

Wave Equation on One-Dimensional Fractals with Spectral Decimation and the Complex Dynamics of Polynomials

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Abstract We study the wave equation on one-dimensional self-similar fractal structures that can be analyzed by the spectral decimation method. We develop efficient numerical approximation techniques and also provide uniform estimates obtained by analytical methods.

Keywords Wave equation · Fractals · Spectral decimation · Infinite speed of propagation · Time change

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1 Introduction

The purpose of this paper is to study, both analytically and numerically, the wave equation on the unit interval endowed with a self-similar fractal measure. Previous studies of wave equation on fractals, including numerical approximations, were published in [20,24–26,33,59]. All these works have some, although not direct, relation to the classical paper [55], but are more directly related to the fractal Fourier analysis, see [57,58]. Our computational methods mostly come from the theoretical papers [9,10,61,62] that develop so-called spectral decimation method in the form applicable for to numerical analysis.

In general, there is a large literature dealing with analysis and probability on fractals in mathematical terms, such as [8,11–14,16,28,32,34–37,39–41,45,53,54,60],[and references therein] and also extensive mathematical physics literature, including [1,2,6,7,21,22,27,29,34,47,48]. Of particular interest are the works studying the appearance of fractals in quantum gravity, including [3–5,17–19,23,31,42,44,49,50].

We consider a situation in which a good enough (fractal) Laplacian Δ is defined on $L^2(K, \mu)$, where a compact set K (the unit interval in our case) equipped with a (fractal) Borel measure μ . This Laplacian Δ is a point-wise limit or as the generator of a Kigami’s resistance form (see Proposition 2.3), and one can extend some of the classical numerical techniques to approximate some (intrinsically smooth) solutions of the wave equation initial value problem

$$\begin{cases} \partial_{tt}u = -\Delta u & \text{on } K \times [0, T], \\ u(\cdot, 0) = \phi & \text{on } K, \\ \partial_t u(\cdot, 0) = \psi & \text{on } K. \end{cases} \tag{1.1}$$

As is well known, if the spectrum of the Laplacian is discrete, then the solution of the wave equation can be represented in terms of $L^2(\mu)$ -eigensolutions $\{\lambda_k, f_k\}_{k=0}^\infty$ of the Laplacian Δ , with $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots \uparrow \infty$ and $\Delta f_k = \lambda_k f_k$. Writing

$$\phi = \sum_{k=0}^\infty \alpha_k f_k \quad \text{and} \quad \psi = \sum_{k=0}^\infty \beta_k f_k, \tag{1.2}$$

where $\alpha_k = \langle \phi, f_k \rangle_{L^2}$ and $\beta_k = \langle \psi, f_k \rangle_{L^2}$, one finds that u admits the series representation

$$u(x, t) = \sum_{k=0}^\infty \alpha_k f_k(x) \cos(t\sqrt{\lambda_k}) + \sum_{k=k_{\min}}^\infty \frac{\beta_k}{\sqrt{\lambda_k}} f_k(x) \sin(t\sqrt{\lambda_k}), \tag{1.3}$$

where $k_{\min} := \min\{j \in \mathbb{N} \cup \{0\} : \lambda_j > 0\}$. It is known that the series point-wise converges poorly and the numerical approximations are very unstable unless the smoothness of solutions can be controlled.

In our setup, $K = I$ and μ is the fractal measure defined in Sect. 2. For simplicity we assume that the initial velocity $\psi \equiv 0$, and so the solution to (1.1) is

$$u(x, t) = \sum_{k=0}^{\infty} \alpha_k f_k(x) \cos \left(t\sqrt{\lambda_k} \right). \tag{1.4}$$

If we theoretically assume that ϕ is given by a δ -impulse at point 0, $\phi = \delta_0$, then we have that $\alpha_k := \int_I f_k(x)\delta_0(x) \mu(dx) = f_k(0)$. Note that $\delta_0(x)$ is not a function by the unit atomic measure at zero, and so the integral in this definition of α_k is to be understood as a formal expression, as in the theory of distributions (for the classical version, see [56], and for the fractal version, see [52]). This approach on a fractal space does not allow an accurate numerical approximation of the solutions.

Therefore we concentrate on a situation where the initial condition is highly localize function, but is smooth in intrinsic sense, and we can show that the approximating series converges uniformly. This is an illustration of the general principle of Strichartz [58]: *Laplacians on fractals with spectral gaps have nicer Fourier series*. However, the abstract result [58] does not include the estimate of the remainder which we obtain in our work.

Numerically, we can only compute the eigensolutions of the fractal Laplacian up to a finite level, so in practice we solve the “approximate” wave equation

$$\begin{cases} \partial_{tt}u_n = -\Delta_n u_n & \text{on } V_n \times [0, T], \\ u_n(\cdot, 0) = \delta_0^{(n_0, n)} & \text{on } V_n, \\ \partial_t u_n(\cdot, 0) = 0 & \text{on } V_n, \end{cases} \tag{1.5}$$

where $\delta_0^{(n, n_0)} = \sum_{k=0}^{3^{n_0}} \alpha_k f_{n,k}$ is the approximate δ -function built up from the first $|V_{n_0}| = (3^{n_0} + 1)$ eigenfunctions of Δ_n (with $\Delta_n f_{n,k} = \lambda_{n,k} f_{n,k}$), and $\alpha_k := \alpha_{n_0,k} \geq 0$ are the coefficients found in Sect. 3.3. Throughout the section n_0 will be fixed, and we will not mention n_0 explicitly unless the context demands it. The solution to (1.5) has the series representation

$$u_n(x, t) = \sum_{k=0}^{3^{n_0}} \alpha_k f_{n,k}(x) \cos \left(t\sqrt{\lambda_{n,k}} \right) \quad \text{for all } x \in V_n \text{ and } t \in [0, T]. \tag{1.6}$$

For each t , we harmonically extend the function $x \mapsto u_n(x, t)$ from V_n to I . This procedure allows us to compare $u_n(x, t)$ with

$$\tilde{u}(x, t) = \sum_{k=0}^{3^{n_0}} \alpha_k f_k(x) \cos \left(t\sqrt{\lambda_k} \right) \quad \text{for all } x \in I \text{ and } t \in [0, T], \tag{1.7}$$

the solution of the wave equation on (I, μ) whose initial condition is the truncated series representation of the δ -impulse. We note that \tilde{u} is differentiable in t and continuous in x . However it is highly localized function at $t = 0$, and therefore it mimics wave propagation from a delta function initial values.

