J Fourier Anal Appl (2017) 23:1195–1226
DOI 10.1007/s00041-016-9503-9

Calderón's First and Second Complex Interpolations of Closed Subspaces of Morrey Spaces

Denny Ivanal Hakim[1] · **Yoshihiro Sawano**[1]

Received: 3 December 2015 / Revised: 2 September 2016 / Published online: 8 October 2016

Abstract In this paper, we are going to describe the first and second complex interpolations of closed subspaces of Morrey spaces, based on our previous results in [11]. Our results will be general enough because we are going to deal with abstract linear subspaces satisfying the lattice condition only. We also consider the closure in Morrey spaces on bounded domains of the set of smooth functions with compact support. Here, we do not require the smoothness condition on domains.

Keywords Morrey spaces · Generalized Morrey spaces · Complex interpolation methods

Mathematics Subject Classification Primary 46B70 · Secondary 42B35 · 46B26

1 Introduction

For $1 \le q \le p < \infty$, the Morrey space $\mathcal{M}^p_q = \mathcal{M}^p_q(\mathbb{R}^n)$ is defined to be the set of all q-locally integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\mathcal{M}^p_q} := \sup_{x\in\mathbb{R}^n, r>0} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |f(y)|^q \, dy\right)^{\frac{1}{q}} < \infty. \qquad (1.1)$$

Communicated by Mieczysław Mastyło.

✉ Denny Ivanal Hakim
dennyivanalhakim@gmail.com

Yoshihiro Sawano
yoshihiro-sawano@celery.ocn.ne.jp

[1] Department of Mathematics and Information Sciences, Tokyo Metropolitan University, Minami-Ohsawa 1-1, Hachioji, Tokyo 192-0397, Japan

Here, $B(x, r)$ denotes the ball centered at $x \in \mathbb{R}^n$ with radius r. The interpolations of Morrey spaces date back to the papers around 1960s. Campanato and Murthy [5], Spanne [31], and Peetre [24] obtained some results on the boundedness of operators on Morrey spaces and the interpolation spaces.

Based on the definition of the complex interpolation functors $(X_0, X_1) \mapsto [X_0, X_1]_\theta$ and $(X_0, X_1) \mapsto [X_0, X_1]^\theta$, introduced by Calderón in [4], whose definition we recall in Sect. 2, the following results are known:

Theorem 1.1 [14,16] *Suppose that* $\theta \in (0, 1)$, $1 \le q_0 \le p_0 < \infty$, $1 \le q_1 \le p_1 < \infty$, *and* $\frac{p_0}{q_0} = \frac{p_1}{q_1}$. *Assume* $q_0 \ne q_1$. *Define*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then

(i) (Lu et al. [16]) $\left[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}\right]_\theta = \overline{\mathcal{M}^{p_0}_{q_0} \cap \mathcal{M}^{p_1}_{q_1}}^{\mathcal{M}^p_q}$

(ii) (Lemarié-Rieusset [14]) $\left[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}\right]^\theta = \mathcal{M}^p_q$.

One of our main results in this note refines Theorem 1.1 (i):

Theorem 1.2 *Keep the same assumption as in Theorem 1.1. Then we have*

$$\left[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}\right]_\theta = \left\{ f \in \overline{\mathcal{M}^p_q} : \lim_{a \to 0^+} \|\chi_{\{|f|<a\}} f\|_{\mathcal{M}^p_q} = 0 \right\}. \tag{1.2}$$

Note that the right-hand side is independent of p_0, p_1, q_0, and q_1. We shall prove (1.2) in a more general framework.

The aim of this paper is to investigate the effect of the first and second complex interpolation functors through closed subspaces.

We use the following notation for closed subspaces of the Morrey space $\mathcal{M}^p_q$:

Definition 1.3 Assume that a linear subspace $U \subset L^0$ enjoys the lattice property: $g \in U$ whenever $f \in U$ and $|g| \le |f|$. For $1 \le q < p < \infty$, define

$$U\mathcal{M}^p_q := \overline{U \cap \mathcal{M}^p_q}^{\mathcal{M}^p_q} \tag{1.3}$$

$$U \bowtie \mathcal{M}^p_q := \left\{ f \in \mathcal{M}^p_q : \chi_{\{a \le |f| \le b\}} f \in U\mathcal{M}^p_q \text{ for all } 0 < a < b < \infty \right\}. \tag{1.4}$$

To investigate further the role of the closed subspace U in the second complex interpolation, we prove the following theorem:

Theorem 1.4 *Suppose that* $\theta \in (0, 1)$, $1 \le q_0 \le p_0 < \infty$, $1 \le q_1 \le p_1 < \infty$, *and* $\frac{p_0}{q_0} = \frac{p_1}{q_1}$. *Assume* $q_0 \ne q_1$. *Define*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then we have

$$\begin{aligned}\left[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}\right]_\theta &= U\mathcal{M}^p_q \cap \left[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}\right]_\theta \\ &= \left\{ f \in U\mathcal{M}^p_q \cap \overline{\mathcal{M}^p_q} : \lim_{a\to 0^+} \|\chi_{\{|f|<a\}} f\|_{\mathcal{M}^p_q} = 0 \right\}. \quad (1.5)\end{aligned}$$

and

$$\left[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}\right]^\theta = U \bowtie \mathcal{M}^p_q. \quad (1.6)$$

We will prove (1.5) in a more general framework.

More and more attention has been paid for the closed subspaces of the Morrey space $\mathcal{M}^p_q$ with $1 \le q < p < \infty$ [29,40]. Some of them are realized as $U\mathcal{M}^p_q$ for some U in Definition 1.3.

Definition 1.5 Let $1 \le q \le p < \infty$.

1. [11, p. 5] The space $\widetilde{\mathcal{M}^p_q}$ is defined to be the closure of L^∞_c in $\mathcal{M}^p_q$.
2. [29, Definition 4.5] A function f in $\mathcal{M}^p_q$ is said to have "absolutely continuous norm" in $\mathcal{M}^p_q$ if $\|f\chi_{E_k}\|_{\mathcal{M}^p_q} \to 0$ for every sequence $\{E_k\}_{k=1}^\infty$ satisfying $\chi_{E_k}(x) \to 0$ a.e. The set of all functions in $\mathcal{M}^p_q$ of absolutely continuous norm is denoted by $\widehat{\mathcal{M}^p_q}$.
3. [40, Definition 2.23] $\overset{\diamond}{\mathcal{M}}{}^p_q$ denotes the closure with respect to $\mathcal{M}^p_q$ of the set of all smooth functions f such that $\partial^\alpha f \in \mathcal{M}^p_q$ for all multi-indexes α.
4. [40, Definition 2.23] $\overset{\circ}{\mathcal{M}}{}^p_q$ denotes the closure with respect to $\mathcal{M}^p_q$ of C^∞_c, or equivalently, the closure of $\mathcal{S}$ in $\mathcal{M}^p_q$.
5. [40, Sect. 2] $\overset{*}{\mathcal{M}}{}^p_q$ denotes the closure with respect to $\mathcal{M}^p_q$ of the set of all compactly supported functions L^0 in $\mathcal{M}^p_q$.
6. [6, p. 1] $\overline{\mathcal{M}^p_q}$ denotes the closure with respect to $\mathcal{M}^p_q$ of the set of all essentially bounded functions in $\mathcal{M}^p_q$.

From the definition, it is easy to see that $\widetilde{\mathcal{M}^p_q} = \overset{\circ}{\mathcal{M}}{}^p_q$ and that $\widetilde{\mathcal{M}^p_q}$, $\overset{*}{\mathcal{M}}{}^p_q$, and $\overline{\mathcal{M}^p_q}$ are realized as $U\mathcal{M}^p_q$ for some linear space U enjoying the lattice property; $U = L^\infty_c$, L^0, and L^∞ do the job, respectively.

The closed subspaces $\widehat{\mathcal{M}^p_q}$ and $\overset{\diamond}{\mathcal{M}}{}^p_q$ arise naturally. We refer to [40, Theorem 2.29] for $\overset{\diamond}{\mathcal{M}}{}^p_q$ and to [29, Theorem 4.3] for $\widehat{\mathcal{M}^p_q}$.

Let $\theta \in (0,1)$, $1 < q \le p < \infty$, $1 < q_0 \le p_0 < \infty$, and $1 < q_1 \le p_1 < \infty$ satisfy $p_0 < p < p_1$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{q_0}{p_0} = \frac{q_1}{p_1}.$$

Then we have the following relations:

$$\overset{\circ}{\mathcal{M}}{}^p_q \subsetneq \widetilde{\mathcal{M}}{}^p_q = \widehat{\mathcal{M}}{}^p_q \subsetneq \overset{*}{\mathcal{M}}{}^p_q, \tag{1.7}$$

$$\overset{\circ}{\mathcal{M}}{}^p_q \subsetneq \overset{\diamond}{\mathcal{M}}{}^p_q, \tag{1.8}$$

$$\widetilde{\mathcal{M}}{}^p_q \subsetneq \overline{\mathcal{M}^{p_0}_{q_0} \cap \mathcal{M}^{p_1}_{q_1}}^{\mathcal{M}^p_q}, \tag{1.9}$$

$$\widetilde{\mathcal{M}}{}^p_q \subsetneq [\widetilde{\mathcal{M}}{}^{p_0}_{q_0}, \widetilde{\mathcal{M}}{}^{p_1}_{q_1}]^\theta, \tag{1.10}$$

$$[\overset{\circ}{\mathcal{M}}{}^{p_0}_{q_0}, \overset{\circ}{\mathcal{M}}{}^{p_1}_{q_1}]_\theta = \overset{\circ}{\mathcal{M}}{}^p_q. \tag{1.11}$$

according to [40, Lemma 2.33], [40, Remark 2.36], [40, Lemma 2.37, Corollary 2.38], [40, Remark 2.36], [29, Theorem 4.3] and [37, Corollary 1.4], respectively. We have no further inclusion; see [11, Sect. 9].

In this paper we also consider the complex interpolations of generalized Morrey spaces, introduced by Nakai [19]. The thrust is that generalized Morrey spaces can be contained in the space L^∞. In fact, in many results the indicator function of the level set of f comes into play as is the case with many results presented in this paper. Recall that for $1 \le q < \infty$ and a function $\varphi : (0, \infty) \to (0, \infty)$, the generalized Morrey space $\mathcal{M}^\varphi_q$ is defined to be the set of all functions $f \in L^q_{\rm loc}$ such that

$$\|f\|_{\mathcal{M}^\varphi_q} := \sup_{x\in\mathbb{R}^n, r>0} \varphi(r) \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|^q \, dy\right)^{\frac{1}{q}}$$

is finite. We assume that φ belongs to $\mathcal{G}_q$, that is, φ is increasing but that $t \mapsto t^{-n/q}\varphi(t)$ is decreasing; see the work [20, p. 446] which justifies this assumption. Note that, for $\varphi(t) := t^{n/p}$, where $1 \le q \le p < \infty$, we have $\mathcal{M}^\varphi_q = \mathcal{M}^p_q$. See Sect. 2.3 for more examples of φ. Our previous results on the complex interpolation of generalized Morrey spaces are given as follows:

Theorem 1.6 [11, Theorem 2] *Let $\theta \in (0, 1)$, $1 \le q_0 < \infty$, $1 \le q_1 < \infty$, $\varphi_0 \in \mathcal{G}_{q_0}$, $\varphi_1 \in \mathcal{G}_{q_1}$, and $\varphi_0^{q_0} = \varphi_1^{q_1}$. Define $\varphi := \varphi_0^{1-\theta}\varphi_1^\theta$ and $\frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Then we have*

$$\left[\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}\right]_\theta = \overline{\mathcal{M}^{\varphi_0}_{q_0} \cap \mathcal{M}^{\varphi_1}_{q_1}}^{\mathcal{M}^\varphi_q} \text{ and } \left[\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}\right]^\theta = \mathcal{M}^\varphi_q.$$

The following is our interpolation result which includes (1.2).

Theorem 1.7 *Let $\theta \in (0, 1)$, $1 \le q_0 < \infty$, $1 \le q_1 < \infty$, $\varphi_0 \in \mathcal{G}_{q_0}$, $\varphi_1 \in \mathcal{G}_{q_1}$, and $\varphi_0^{q_0} = \varphi_1^{q_1}$. Define $\varphi := \varphi_0^{1-\theta}\varphi_1^\theta$ and $\frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Then we have*

$$\left[\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}\right]_\theta = \left\{ f \in \overline{\mathcal{M}^\varphi_q} : \lim_{a\to 0^+} \|\chi_{\{|f|<a\}} f\|_{\mathcal{M}^\varphi_q} = 0 \right\}. \tag{1.12}$$

Note again that the right-hand side is independent of φ_0, φ_1, q_0, and q_1.

We remark that (1.12) refines the general result asserting that $[\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}]_\theta$ is the closure of $\mathcal{M}^{\varphi_0}_{q_0} \cap \mathcal{M}^{\varphi_1}_{q_1}$ in $\mathcal{M}^{\varphi}_{q}$.

