϕ, $x_0 \in \mathbb{R}^n$ and $t > 0$, denote $\phi^{x_0,t}(x) := \phi\big(\frac{x - x_0}{t}\big)$. As proved in [3, Lemma 9], a bilinear operators T with Calderón–Zygmund kernel satisfies the weak boundedness property (as stated in [14, Theorem 1.1]) if there exists $M \in \mathbb{N}_0$ such that

$$\left|\langle T(\phi_1^{x_0,t}, \phi_2^{x_0,t}), \phi_3^{x_0,t}\rangle\right| \lesssim t^n,$$

for all normalized bump functions ϕ_1, ϕ_2, and ϕ_3 of order M, $x_0 \in \mathbb{R}^n$ and $t > 0$.

We will show that if $\sigma \in \mathcal{B}BS^1_{1,1}$ and a is Lipschitz continuous in $\mathbb{R}^n$ then the first commutator $[T_\sigma, a]_1$ satisfies the above inequality for normalized bump functions of order 1 and with a constant controlled by $\|a\|_{\mathrm{Lip}^1}$. An analogous proof is valid for the second commutator. Let $x_0 \in \mathbb{R}^n$, $t > 0$ and ϕ_1, ϕ_2, ϕ_3 be normalized bump functions of order 1 defined on $\mathbb{R}^n$. Without loss of generality, we can assume that $a(x_0) = 0$ (replacing a with $a - a(x_0)$ does not change the commutator); we then have $\|a\|_{L^\infty(B_{x_0}(t))} \lesssim t\|a\|_{\mathrm{Lip}^1}$ and

$$\left|\langle [T_\sigma, a]_1(\phi_1^{x_0,t}, \phi_2^{x_0,t}), \phi_3^{x_0,t}\rangle\right| \le \left|\langle T_\sigma(a\phi_1^{x_0,t}, \phi_2^{x_0,t}), \phi_3^{x_0,t}\rangle\right| + \left|\langle a T_\sigma(\phi_1^{x_0,t}, \phi_2^{x_0,t}), \phi_3^{x_0,t}\rangle\right|.$$

We estimate the second expression on the left by

$$\begin{aligned}\|aT_\sigma(\phi_1^{x_0,t}, \phi_2^{x_0,t})\|_{L^2(B_{x_0}(t))}\|\phi_3^{x_0,t}\|_{L^2} &\lesssim t^{\frac{n}{2}}\|a\|_{L^\infty(B_{x_0}(t))}\|T_\sigma(\phi_1^{x_0,t}, \phi_2^{x_0,t})\|_{L^2(B_{x_0}(t))}\\ &\lesssim t^{\frac{n}{2}+1}\|a\|_{\mathrm{Lip}^1}\left\|\sum_{j=1}^n \left[T_{\sigma_j}(\partial_j\phi_1^{x_0,t}, \phi_2^{x_0,t}) + T_{\widetilde{\sigma}_j}(\phi_1^{x_0,t}, \partial_j\phi_2^{x_0,t})\right]\right\|_{L^2}.\end{aligned}$$

Recalling now that $\sigma_j, \widetilde{\sigma_j} \in \widetilde{BS}^0_{1,1}$ and using again [13, Corollary 1], we can further estimate the above by

$$t^{\frac{n}{2}+1}\|a\|_{\mathrm{Lip}^1}\sum_{j=1}^n\left[\|\partial_j\phi_1^{x_0,t}\|_{L^4}\|\phi_2^{x_0,t}\|_{L^4} + \|\phi_1^{x_0,t}\|_{L^4}\|\partial_j\phi_2^{x_0,t}\|_{L^4}\right] \lesssim t^n\|a\|_{\mathrm{Lip}^1}.$$

The estimate for the first expression $\left|\langle T_\sigma(a\phi_1^{x_0,t}, \phi_2^{x_0,t}), \phi_3^{x_0,t}\rangle\right|$ follows a similar pattern. It is controlled by