Our paper is organized as follows. Section 2 contains the construction of the unit interval as a p.c.f. fractal, definition of the Dirichlet energy form, the definition of the corresponding Laplacian and its associated eigenvalues. In Sect. 3 we use spectral decimation to construct the eigenfunctions of the discrete Laplacian and prove that their limit is continuous. The section concludes with the spectral decomposition of the delta function. Section 4 contains various technical estimates needed to show the convergence of solutions of the wave equation. In Sect. 5 we give theoretical bounds on the approximations to the wave equations solutions and convergence information. Section 6 contains the numerical computation of the wave equation solutions, their associated eigenfunctions, and the Fourier approximations for the delta function.

Remark 1.1 Theoretically, the infinite propagation speed for wave equation solutions was established in [43] on some p.c.f. fractals with heat kernel estimates

$$\frac{c_1}{V(x, t^{1/\beta})} \exp\left(-c_2 \left(\frac{d(x, y)^\beta}{t}\right)^{1/(\beta-1)}\right) \leq p(t, x, y) \leq \frac{c_3}{V(x, t^{1/\beta})} \exp\left(-c_4 \left(\frac{d(x, y)^\beta}{t}\right)^{1/(\beta-1)}\right) \tag{1.8}$$

for positive constants $c_1, c_2, c_3, c_4, x, y \in I, t \in (0, 1]$, where $\beta = 2/d_S$ and $V(x, r) = \mu(B_r(x))$. Kigami in [38] obtained such estimates in a situation which resembles, but is technically different, from ours. We conjecture that an analogue (1.8) holds in our situation, but proving this would lie outside of the scope of our paper.

2 Eigenvalues of the Fractal Laplacian on an Interval

In this section we define a particular self-similar structure on the unit interval. In this way, it can be seen as a p.c.f fractal (see [11, 16, 36, 37, 62, 63]). In these papers the reader can find these definitions and an exposition of the general theory of Dirichlet forms on fractals, as well as further references on the subject. Herein we will use three contractions for simplicity. However, one could perform the same construction using any number of contractions in order to obtain a fractal Laplacian on the unit interval.

To define the standard Laplacian, we can use three contractions $F_1, F_2, F_3 : \mathbb{R} \rightarrow \mathbb{R}$ $F_j(x) = \frac{1}{3}x + \frac{2}{3}p_j$ with respective fixed points $p_1 = 0, p_2 = \frac{1}{2}, p_3 = 1$. Then the interval $I=[0, 1]$ is a unique compact set such that $I = \bigcup_{j=1,2,3} F_j(I)$. The discrete

approximations to I are defined inductively by $V_n = \bigcup_{j=1,2,3} F_j(V_{n-1}) = \left\{\frac{k}{3^n}\right\}_{k=0}^{3^n}$,

where $V_0 = \partial I = \{0, 1\}$ is the boundary of I . For $x, y \in V_n$ we write $y \sim x$ if $|x - y| = 3^{-n}$. Then the standard discrete Dirichlet (energy) form on V_n is

$$\mathcal{E}_n(f, f) = 3^n \sum_{\substack{x, y \in V_n \\ y \sim x}} (f(y) - f(x))^2,$$

and the standard Dirichlet (energy) form on I is $\mathcal{E}(f, f) = \lim_{n \rightarrow \infty} \mathcal{E}_n(f, f)$ if this limit exists. We call a function h harmonic if it minimizes the energy subject to the constraint of the given boundary values. Then we have that $\mathcal{E}_{n+1}(f, f) \geq \mathcal{E}_n(f, f)$ for any function f , and $\mathcal{E}_{n+1}(h, h) = \mathcal{E}_n(h, h) = \mathcal{E}(h, h)$ for a harmonic h . A function h is harmonic if and only if it is linear. If f is continuously differentiable then

$$\mathcal{E}(f, f) = \int_0^1 |f'(x)|^2 dx.$$

The domain \mathcal{F} of this standard Dirichlet (energy) form \mathcal{E} on I coincides with the usual Sobolev space $H^1[0, 1]$. Moreover \mathcal{E} on I is self-similar in the sense that

$$\mathcal{E}(f, f) = 3 \sum_{j=1,2,3} \mathcal{E}(f \circ F_j, f \circ F_j).$$

The corresponding standard discrete Laplacians on V_n are

$$\Delta_n f(x) = \frac{1}{2} \sum_{\substack{y \in V_n \\ y \sim x}} (f(x) - f(y)), \quad x \in V_n \setminus V_0,$$

and the (renormalized) Laplacian on I is

$$\Delta f(x) = \lim_{n \rightarrow \infty} 9^n \Delta_n f(x) = -\frac{1}{2} f''(x)$$

for any twice differentiable function. In our convention the Laplacian is a nonnegative operator. For any twice differentiable function f , the Gauss–Green (integration by parts) formula applies

$$\mathcal{E}(f, f) = 2 \int_0^1 f \Delta f dx + f f' \Big|_0^1.$$

We can modify the above construction with the introduction of the parameter p , where $0 < p < 1$, and write $q = 1 - p$. Later we will show that these parameters give the transition probabilities of a random walk on the unit interval. Now, we define contraction factors (or resistance weights)

$$r_1 = r_3 = \frac{p}{1+p} \quad \text{and} \quad r_2 = \frac{q}{1+p}, \tag{2.1}$$

and measure weights

$$m_1 = m_3 = \frac{q}{1+q} \quad \text{and} \quad m_2 = \frac{p}{1+q}. \tag{2.2}$$