We use the following notation for closed subspaces of generalized Morrey spaces:

Definition 1.8 Let U be the same as in Definition 1.3, $1 \le q < \infty$, and $\varphi \in \mathcal{G}_q$. Define $U\mathcal{M}^\varphi_q := \overline{U \cap \mathcal{M}^\varphi_q}^{\mathcal{M}^\varphi_q}$ and

$$U \bowtie \mathcal{M}^\varphi_q := \left\{ f \in \mathcal{M}^\varphi_q : \chi_{\{a \le |f| \le b\}} f \in U\mathcal{M}^\varphi_q \text{ for all } 0 < a < b < \infty \right\}.$$

The complex interpolation result for $U\mathcal{M}^\varphi_q$ is given in the following theorem:

Theorem 1.9 *Suppose that $\theta \in (0,1)$, $1 \le q_0 < \infty$, $1 \le q_1 < \infty$, and $\varphi_0^{q_0} = \varphi_1^{q_1}$. Define*

$$\varphi := \varphi_0^{1-\theta} \varphi_1^\theta \quad \text{and} \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then we have

$$\left[U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}\right]_\theta = U\mathcal{M}^\varphi_q \cap \left[\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}\right]_\theta \tag{1.13}$$

$$= \left\{ f \in U\mathcal{M}^\varphi_q \cap \overline{\mathcal{M}^\varphi_q} : \lim_{a \to 0^+} \|\chi_{\{|f|<a\}} f\|_{\mathcal{M}^\varphi_q} = 0 \right\}, \tag{1.14}$$

$$\left[U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}\right]^\theta = U \bowtie \mathcal{M}^\varphi_q. \tag{1.15}$$

As a special case for these examples, we have the following results:

Corollary 1.10 [11, Theorems 5.2 and 5.12] *Suppose that $\theta \in (0,1)$, $1 \le q_0 < \infty$, $1 \le q_1 < \infty$, and $\varphi_0^{q_0} = \varphi_1^{q_1}$. Define $\varphi := \varphi_0^{1-\theta}\varphi_1^\theta$ and $\frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.*

1. *The description of the first interpolation functor of these closed subspaces is as follows:*

$$\left[\widetilde{\mathcal{M}^{\varphi_0}_{q_0}}, \widetilde{\mathcal{M}^{\varphi_1}_{q_1}}\right]_\theta = \widetilde{\mathcal{M}^\varphi_q}, \quad \left[\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}, \overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}\right]_\theta = \widetilde{\mathcal{M}^\varphi_q}, \quad \left[\overline{\mathcal{M}^{\varphi_0}_{q_0}}, \overline{\mathcal{M}^{\varphi_1}_{q_1}}\right]_\theta = \overline{\mathcal{M}^\varphi_q}.$$

2. *The description of the second interpolation functor of these closed subspaces is as follows:*

$$\left[\widetilde{\mathcal{M}^{\varphi_0}_{q_0}}, \widetilde{\mathcal{M}^{\varphi_1}_{q_1}}\right]^\theta = \left[\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}, \overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}\right]^\theta = \bigcap_{0<a<b<\infty} \left\{ f \in \mathcal{M}^\varphi_q : \chi_{\{a \le |f| \le b\}} f \in \widetilde{\mathcal{M}^\varphi_q} \right\}, \tag{1.16}$$

$$\left[\overline{\mathcal{M}^{\varphi_0}_{q_0}}, \overline{\mathcal{M}^{\varphi_1}_{q_1}}\right]^\theta = \mathcal{M}^\varphi_q. \tag{1.17}$$

To investigate the effect of the finiteness of the ambient spaces, we consider Morrey spaces on bounded connected open set $\Omega \subseteq \mathbb{R}^n$. For $1 \le q < \infty$ and $\varphi \in \mathcal{G}_q$, the space $\mathcal{M}_q^\varphi(\Omega)$ is defined to be the set of all functions $f \in L^q(\Omega)$ such that

$$\|f\|_{\mathcal{M}_q^\varphi(\Omega)} := \sup_{x\in\Omega, 0<r<\mathrm{diam}(\Omega)} \varphi(r)\left(\frac{1}{|B(x,r)|}\int_{B(x,r)\cap\Omega} |f(y)|^q\,dy\right)^{\frac{1}{q}} < \infty.$$

Here, we do not require that Ω is smooth. Let $\overset{\circ}{\mathcal{M}}{}_q^\varphi(\Omega)$ be the closure of $C_c^\infty(\Omega)$ in $\mathcal{M}_q^\varphi(\Omega)$. In the special case of $\varphi := 1$, one defines $\overset{\circ}{L}{}^\infty(\Omega) := \overset{\circ}{\mathcal{M}}{}_q^\varphi(\Omega)$. Via the mollification, we shall show that $\overset{\circ}{\mathcal{M}}{}_q^\varphi(\Omega)$ is the closure of $C_c(\Omega)$ in $\mathcal{M}_q^\varphi(\Omega)$. We remark that, for $U := L^0(\Omega)$, we have $U\mathcal{M}_q^\varphi = \mathcal{M}_q^\varphi(\Omega)$.

The interpolation result for these spaces is presented in the following theorem:

Theorem 1.11 *Let $\theta \in (0,1)$, $1 \le q_0 < \infty$, $1 \le q_1 < \infty$, $\varphi_0 \in \mathcal{G}_{q_0}$, and $\varphi_1 \in \mathcal{G}_{q_1}$. Assume $\varphi_0^{q_0} = \varphi_1^{q_1}$. Define $\frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ and $\varphi := \varphi_0^{1-\theta}\varphi_1^\theta$. Then we have*

$$\left[\overset{\circ}{\mathcal{M}}{}_{q_0}^{\varphi_0}(\Omega), \overset{\circ}{\mathcal{M}}{}_{q_1}^{\varphi_1}(\Omega)\right]_\theta = \overset{\circ}{\mathcal{M}}{}_q^\varphi(\Omega).$$

The related result in $\mathbb{R}^n$ can be seen in [37, Corollary 1.4].

Let us explain why the interpolations of Morrey spaces are complicated unlike Lebesgue spaces. Noteworthy is the fact that the first complex interpolation functor behaves differently from Lebesgue spaces. This problem comes basically from the fact that the Morrey norm $\mathcal{M}_q^p$ involves the supremum over all balls $B(a,r)$. Due to this fact, we have many difficulties when $1 < q < p < \infty$, namely:

1. The Morrey space $\mathcal{M}_q^p$ is not reflexive; see [29, Example 5.2] and [36, Theorem 1.3].
2. The Morrey space $\mathcal{M}_q^p$ does not have C_c^∞ as a dense closed subspace; see [33, Proposition 2.16].
3. The Morrey space $\mathcal{M}_q^p$ is not separable; see [33, Proposition 2.16].
4. The Morrey space $\mathcal{M}_q^p$ is not included in $L^1 + L^\infty$; see Sect. 6 for the proof.

The non-density of C_c^∞ and the failure of reflexivity and separability influence many other related function spaces such as Besov–Morrey spaces, Triebel–Lizorkin–Morrey spaces, Besov-type spaces, and Triebel–Lizorkin type spaces. These spaces are nowadays called smoothness Morrey spaces. We remark that these spaces cover Morrey spaces as a special case as is shown in [17, Proposition 4.1]. We refer to [15, Theorem 9.6] and [23, Corollary 6.2] for the counterpart of generalized Morrey spaces. We refer to [37,38,40] for the results of this direction. Since we do not deal with smoothness Morrey spaces in this paper, we content ourselves with listing the papers containing the definition of the function spaces [12,17,18,27,32,34,35] as well as the textbooks [33,39] without stating the precise definitions. As is pointed out in [37, Remark 1.9], the second author made a careless claim in [28, Theorem 5.4] that (homogeneous) Besov–Morrey spaces are closed under taking the first complex interpolation. However, this mistake comes essentially from the misunderstanding that $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta = \mathcal{M}^p_q$.

Despite a counterexample by Blasco et al. [2,26], the interpolation theory of Morrey spaces progressed so much. As for the real interpolation results, Burenkov and Nursultanov obtained an interpolation result in local Morrey spaces [3]. Nakai and Sobukawa generalized their results to B^w_u setting [22], where B^w_u denotes the weighted B_σ-space. We made a significant progress in the complex interpolation theory of Morrey spaces. Denote by $[X_0, X_1]_\theta$ the first complex interpolation; see Definition 2.1. In [8, p. 35] Cobos, Peetre and Persson pointed out that

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta \subset \mathcal{M}^p_q$$

as long as $1 \le q_0 \le p_0 < \infty$, $1 \le q_1 \le p_1 < \infty$, and $1 \le q \le p < \infty$ satisfy

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{1.18}$$

As is shown in [13, Theorem 3(ii)], when an interpolation functor F satisfies

$$F\left[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}\right] = \mathcal{M}^p_q$$

under the condition (1.18), then

$$\frac{q_0}{p_0} = \frac{q_1}{p_1} \tag{1.19}$$

holds. Lemarié-Rieusset showed this assertion by using the counterexample, given by Ruiz and Vega in [26]. Lemarié-Rieusset also proved that we can choose the second complex interpolation functor, introduced by Calderón [4]. Meanwhile, as for the interpolation result under (1.18) and (1.19) by the first complex interpolation functor, also introduced by Calderón [4], Lu et al. [16, Theorem 1.2] obtained Theorem 1.1 (i). They also extended this result by placing themselves in the setting of a metric measure space. Their technique is to calculate the Calderón product; see [16].

We organize the remaining part of this paper as follows: Sect. 2 collects some fundamental facts on complex interpolation functors. Section 3 is dedicated to Morrey spaces and Sect. 4 generalizes what we obtained to generalized Morrey spaces. Generalized Morrey spaces can be a proper subspace of L^∞. This result forces the result in [11] to be decomposed into two cases. Here we can unify them. In Sect. 5 we consider the function spaces on bounded domains.

2 Preliminaries

2.1 Complex Interpolation Functors

We recall the definition of the complex interpolation functors (see [1,4]). We write $\overline{S} := \{z \in \mathbb{C} : 0 \le \mathrm{Re}(z) \le 1\}$ and let S be its interior.

Definition 2.1 [Calderón's first complex interpolation space] Let (X_0, X_1) be a compatible couple of Banach spaces.

1. The space $\mathcal{F}(X_0, X_1)$ is defined as the set of all functions $F : \overline{S} \to X_0 + X_1$ such that
   (a) F is continuous on $\bar{S}$ and $\sup_{z \in \bar{S}} \|F(z)\|_{X_0+X_1} < \infty$,
   (b) F is holomorphic on S,
   (c) the functions $t \in \mathbb{R} \mapsto F(j + it) \in X_j$ are bounded and continuous on $\mathbb{R}$ for $j = 0, 1$.

   The space $\mathcal{F}(X_0, X_1)$ is equipped with the norm
$$\|F\|_{\mathcal{F}(X_0,X_1)} := \max\left\{\sup_{t\in\mathbb{R}} \|F(it)\|_{X_0}, \sup_{t\in\mathbb{R}} \|F(1+it)\|_{X_1}\right\}.$$
2. Let $\theta \in (0, 1)$. Define the complex interpolation space $[X_0, X_1]_\theta$ with respect to (X_0, X_1) to be the set of all functions $f \in X_0 + X_1$ such that $f = F(\theta)$ for some $F \in \mathcal{F}(X_0, X_1)$. The norm on $[X_0, X_1]_\theta$ is defined by
$$\|f\|_{[X_0,X_1]_\theta} := \inf\left\{\|F\|_{\mathcal{F}(X_0,X_1)} : f = F(\theta) \text{ for some } F \in \mathcal{F}(X_0, X_1)\right\}.$$

Let X be a Banach space. The space $\mathrm{Lip}(\mathbb{R}, X)$ is defined to be the set of all functions $F : \mathbb{R} \to X$ for which the quantity
$$\|F\|_{\mathrm{Lip}(\mathbb{R},X)} := \sup_{-\infty<s<t<\infty} \frac{\|F(t) - F(s)\|_X}{|t - s|} < \infty.$$

Definition 2.2 (Calderón's second complex interpolation space) Let (X_0, X_1) be a compatible couple of Banach spaces.

1. Define $\mathcal{G}(X_0, X_1)$ as the set of all functions $G : \bar{S} \to X_0 + X_1$ such that
   (a) G is continuous on $\bar{S}$ and $\sup_{z\in\bar{S}} \left\|\frac{G(z)}{1+|z|}\right\|_{X_0+X_1} < \infty$,
   (b) G is holomorphic on S,
   (c) the functions
$$t \in \mathbb{R} \mapsto G(j + it) - G(j) \in X_j$$
   are Lipschitz continuous on $\mathbb{R}$ for $j = 0, 1$.

   The space $\mathcal{G}(X_0, X_1)$ is equipped with the norm
$$\|G\|_{\mathcal{G}(X_0,X_1)} := \max\left\{\|G(i\cdot)\|_{\mathrm{Lip}(\mathbb{R},X_0)}, \|G(1 + i\cdot)\|_{\mathrm{Lip}(\mathbb{R},X_1)}\right\}. \tag{2.1}$$
2. Let $\theta \in (0, 1)$. Define the complex interpolation space $[X_0, X_1]^\theta$ with respect to (X_0, X_1) to be the set of all functions $f \in X_0 + X_1$ such that $f = G'(\theta)$ for some $G \in \mathcal{G}(X_0, X_1)$. The norm on $[X_0, X_1]^\theta$ is defined by
$$\|f\|_{[X_0,X_1]^\theta} := \inf\left\{\|G\|_{\mathcal{G}(X_0,X_1)} : f = G'(\theta) \text{ for some } G \in \mathcal{G}(X_0, X_1)\right\}.$$

The key tool used for proving our results is the three-lines lemma for Banach space-valued function which we invoke as follows:

Lemma 2.3 [41, Corollary 2.3] *Let X be a Banach space. Suppose that $F : \overline{S} \to X$ is continuous and bounded and also $F|_S : S \to X$ is holomorphic. Then we have*

$$\sup_{t\in\mathbb{R}} \|F(\theta+it)\|_X \le \left(\sup_{t\in\mathbb{R}} \|F(it)\|_X\right)^{1-\theta} \left(\sup_{t\in\mathbb{R}} \|F(1+it)\|_X\right)^{\theta}$$

for all $\theta \in (0,1)$.

The following lemma can be seen as a tool to relate the first and the second complex interpolations:

Lemma 2.4 *Let (X_0, X_1) be a compatible couple. Suppose that $G \in \mathcal{G}(X_0, X_1)$ and $\theta \in (0,1)$. For $z \in \overline{S}$ and $k \in \mathbb{N}$, define*

$$H_k(z) := \frac{G(z+2^{-k}i) - G(z)}{2^{-k}i}. \tag{2.2}$$

Then $H_k(\theta) \in [X_0, X_1]_\theta$.