$$\begin{aligned}&t^{\frac{n}{2}}\left\|\sum_{j=1}^n\left[T_{\sigma_j}(\partial_j(a\phi_1^{x_0,t}), \phi_2^{x_0,t}) + T_{\widetilde{\sigma}_j}(a\phi_1^{x_0,t}, \partial_j\phi_2^{x_0,t})\right]\right\|_{L^2}\\ &\quad\lesssim t^{\frac{n}{2}}\sum_{j=1}^n\left[\|\partial_j(a\phi_1^{x_0,t})\|_{L^4}\|\phi_2^{x_0,t}\|_{L^4} + \|a\phi_1^{x_0,t}\|_{L^4}\|\partial_j\phi_2^{x_0,t}\|_{L^4}\right]\\ &\quad\lesssim t^{\frac{n}{2}}\sum_{j=1}^n\Big[\|\partial_j a\,\phi_1^{x_0,t}\|_{L^4}\|\phi_2^{x_0,t}\|_{L^4} + \|a\,\partial_j\phi_1^{x_0,t}\|_{L^4}\|\phi_2^{x_0,t}\|_{L^4}\\ &\qquad\qquad + \|a\phi_1^{x_0,t}\|_{L^4}\|\partial_j\phi_2^{x_0,t}\|_{L^4}\Big]\\ &\quad\lesssim t^n\|a\|_{\mathrm{Lip}^1}.\end{aligned}$$

4.4 End of Proof

By [14, Theorem 1.1], we conclude that $[T_\sigma, a]_j$, $j = 1, 2$, are bilinear Calderón–Zygmund operators for any $\sigma \in \mathcal{B}BS^1_{1,1}$ and any Lipschitz continuous function a on $\mathbb{R}^n$. Moreover, the proof of [14, Theorem 1.1] gives that $\left\|[T_\sigma, a]_j\right\|_{L^2\times L^2\to L^1}$ is controlled by the sum of the constant in the estimates of the kernel of $[T_\sigma, a]_j$, the constant in the weak boundedness property condition for $[T_\sigma, a]_j$ and the BMO norms of the evaluations of $[T_\sigma, a]_j$ and its transposes at $(1, 1)$. Since each of these are controlled by $\|a\|_{\mathrm{Lip}^1}$, we have that $\left\|[T_\sigma, a]_j\right\|_{L^2\times L^2\to L^1} \lesssim \|a\|_{\mathrm{Lip}^1}$. All boundedness properties satisfied by bilinear Calderón–Zygmund operators then follow for the bilinear commutators with the corresponding norms bounded by a multiple of $\|a\|_{\mathrm{Lip}^1}$. □

Our arguments above made use in an essential way of the fact that the symbols σ_j and $\widetilde{\sigma}_j$ defined in (4.20) behave well with respect to transposition. This is the content of the next result.

Lemma 4.1 *If $\sigma \in \mathcal{B}_r BS^1_{1,1}$ for some $0 < r < \frac{1}{3}$ or $\sigma \in BS^1_{1,\delta}$ for some $0 \le \delta < 1$, then the symbols σ_j and $\widetilde{\sigma}_j$ defined in (4.20) belong to $\widetilde{BS}^0_{1,1}$ for all $j \in \{1, \dots, n\}$.*

Proof The facts $\sigma \in BS^1_{1,\delta}$ for some $0 \le \delta \le 1$ and $t\big(1 + t(|\xi| + |\eta|)\big)^{-1} \lesssim (1 + |\xi| + |\eta|)^{-1}$ for all $t \in [0, 1]$ easily imply that σ_j and $\widetilde{\sigma}_j$ belong to $BS^0_{1,\delta}$ for all j. With the additional assumption $\delta < 1$, the class $BS^0_{1,\delta}$ is closed under transposition and since $BS^0_{1,\delta} \subset BS^0_{1,1}$, the result follows in this case.

Let $\sigma \in \mathcal{B}_r BS^1_{1,1}$ for some $0 < r < \frac{1}{3}$; assume first that σ is supported on $\{(x, \xi, \eta) \in \mathbb{R}^{3n} : |\xi| + |\eta| > 1\}$. We will prove that $(\sigma_j)^{*k}, (\widetilde{\sigma}_j)^{*k} \in BS^0_{1,1}$ for $k = 1, 2$ and all j. We will work with $(\sigma_j)^{*1}$; an analogous reasoning holds for $(\widetilde{\sigma}_j)^{*1}$ and the second transposes of σ_j and $\widetilde{\sigma}_j$. Formal computations show that

$$(\sigma_j)^{*1}(x, \xi, \eta) = -\int_0^1 \partial_{\xi'_j}[(\sigma^t)^{*1}(\tfrac{x}{t}, \xi', t\eta)]\Big|_{\xi'=t\xi} dt,$$