Note that in general the choices of resistance and measure weights are essentially free, up to constant multiples, according to Kigami’s theory of Harmonic calculus on

p.c.f. self-similar sets [36,37], but we make a unique choice that leads to a manageable spectral analysis, as explained in [9,53,60–63]. We do not give a complete explanation here because it would require too much space. In short, the spectral decimation requires a symmetry $m_1 = m_3$. Moreover, the spectral decimation also requires that the resistance weights are, up to a constant, reciprocals of the measure weights, and

$$m_1 + m_2 + m_3 = r_1 + r_2 + r_3 = 1. \tag{2.3}$$

Thus, our system essentially has one independent parameter, which we denote p and express everything else in terms of this parameter.

We may now define the three contractions: $F_1, F_2, F_3 : \mathbb{R} \rightarrow \mathbb{R}$ with respective fixed points $p_1 = 0, p_2 = \frac{1}{2}, p_3 = 1$ in terms of resistances which depend on our parameter p

$$F_j(x) = r_j x + (1 - r_j) p_j. \tag{2.4}$$

Then the interval $I=[0, 1]$ is the unique compact set such that

$$I = \bigcup_{j=1,2,3} F_j(I). \tag{2.5}$$

The discrete approximations to I are defined inductively by

$$V_n = \bigcup_{j=1,2,3} F_j(V_{n-1}), \tag{2.6}$$

where $V_0 = \partial I = \{0, 1\}$ is the boundary of I .

The following definitions and results come directly from the more general theory in [11,16,36,37], so we omit the proofs.

Definition 2.1 The discrete Dirichlet (energy) form on V_n is defined inductively

$$\mathcal{E}_n(f, f) = \sum_{j=1,2,3} \frac{1}{r_j} \mathcal{E}_{n-1}(f \circ F_j). \tag{2.7}$$

with $\mathcal{E}_0(f, f) = (f(1) - f(0))^2$, and the Dirichlet (energy) form on I is

$$\mathcal{E}(f, f) = \lim_{n \rightarrow \infty} \mathcal{E}_n(f, f) = \int_0^1 |f'(x)|^2 dx \tag{2.8}$$

The domain \mathcal{F} of \mathcal{E} consists of continuous functions for which the limit is finite, and coincides with the usual Sobolev space $H^1[0, 1]$.

The existence of this limit is justified by the next proposition.

Proposition 2.2 We have that $\mathcal{E}_{n+1}(f, f) \geq \mathcal{E}_n(f, f)$ for any function f , and

$$\mathcal{E}_{n+1}(h, h) = \mathcal{E}_n(h, h) = \mathcal{E}(h, h) \tag{2.9}$$

for a harmonic function h .

Proposition 2.3 *The Dirichlet (energy) form \mathcal{E} on I is local and regular, and is self-similar in the sense that*

$$\mathcal{E}(f, f) = \sum_{j=1,2,3} \frac{1}{r_j} \mathcal{E}(f \circ F_j, f \circ F_j). \quad (2.10)$$

The domain of \mathcal{E} , see Definition 2.1, is dense in the space of continuous functions on I .

The μ -Laplacian Δ_μ , satisfying the following Gauss–Green (integration by parts) formula

$$\mathcal{E}(f, f) = C \int_0^1 f \Delta_\mu f d\mu + ff'|_0^1, \quad (2.11)$$

where μ is a unique probability self-similar measure with weights m_1, m_2, m_3 , that is

$$\mu = \sum_{j=1,2,3} m_j \mu \circ F_j. \quad (2.12)$$

can be defined by

$$\Delta_\mu f(x) = \lim_{n \rightarrow \infty} \left(1 + \frac{2}{pq}\right)^n \Delta_n f(x), \quad (2.13)$$

where the discrete Laplacians

$$\Delta_n f(x_k) = \begin{cases} f(x_k) - pf(x_{k-1}) - qf(x_{k+1}) \\ \text{or} \\ f(x_k) - qf(x_{k-1}) - pf(x_{k+1}) \end{cases} \quad (2.14)$$

are defined as the generators of the nearest neighbor random walks on V_n with transition probabilities p and q assigned according to the weights of the corresponding intervals. The domain of the corresponding continuous Laplacian Δ_μ , defined to be the set of all continuous function f for which the limit (2.13) exists and is continuous, is dense in the space of continuous functions on I .

Note that by definition $p = \frac{m_2}{m_1+m_2}$ and $q = \frac{m_1}{m_1+m_2}$. The transition probabilities p and q can be assigned inductively as shown on Fig. 1.

Proposition 2.4 (Self-similarity of the Laplacian)

$$\Delta_\mu(u \circ F_w) = \left(1 + \frac{2}{pq}\right)^{-|w|} (\Delta_\mu u) \circ F_w. \quad (2.15)$$

The above construction of the standard Laplacian and the associated Dirichlet form on I corresponds to the case $p = \frac{1}{2}$. In the $p \neq \frac{1}{2}$ case, a change of variables can either turn the Dirichlet form into the standard one, or turn the μ -measure into Lebesgue measure, but *not both* at the same time. For this reason, different values of p give different μ -Laplacians even up to a change of variable.