Proof Note that, H_k inherits continuity and holomorphicity from G. By Lipschitz-continuity of $t \in \mathbb{R} \mapsto G(it) - G(0) \in X_0$ and $t \in \mathbb{R} \mapsto G(1+it) - G(1) \in X_1$, we have

$$\sup_{t\in\mathbb{R}} \|H_k(it)\|_{X_0+X_1} \le \sup_{t\in\mathbb{R}} \left\| \frac{G((2^{-k}+t)i) - G(it)}{2^{-k}i} \right\|_{X_0} \le \|G\|_{\mathcal{G}(X_0,X_1)}$$

and likewise $\sup_{t\in\mathbb{R}} \|H_k(1+it)\|_{X_0+X_1} \le \|G\|_{\mathcal{G}(X_0,X_1)}$. By Lemma 2.3, we have

$$\|H_k(z)\|_{X_0+X_1} \le \left(\|G\|_{\mathcal{G}(X_0,X_1)}\right)^{1-\mathrm{Re}(z)} \left(\|G\|_{\mathcal{G}(X_0,X_1)}\right)^{\mathrm{Re}(z)} \le \|G\|_{\mathcal{G}(X_0,X_1)}. \tag{2.3}$$

This shows that $H_k(z) \in \mathcal{F}(X_0, X_1)$. Thus, $H_k(\theta) \in [X_0, X_1]_\theta$. □

2.2 Some Elementary Facts on Closed Subspaces

Lemma 2.5 *Let $1 \le q < \infty$ and $\varphi \in \mathcal{G}_q$. Define*

$$A := \left\{ f \in \overline{\mathcal{M}_q^\varphi} : \lim_{a\to 0^+} \|\chi_{\{|f|<a\}} f\|_{\mathcal{M}_q^\varphi} = 0 \right\}. \tag{2.4}$$

Then, A is a closed subset of $\mathcal{M}_q^\varphi$.

Proof Let $\{f_j\}_{j=1}^\infty \subset A$ such that f_j converges to f in $\mathcal{M}_q^\varphi$. Fix $j \in \mathbb{N}$. For every $a > 0$, we have

$$\|\chi_{\{|f|<a\}} f\|_{\mathcal{M}_q^\varphi} \le \|f - f_j\|_{\mathcal{M}_q^\varphi} + \|\chi_{\{|f|<a\}\cap\{|f_j|\ge 2a\}} f_j\|_{\mathcal{M}_q^\varphi} + \|\chi_{\{|f_j|<2a\}} f_j\|_{\mathcal{M}_q^\varphi}.$$

On the set $\{|f| < a\} \cap \{|f_j| \geq 2a\}$, we have

$$|f_j| \leq |f_j - f| + |f| < |f_j - f| + a \leq |f_j - f| + \frac{1}{2}|f_j|,$$

hence $|f_j| \leq 2|f - f_j|$. Consequently,

$$\|\chi_{\{|f|<a\}} f\|_{\mathcal{M}^{\varphi}_q} \leq 3\|f - f_j\|_{\mathcal{M}^{\varphi}_q} + \|\chi_{\{|f_j|<2a\}} f_j\|_{\mathcal{M}^{\varphi}_q}.$$

Since $f_j \in A$, we have

$$\limsup_{a\to 0^+} \|\chi_{\{|f|<a\}} f\|_{\mathcal{M}^{\varphi}_q} \leq 3\|f - f_j\|_{\mathcal{M}^{\varphi}_q}.$$

By taking $j \to \infty$, we have $\lim_{a\to 0^+} \|\chi_{\{|f|<a\}} f\|_{\mathcal{M}^{\varphi}_q} = 0$, hence, $f \in A$. □

We prove the following lemma:

Lemma 2.6 *Let* $1 \leq q < \infty$ *and* $\varphi \in \mathcal{G}_q$. *If* $f \in \overline{\mathcal{M}^{\varphi}_q}$, *then*

$$\lim_{R\to\infty} \|\chi_{\{|f|>R\}} f\|_{\mathcal{M}^{\varphi}_q} = 0. \tag{2.5}$$

Proof Here, we do not assume that $\inf \varphi = 0$. For every $\varepsilon > 0$, choose $g = g_\varepsilon \in L^\infty \cap \mathcal{M}^{\varphi}_q$ such that $\|f - g\|_{\mathcal{M}^{\varphi}_q} < \varepsilon$. Observe that, for each $R > 0$, we have

$$|\chi_{\{|f|>R\}} f| \leq |\chi_{\{|f|>R\}\cap\{|g|\leq R/2\}} f| + |f - g| + |\chi_{\{|g|>R/2\}} g|.$$

On the set $\{|f| > R\} \cap \{|g| \leq R/2\}$, we see that $|f| \leq |f - g| + \frac{R}{2} \leq |f - g| + \frac{|f|}{2}$, so $|\chi_{\{|f|>R\}\cap\{|g|\leq R/2\}} f| \leq 2|f - g|$. Consequently, for $R > 2\|g\|_{L^\infty}$, we have

$$|\chi_{\{|f|>R\}} f| \leq 3|f - g|.$$

Hence, $\|\chi_{\{|f|>R\}} f\|_{\mathcal{M}^{\varphi}_q} \leq 3\|f - g\|_{\mathcal{M}^{\varphi}_q} < 3\varepsilon$. Thus, we have showed that (2.5) holds. □

Under the conditions in Theorem 1.7, we have the following approximation formula:

Lemma 2.7 *Maintain the same conditions as Theorem 1.7. Let* $f \in \mathcal{M}^{\varphi_0}_{q_0} \cap \mathcal{M}^{\varphi_1}_{q_1}$. *Then, we have* $f \in \mathcal{M}^{\varphi}_q$ *and* $f = \lim_{a\to 0^+} \chi_{\{a\leq|f|\leq a^{-1}\}} f$ *in* $\mathcal{M}^{\varphi}_q$.

Proof Without any loss of generality, we may assume $q_1 < q_0$. The proof is immediate from $|f - \chi_{\{a\leq|f|\leq a^{-1}\}} f| \leq a^{\frac{q_0}{q}-1}|f|^{\frac{q_0}{q}} + a^{1-\frac{q_1}{q}}|f|^{\frac{q_1}{q}}$. □

2.3 Example of φ in $\mathcal{M}_q^\varphi$

As we mentioned in the introduction, the case when $\varphi(t) = t^{n/p}$ boils down to $\mathcal{M}_q^p$. However, considering generalized Morrey spaces is not a mere quest to generalization for its own sake. This applies to the point of applications of generalized Morrey spaces and to the context of interpolations. First, we give an example showing that generalized Morrey spaces are useful.

Example 2.8 In this example, we claim that generalized Morrey spaces are useful. In [30, Theorem 5.1] the following estimate is shown:

$$\|(1-\Delta)^{-\frac{n}{p}} f\|_{\mathcal{M}_1^\varphi} \le C\|f\|_{\mathcal{M}_q^p}.$$

when $1 < q \le p < \infty$ and $\varphi(t) = (1+t)^{n/p}/\log(3+t)$ for $t > 0$. We know that log here can not be removed. See [9, Sect. 5] and [23, Proposition 7.3] for more generalizations.

Generalized Morrey spaces seem to reflect the interpolation property as the following two examples show.

Example 2.9 Let $1 \le q < \infty$ and $\varphi_0, \varphi_1 \in \mathcal{G}_q$. Define $\varphi = \varphi_0 + \varphi_1$. Then $\mathcal{M}_q^\varphi = \mathcal{M}_q^{\varphi_0} \cap \mathcal{M}_q^{\varphi_1}$ with norm equivalence.

Example 2.10 As is seen from Sect. 1, it seems that the first and second complex interpolation functors seem to control the modulus of the function. From this point, it is important to pay attention to the following proposition for $1 \le q < \infty$ and $\varphi \in \mathcal{G}_q$. $L^\infty \subset \mathcal{M}_q^\varphi$ holds if and only if $\inf \varphi > 0$. See [21, Proposition 3.3].

3 The Interpolations of Closed Subspaces of Morrey Space

First, we prove the following lemma:

Lemma 3.1 *Suppose that $\theta \in (0,1)$, $1 \le q_0 \le p_0 < \infty$, $1 \le q_1 \le p_1 < \infty$, and $\frac{p_0}{q_0} = \frac{p_1}{q_1}$. Assume $q_0 \ne q_1$. Define*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Let E be a measurable set such that $\chi_E \in U\mathcal{M}_q^p$. Then $\chi_E \in U\mathcal{M}_{q_0}^{p_0} \cap U\mathcal{M}_{q_1}^{p_1}$.

Proof Let $\chi_E \in U\mathcal{M}_q^p$ and $\varepsilon > 0$. Choose $g_\varepsilon \in U \cap \mathcal{M}_q^p$ such that

$$\|\chi_E - g_\varepsilon\|_{\mathcal{M}_q^p} < \varepsilon.$$

Define $h_\varepsilon := \chi_{\{g_\varepsilon \ne 0\} \cap E}$. Then

$$|\chi_E - h_\varepsilon| = \chi_E - h_\varepsilon \le |\chi_E - g_\varepsilon|.$$

Consequently, for $j = 0, 1$, we have

$$\|\chi_E - h_\varepsilon\|_{\mathcal{M}^{p_j}_{q_j}} = \|\chi_E - h_\varepsilon\|^{q/q_j}_{\mathcal{M}^p_q} < \varepsilon^{q/q_j}.$$

This shows that $\chi_E \in U\mathcal{M}^{p_0}_{q_0} \cap U\mathcal{M}^{p_1}_{q_1}$. □

Lemma 3.2 *Keep the assumption in* Lemma 3.1. *Then we have*

$$U \bowtie \mathcal{M}^p_q \subseteq \left[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}\right]^\theta.$$

Proof Without loss of generality, assume that $q_0 > q_1$. Let $f \in U \bowtie \mathcal{M}^p_q$. Since $\chi_{\{a \le |f| \le b\}} \le \frac{1}{a}\chi_{\{a \le |f| \le b\}}|f|$, we have $\chi_{\{a \le |f| \le b\}} \in U\mathcal{M}^p_q$. From Lemma 3.1, we have $\chi_{\{a \le |f| \le b\}} \in U\mathcal{M}^{p_0}_{q_0} \cap U\mathcal{M}^{p_1}_{q_1}$. For $z \in \overline{S}$, define

$$F(z) := \operatorname{sgn}(f)|f|^{\frac{qz}{q_0} + \frac{q(1-z)}{q_1}} \text{ and } G(z) := (z-\theta)\int_0^1 F(\theta + (z-\theta)t)\,dt. \quad (3.1)$$

Decompose $G(z) = G_0(z) + G_1(z)$ where $G_0(z) := \chi_{\{|f| \le 1\}}G(z)$. Let $0 < \varepsilon < 1$. Since $\chi_{\{\varepsilon \le |f| \le 1\}} \in U\mathcal{M}^{p_0}_{q_0}$ and

$$\chi_{\{\varepsilon \le |f| \le 1\}}|G_0(z)| \le (1+|z|)(|f|^{q/q_0} + |f|^{q/q_1})\chi_{\{\varepsilon \le |f| \le 1\}} \le 2(1+|z|)\chi_{\{\varepsilon \le |f| \le 1\}}, \quad (3.2)$$

we have $\chi_{\{\varepsilon \le |f| \le 1\}}G_0(z) \in U\mathcal{M}^{p_0}_{q_0}$. Observe that

$$\begin{aligned}\|G_0(z) - \chi_{\{\varepsilon \le |f| \le 1\}}G_0(z)\|_{\mathcal{M}^{p_0}_{q_0}} &= \left\|\chi_{\{|f|<\varepsilon\}}\frac{F(z) - F(\theta)}{\left(\frac{q}{q_1} - \frac{q}{q_0}\right)\log|f|}\right\|_{\mathcal{M}^{p_0}_{q_0}} \\ &\le \left\|\frac{2|f|^{q/q_0}}{\left(\frac{q}{q_1} - \frac{q}{q_0}\right)\log(\varepsilon^{-1})}\right\|_{\mathcal{M}^{p_0}_{q_0}} \\ &\le \frac{2\|f\|^{q/q_0}_{\mathcal{M}^p_q}}{\left(\frac{q}{q_1} - \frac{q}{q_0}\right)\log\varepsilon^{-1}} \to 0 \qquad (3.3)\end{aligned}$$

as $\varepsilon \to 0^+$. Hence $G_0(z) \in U\mathcal{M}^{p_0}_{q_0}$. Similarly, $G_1(z) \in U\mathcal{M}^{p_1}_{q_1}$. Thus $G(z) \in U\mathcal{M}^{p_0}_{q_0} + U\mathcal{M}^{p_1}_{q_1}$. Let $t \in \mathbb{R}$ and $R > 1$. Since $\chi_{\{R^{-1} \le |f| \le R\}} \in U\mathcal{M}^{p_0}_{q_0}$ and

$$|(G(it) - G(0))|\chi_{\{R^{-1} \le |f| \le R\}} \le (2+|t|)(R^{q/q_0} + R^{q/q_1})\chi_{\{R^{-1} \le |f| \le R\}}, \quad (3.4)$$

we have $[G(it) - G(0)]\chi_{\{R^{-1} \le |f| \le R\}} \in U\mathcal{M}^{p_0}_{q_0}$. Note that

$$\|[G(it) - G(0)]\chi_{\mathbb{R}^n \setminus \{R^{-1} \le |f| \le R\}}\|_{\mathcal{M}^{p_0}_{q_0}} \le \frac{2\|f\|^{q/q_0}_{\mathcal{M}^p_q}}{\left(\frac{q}{q_1} - \frac{q}{q_0}\right)\log R} \to 0 \tag{3.5}$$

as $R \to \infty$. Thus $G(it) - G(0) \in U\mathcal{M}^{p_0}_{q_0}$. Similarly, $G(1+it) - G(1) \in U\mathcal{M}^{p_1}_{q_1}$. Since $G \in \mathcal{G}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$, we have $G \in \mathcal{G}(U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1})$. From $f = G'(\theta)$, it follows that $f \in [U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]^\theta$. □

Lemma 3.3 *Let $G \in \mathcal{G}(U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1})$ and $\theta \in (0, 1)$. For $z \in \overline{S}$ and $k \in \mathbb{N}$, define $H_k(z)$ by (2.2). Then we have $H_k(\theta) \in \overline{U\mathcal{M}^{p_0}_{q_0} \cap U\mathcal{M}^{p_1}_{q_1}}^{\mathcal{M}^p_q}$.*

Proof Let $\varepsilon > 0$. By Lemma 2.4, we have $H_k(\theta) \in [U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]_\theta$. Since $U\mathcal{M}^{p_0}_{q_0} \cap U\mathcal{M}^{p_1}_{q_1}$ is dense in $[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]_\theta$, we can find $J_k(\theta) \in U\mathcal{M}^{p_0}_{q_0} \cap U\mathcal{M}^{p_1}_{q_1}$ such that

$$\|H_k(\theta) - J_k(\theta)\|_{\left[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}\right]_\theta} < \varepsilon.$$

Since $[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]_\theta \subseteq [\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta \subseteq \mathcal{M}^p_q$, we have

$$\|H_k(\theta) - J_k(\theta)\|_{\mathcal{M}^p_q} \lesssim \|H_k(\theta) - J_k(\theta)\|_{[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]_\theta} < \varepsilon.$$