where we recall that $\sigma^t(x, \xi, \eta) := \sigma(tx, \xi, \eta)$ for all $x, \xi, \eta \in \mathbb{R}^n$. In view of Remark 2.1, $(\sigma^t)^{*1} \in BS^1_{1,1}$ and (2.6) holds; we then obtain

$$\begin{aligned}
&|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma (\sigma_j)^{*1}(x, \xi, \eta)| \\
&\quad = \left| \int_0^1 t^{-|\alpha|+|\beta|+|\gamma|} \partial_{x'}^\alpha \partial_{\xi'}^\beta \partial_{\eta'}^\gamma \partial_{\xi'_j} (\sigma^t)^{*1}(x', \xi', \eta')\Big|_{(x',\xi',\eta')=(\frac{x}{t}, t\xi, t\eta)} dt \right| \\
&\quad \lesssim \int_0^1 t^{|\beta|+|\gamma|}\big(1 + t(|\xi| + |\eta|)\big)^{1+|\alpha|-(|\beta|+1+|\gamma|)} dt \\
&\quad \lesssim (1 + |\xi| + |\eta|)^{|\alpha|-|\beta+\gamma|},
\end{aligned}$$

where we used again that $t\big(1 + t(|\xi| + |\eta|)\big)^{-1} \lesssim (1 + |\xi| + |\eta|)^{-1}$ for $t \in [0, 1]$. Therefore $(\sigma_j)^{*1} \in BS^0_{1,1}$.

For a general $\sigma \in \mathcal{B}_r BS^1_{1,1}$, let χ be an infinitely differentiable function defined on $\mathbb{R}^{2n}$, supported on $\{(\xi, \eta) \in \mathbb{R}^{2n} : |\xi| + |\eta| \le 2\}$ and identically equal to one on $\{(\xi, \eta) \in \mathbb{R}^{2n} : |\xi| + |\eta| \le 1\}$; write $\sigma(x, \xi, \eta) = \chi(\xi, \eta)\sigma(x, \xi, \eta) + (1 - \chi(\xi, \eta))\sigma(x, \xi, \eta)$ for $x, \xi, \eta \in \mathbb{R}^n$. Then $\sigma_j = (\chi\sigma)_j + ((1-\chi)\sigma)_j$ for $j = 1, \cdots, n$, and similarly for $\widetilde{\sigma}_j$. The symbol $(1 - \chi)\sigma$ belongs to $\mathcal{B}_r BS^1_{1,1}$ and the symbol $\chi\sigma$ belongs to $BS^1_{1,0}$. By the previous cases, we conclude that σ_j and $\widetilde{\sigma}_j$ are in $\widetilde{BS}^0_{1,1}$ for all $j = 1, \dots, n$. □

Remark 4.1 Consider the class of symbols

$$\widetilde{\widetilde{BS}}{}^1_{1,1} := \{\sigma \in \widetilde{BS}^1_{1,1} : \sigma_j, \widetilde{\sigma}_j \in \widetilde{BS}^0_{1,1} \text{ for } j = 1, \dots, n\},$$

where σ_j and $\widetilde{\sigma}_j$ are as in (4.20). Lemma 2.2 and Lemma 4.1 imply that $\mathcal{B}BS^1_{1,1}$ is contained in such class. The proof of Lemma 4.1 can be used to show that $\sigma \in \widetilde{BS}^1_{1,1}$ is in $\widetilde{\widetilde{BS}}^1_{1,1}$ if (2.6) holds for $(1-\chi)\sigma$ with χ as in the proof of Lemma 4.1. Therefore, if the latter is true for every $\sigma \in \widetilde{BS}^1_{1,1}$, the two classes of symbols coincide and then the thesis of Theorem 1.1 would be true for every symbol in $\widetilde{BS}^1_{1,1}$ (the proof of Theorem 1.1 uses that σ and its adjoints are in $\widetilde{\widetilde{BS}}^1_{1,1}$).

Acknowledgements The authors initiated this collaboration while visiting the Pacific Institute for the Mathematical Sciences in June 2015, during the Western International Workshop on Harmonic Analysis and PDE. They thank the institute and the organizers of the workshop for their hospitality.