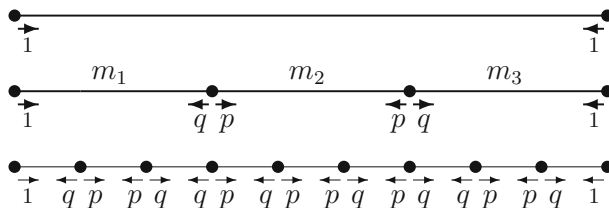


Fig. 1 Random walks corresponding to the discrete Laplacians Δ_n

We can apply the classical result of Kigami and Lapidus [39] to show that both the Dirichlet and the Neumann Laplacians Δ_μ satisfy the spectral asymptotics

$$0 < \liminf_{\lambda \rightarrow \infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} \leq \limsup_{\lambda \rightarrow \infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} < \infty, \tag{2.16}$$

where as before $\rho(\lambda)$ is the eigenvalue counting function, and the spectral dimension is

$$d_s = \frac{\log 9}{\log(1 + \frac{2}{pq})} \leq 1, \tag{2.17}$$

where the inequality is strict if and only if $p \neq q$.

In the lemma below, $\sigma(\Delta_n)$ is the spectrum n of the level n Laplacian Δ_n .

Lemma 2.5 *If $z \neq 1 \pm p$, then $R(z) \in \sigma(\Delta_n)$ if and only if $z \in \sigma(\Delta_{n+1})$, with the same multiplicities. Here*

$$R(z) = \frac{z(z^2 - 3z + 2 + pq)}{pq}. \tag{2.18}$$

Moreover, the Neumann discrete Laplacians have simple spectrum with $\sigma(\Delta_0) = \{0, 2\}$ and

$$\sigma(\Delta_n) = \{0, 2\} \cup_{m=0}^{n-1} R^{-m}\{1 \pm q\} \tag{2.19}$$

for all $n > 0$. In particular, for all $n > 0$ we have $0, 1 \pm q, 2 \in \sigma(\Delta_n)$. Also, for all $n > 0$ we have $1 \pm p \in \sigma(\Delta_n)$ if and only if $p = q$.

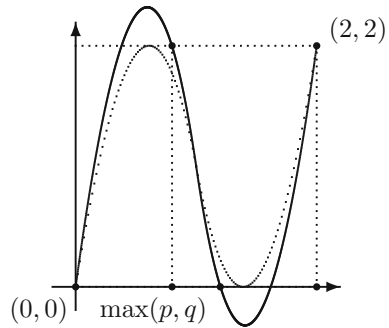
Proof In this case, according to [61, Lemma 3.4], [45, (3.2)], we have that $R(z) = \frac{\varphi_1(z)}{\varphi_0(z)}$, where φ_0 and φ_1 solve the matrix equation

$$S - zI_0 - \bar{X}(Q - zI_1)^{-1}X = \varphi_0(z)H_0 - \varphi_1(z)I_0. \tag{2.20}$$

with $S = I_0 = I_1 = I_{2 \times 2}$, $X = -qI_{2 \times 2}$, $\bar{X} = -I_{2 \times 2}$,

$$Q = \begin{pmatrix} 1 & -p \\ -p & 1 \end{pmatrix}, \tag{2.21}$$

Fig. 2 Sketch of the cubic polynomial $R(z)$ associated with the fractal Laplacians on the interval



and

$$H_0 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{2.22}$$

Solving this we obtain

$$\varphi_0(z) = \frac{pq}{z^2 - 2z + 1 - p^2} \tag{2.23}$$

and

$$\varphi_1(z) = \frac{z(z^2 - 3z + 2 + pq)}{z^2 - 2z + 1 - p^2}. \tag{2.24}$$

Then we use the abstract spectral self-similarity results (see [45,61]) to find that $\sigma(\Delta_{n+1}) = R^{-1}\{\sigma(\Delta_n)\}$. Note that 0 and 2 are fixed points of $R(z)$. The preimages of 0 are 0, $1 + p$ and $1 + q$. The preimages of 2 are 2, $1 - p$ and $1 - q$. If $p \neq q$ then $1 \pm p$ are not eigenvalues because they are poles of $\varphi_0(z)$ (see [45,61]). \square

Remark 2.6 In Fig. 2 we give a sketch that describes the complex dynamics of the family of cubic polynomials associated with the fractal Laplacians on the interval (see [45,61]). The curved dotted line corresponds to the case when $p = \frac{1}{2}$ and the Julia set is the interval $[0, 2]$. For any other value of p ($0 < p < 1, p \neq \frac{1}{2}$), the graph of the polynomial $R(z)$ behaves like the shown solid curved line. It is easy to see that then the Julia set of $R(z)$ is a Cantor set of Lebesgue measure zero. Note that the transformation $p \mapsto 1 - p$ does not change the polynomial $R(z)$, although the Laplacians Δ_μ are different.

3 Spectral Decimation and Eigenfunction Approximations in the Limit

Thus far we have described the spectral decimation which allows us to characterize the eigenvalues of the fractal Laplacian (Lemma 2.5). We now turn to the eigenfunctions.

3.1 Eigenfunction Extension

In this subsection we demonstrate how to extend an eigenfunction $f_{n,*}$ to an eigenfunction $f_{n+1,*}$ using spectral decimation.