This shows that $H_k(\theta) \in \overline{U\mathcal{M}^{p_0}_{q_0} \cap U\mathcal{M}^{p_1}_{q_1}}^{\mathcal{M}^p_q}$. □

Lemma 3.4 *Assume the same conditions on the paramaters as in Lemma 3.1. Then $U\mathcal{M}^{p_0}_{q_0} \cap U\mathcal{M}^{p_1}_{q_1} \subseteq U\mathcal{M}^p_q$.*

Proof Without loss of generality assume that $q_0 > q_1$. Let $f \in U\mathcal{M}^{p_0}_{q_0} \cap U\mathcal{M}^{p_1}_{q_1}$. In view of Lemma 2.7, we may assume $f = \chi_{\{a \le |f| \le a^{-1}\}} f$ for some $a > 0$. By the lattice property of the spaces $U\mathcal{M}^{p_0}_{q_0}$, $U\mathcal{M}^{p_1}_{q_1}$, and $U\mathcal{M}^p_q$, we may assume $f = \chi_E$ for some measurable set E. Choose a sequence $\{g_j\}_{j=1}^\infty \subseteq U \cap \mathcal{M}^{p_1}_{q_1}$ such that

$$\lim_{j \to \infty} \|f - g_j\|_{\mathcal{M}^{p_1}_{q_1}} = 0.$$

Define $F_j = \{g_j \ne 0\} \cap E$. Hence $|f - \chi_{F_j}| \le 2$ and $|f - \chi_{F_j}| \le |f - g_j|$. Consequently,

$$\|f - \chi_{F_j}\|_{\mathcal{M}^p_q} = \left\| |f - \chi_{F_j}|^{1 - \frac{q_1}{q}} |f - \chi_{F_j}|^{\frac{q_1}{q}} \right\|_{\mathcal{M}^p_q} \le 2^{1 - \frac{q_1}{q}} \|f - g_j\|^{\frac{q_1}{q}}_{\mathcal{M}^{p_1}_{q_1}}.$$

This shows that $f \in U\mathcal{M}^p_q$. □

Lemma 3.5 *Under the assumption of* Lemma 3.1,

$$\mathcal{M}^p_q \cap \overline{U\mathcal{M}^p_q}^{\mathcal{M}^{p_0}_{q_0} + \mathcal{M}^{p_1}_{q_1}} \subseteq U \bowtie \mathcal{M}^p_q.$$

Proof Let $f \in \mathcal{M}^p_q \cap \overline{U\mathcal{M}^p_q}^{\mathcal{M}^{p_0}_{q_0}+\mathcal{M}^{p_1}_{q_1}}$. We may assume that $0 < a < 1 < b < \infty$ for the purpose of showing $\chi_{\{a \le |f| \le b\}} f \in U\mathcal{M}^p_q$ for all $0 < a < b < \infty$.

Choose $\{f_j\}_{j=1}^\infty \subseteq U\mathcal{M}^p_q$ such that

$$\lim_{j\to\infty} \|f - f_j\|_{\mathcal{M}^{p_0}_{q_0}+\mathcal{M}^{p_1}_{q_1}} = 0.$$

Let $\Theta(t)$ be a piecewise linear function such that $\Theta(1) = 1$ and that

$$\Theta'(t) := \frac{2}{a}\chi_{(a/2,a)}(t) - \frac{1}{b}\chi_{(b,2b)}(t) \tag{3.6}$$

except at $t = \frac{a}{2}, a, b, 2b$. According to [11, Lemma 3.3], we have

$$\lim_{j\to\infty} \|\chi_{\{a\le|f|\le b\}}\Theta(|f_j|) - \chi_{\{a\le|f|\le b\}}\Theta(|f|)\|_{\mathcal{M}^p_q} = 0.$$

Since $\chi_{\{a\le|f|\le b\}}\Theta(|f_j|) \le a^{-1}|f_j|$, we have $\chi_{\{a\le|f|\le b\}}\Theta(|f|) \in U\mathcal{M}^p_q$. From the inequality $\chi_{\{a\le|f|\le b\}}|f| \le b\chi_{\{a\le|f|\le b\}}\Theta(|f|)$, it follows that $\chi_{\{a\le|f|\le b\}} f \in U\mathcal{M}^p_q$. □

Now, the proof of (1.6) is given as follows:

Proof of (1.6) In view of Lemma 3.2, we only need to show $[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]^\theta \subseteq U \bowtie \mathcal{M}^p_q$. Let $f \in [U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]^\theta$. Then there exists $G \in \mathcal{G}(U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1})$ such that $G'(\theta) = f$. Define $H_k(z)$ by (2.2) for $z \in \overline{S}$ and $k \in \mathbb{N}$. By Lemmas 3.3 and 3.4, we have $H_k(\theta) \in U\mathcal{M}^p_q$. Since $H_k(\theta)$ converges to $G'(\theta) = f$ in $\mathcal{M}^{p_0}_{q_0} + \mathcal{M}^{p_1}_{q_1}$, it follows from Lemma 3.5 that $f \in U \bowtie \mathcal{M}^p_q$. □

4 The Interpolations of Closed Subspaces of Generalized Morrey Spaces

We remark that the inclusion $U\mathcal{M}^{\varphi_0}_{q_0} \cap U\mathcal{M}^{\varphi_1}_{q_1} \subseteq U\mathcal{M}^{\varphi}_q$ is the important part for the first and second complex interpolations of closed subspaces of Morrey spaces.

Lemma 4.1 *Suppose that $\theta \in (0, 1)$, $1 \le q_0 < \infty$, $1 \le q_1 < \infty$, and $\varphi_0^{q_0} = \varphi_1^{q_1}$. Define*

$$\varphi := \varphi_0^{1-\theta}\varphi_1^\theta \quad \text{and} \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then $U\mathcal{M}^{\varphi_0}_{q_0} \cap U\mathcal{M}^{\varphi_1}_{q_1} \subseteq U\mathcal{M}^{\varphi}_q$.

Proof The proof is similar to the proof of Lemma 3.4. □

We prove the generalization of Lemma 3.1 as follows:

Lemma 4.2 *Keep the same assumption as in* Lemma 4.1. *Let E be a measurable set such that $\chi_E \in U\mathcal{M}^{\varphi}_q$. Then we have*

$$\chi_E \in U\mathcal{M}^{\varphi_0}_{q_0} \cap U\mathcal{M}^{\varphi_1}_{q_1}.$$

Proof Let $\chi_E \in U\mathcal{M}^{\varphi}_q$ and choose $\{g_k\}_{k=1}^{\infty} \subseteq U \cap \mathcal{M}^{\varphi}_q$ for which

$$\lim_{k\to\infty} \|\chi_E - g_k\|_{\mathcal{M}^{\varphi}_q} = 0.$$

Define $h_k := \chi_{\{g_k \neq 0\}\cap E}$. Then, for each $k = 0, 1$, we have

$$\|\chi_E - h_k\|_{\mathcal{M}^{\varphi_j}_{q_j}} = \|\chi_E - h_k\|^{q/q_j}_{\mathcal{M}^{\varphi}_q} \leq \|\chi_E - g_k\|^{q/q_j}_{\mathcal{M}^{\varphi}_q} \to 0$$

as $k \to \infty$. Thus, $\chi_E \in U\mathcal{M}^{\varphi_0}_{q_0} \cap U\mathcal{M}^{\varphi_1}_{q_1}$. □

4.1 The First Complex Interpolation Method

We prove Theorem 1.7, which includes Theorem 1.2 as a special case.

Proof Without loss of generality, assume that $q_0 > q_1$. Define A by (2.4). Suppose that $f \in \overline{\mathcal{M}^{\varphi}_q}$ satisfies

$$\lim_{a\to 0^+} \|\chi_{\{|f|<a\}} f\|_{\mathcal{M}^{\varphi}_q} = 0. \tag{4.1}$$

Note that, for every $a > 0$, we have

$$\begin{aligned}\|\chi_{\{a\leq|f|\}} f\|_{\mathcal{M}^{\varphi_1}_{q_1}} &\leq \sup_{B=B(x_0,r)} \left[\frac{\varphi(r)}{|B|^{1/q}} \left(\int_{B\cap\{a\leq|f|\}} a^{q_1-q}|f(x)|^q\,dx\right)^{\frac{1}{q}}\right]^{\frac{q}{q_1}} \\ &\leq a^{\frac{q_1-q}{q_1}} \|f\|^{\frac{q}{q_1}}_{\mathcal{M}^{\varphi}_q} < \infty.\end{aligned} \tag{4.2}$$

Given $\varepsilon > 0$, choose $g_\varepsilon \in L^\infty \cap \mathcal{M}^{\varphi}_q$ such that

$$\|f - g_\varepsilon\|_{\mathcal{M}^{\varphi}_q} < \varepsilon. \tag{4.3}$$

Since $g_\varepsilon \in L^\infty$, we have

$$\|g_\varepsilon\|_{\mathcal{M}^{\varphi_0}_{q_0}} \leq \|g_\varepsilon\|^{\frac{q_0-q}{q_0}}_{L^\infty} \|g_\varepsilon\|^{\frac{q}{q_0}}_{\mathcal{M}^{\varphi}_q} < \infty. \tag{4.4}$$

Define

$$g_{\varepsilon,a} := \begin{cases} \chi_{\{|f|\geq a\}} f, & |g_\varepsilon| > |f|, \\ \chi_{\{|f|\geq a\}} g_\varepsilon, & |g_\varepsilon| \leq |f|. \end{cases} \tag{4.5}$$

Since $|g_{\varepsilon,a}| \leq |g_\varepsilon|$ and $|g_{\varepsilon,a}| \leq \chi_{\{|f|\geq a\}}|f|$, by (4.2) and (4.4), it follows that $g_{\varepsilon,a} \in \mathcal{M}^{\varphi_0}_{q_0} \cap \mathcal{M}^{\varphi_1}_{q_1}$. Using the following inequality:

$$|f - g_{\varepsilon,a}| \leq |f - \chi_{\{a\leq|f|\}} f| + |f - g_\varepsilon|, \tag{4.6}$$

we have

$$\lim_{\varepsilon,a\to 0^+} \|f - g_{\varepsilon,a}\|_{\mathcal{M}^\varphi_q} = 0. \tag{4.7}$$

This shows that $f \in \overline{\mathcal{M}^{\varphi_0}_{q_0} \cap \mathcal{M}^{\varphi_1}_{q_1}}^{\mathcal{M}^\varphi_q} = [\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}]_\theta$.

Conversely, let $g \in \mathcal{M}^{\varphi_0}_{q_0} \cap \mathcal{M}^{\varphi_1}_{q_1}$. Then $g \in \overline{\mathcal{M}^\varphi_q}$ thanks to Lemma 2.7. Since

$$\|\chi_{\{|g|<a\}} g\|_{\mathcal{M}^\varphi_q} \leq a^{\frac{q-q_1}{q}} \|g\|_{\mathcal{M}^{\varphi_1}_{q_1}} \to 0 \tag{4.8}$$

as $a \to 0^+$, we conclude that

$$\mathcal{M}^{\varphi_0}_{q_0} \cap \mathcal{M}^{\varphi_1}_{q_1} \subseteq \left\{ f \in \overline{\mathcal{M}^\varphi_q} : \lim_{a\to 0^+} \|\chi_{\{|f|<a\}} f\|_{\mathcal{M}^\varphi_q} = 0 \right\}.$$

Using [11, Theorem 4.5], we get

$$\left[\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}\right]_\theta = \overline{\mathcal{M}^{\varphi_0}_{q_0} \cap \mathcal{M}^{\varphi_1}_{q_1}}^{\mathcal{M}^\varphi_q} \subseteq \left\{ f \in \overline{\mathcal{M}^\varphi_q} : \lim_{a\to 0^+} \|\chi_{\{|f|<a\}} f\|_{\mathcal{M}^\varphi_q} = 0 \right\}$$

as desired. □

Next, we prove (1.13).

Proof of (1.13) Without loss of generality, assume that $q_0 > q_1$. By Theorem 1.7, we have

$$\left[U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}\right]_\theta \subseteq \left[\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}\right]_\theta = \left\{ f \in \overline{\mathcal{M}^\varphi_q} : \lim_{a\to 0^+} \|\chi_{\{|f|<a\}} f\|_{\mathcal{M}^\varphi_q} = 0 \right\}.$$

Let $g \in [U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}]_\theta$ and $\varepsilon > 0$. Choose $g_\varepsilon \in U\mathcal{M}^{\varphi_0}_{q_0} \cap U\mathcal{M}^{\varphi_1}_{q_1}$ such that

$$\|g - g_\varepsilon\|_{[U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}]_\theta} < \varepsilon. \tag{4.9}$$

Since $U\mathcal{M}_{q_0}^{\varphi_0} \cap U\mathcal{M}_{q_1}^{\varphi_1} \subseteq U\mathcal{M}_q^{\varphi}$, we have $g_\varepsilon \in U\mathcal{M}_q^{\varphi}$. From $[U\mathcal{M}_{q_0}^{\varphi_0}, U\mathcal{M}_{q_1}^{\varphi_1}]_\theta \subseteq [\mathcal{M}_{q_0}^{\varphi_0}, \mathcal{M}_{q_1}^{\varphi_1}]_\theta \subseteq \mathcal{M}_q^{\varphi}$, it follows that

$$\|g - g_\varepsilon\|_{\mathcal{M}_q^{\varphi}} \lesssim \varepsilon, \tag{4.10}$$

and hence, $g \in U\mathcal{M}_q^{\varphi}$.