References

1. Auscher, P., Taylor, M.: Paradifferential operators and commutator estimates. Commun. Partial Differ. Equ. **20**(9–10), 1743–1775 (1995)
2. Bényi, Á.: Bilinear pseudodifferential operators with forbidden symbols on Lipschitz and Besov spaces. J. Math. Anal. Appl. **284**(1), 97–103 (2003)
3. Bényi, Á., Oh, T.: Smoothing of commutators for a Hörmander class of bilinear pseudodifferential operators. J. Fourier Anal. Appl. **20**(2), 282–300 (2014)
4. Bényi, Á., Torres, R.H.: Symbolic calculus and the transposes of bilinear pseudodifferential operators. Commun. Partial Differ. Equ. **28**(5–6), 1161–1181 (2003)
5. Bényi, Á., Torres, R.H.: Almost orthogonality and a class of bounded bilinear pseudodifferential operators. Math. Res. Lett. **11**(1), 1–11 (2004)
6. Bényi, Á., Maldonado, D., Naibo, V., Torres, R.H.: On the Hörmander classes of bilinear pseudodifferential operators. Integr. Equ. Oper. Theory **67**(3), 341–364 (2010)
7. Bényi, Á., Bernicot, F., Maldonado, D., Naibo, V., Torres, R.H.: On the Hörmander classes of bilinear pseudodifferential operators II. Indiana Univ. Math. J. **62**(6), 1733–1764 (2013)
8. Bourdaud, G.: Une algèbre maximale d'opérateurs pseudo-différentiels. Commun. Partial Differ. Equ. **13**(9), 1059–1083 (1988)
9. Calderón, A.: Commutators of singular integral operators. Proc. Natl. Acad. Sci. USA **53**, 1092–1099 (1965)
10. Christ, M., Journé, J.-L.: Polynomial growth estimates for multilinear singular integral operators. Acta Math. **159**(1–2), 51–80 (1987)
11. Coifman, R., Meyer, Y.: Au delà des opérateurs pseudo-différentiels. Volume 57 of Astérisque. Société Mathématique de France, Paris (1978)
12. Coifman, R., Meyer, Y.: Commutateurs d'intégrales singulières et opérateurs multilinéaires. Ann. Inst. Fourier (Grenoble) **28**(3), 177–202 (1978)
13. Grafakos, L., Torres, R.H.: Multilinear Calderón–Zygmund theory. Adv. Math. **165**(1), 124–164 (2002)
14. Hart, J.: A new proof of the bilinear T(1) theorem. Proc. Am. Math. Soc. **142**(9), 3169–3181 (2014)
15. Herbert, J., Naibo, V.: Bilinear pseudodifferential operators with symbols in Besov spaces. J. Pseudo-Differ. Oper. Appl. **5**(2), 231–254 (2014)
16. Hörmander, L.: Pseudo-differential operators of type 1, 1. Commun. Partial Differ. Equ. **13**(9), 1085–1111 (1988)
17. Meyer, Y: Remarques sur un théorème de J.-M. Bony. In: Proceedings of the Seminar on Harmonic Analysis (Pisa, 1980), number suppl. 1, pp. 1–20 (1981)
18. Michalowski, N., Rule, D., Staubach, W.: Multilinear pseudodifferential operators beyond Calderón–Zygmund theory. J. Math. Anal. Appl. **414**(1), 149–165 (2014)
19. Miyachi, A., Tomita, N.: Calderón–Vaillancourt-type theorem for bilinear operators. Indiana Univ. Math. J. **62**(4), 1165–1201 (2013)
20. Naibo, V.: On the bilinear Hörmander classes in the scales of Triebel–Lizorkin and Besov spaces. J. Fourier Anal. Appl. **21**(5), 1077–1104 (2015)

21. Naibo, V.: On the $L^\infty \times L^\infty \to BMO$ mapping property for certain bilinear pseudodifferential operators. Proc. Am. Math. Soc. **143**(12), 5323–5336 (2015)
22. Petersen, B: Introduction to the Fourier transform & pseudodifferential operators, volume 19 of Monographs and Studies in Mathematics. Pitman (Advanced Publishing Program), Boston (1983)
23. Rodríguez-López, S., Staubach, W.: Estimates for rough Fourier integral and pseudodifferential operators and applications to the boundedness of multilinear operators. J. Funct. Anal. **264**(10), 2356–2385 (2013)
24. Taylor, M.E.: Commutator estimates. Proc. Am. Math. Soc. **131**(5), 1501–1507 (2003)
25. Taylor, M.E.: Commutator estimates for Hölder continuous and bmo-Sobolev multipliers. Proc. Am. Math. Soc. **143**(12), 5265–5274 (2015)