To fix notation, let $x_0 < y_0 < y_1 < x_1$ be four consecutive vertices in V_{n+1} with $x_0, x_1 \in V_n$ and $y_0, y_1 \in V_{n+1} \setminus V_n$. Given an eigenfunction $f_{n,*}$ of Δ_n with eigenvalue $\lambda_{n,*}$, we define its extension $f_{n+1,*}$ to V_{n+1} according to the formulas

$$f_{n+1,*}(y_0) = \frac{q(1-z)f_{n,*}(x_0) + pqf_{n,*}(x_1)}{(1-p-z)(1+p-z)}, \tag{3.1}$$

$$f_{n+1,*}(y_1) = \frac{q(1-z)f_{n,*}(x_1) + pqf_{n,*}(x_0)}{(1-p-z)(1+p-z)}. \tag{3.2}$$

Here we assume $z \neq 1 \pm p$. The claim is that $f_{n+1,*}$ is an eigenfunction of Δ_{n+1} with eigenvalue $z = R^{-1}(\lambda_{n,*})$, where R is the cubic polynomial which appeared in Lemma 2.5. As explained in the proof of 2.5, the preimage $R^{-1}([0, 2])$ has three branches, so each eigenvalue $\lambda_{n,*}$ on level n generates three new eigenvalues $\lambda_{n+1,*}$ on level $(n + 1)$. The only exceptions are the eigenvalues 0 and 2, each of which generates two new eigenvalues because $1 \pm p$ are forbidden (see Fig. 3). This means that each eigenfunction extends to either two or three eigenfunctions at the next level.

Theorem 3.1 (Eigenfunction extension) *Suppose $f_{n,*} : V_n \rightarrow \mathbb{R}$ is an eigenfunction of Δ_n with eigenvalue $\lambda_{n,*}$. Let $f_{n+1,*} : V_{n+1} \rightarrow \mathbb{R}$ be an extension of $f_{n,*}$ to V_{n+1} defined via (3.1) and (3.2), with $z \neq 1 \pm p$. If*

$$\lambda_{n,*} = R(z) = \frac{z^3 - 3z^2 + (2 + pq)z}{pq}, \tag{3.3}$$

then $f_{n+1,*}$ is an eigenfunction of Δ_{n+1} with eigenvalue $z = R^{-1}(\lambda_{n,*})$.

Proof We break the proof into two parts. Given $f_{n,*}$, we first show that the following are equivalent for an extension $f_{n+1,*}$ of $f_{n,*}$:

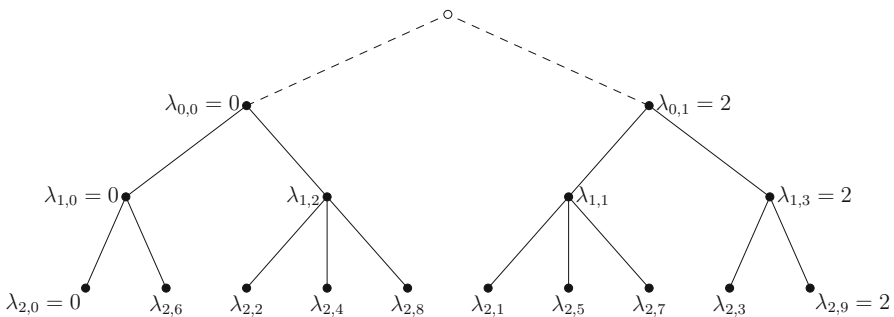


Fig. 3 Eigenvalues of the first three fractal levels arranged in a rooted tree and numbered in increasing order

- (1) $f_{n+1,*}$ is defined via the extension formulas (3.1) and (3.2).
- (2) $f_{n+1,*}$ satisfies the eigenvalue equation $\Delta_{n+1} f_{n+1,*} = z f_{n+1,*}$ on $V_{n+1} \setminus V_n$.

After establishing this equivalence, we proceed to show that $f_{n+1,*}$ is an eigenfunction of Δ_{n+1} on all of V_{n+1} , provided that (3.3) holds.

First we show the equivalence of (1) and (2). Assuming (2), we apply the eigenvalue equation $\Delta_{n+1} f_{n+1,*} = z f_{n+1,*}$ at the points $y_0, y_1 \in V_{n+1} \setminus V_n$ to obtain, by using both formulae in (2.14) depending on the point x (in fact, to cover these cases as well as the case $x \in V_0$, i.e. x is a boundary vertex, below we use the parameters $a \in \{p, q, 0, 1\}$, and $b = 1 - a$, instead of p and $q = 1 - p$),

$$(1 - z) f_{n+1,*}(y_0) = p f_{n+1,*}(y_1) + q f_{n+1,*}(x_0), \tag{3.4}$$

$$(1 - z) f_{n+1,*}(y_1) = q f_{n+1,*}(x_1) + p f_{n+1,*}(y_0). \tag{3.5}$$

This is a linear system of 2 equations with 2 unknowns ($f_{n+1,*}(x_0)$ and $f_{n+1,*}(x_1)$ are known, $f_{n+1,*}(y_0)$ and $f_{n+1,*}(y_1)$ are unknown), which has a unique solution. After some elementary calculation, and using the fact that $f_{n+1,*}|_{V_n} = f_{n,*}$, it is easy to verify that $f_{n+1,*}(y_0)$ and $f_{n+1,*}(y_1)$ are uniquely expressed in terms of $f_{n,*}(x_0)$ and $f_{n,*}(x_1)$ according to the extension formulas (3.1) and (3.2), which shows (1). The reverse implication (1) \Rightarrow (2) is straightforward.

At this point we have proved that the eigenvalue equation $\Delta_{n+1} f_{n+1,*}(x) = z f_{n+1,*}(x)$ holds for $x \in V_{n+1} \setminus V_n$. However, we have neither used the property of the eigenfunction $f_{n,*}$, nor related z to the eigenvalue $\lambda_{n,*}$. To do so we must check the Δ_{n+1} -eigenvalue equation on V_n .