Conversely, let $f \in \overline{\mathcal{M}_q^{\varphi}} \cap U\mathcal{M}_q^{\varphi}$ such that

$$\lim_{a \to 0^+} \|\chi_{\{|f|<a\}} f\|_{\mathcal{M}_q^{\varphi}} = 0.$$

From Lemma 2.6, we have

$$\lim_{a \to 0^+} \|\chi_{\{|f|>a^{-1}\}} f\|_{\mathcal{M}_q^{\varphi}} = 0.$$

Thus,

$$\|f - \chi_{\{a \le |f| \le a^{-1}\}} f\|_{\mathcal{M}_q^{\varphi}} \le \|\chi_{\{|f|<a\}} f\|_{\mathcal{M}_q^{\varphi}} + \|\chi_{\{|f|>a^{-1}\}} f\|_{\mathcal{M}_q^{\varphi}} \to 0$$

as $a \to 0^+$. Observe also that $\chi_{\{a \le |f| \le a^{-1}\}} f \in \overline{\mathcal{M}_q^{\varphi}} \cap U\mathcal{M}_q^{\varphi}$ thanks to the lattice property of U. As a result, we may assume that $f = \chi_{\{a \le |f| \le a^{-1}\}} f$ for some $0 < a < 1$. For every $z \in \overline{S}$, define

$$F(z) := \operatorname{sgn}(f)|f|^{q\left(\frac{1-z}{q_0} + \frac{z}{q_1}\right)}.$$

Decompose $F(z)$ as $F_0(z) := F(z)\chi_{\{|f| \le 1\}}$ and $F_1(z) := F(z)\chi_{\{|f|>1\}}$. Note that, for any $0 < b < c < \infty$, we have a pointwise estimate:

$$\chi_{\{b \le |f| \le c\}} \le \frac{1}{b} \chi_{\{b \le |f| \le c\}} |f| \le \frac{|f|}{b}, \tag{4.11}$$

so $\chi_{\{b \le |f| \le c\}} \in U\mathcal{M}_q^{\varphi}$. From Lemma 4.2, it follows that $\chi_{\{b \le |f| \le c\}} \in U\mathcal{M}_{q_0}^{\varphi_0} \cap U\mathcal{M}_{q_1}^{\varphi_1}$. Since

$$|F_0(z)| \le \chi_{\{a \le |f| \le 1\}} \quad \text{and} \quad |F_1(z)| \le \left(a^{-\frac{q}{q_0}} + a^{-\frac{q}{q_1}}\right) \chi_{\{1 \le |f| \le a^{-1}\}},$$

we have $F(z) = F_0(z) + F_1(z) \in U\mathcal{M}_{q_0}^{\varphi_0} + U\mathcal{M}_{q_1}^{\varphi_1}$. Moreover, we also have

$$\begin{aligned} \sup_{z \in \overline{S}} \|F(z)\|_{U\mathcal{M}_{q_0}^{\varphi_0} + U\mathcal{M}_{q_1}^{\varphi_1}} &\le \|\chi_{\{a \le |f| \le 1\}}\|_{U\mathcal{M}_{q_0}^{\varphi_0}} \\ &\quad + \left(a^{-\frac{q}{q_0}} + a^{-\frac{q}{q_1}}\right) \|\chi_{\{1 \le |f| \le a^{-1}\}}\|_{U\mathcal{M}_{q_1}^{\varphi_1}}. \end{aligned}$$

Next, we shall check that $F|_S : S \to U\mathcal{M}^{\varphi_0}_{q_0} + U\mathcal{M}^{\varphi_1}_{q_1}$ is a holomorphic function. For every $z \in S$, set $H(z) := \left(\frac{q}{q_1} - \frac{q}{q_0}\right)(\log|f|)F(z)$. Then $H(z) \in U\mathcal{M}^{\varphi_0}_{q_0} + U\mathcal{M}^{\varphi_1}_{q_1}$ with

$$\|H(z)\|_{U\mathcal{M}^{\varphi_0}_{q_0}+U\mathcal{M}^{\varphi_1}_{q_1}} \le \left(\frac{q}{q_1} - \frac{q}{q_0}\right)(\log a^{-1})$$
$$\left(\|\chi_{\{a \le |f| \le 1\}}\|_{U\mathcal{M}^{\varphi_0}_{q_0}} + (a^{-q/q_0} + a^{-q/q_1})\|\chi_{\{1 \le |f| \le a^{-1}\}}\|_{U\mathcal{M}^{\varphi_1}_{q_1}}\right).$$

For each $0 < \varepsilon \ll 1$, define $S_\varepsilon := \{z \in S : \varepsilon < \mathrm{Re}(z) < 1 - \varepsilon\}$. Let $z \in S_\varepsilon$ be fixed and let $w \in S_\varepsilon$ be such that $z + w \in S_\varepsilon$. As a consequence of the following inequalities

$$\begin{aligned}\left|\frac{F(z+w) - F(z)}{w} - H(z)\right| &= \left|\frac{|f|^{w\left(\frac{q}{q_1} - \frac{q}{q_0}\right)} - 1 - w\left(\frac{q}{q_1} - \frac{q}{q_0}\right)\log|f|}{w}\right| |F(z)| \\ &\le \left|\left(\frac{q}{q_1} - \frac{q}{q_0}\right)\log|f|\right| \left(\sum_{k=2}^{\infty} \frac{|w \log|f||^{k-1}}{k!}\right)|F(z)| \\ &\le \left(\frac{q}{q_1} - \frac{q}{q_0}\right)\log(a^{-1})\left(e^{|w|\log(a^{-1})} - 1\right)|F(z)|\end{aligned}$$

and $\|F(z)\|_{U\mathcal{M}^{\varphi_0}_{q_0}+U\mathcal{M}^{\varphi_1}_{q_1}} < \infty$, we have

$$\begin{aligned}&\left\|\frac{F(z+w) - F(z)}{w} - H(z)\right\|_{U\mathcal{M}^{\varphi_0}_{q_0}+U\mathcal{M}^{\varphi_1}_{q_1}} \\ &\lesssim \left(e^{|w|\log(a^{-1})} - 1\right)\|F(z)\|_{U\mathcal{M}^{\varphi_0}_{q_0}+U\mathcal{M}^{\varphi_1}_{q_1}} \to 0\end{aligned}$$

as $w \to 0$. Hence, $F : S_\varepsilon \to U\mathcal{M}^{\varphi_0}_{q_0} + U\mathcal{M}^{\varphi_1}_{q_1}$ is holomorphic. Since $\varepsilon > 0$ is arbitrary, we conclude that $F : S \to U\mathcal{M}^{\varphi_0}_{q_0} + U\mathcal{M}^{\varphi_1}_{q_1}$ is holomorphic.
Observe that for every $w \in S$, we have

$$|F'(w)| \le \left(\frac{q}{q_1} - \frac{q}{q_0}\right)\max\left(a^{-\frac{q}{q_0}}, a^{-\frac{q}{q_1}}\right)\log\frac{1}{a} \times \chi_{\{a \le |f| \le a^{-1}\}}. \tag{4.12}$$

Then we have

$$
\begin{aligned}
&\|F(z)-F(z')\|_{U\mathcal{M}^{\varphi_0}_{q_0}+U\mathcal{M}^{\varphi_1}_{q_1}}\\
&=\left\|\int_{z'}^{z} F'(w)\,dw\right\|_{U\mathcal{M}^{\varphi_0}_{q_0}+U\mathcal{M}^{\varphi_1}_{q_1}}\\
&\le \max\left(\frac{q}{q_0},\frac{q}{q_1}\right)\max\left(a^{-\frac{q}{q_0}},a^{-\frac{q}{q_1}}\right)\log\frac{1}{a}\\
&\quad\times\left(\|\chi_{\{a\le|f|\le 1\}}+\chi_{\{1<|f|\le a^{-1}\}}\|_{U\mathcal{M}^{\varphi_0}_{q_0}+U\mathcal{M}^{\varphi_1}_{q_1}}\right)|z-z'|\\
&\le \max\left(\frac{q}{q_0},\frac{q}{q_1}\right)\max\left(a^{-\frac{q}{q_0}},a^{-\frac{q}{q_1}}\right)\log\frac{1}{a}\\
&\quad\times\left(\|\chi_{\{a\le|f|\le 1\}}\|_{U\mathcal{M}^{\varphi_0}_{q_0}}+\|\chi_{\{1<|f|\le a^{-1}\}}\|_{U\mathcal{M}^{\varphi_1}_{q_1}}\right)|z-z'|
\end{aligned}
$$

for all $z,z'\in\overline{S}$. Thus, $F:\overline{S}\to U\mathcal{M}^{\varphi_0}_{q_0}+U\mathcal{M}^{\varphi_1}_{q_1}$ is a continuous function.

Note that, for all $t\in\mathbb{R}$ and $j=0,1$, we have

$$
|F(j+it)|=|f|^{\frac{q}{q_j}}\le a^{-\frac{q}{q_j}}\chi_{\{a\le|f|\le a^{-1}\}},
$$

so, $F(j+it)\in U\mathcal{M}^{\varphi_j}_{q_j}$. Furthermore, using (4.12), we get

$$
\begin{aligned}
\|F(j+it)-F(j+it')\|_{U\mathcal{M}^{\varphi_j}_{q_j}}&=\left\|\int_{j+it'}^{j+it} F'(w)dw\right\|_{U\mathcal{M}^{\varphi_j}_{q_j}}\\
&\le\left(\frac{q}{q_1}-\frac{q}{q_0}\right)\max(a^{-\frac{q}{q_0}},a^{-\frac{q}{q_1}})\log\frac{1}{a}\\
&\quad\times\|\chi_{\{a\le|f|\le a^{-1}\}}\|_{U\mathcal{M}^{\varphi_j}_{q_j}}|t-t'|
\end{aligned}
$$

for all $t,t'\in\mathbb{R}$. This shows that $t\in\mathbb{R}\mapsto F(j+it)\in U\mathcal{M}^{\varphi_j}_{q_j}$, $j=0,1$ are continuous functions. In total, we have showed that $F\in\mathcal{F}(U\mathcal{M}^{\varphi_0}_{q_0},U\mathcal{M}^{\varphi_1}_{q_1})$. Since $F(\theta)=f$, we have $f\in[U\mathcal{M}^{\varphi_0}_{q_0},U\mathcal{M}^{\varphi_1}_{q_1}]_\theta$ as desired. □

4.2 The Second Complex Interpolation Method

From now on, we shall always use the assumption of Theorem 1.9. To prove Theorem 1.9, we shall invoke and prove several lemmas:

Lemma 4.3 *Keep the assumption in Theorem 1.9. Then we have*

$$
U\bowtie\mathcal{M}^{\varphi}_{q}\subseteq\left[U\mathcal{M}^{\varphi_0}_{q_0},U\mathcal{M}^{\varphi_1}_{q_1}\right]^{\theta}. \tag{4.13}
$$

Proof Assume that $q_0 > q_1$. We go through a similar argument as in the proof of Lemma 3.2 to obtain (4.13). □

Lemma 4.4 *Let* $G \in \mathcal{G}(\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1})$ *and* $\theta \in (0,1)$. *For* $z \in \overline{S}$ *and* $k \in \mathbb{N}$, *define* $H_k(z)$ *by* (2.2). *Then* $H_k(\theta) \in \overline{U\mathcal{M}^{\varphi_0}_{q_0} \cap U\mathcal{M}^{\varphi_1}_{q_1}}^{\mathcal{M}^{\varphi}_q}$.

Proof From Lemma 2.4, it follows that $H_k(\theta) \in [U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}]_\theta$. Let $\varepsilon > 0$. Since $U\mathcal{M}^{\varphi_0}_{q_0} \cap U\mathcal{M}^{\varphi_1}_{q_1}$ is dense in $[U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}]_\theta$, we can find $J_k(\theta) \in U\mathcal{M}^{\varphi_0}_{q_0} \cap U\mathcal{M}^{\varphi_1}_{q_1}$ such that

$$\|H_k(\theta) - J_k(\theta)\|_{[U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}]_\theta} < \varepsilon.$$

Since $[U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}]_\theta \subseteq [\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}]_\theta \subseteq \mathcal{M}^{\varphi}_q$, we have

$$\|H_k(\theta) - J_k(\theta)\|_{\mathcal{M}^{\varphi}_q} \lesssim \|H_k(\theta) - J_k(\theta)\|_{[U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}]_\theta} < \varepsilon.$$

This shows that $H_k(\theta) \in \overline{U\mathcal{M}^{\varphi_0}_{q_0} \cap U\mathcal{M}^{\varphi_1}_{q_1}}^{\mathcal{M}^{\varphi}_q}$. □

Lemma 4.5 *We use the assumption of* Theorem 4.2. *Then we have*

$$\mathcal{M}^{\varphi}_q \cap \overline{U\mathcal{M}^{\varphi}_q}^{\mathcal{M}^{\varphi_0}_{q_0} + \mathcal{M}^{\varphi_1}_{q_1}} \subseteq U \bowtie \mathcal{M}^{\varphi}_q.$$

Proof Let $f \in \mathcal{M}^{\varphi}_q \cap \overline{U\mathcal{M}^{\varphi}_q}^{\mathcal{M}^{\varphi_0}_{q_0} + \mathcal{M}^{\varphi_1}_{q_1}}$. Assume $0 < a < 1 < b < \infty$ as before. Choose $\{f_j\}_{j=1}^\infty \subseteq U\mathcal{M}^{\varphi}_q$ such that

$$\lim_{j\to\infty} \|f - f_j\|_{\mathcal{M}^{\varphi_0}_{q_0} + \mathcal{M}^{\varphi_1}_{q_1}} = 0.$$

Let $\Theta(t)$ be a function defined by (3.6). By a similar argument as in the proof of [11, Lemma 3.3], we have

$$\lim_{j\to\infty} \|\chi_{\{a\le|f|\le b\}}\Theta(|f_j|) - \chi_{\{a\le|f|\le b\}}\Theta(|f|)\|_{\mathcal{M}^{\varphi}_q} = 0.$$

Since $\chi_{\{a\le|f|\le b\}}\Theta(|f_j|) \le a^{-1}|f_j|$, we have $\chi_{\{a\le|f|\le b\}}\Theta(|f|) \in U\mathcal{M}^{\varphi}_q$. From the inequality $\chi_{\{a\le|f|\le b\}}|f| \le b\chi_{\{a\le|f|\le b\}}\Theta(|f|)$, it follows that $\chi_{\{a\le|f|\le b\}} f \in U\mathcal{M}^{\varphi}_q$. □

Now, we are ready to prove Theorem 1.9.

Proof of (1.15) In view of Lemma 4.3, we only need to show that

$$\left[U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}\right]^\theta \subseteq U \bowtie \mathcal{M}^{\varphi}_q.$$

Let $f \in [U\mathcal{M}_{q_0}^{\varphi_0}, U\mathcal{M}_{q_1}^{\varphi_1}]^\theta$. Then there exists $G \in \mathcal{G}(U\mathcal{M}_{q_0}^{\varphi_0}, U\mathcal{M}_{q_1}^{\varphi_1})$ such that $G'(\theta) = f$. For $z \in \overline{S}$ and $k \in \mathbb{N}$, define $H_k(z)$ by (2.2). By Lemmas 4.1 and 4.4, we have $H_k(\theta) \in U\mathcal{M}_q^\varphi$. Since $H_k(\theta)$ converges to $G'(\theta) = f$ in $\mathcal{M}_{q_0}^{\varphi_0} + \mathcal{M}_{q_1}^{\varphi_1}$, by Lemma 4.5, it follows that $f \in U \bowtie \mathcal{M}_q^\varphi$. □

We compare Theorem 1.9 with our previous result.