We introduce some additional notation. Fix an $x \in V_n$. Let $x'_0, x'_1 \in V_n$ be adjacent to x on level n , and $y'_0, y'_1 \in V_{n+1} \setminus V_n$ be adjacent to x on level $(n + 1)$, as shown in Fig. 8. (If $x \in V_0 = \{0, 1\}$, then there is only one adjacent vertex on level n . This will be taken care of in the next argument.) We also label the transition probabilities according to (2.14); see also Fig. 1. The parameter a can be one of $\{p, q, 0, 1\}$ depending on x . In particular, to take into account that $x \in V_0$ has only 1 adjacent vertex, we set $a = 0$ if $x = 1$ and $a = 1$ if $x = 0$. The parameter b is set to equal $1 - a$.

Now we show that if $z \neq 1 \pm p$ and (3.3) holds, then $\Delta_{n+1} f_{n+1,*}(x) = z f_{n+1,*}(x)$ for $x \in V_n$. By (2.14),

$$\Delta_{n+1} f_{n+1,*}(x) = f_{n+1,*}(x) - a f_{n+1,*}(y'_1) - b f_{n+1,*}(y'_0), \tag{3.6}$$

$$\Delta_n f_{n,*}(x) = f_{n,*}(x) - a f_{n,*}(x'_1) - b f_{n,*}(x'_0). \tag{3.7}$$

Using the extension formulas (3.1), (3.2), $f_{n+1,*}(x) = f_{n,*}(x)$, and (3.6), we find

$$\begin{aligned} (\Delta_{n+1} - z) f_{n+1,*}(x) &= (1 - z) f_{n,*}(x) - a \left(\frac{q(1 - z) f_{n,*}(x) + pq f_{n,*}(x'_1)}{(1 - p - z)(1 + p - z)} \right) \\ &\quad - b \left(\frac{q(1 - z) f_{n,*}(x) + pq f_{n,*}(x'_0)}{(1 - p - z)(1 + p - z)} \right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{(1-z)(1-p-z)(1+p-z) - q(1-z)}{(1-p-z)(1+p-z)} f_{n,*}(x) \\
 &\quad - \frac{pq (af_{n,*}(x'_1) + bf_{n,*}(x'_0))}{(1-p-z)(1+p-z)}. \tag{3.8}
 \end{aligned}$$

Using (3.7) we can write

$$af_{n,*}(x'_1) + bf_{n,*}(x'_0) = -(\Delta_n - 1)f_{n,*}(x) = -(\lambda_{n,*} - 1)f_{n,*}(x). \tag{3.9}$$

This allows us to replace the second term of (3.8), so that the entire (3.8) equals

$$\begin{aligned}
 &\frac{(1-z)[(1-p-z)(1+p-z) - q] - pq(1-\lambda_{n,*})}{(1-p-z)(1+p-z)} f_{n,*}(x) \\
 &= \frac{-(z^3 - 3z^2 + (2 + pq)z) + pq - pq(1-\lambda_{n,*})}{(1-p-z)(1+p-z)} f_{n,*}(x). \tag{3.10}
 \end{aligned}$$

Infer that $(\Delta_{n+1} - z)f_{n+1,*} = 0$ on V_n , and in turn on V_{n+1} , if $z \neq 1 \pm p$ and

$$\lambda_{n,*} = \frac{z^3 - 3z^2 + (2 + pq)z}{pq} = R(z). \tag{3.11}$$

□

3.2 Continuity in the Limit

In this subsection, we show that the eigenfunction extension algorithm (Theorem 3.1) produces a continuous eigenfunction of the fractal Laplacian in the limit $n \rightarrow \infty$, provided that one always chooses the lowest branch of the inverse map R^{-1} at all levels $n \geq n_0$.

Lemma 3.2 Fix $n_0, k \in \mathbb{N} \cup \{0\}$. Let $f_{n_0,k} : V_{n_0} \rightarrow \mathbb{R}$ be an eigenfunction of Δ_{n_0} with eigenvalue $\lambda_{n_0,k}$. Let $\{f_{n_0+i,k}\}_{i=1}^\infty$ be the sequence of Δ_{n_0+i} -eigenfunctions extended from $f_{n_0,k}$ via successive applications of Theorem 3.1, where one always chooses the lowest branch of the inverse cubic polynomial $R^{-1}(z)$ [see (3.3)]. Then

$$\limsup_{i \rightarrow \infty} \max_{x \in V_{n_0+i}} |f_{n_0+i,k}(x)| \tag{3.12}$$

is bounded.

Proof From Lemma 2.5 we know that $\lambda_{n,*} = R(\lambda_{n+1,*})$. Assume that the lowest branch of R^{-1} is chosen to generate $\lambda_{n+1,k} = R^{-1}(\lambda_{n,k})$ from $\lambda_{n,k}$. Observe that R is concave on $[0, \min(p, q)]$; therefore the graph of R on $[0, \min(p, q)]$ lies above the secant line connecting $(0, 0)$ and $(\min(p, q), 2)$ (see Fig. 2). This implies the inequality

$$\lambda_{n,k} = R(\lambda_{n+1,k}) \geq \frac{2}{\min(p, q)} \lambda_{n+1,k}. \tag{3.13}$$

By iterating this inequality, we see that the i -fold preimage $\lambda_{n_0+i,k} = R^{-i}(\lambda_{n_0,k})$, where the lowest branch of R^{-1} is always chosen, satisfies

$$\lambda_{n_0+i,k} \leq \left(\frac{\min(p, q)}{2}\right)^i \lambda_{n_0,k}. \tag{3.14}$$

The corresponding eigenfunctions $f_{n_0+i,k}$ are generated via Theorem 3.1.

Let $M_{n,k} = \max_{x \in V_n} |f_{n,k}(x)|$. For each $n > n_0$ and each $y \in V_{n+1} \setminus V_n$, we use the eigenfunction extension algorithm (3.1) and (3.2) to arrive at the following estimate: there exist $x_0, x_1 \in V_n$ such that

$$|f_{n+1,k}(y)| = \left| \frac{q(1 - \lambda_{n+1,k})f_{n+1,k}(x_0) + pqf_{n+1,k}(x_1)}{(1 - p - \lambda_{n+1,k})(1 + p - \lambda_{n+1,k})} \right| \tag{3.15}$$

$$\leq \frac{q(1 - \lambda_{n+1,k})|f_{n+1,k}(x_0)| + pq|f_{n+1,k}(x_1)|}{(1 - p - \lambda_{n+1,k})(1 + p - \lambda_{n+1,k})} \tag{3.16}$$

$$\leq \frac{q}{q - \lambda_{n+1,k}} M_{n,k}. \tag{3.17}$$

In the second line we used the triangle inequality and the bound $\lambda_{n+1,k} < 1$, which can be seen from (3.14). This then implies the estimate

$$M_{n+1,k} \leq \frac{q}{q - \lambda_{n+1,k}} M_{n,k}. \tag{3.18}$$

for all $n \geq n_0$. Applying (3.18) inductively and using (3.14), we see that for all $i \in \mathbb{N}$,

$$M_{n_0+i,k} \leq M_{n_0,k} \prod_{j=1}^i \frac{q}{q - \lambda_{n_0+j,k}} \leq M_{n_0,k} \prod_{j=1}^i \left(1 - \left(\frac{\min(p, q)}{2}\right)^j \frac{\lambda_{n_0,k}}{q}\right)^{-1}. \tag{3.19}$$

Setting $\gamma_j = \left(\frac{\min(p, q)}{2}\right)^j \frac{\lambda_{n_0,k}}{q}$ and taking the limit, we obtain

$$\limsup_{i \rightarrow \infty} M_{n_0+i,k} \leq M_{n_0,k} \limsup_{i \rightarrow \infty} \prod_{j=1}^i (1 - \gamma_j)^{-1}. \tag{3.20}$$

It remains to verify the convergence of the infinite product $\prod_{j=1}^{\infty} (1 - \gamma_j)^{-1}$, which is equivalent to showing the convergence of the series $\sum_{j=1}^{\infty} \log(1 - \gamma_j)^{-1}$. Observe that if we set ϵ_j to satisfy $(1 - \gamma_j)^{-1} = 1 + \epsilon_j$, then

$$\log(1 - \gamma_j)^{-1} = \log(1 + \epsilon_j) \leq \epsilon_j \tag{3.21}$$

by the inequality $1 + x \leq e^x$. Moreover, since $\gamma_j = Kr^j$ for suitable positive constants K and $r \leq \frac{1}{2}$, we can always find a constant K_0 such that

$$\epsilon_j = \frac{\gamma_j}{1 - \gamma_j} = \frac{Kr^j}{1 - Kr^j} \leq K_0r^j \tag{3.22}$$

for all sufficiently large j . Since the geometric series $\sum_j K_0r^j$ converges, this implies that the series $\sum_{j=1}^\infty \log(1 - \alpha_j)^{-1}$ converges. \square

Now we prove the continuity of the eigenfunction in the limit.

Theorem 3.3 *Let $\{f_{n_0+i,k}\}_{i=1}^\infty$ be the sequence of Δ_{n_0+i} -eigenfunctions extended from $f_{n_0,k}$ as in Lemma 3.2. Then the limit $f_k := \lim_{i \rightarrow \infty} f_{n_0+i,k}$ is uniform on I , and can be extended to a continuous function on I .*

Proof The key argument is that since the eigenvalues $\lambda_{n_0+i,k} = R^{-i}(\lambda_{n_0,k})$ tend to 0 as $i \rightarrow \infty$, the eigenfunction extension (3.1) and (3.2) of $f_{n_0+i,k}$ to V_{n_0+i+1} can be approximated by the harmonic extension of $f_{n_0+i,k}$ to V_{n_0+i+1} as $i \rightarrow \infty$, uniformly on I . Since a harmonic extension on I is continuous in the limit, we deduce that the limit f_k can also be extended to a continuous function.

Define, for each $n \geq n_0$ and each k , the harmonic extension $\tilde{f}_{n+1,k}$ of $f_{n,k}$ to V_{n+1} . Using the coordinates x_0, x_1, y_0, y_1 introduced before Theorem 3.1,

$$\tilde{f}_{n+1,k}(y_0) = \frac{f_{n,k}(x_0) + pf_{n,k}(x_1)}{1 + p}, \tag{3.23}$$

$$\tilde{f}_{n+1,k}(y_1) = \frac{f_{n,k}(x_1) + pf_{n,k}(x_0)}{1 + p}. \tag{3.24}$$

Note that these are (3.1) and (3.2) with $z = 0$.

Let us now estimate $|f_{n+1,k}(y_0) - \tilde{f}_{n+1,k}(y_0)|$, which equals

$$\left| \frac{q(1 - \lambda_{n,k})f_{n,k}(x_0) + pqf_{n,k}(x_1)}{(1 - p - \lambda_{n,k})(1 + p - \lambda_{n,k})} - \frac{f_{n,k}(x_0) + pf_{n,k}(x_1)}{1 + p} \right| \tag{3.25}$$

$$= \left| \frac{\lambda_{n,k}(1 + p - pq - \lambda_{n,k})f_{n,k}(x_0) + p\lambda_{n,k}(2 - \lambda_{n,k})f_{n,k}(x_1)}{(q - \lambda_{n,k})(1 + p - \lambda_{n,k})(1 + p)} \right| \tag{3.26}$$