Remark 4.6 Assume that $\inf \varphi > 0$. According to [11, Theorem 5.12],

$$\left[\widetilde{\mathcal{M}_{q_0}^{\varphi_0}}, \widetilde{\mathcal{M}_{q_1}^{\varphi_1}}\right]^\theta = \bigcap_{0<a<b<\infty} \left\{ f \in \mathcal{M}_q^\varphi \cap \widetilde{L^\infty} \ : \ \chi_{\{a \le |f| \le b\}} f \in \widetilde{\mathcal{M}_q^\varphi} \right\}. \tag{4.14}$$

Meanwhile, in the light of Theorem 1.9, we have

$$[\widetilde{\mathcal{M}_{q_0}^{\varphi_0}}, \widetilde{\mathcal{M}_{q_1}^{\varphi_1}}]^\theta = \bigcap_{0<a<b<\infty} \left\{ f \in \mathcal{M}_q^\varphi \ : \ \chi_{\{a \le |f| \le b\}} f \in \widetilde{\mathcal{M}_q^\varphi} \right\}. \tag{4.15}$$

Thus, the sets in the right-hand side of (4.14) and (4.15) coincide. In fact, this can be verified directly from the fact that $\mathcal{M}_q^\varphi \subset L^\infty$ (see [11, Theorem 5.9]).

5 The Closure of Compactly Supported Functions in Morrey Spaces on Bounded Connected Open Sets

We recall that we do not require that the domain Ω is smooth. In view of Theorem 5.1 below and the fact that $\mathcal{M}_q^\varphi \supset L^\infty$ if and only if $\inf \varphi > 0$; see [21, Proposition 3.3], we shall concentrate on the case $\inf \varphi = 0$.

Lemma 5.1 *Let $1 \le q < \infty$, $\varphi \in \mathcal{G}_q$, and Ω be bounded. Then we have $L^\infty(\Omega) \subseteq \mathcal{M}_q^\varphi(\Omega)$. In particular, when* $\inf \varphi > 0$, *we have $\mathcal{M}_q^\varphi(\Omega) = L^\infty(\Omega)$.*

Proof Let $f \in L^\infty(\Omega)$. For $x \in \Omega$ and $0 < r < \operatorname{diam}(\Omega)$, we have

$$\varphi(r) \left(\frac{1}{|B(x,r)|} \int_{B(x,r)\cap\Omega} |f(y)|^q \, dy \right)^{\frac{1}{q}} \le \varphi(\operatorname{diam}(\Omega)) \|f\|_{L^\infty(\Omega)}. \tag{5.1}$$

Consequently, $f \in \mathcal{M}_q^\varphi(\Omega)$ with $\|f\|_{\mathcal{M}_q^\varphi(\Omega)} \le \varphi(\operatorname{diam}(\Omega)) \|f\|_{L^\infty(\Omega)}$. This shows that $L^\infty(\Omega) \subseteq \mathcal{M}_q^\varphi(\Omega)$. When $\inf \varphi > 0$, we combine $L^\infty(\Omega) \subseteq \mathcal{M}_q^\varphi(\Omega)$ with [21, Proposition 3.3] to obtain $\mathcal{M}_q^\varphi(\Omega) = L^\infty(\Omega)$. □

We shall prove Theorem 1.11. Our proof will use the identification of $\overset{\circ}{\mathcal{M}}{}_q^\varphi(\Omega)$ as the vanishing generalized Morrey spaces. The definition of these spaces is given as follows (see also [7,10,25]).

Definition 5.2 Let $1 \le q < \infty$, $\varphi \in \mathcal{G}_q$, and $f \in \mathcal{M}_q^\varphi(\Omega)$. For $0 < r < \mathrm{diam}(\Omega)$, define

$$\eta_{f,\varphi,q,\Omega}(r) := \sup_{x\in\Omega, 0<R<r} \frac{\varphi(R)}{|B(x,R)|^{\frac{1}{q}}} \left(\int_{B(x,R)\cap\Omega} |f(y)|^q \, dy \right)^{\frac{1}{q}}.$$

The generalized vanishing Morrey space $V\mathcal{M}_q^\varphi(\Omega)$ is defined to be the subset of $\mathcal{M}_q^\varphi(\Omega)$ such that

$$\lim_{r\to 0^+} \eta_{f,\varphi,q,\Omega}(r) = 0.$$

For the setting in $\mathbb{R}^n$, we also define

$$\eta_{f,\varphi,q,\mathbb{R}^n}(r) := \sup_{x\in\mathbb{R}^n, 0<R<r} \frac{\varphi(R)}{|B(x,R)|^{\frac{1}{q}}} \left(\int_{B(x,R)} |f(y)|^q \, dy \right)^{\frac{1}{q}} \quad (0 < r < \infty)$$

and $VM_q^\varphi(\mathbb{R}^n) := \left\{ f \in \mathcal{M}_q^\varphi(\mathbb{R}^n) : \lim_{r\to 0^+} \eta_{f,\varphi,q,\mathbb{R}^n}(r) = 0 \right\}$.

Before we go further, a helpful remark may be in order.

Remark 5.3 When $\inf \varphi > 0$, $V\mathcal{M}_q^\varphi(\Omega) = \{0\}$ by the Lebesgue differentiation theorem.

The fact that vanishing Morrey spaces and the closure of test functions in Morrey spaces coincide can be traced back to [7, Lemma 1.2]. We generalize this fact in the following lemmas:

Lemma 5.4 *Let* $1 \le q < \infty$, $\varphi \in \mathcal{G}_q$, $\inf \varphi = 0$, *and* $f \in V\mathcal{M}_q^\varphi(\Omega)$. *Define*

$$\tilde{f}(x) := \begin{cases} f(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Then we have $\lim_{|h|\to 0^+} \|\tilde{f}(\cdot + h) - \tilde{f}\|_{\mathcal{M}_q^\varphi(\mathbb{R}^n)} = 0.$

Proof Fix $r > 0$. Since $\tilde{f} = 0$ outside Ω, we have

$$\eta_{\tilde{f},\varphi,q,\mathbb{R}^n}(r) = \sup_{x\in\mathbb{R}^n, 0<R<r} \varphi(R) \left(\frac{1}{|B(x,R)|} \int_{B(x,R)\cap\Omega} |f(y)|^q \, dy \right)^{1/q}.$$

Let $\Omega_r := \cup_{z\in\Omega} B(z,r)$. Since $\varphi \in \mathcal{G}_q$, we have

$$\begin{aligned}\eta_{\tilde{f},\varphi,q,\mathbb{R}^n}(r) &= \sup_{x\in\Omega_r, 0<R<r} \varphi(R)\left(\frac{1}{|B(x,R)|}\int_{B(x,R)\cap\Omega}|f(y)|^q\,dy\right)^{1/q}\\ &\le 3^{n/q}\sup_{x\in\Omega,0<R<3r}\varphi(R)\left(\frac{1}{|B(x,R)|}\int_{B(x,R)\cap\Omega}|f(y)|^q\,dy\right)^{1/q}\\ &\le 3^{n/q}\eta_{f,\varphi,q,\Omega}(3r).\end{aligned}$$

Since $f \in V\mathcal{M}_q^\varphi(\Omega)$, we have $\lim_{r\to 0^+}\eta_{f,\varphi,q,\Omega}(3r) = 0$, and hence

$$\lim_{r\to 0^+}\eta_{\tilde{f},\varphi,q,\mathbb{R}^n}(r) = 0.$$

Let $h \in \mathbb{R}^n$. Then we have

$$\begin{aligned}&\|\tilde{f}(\cdot+h)-\tilde{f}\|_{\mathcal{M}_q^\varphi(\mathbb{R}^n)}\\ &\le \sup_{x\in\mathbb{R}^n, R\ge r}\varphi(R)\left(\frac{1}{|B(x,R)|}\int_{B(x,R)}|\tilde{f}(y+h)-\tilde{f}(y)|^q\,dy\right)^{1/q}\\ &\quad + \sup_{x\in\Omega,0<R<r}\varphi(R)\left(\frac{1}{|B(x,R)|}\int_{B(x,R)}|\tilde{f}(y+h)-\tilde{f}(y)|^q\,dy\right)^{1/q}\\ &\le \frac{\varphi(r)}{|B(x,r)|^{1/q}}\|\tilde{f}(\cdot-h)-\tilde{f}\|_{L^q(\mathbb{R}^n)}\\ &\quad +2\sup_{x\in\mathbb{R}^n,0<R<r}\frac{\varphi(R)}{|B(x,R)|^{1/q}}\left(\int_{B(x,R)}|\tilde{f}(y)|^q\,dy\right)^{1/q}.\end{aligned}$$

By the L^q-continuity of translation, we get

$$\limsup_{|h|\to 0^+}\|\tilde{f}(\cdot+h)-\tilde{f}\|_{\mathcal{M}_q^\varphi(\mathbb{R}^n)} \le 2\eta_{\tilde{f},\varphi,q,\mathbb{R}^n}(r).$$

Finally, taking $r\to 0^+$, we get $\lim_{|h|\to 0^+}\|\tilde{f}(\cdot+h)-\tilde{f}\|_{\mathcal{M}_q^\varphi(\mathbb{R}^n)} = 0$.

Lemma 5.5 *Let $1 \le q < \infty$, $\varphi \in \mathcal{G}_q$, and $f \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$ be such that f vanishes almost everywhere outside Ω. If*

$$\lim_{|y|\to 0^+}\|f(\cdot-y)-f\|_{\mathcal{M}_q^\varphi(\Omega)} = 0,$$

then $f \in \overline{C^\infty(\mathbb{R}^n)\cap\mathcal{M}_q^\varphi(\mathbb{R}^n)}^{\mathcal{M}_q^\varphi(\Omega)}$.

Proof By the translation, we may assume that $0 \in \Omega$. Let $r_0 > 0$ be so small that $B(0,r_0) \subset \Omega$. Let ψ be a smooth function supported on the unit ball $B(0,r_0)$, $0 \le \psi \le 1$, and $\|\psi\|_{L^1(\Omega)} = 1$. For every $x \in \Omega$ and $j \in \mathbb{N}$, define $\psi_j(x) := j^n\psi(jx)$. Note that $f * \psi_j \in C^\infty(\mathbb{R}^n)$, since f is locally integrable.

Let $x_0 \in \Omega$ and $r > 0$ be fixed. By the Minkowski integral inequality, we have

$$\begin{aligned}
&\left(\int_{B(x_0,r)\cap\Omega} \left|f * \psi_j(x) - f(x)\right|^q dx\right)^{\frac{1}{q}} \\
&\quad = \left(\int_{B(x_0,r)\cap\Omega} \left|\int_{B(0,1/j)} (f(x-y) - f(x))\psi_j(y)\, dy\right|^q dx\right)^{\frac{1}{q}} \\
&\quad \le \int_{B(0,1/j)} \psi_j(y) \left(\int_{B(x_0,r)\cap\Omega} |f(x-y) - f(x)|^q\, dx\right)^{1/q} dy \\
&\quad \le \frac{|B(0,r)|^{1/q}}{\varphi(r)} \int_{B(0,1/j)} \psi_j(y) \|f(\cdot - y) - f\|_{\mathcal{M}_q^\varphi(\Omega)}\, dy \\
&\quad \le \frac{|B(0,r)|^{1/q}}{\varphi(r)} \sup_{y\in B(0,1/j)} \|f(\cdot - y) - f\|_{\mathcal{M}_q^\varphi(\Omega)}.
\end{aligned}$$

Consequently, r and x_0 being arbitrary, we have

$$\|f - f * \psi_j\|_{\mathcal{M}_q^\varphi(\Omega)} \le \sup_{y\in B(0,1/j)} \|f(\cdot - y) - f\|_{\mathcal{M}_q^\varphi(\Omega)}.$$

Finally, by taking $j \to \infty$, we get $|y| \to 0$, and hence $\lim_{j\to\infty} \|f - f * \psi_j\|_{\mathcal{M}_q^\varphi(\Omega)} = 0$. This shows that $f \in \overline{C^\infty(\mathbb{R}^n) \cap \mathcal{M}_q^\varphi(\mathbb{R}^n)}^{\mathcal{M}_q^\varphi(\Omega)}$ as desired. □

Recall that we are assuming $\inf \varphi = 0$. This assumption is necessary when we derive (5.3) from (5.2) below.

Lemma 5.6 *Let* $1 \le q < \infty$ *and* $\varphi \in \mathcal{G}_q$ *be such that* $\inf \varphi = 0$. *Then we have* $\overset{\circ}{\mathcal{M}}{}_q^\varphi(\Omega) = V\mathcal{M}_q^\varphi(\Omega)$.

Proof As before a translation allows us to assume $B(0, r_0) \subset \Omega$. Let $f \in \overset{\circ}{\mathcal{M}}{}_q^\varphi(\Omega)$. For any $\varepsilon > 0$, choose $g \in C_c^\infty(\Omega)$ such that

$$\|f - g\|_{\mathcal{M}_q^\varphi(\Omega)} < \varepsilon.$$

Let $0 < r < \operatorname{diam}(\Omega)$. Note that, for every $R \in (0, r)$, we have

$$\begin{aligned}
&\varphi(R)\left(\frac{1}{|B(x,R)|}\int_{B(x,R)\cap\Omega}|f(y)|^q\,dy\right)^{1/q}\\
&\le \frac{\varphi(R)}{|B(x,R)|^{1/q}}\left[\left(\int_{B(x,R)\cap\Omega}|f(y)-g(y)|^q\,dy\right)^{1/q}\right.\\
&\quad\left.+\left(\int_{B(x,R)\cap\Omega}|g(y)|^q\,dy\right)^{1/q}\right]\\
&\le \|f-g\|_{\mathcal{M}_q^\varphi(\Omega)}+\|g\|^q_{L^\infty(\Omega)}\varphi(r)\\
&\le \varepsilon+\|g\|_{L^\infty(\Omega)}\varphi(r).
\end{aligned}$$

Consequently,

$$\eta_{f,\varphi,q,\Omega}(r) \le \|g\|_{L^\infty(\Omega)}\varphi(r). \tag{5.2}$$

By taking $r \to 0^+$, we get

$$\lim_{r\to 0^+}\eta_{f,\varphi,q,\Omega}(r) = 0. \tag{5.3}$$

This shows that $f \in V\mathcal{M}_q^\varphi(\Omega)$.

Conversely, by assuming $f \in V\mathcal{M}_q^\varphi(\Omega)$, we shall show that $f \in \overset{\circ}{\mathcal{M}}{}_q^\varphi(\Omega)$. Define

$$\tilde{f}(x) := \begin{cases} f(x), & x \in \Omega,\\ 0, & x \notin \Omega.\end{cases}$$

By Lemmas 5.4 and 5.5, we can find $\{\tilde{g_j}\}_{j=1}^\infty \subset C^\infty(\mathbb{R}^n)\cap\mathcal{M}_q^\varphi(\mathbb{R}^n)$ such that

$$\|\tilde{f}-\tilde{g_j}\|_{\mathcal{M}_q^\varphi(\Omega)} \le \frac{1}{j}.$$

Define $g_j := \chi_\Omega\tilde{g_j}$. Since Ω is bounded, we have $\|g_j\|_{L^\infty(\Omega)} \lesssim 1$. Write $\Omega = \bigcup_{k=1}^\infty K_k$ where $\{K_k\}_{k=1}^\infty$ is a collection of compact sets with property $K_k \subseteq \operatorname{int} K_{k+1}$. Let $g_{j,k} := g_j\chi_{K_k}$. Note that $g_{j,k} \in L_{\rm c}^\infty(\Omega)$. Let $\psi \in C_{\rm c}^\infty(\Omega)$ with $\operatorname{supp}(\psi) \subset \mathrm{B(0, r_0)}$, $0 \le \psi \le 1$, and $\|\psi\|_{L^1} = 1$. For every $l \in \mathbb{N}$, define

$$\psi_l(x) := l^n\psi(lx).$$

For large $l \in \mathbb{N}$, observe that $g_{j,k} * \psi_l \in C_{\rm c}^\infty(\Omega)$ in view of the size of the support of $g_{j,k}$. Note that

$$\begin{aligned}
\|f-g_{j,k}*\psi_l\|_{\mathcal{M}_q^\varphi(\Omega)} &\le \|f-g_j\|_{\mathcal{M}_q^\varphi(\Omega)}+\|g_j-g_{j,k}\|_{\mathcal{M}_q^\varphi(\Omega)}\\
&\quad+\|g_{j,k}-g_{j,k}*\psi_l\|_{\mathcal{M}_q^\varphi(\Omega)}.
\end{aligned}$$

Since $g_{j,k} \in L^\infty_{\rm c}(\Omega) \subseteq VM^\varphi_q(\Omega)$ and

$$\|g_{j,k} - g_{j,k} * \psi_l\|_{\mathcal{M}^\varphi_q(\Omega)} \le \sup_{y \in B(0,\frac{1}{l})} \|g_{j,k} - g_{j,k}(\cdot - y)\|_{\mathcal{M}^\varphi_q(\Omega)},$$

we have $\lim_{l\to\infty} \|g_{j,k} - g_{j,k} * \psi_l\|_{\mathcal{M}^\varphi_q(\Omega)} = 0$. For any $\varepsilon > 0$, choose $\delta > 0$ such that $\varphi(r) < \varepsilon$ for every $0 < r < \delta$. Since $g_j \in L^\infty_c(\mathbb{R}^n) \subseteq L^q(\mathbb{R}^n)$ and

$$\begin{aligned}\|g_j - g_{j,k}\|_{\mathcal{M}^\varphi_q(\Omega)} &\le \sup_{x\in\Omega, 0<r<\delta} \varphi(r) \left(\frac{1}{|B(x,r)|}\int_{B(x,r)\cap\Omega} |g_j(y) - g_{j,k}(y)|^q\, dy\right)^{1/q}\\ &\quad + \sup_{x\in\Omega, r\ge\delta} \varphi(r) \left(\frac{1}{|B(x,r)|}\int_{B(x,r)\cap\Omega} |g_j(y) - g_{j,k}(y)|^q\, dy\right)^{1/q}\\ &\le \varepsilon\|g_j\|_{L^\infty(\Omega)} + \frac{\varphi(\delta)}{|B(x,\delta)|^{1/q}}\|\chi_{\Omega\setminus K_k} g_j\|_{L^q(\mathbb{R}^n)},\end{aligned}$$

by the dominated convergence theorem, we have

$$\limsup_{k\to\infty} \|g_j - g_{j,k}\|_{\mathcal{M}^\varphi_q(\Omega)} \le \varepsilon\|g_j\|_{L^\infty(\Omega)}.$$

Since $\varepsilon > 0$ is arbitrary, we have $\lim_{k\to\infty} \|g_j - g_{j,k}\|_{\mathcal{M}^\varphi_q(\Omega)} = 0$. Consequently,

$$\limsup_{k,l\to\infty} \|f - g_{j,k} * \psi_l\|_{\mathcal{M}^\varphi_q(\Omega)} \le \|f - g_j\|_{\mathcal{M}^\varphi_q(\Omega)} \le \|\tilde{f} - \tilde{g}_j\|_{\mathcal{M}^\varphi_q(\mathbb{R}^n)} \le \frac{1}{j}.$$

By taking $j \to \infty$, we see that $f \in \overset{\circ}{\mathcal{M}}{}^\varphi_q(\Omega)$. □

Before proving Theorem 1.11, we shall prove the following lemmas:

Lemma 5.7 *For all $f \in \mathcal{M}^\varphi_q(\Omega)$, we have*

$$\lim_{a\to 0^+} \|\chi_{\{|f|<a\}} f\|_{\mathcal{M}^\varphi_q(\Omega)} = 0.$$

Proof For every $x \in \Omega$ and $0 < r < \mathrm{diam}(\Omega)$, we have

$$\varphi(r)\left(\frac{1}{|B(x,r)|}\int_{B(x,r)\cap\Omega} \chi_{\{|f|<a\}}(y)|f(y)|^q\, dy\right)^{\frac{1}{q}} \le \varphi(\mathrm{diam}(\Omega))a.$$

Thus,

$$\|\chi_{\{|f|<a\}} f\|_{\mathcal{M}^\varphi_q(\Omega)} \le \varphi(\mathrm{diam}(\Omega))a \to 0$$

as $a \to 0^+$. □

Lemma 5.8 *Let* $g \in V\mathcal{M}^{\varphi}_{q}(\Omega)$ *and* $|f| \le |g|$. *Then we have* $f \in V\mathcal{M}^{\varphi}_{q}(\Omega)$.

Proof This is a direct consequence of $\eta_{f,\varphi,q,\Omega}(r) \le \eta_{g,\varphi,q,\Omega}(r)$ for every $r > 0$. □

Lemma 5.9 *Keep using the same assumption as in* Theorem 1.11. *Let* E *be a measurable set such that* $\chi_E \in V\mathcal{M}^{\varphi}_{q}(\Omega)$. *Then* χ_E *belongs to* $V\mathcal{M}^{\varphi_0}_{q_0}(\Omega) \cap V\mathcal{M}^{\varphi_1}_{q_1}(\Omega)$.

Proof From our assumption, we have $\varphi_0^{q_0} = \varphi_1^{q_1} = \varphi^q$. This implies

$$\eta_{\chi_E,\varphi_0,q_0,\Omega}(r) = \eta_{\chi_E,\varphi,q,\Omega}(r)^{q/q_0} \quad \text{and} \quad \eta_{\chi_E,\varphi_1,q_1,\Omega}(r) = \eta_{\chi_E,\varphi,q,\Omega}(r)^{q/q_1}.$$

By taking $r \to 0^+$, we see that $\chi_E \in V\mathcal{M}^{\varphi_0}_{q_0}(\Omega) \cap V\mathcal{M}^{\varphi_1}_{q_1}(\Omega)$. □

Finally, we give the proof of Theorem 1.11 as follows.

Proof of Theorem 1.11 Without loss of generality, we may assume that $q_0 > q_1$. By a similar argument as in the proof of Theorem 1.7, we have

$$\begin{aligned}\left[\overset{\circ}{\mathcal{M}}{}^{\varphi_0}_{q_0}(\Omega), \overset{\circ}{\mathcal{M}}{}^{\varphi_1}_{q_1}(\Omega)\right]_\theta &\subseteq \left[\mathcal{M}^{\varphi_0}_{q_0}(\Omega), \mathcal{M}^{\varphi_1}_{q_1}(\Omega)\right]_\theta \\ &\subseteq \overline{L^\infty(\Omega) \cap \mathcal{M}^{\varphi}_{q}(\Omega)}^{\mathcal{M}^{\varphi}_{q}(\Omega)} \subseteq V\mathcal{M}^{\varphi}_{q}(\Omega).\end{aligned}$$

Conversely let $f \in \overset{\circ}{\mathcal{M}}{}^{\varphi}_{q}(\Omega)$. For every $z \in \overline{S}$, define

$$F(z) := \operatorname{sgn}(f)|f|^{q\left(\frac{1-z}{q_0}+\frac{z}{q_1}\right)}, \quad F_0(z) := \chi_{\{|f|\le 1\}}F(z), \quad \text{and} \quad F_1(z) := \chi_{\{|f|>1\}}F(z).$$

Since $C_c^\infty(\Omega) \subseteq L^\infty(\Omega)$, we can combine Lemmas 2.6 and 5.7 to obtain

$$\lim_{a\to 0^+} \|f - \chi_{\{a\le|f|\le a^{-1}\}}f\|_{\mathcal{M}^{\varphi}_{q}(\Omega)} = 0.$$

Therefore, we may assume that

$$f = \chi_{\{a\le|f|\le a^{-1}\}}f. \tag{5.4}$$

for some $a \in (0, 1)$.

By Lemma 5.6, we have $f \in V\mathcal{M}^{\varphi}_{q}(\Omega)$. Meanwhile, for any $0 < b < c < \infty$, we have (4.11). From Lemmas 5.6, 5.8, and 5.9, it follows that $\chi_{\{b\le|f|\le c\}} \in \overset{\circ}{\mathcal{M}}{}^{\varphi_0}_{q_0}(\Omega) \cap \overset{\circ}{\mathcal{M}}{}^{\varphi_1}_{q_1}(\Omega)$. Since

$$|F_0(z)| = |f|^{q\left(\frac{1-\mathrm{Re}(z)}{q_0}+\frac{\mathrm{Re}(z)}{q_1}\right)}\chi_{\{|f|\le 1\}} = |f|^{\frac{q}{q_0}}|f|^{q\,\mathrm{Re}(z)\left(\frac{1}{q_1}-\frac{1}{q_0}\right)}\chi_{\{a\le|f|\le 1\}} \le \chi_{\{a\le|f|\le 1\}}$$

and

$$
\begin{aligned}
|F_1(z)| &= \left(|f|^{\frac{q}{q_0}}\right)^{1-\mathrm{Re}(z)} \left(|f|^{\frac{q}{q_1}}\right)^{\mathrm{Re}(z)} \chi_{\{1<|f|<a^{-1}\}} \\
&\le \left(|f|^{q/q_0} + |f|^{q/q_1}\right) \chi_{\{1<|f|<a^{-1}\}} \\
&\le \left(a^{-q/q_0} + a^{-q/q_1}\right) \chi_{\{1<|f|<a^{-1}\}},
\end{aligned}
$$

we have $F_0(z) \in \overset{\circ}{\mathcal{M}}{}^{\varphi_0}_{q_0}(\Omega)$ and $F_1(z) \in \overset{\circ}{\mathcal{M}}{}^{\varphi_1}_{q_1}(\Omega)$, and hence $F(z) \in \overset{\circ}{\mathcal{M}}{}^{\varphi_0}_{q_0}(\Omega) + \overset{\circ}{\mathcal{M}}{}^{\varphi_1}_{q_1}(\Omega)$. Moreover, we also have

$$
\begin{aligned}
&\sup_{z\in\overline{S}} \|F(z)\|_{\overset{\circ}{\mathcal{M}}{}^{\varphi_0}_{q_0}(\Omega)+\overset{\circ}{\mathcal{M}}{}^{\varphi_1}_{q_1}(\Omega)} \\
&\quad \le \sup_{z\in\overline{S}} \|F_0(z)\|_{\overset{\circ}{\mathcal{M}}{}^{\varphi_0}_{q_0}(\Omega)} + \|F_1(z)\|_{\overset{\circ}{\mathcal{M}}{}^{\varphi_1}_{q_1}(\Omega)} \\
&\quad \le \|\chi_{\{a<|f|\le 1\}}\|_{\overset{\circ}{\mathcal{M}}{}^{\varphi_0}_{q_0}(\Omega)} + (a^{-\frac{q}{q_0}} + a^{-\frac{q}{q_1}})\|\chi_{\{1\le|f|\le a^{-1}\}}\|_{\overset{\circ}{\mathcal{M}}{}^{\varphi_1}_{q_1}(\Omega)} < \infty.
\end{aligned}
$$

Observe that for all $w \in S$, we have

$$
|F'(w)| \le \left(\frac{q}{q_1} - \frac{q}{q_0}\right) \left(|f|^{\frac{q}{q_0}} + |f|^{\frac{q}{q_1}}\right) \chi_{\{a\le|f|\le a^{-1}\}} |\log|f|| \le C_{a,q,q_0,q_1} \chi_{\{a\le|f|\le a^{-1}\}}
$$

where $C_{a,q,q_0,q_1} := \left(\frac{q}{q_0} - \frac{q}{q_1}\right)\left(a^{-q/q_0} + a^{-q/q_1}\right) \log\frac{1}{a}$. Consequently, for all $z_1, z_2 \in \overline{S}$, we have

$$
\begin{aligned}
&\|F(z_2) - F(z_1)\|_{\overset{\circ}{\mathcal{M}}{}^{\varphi_0}_{q_0}(\Omega)+\overset{\circ}{\mathcal{M}}{}^{\varphi_1}_{q_1}(\Omega)} \\
&\quad = \left\| \int_{z_1}^{z_2} F'(w)\, dw \right\|_{\overset{\circ}{\mathcal{M}}{}^{\varphi_0}_{q_0}(\Omega)+\overset{\circ}{\mathcal{M}}{}^{\varphi_1}_{q_1}(\Omega)} \\
&\quad \le C_{a,q,q_0,q_1} \left(\|\chi_{\{a\le|f|\le 1\}}\|_{\overset{\circ}{\mathcal{M}}{}^{\varphi_0}_{q_0}(\Omega)} + \|\chi_{\{1\le|f|\le a^{-1}\}}\|_{\overset{\circ}{\mathcal{M}}{}^{\varphi_1}_{q_1}(\Omega)} \right) |z_2 - z_1|.
\end{aligned}
$$

This shows that $F : \overline{S} \to \overset{\circ}{\mathcal{M}}{}^{\varphi_0}_{q_0}(\Omega) + \overset{\circ}{\mathcal{M}}{}^{\varphi_1}_{q_1}(\Omega)$ is a continuous function. Likewise, we also can verify that $F|_S : S \to \overset{\circ}{\mathcal{M}}{}^{\varphi_0}_{q_0}(\Omega) + \overset{\circ}{\mathcal{M}}{}^{\varphi_1}_{q_1}(\Omega)$ is a holomorphic function by the same argument as in the proof of (1.13). On the boundary of $\overline{S}$, we have

$$
|F(j+it)| = |f|^{\frac{q}{q_j}} \le a^{-\frac{q}{q_j}} \chi_{\{a\le|f|\le a^{-1}\}}
$$

for $j = 0, 1$ and $t \in \mathbb{R}$ from (4.11), so $F(j+it) \in \overset{\circ}{\mathcal{M}}{}^{\varphi_j}_{q_j}(\Omega)$. By a similar argument for showing the continuity of $F(z)$, we also have

$$
\|F(j+it_1) - F(j+it_2)\|_{\overset{\circ}{\mathcal{M}}{}^{\varphi_j}_{q_j}(\Omega)} \le C_{a,q,q_0,q_1} \|\chi_{\{a\le|f|\le a^{-1}\}}\|_{\overset{\circ}{\mathcal{M}}{}^{\varphi_j}_{q_j}(\Omega)} |t_1 - t_2|
$$

for all $t_1, t_2 \in \mathbb{R}$. Hence $F \in \mathcal{F}(\overset{\circ}{\mathcal{M}}{}^{\varphi_0}_{q_0}(\Omega), \overset{\circ}{\mathcal{M}}{}^{\varphi_1}_{q_1}(\Omega))$. Since $F(\theta) = f$, we conclude that $f \in [\overset{\circ}{\mathcal{M}}{}^{\varphi_0}_{q_0}(\Omega), \overset{\circ}{\mathcal{M}}{}^{\varphi_1}_{q_1}(\Omega)]_\theta$ as desired. □

Acknowledgements The second author is supported by JSPS Grand-in-Aid for Scientific Research (C) No. 16K05209.

6 Appendix: a function $f \in \mathcal{M}^p_q \setminus (L^1 + L^\infty)$

We aim here to present an example of a function $f \in \mathcal{M}^p_q \setminus (L^1 + L^\infty)$. Let $n = 1$ for simplicity. Define

$$f = f_p := \sum_{j=100}^{\infty} [\log_2 \log_2 j]^{1/p} \chi_{[j!, j!+[\log_2 \log_2 j]^{-1}]}. \tag{6.1}$$

Lemma 6.1 *Let $1 \le q < p < \infty$. Then f given by (6.1) belongs to $\mathcal{M}^p_q$ but does not belong to $L^1 + L^\infty$.*

Proof Let (a, b) be an interval which intersects the support of f.

1. Case $1 : b - a < 2$. In this case, there exists uniquely $j \in \mathbb{N} \cap [100, \infty)$ such that $[a, b] \cap [j!, j! + [\log_2 \log_2 j]^{-1}] \neq \emptyset$. Thus,

$$\begin{aligned}
&(b-a)^{\frac{1}{p}-\frac{1}{q}} \left(\int_a^b f(t)^q \, dt \right)^{\frac{1}{q}} \\
&= (b-a)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{\max(a,j!)}^{\min(b,j!+[\log_2 \log_2 j]^{-1})} f(t)^q \, dt \right)^{\frac{1}{q}} \\
&\le (\min(b, j! + [\log_2 \log_2 j]^{-1}) - \max(a, j!))^{\frac{1}{p}-\frac{1}{q}} \\
&\quad \left(\int_{\max(a,j!)}^{\min(b,j!+[\log_2 \log_2 j]^{-1})} f(t)^q \, dt \right)^{\frac{1}{q}} \\
&= [\log_2 \log_2 j]^{\frac{1}{p}} (\min(b, j! + [\log_2 \log_2 j]^{-1}) - \max(a, j!))^{\frac{1}{p}} \\
&\le 1.
\end{aligned}$$

2. Case $2 : b - a > 2$. Set

$$m := \min([a, b] \cap \mathrm{supp}(f)), \quad M := \max([a, b] \cap \mathrm{supp}(f)).$$

Choose $j_m, j_M \in \mathbb{N} \cap [100, \infty)$ so that $m \in [j_m!, j_m! + j_m{}^{-1}]$ and $M \in [j_M!, j_M! + j_M{}^{-1}]$. If $j_M - j_m \le 2$, then we go through a similar argument as before. Assume $j_M - j_m \ge 3$. Then we have

$$b - a \ge M - m \ge j_M! - j_m! - j_m{}^{-1} \ge j_M! - j_m! - 1.$$

Thus,

$$(b-a)^{\frac{1}{p}-\frac{1}{q}}\left(\int_a^b f(t)^q\,dt\right)^{\frac{1}{q}} \leq (j_M!-j_m!-1)^{\frac{1}{p}-\frac{1}{q}}\left(\int_{j_m!}^{j_M!+1} f(t)^q\,dt\right)^{\frac{1}{q}}$$
$$\leq C j_M!^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{j=j_m}^{j_M}(\log_2\log_2 j)^{\frac{q-p}{p}}\right)^{\frac{1}{q}}$$
$$\leq C.$$

Thus, $f\in\mathcal{M}^p_q$.

Now we disprove $f\in L^1+L^\infty$. Let R be fixed. Then a geometric observation shows that

$$\|f-\min(f,R)\|_{L^1}\leq\|f-h\|_{L^\infty}$$

for any $h\in L^\infty$ with $\|h\|_{L^\infty}\leq R$.

Let $S>2R+2$ be an integer. Then

$$\int_{f=S}(f-\min(f,R))=|\{f=S\}|(S-R)\geq\frac{S}{2}|\{f=S\}|$$
$$=\frac{S}{2}\sum_{k=2^{2^S}}^{2^{2^{S+1}}}\frac{1}{k}\geq CS(2^{S+1}-2^S).$$

Thus, $\|f-\min(f,R)\|_{L^1}=\infty$. Hence, $f\notin L^1+L^\infty$.

Remark 6.2 For the case in $\mathbb{R}^n$ with $n>1$, we can consider

$$f(x)=f(x_1,\ldots,x_n):=\prod_{j=1}^n f_p(x_j),$$

where $f_p(x_j)$ is defined in (6.1).

References

1. Bergh, J., Löfström, J.: Interpolation spaces. An introduction, Grundlehren der Mathematischen Wissenschaften, no. 223. Springer, Berlin (1976)
2. Blasco, O., Ruiz, A., Vega, L.: Non-interpolation in Morrey–Campanato and block spaces. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **28**, 31–40 (1999)
3. Burenkov, V. I., Nursultanov, E. D.: Description of interpolation spaces for local Morrey-type spaces. (Russian), Tr. Mat. Inst. Steklova **269** (2010), Teoriya Funktsii i Differentsialnye Uravneniya, 52–62; translation in Proc. Steklov Inst. Math. **269** (2010)
4. Calderón, A. P.: Intermediate spaces and interpolation, the complex method. Studia Math. **14**, 113–190. no. 1, 46–56 (1964)

5. Campanato, S., Murthy, M.K.V.: Una generalizzazione del teorema di Riesz-Thorin. Ann. Scuola Norm. Sup. Pisa **19**, 87–100 (1965). (Italian)
6. Caso, L., D'Ambrosio, R., Monsurrò, S.: Some remarks on spaces of Morrey type. Abstr. Appl. Anal., pp. 22 (2010) **(Art. ID 242079)**
7. Chiarenza, F., Franciosi, M.: A generalization of a theorem by C. Miranda. Ann. Mat. Pura Appl. **161**(1), 285–297 (1992)
8. Cobos, F., Peetre, J., Persson, L.E.: On the connection between real and complex interpolation of quasi-Banach spaces. Bull. Sci. Math. **122**, 17–37 (1998)
9. Eridani, Gunawan, H., Nakai, E., Sawano, Y.: Characterizations for the generalized fractional integral operators on Morrey spaces. Math. Ineq. Appl. **17**(2), 761–777 (2014)
10. Guliyev, V.S., Deringoz, F., Hasanov, J.J.: Φ-Admissible singular operators and their commutators on vanishing generalized Orlicz–Morrey spaces. J. Inequal. Appl. **2014**(1), 1–18 (2014)
11. Hakim, D.I., Sawano, Y.: Interpolation of generalized Morrey spaces. Rev. Mat. Complut. **29**(2), 295–340 (2016)
12. Kozono, H., Yamazaki, M.: Semilinear heat equations and the Navier–Stokes equation with distributions in new function spaces as initial data. Comm. PDE **19**, 959–1014 (1994)
13. Lemarié-Rieusset, P.G.: Multipliers and Morrey spaces. Potential Anal. **38**(3), 741–752 (2013)
14. Lemarié-Rieusset, P.G.: Erratum to: Multipliers and Morrey spaces. Potential Anal. **41**, 1359–1362 (2014)
15. Liang, Y., Sawano, Y., Ullrich, T., Yang, D., Yuan, W.: A new framework for generalized Besov-type and Triebel–Lizorkin-type spaces. Dissertationes Math. (Rozprawy Mat.) **489**, pp. 114 (2013)
16. Lu, Y., Yang, D., Yuan, W.: Interpolation of Morrey spaces on metric measure spaces. Can. Math. Bull. **57**, 598–608 (2014)
17. Mazzucato, A. L.: Decomposition of Besov–Morrey spaces. Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001), 279–294. Contemp. Math. **320**, Am. Math. Soc., Providence, RI (2003)
18. Mazzucato, A.L.: Besov–Morrey spaces: function space theory and applications to non-linear PDE. Trans. Am. Math. Soc. **355**(4), 1297–1364 (2003)
19. Nakai, E.: Hardy–Littlewood maximal operator, singular integral operators, and the Riesz potential on generalized Morrey spaces. Math. Nachr. **166**, 95–103 (1994)
20. Nakai, E.: A characterization of pointwise multipliers on the Morrey spaces. Sci. Math. **3**, 445–454 (2000)
21. Nakai, E.: Orlicz–Morrey spaces and the Hardy–Littlewood maximal function. Studia Math. **188**(3), 193–221 (2008)
22. Nakai, E., Sobukawa, T.: B_u^w-function spaces and their interpolation. Tokyo Math. J. **(online)**
23. Nakamura, S., Noi, T., Sawano, Y.: Generalized Morrey spaces and trace operator. Sci. Math. China **59**(2), 281–336 (2016)
24. Peetre, J.: On the theory of $\mathcal{L}_{p,\lambda}$ spaces. J. Funct. Anal. **4**, 71–87 (1969)
25. Ragusa, M.A.: Commutators of fractional integral operators on vanishing-Morrey spaces. J. Glob. Optim. **40**(1–3), 361–368 (2008)
26. Ruiz, A., Vega, L.: Corrigenda to Unique continuation for Schrödinger operators with potential in Morrey spaces and a remark on interpolation of Morrey spaces. Publ. Mat. **39**, 405–411 (1995)
27. Sawano, Y., Tanaka, H.: Decompositions of Besov–Morrey spaces and Triebel–Lizorkin–Morrey spaces. Math. Z. **257**(4), 871–905 (2007)
28. Sawano, Y., Tanaka, H.: Besov–Morrey spaces and Triebel–Lizorkin–Morrey spaces for non-doubling measures. Math. Nachr. **282**(12), 1788–1810 (2009)
29. Sawano, Y., Tanaka, H.: The Fatou property of block spaces. J. Math. Sci. Univ. Tokyo. **22**, 663–683 (2015)
30. Sawano, Y., Wadade, H.: On the Gagliardo–Nirenberg type inequality in the critical Sobolev–Morrey space. J. Fourier Anal. Appl. **19**(1), 20–47 (2013)
31. Spanne, S.: Sur l'interpolation entres les espaces $\mathcal{L}_k^{(p,\Phi)}$. Ann. Scuola Norm. Sup. Pisa **20**, 625–648 (1966)
32. Tang, L., Xu, J.: Some properties of Morrey type Besov–Triebel spaces. Math. Nachr. **278**, 904–917 (2005)
33. Triebel, H.: Hybrid function spaces, heat and Navier–Stokes equations, tracts in Mathematics 24, European Mathematical Society (2014)
34. Yang, D., Yuan, W.: A new class of function spaces connecting Triebel–Lizorkin spaces and Q spaces. J. Funct. Anal. **255**, 2760–2809 (2008)

35. Yang, D., Yuan, W.: New Besov-type spaces and Triebel–Lizorkin-type spaces including Q spaces. Math. Z. **265**, 451–480 (2010)
36. Yang, D., Yuan, W.: Dual properties of Triebel–Lizorkin-type spaces and their applications. Z. Anal. Anwend. **30**, 29–58 (2011)
37. Yang, D., Yuan, W., Zhuo, C.: Complex interpolation on Besov-type and Triebel–Lizorkin-type spaces. Anal. Appl. (Singap.) **11**(5), 1350021, pp. 45 (2013)
38. Yuan, W.: Complex interpolation for predual spaces of Morrey-type spaces. Taiwan. J. Math. **18**(5), 1527–1548 (2014)
39. Yuan, W., Sickel, W., Yang, D.: Morrey and Campanato Meet Besov, Lizorkin and Triebel. Lecture notes in mathematics. Springer, Berlin (2010)
40. Yuan, W., Sickel, W., Yang, D.: Interpolation of Morrey–Campanato and related smoothness spaces. Sci. China Math. **58**(9), 1835–1908 (2015)
41. Zhu, K.: Operator theory in function spaces. No. 138. American Mathematical Society, Providence (2007)