Using the triangle inequality and then replacing $|f_{n,k}(x_0)|$ and $|f_{n,k}(x_1)|$ by the sup $M_{n,k} = \sup_{x \in V_n} |f_{n,k}(x)|$, we can bound (3.26) from above by

$$\frac{\lambda_{n,k}(1 + p - pq - \lambda_{n,k})M_{n,k} + p\lambda_{n,k}(2 - \lambda_{n,k})M_{n,k}}{(q - \lambda_{n,k})(1 + p - \lambda_{n,k})(1 + p)} = \frac{\lambda_{n,k}}{q - \lambda_{n,k}}M_{n,k} \tag{3.27}$$

Since $\lim_{n \rightarrow \infty} M_{n,k}$ is bounded by Lemma 3.2 and $\lambda_{n,k} \rightarrow 0$ as $n \rightarrow \infty$, the right-hand side of this inequality tends to 0. The same estimate holds for $|f_{n+1,k}(y_1) - \tilde{f}_{n+1,k}(y_1)|$. Since y_0 and y_1 are arbitrary, we conclude that $|f_{n,k} - \tilde{f}_{n,k}|$ converges to 0 uniformly on I . \square

3.3 Spectral Decomposition of the Delta Function

Let $\{f_{n,k}\}_{k=0}^{3^n}$ be a complete set of eigenfunctions of Δ_n with corresponding eigenvalues $\{\lambda_{n,k}\}_{k=0}^{3^n}$. Consider the level- n delta function $\delta_0^{(n)} : V_n \rightarrow \mathbb{R}$ defined by

$$\delta_0^{(n)}(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \in V_n \setminus \{0\}. \end{cases} \quad (3.28)$$

By the spectral theorem, we can find a set of real numbers (or weights) $\{\alpha_{n,k}\}_{k=0}^{3^n}$ such that

$$\delta_0^{(n)}(x) = \sum_{k=0}^{3^n} \alpha_{n,k} f_{n,k}(x). \quad (3.29)$$

The sequence $\delta_0^{(n)}$ approximates a delta function at 0 in the limit $n \rightarrow \infty$.

In order to study the wave equation in Sect. 5, we need estimates on the eigensolutions $(f_{n,*}, \lambda_{n,*})$, as well as information about the weights $\alpha_{n,*}$. We will address the former in Sect. 4, and the latter in the following proposition.

Proposition 3.4 *The weights $\alpha_{n+1,*}$ can be obtained inductively from $\alpha_{n,*}$.*

Proof First we fix our convention at $n = 0$. The two (non- ℓ^2 -normalized) eigenfunctions $f_{0,1}$ and $f_{0,2}$ of Δ_0 are

$$(f_{0,1}(0), f_{0,1}(1)) = (1, 1) \text{ and } (f_{0,2}(0), f_{0,2}(1)) = (1, -1), \quad (3.30)$$

with corresponding eigenvalue 0 and 2, respectively. It is then easy to see that $\delta_0^{(0)} = \frac{1}{2}f_{0,1} + \frac{1}{2}f_{0,2}$, i.e., $\alpha_{0,1} = \alpha_{0,2} = \frac{1}{2}$.

For the iteration step, suppose the weights $\alpha_{n,*}$ are known at level n , and we want to determine the weights $\alpha_{n+1,*}$. The idea is to write each contribution $\alpha_{n,k}f_{n,k}$ in terms of a linear combination $\sum_j \alpha_{n+1,k_j}f_{n+1,k_j}$ of the (2 or 3) eigenfunctions f_{n+1,k_j} which are extensions of $f_{n,k}$ given by Theorem 3.1.

To make this idea precise without adding too much notation, we fix n and k , and write f , λ , and α as respective shorthands for $f_{n,k}$, $\lambda_{n,k}$, and $\alpha_{n,k}$. If $\lambda \notin \{0, 2\}$, then spectral decimation (Theorem 3.1) implies that f has 3 extensions f_1 , f_2 , and f_3 to V_{n+1} which are eigenfunctions of Δ_{n+1} with respective eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3$. We would like to find the corresponding weights α_1 , α_2 , and α_3 by imposing the following matching condition: For any four consecutive vertices $x_0 < x_1 < x_2 < x_3$ in V_{n+1} with $x_0, x_1 \in V_n$ and $x_2, x_3 \in V_{n+1} \setminus V_n$,

$$\sum_{i=1}^3 \alpha_i f_i(x) = \begin{cases} \alpha f(x_0), & \text{if } x = x_0, \\ \alpha f(x_1), & \text{if } x \in \{x_1, x_2, x_3\}. \end{cases} \quad (3.31)$$

An explicit calculation verifies that with the weights $\alpha_{n+1,*}$ generated from this matching condition, we have

$$\sum_{k=0}^{3^{n+1}} \alpha_{n+1,k} f_{n+1,k} = \delta_0^{(n+1)}. \tag{3.32}$$

We now determine the weights. Observe that f_i and f agree on V_n by construction. This together with the matching condition (3.31) at x_0 (or at x_1) implies that

$$\sum_{i=1}^3 \alpha_i = \alpha. \tag{3.33}$$

Next, using the eigenfunction extension formula (3.1) and the matching condition at y_0 in (3.31), we get

$$\sum_{i=1}^3 \frac{q(1 - \lambda_i)}{(1 - p - \lambda_i)(1 + p - \lambda_i)} \alpha_i = 0. \tag{3.34}$$

Notice that there is no dependence on f . Similarly, using (3.2) and the matching condition at y_1 in (3.31), we arrive at a third relation

$$\sum_{i=1}^3 \frac{pq}{(1 - p - \lambda_i)(1 + p - \lambda_i)} \alpha_i = 0. \tag{3.35}$$

Equations (3.33), (3.34), and (3.35) form a linear system of 3 equations with 3 unknowns $(\alpha_1, \alpha_2, \alpha_3)$. It has the unique solution

$$\alpha_1 = \frac{(\lambda_3 - \lambda_2)(q - \lambda_1)(1 + p - \lambda_1)}{(\lambda_2 - \lambda_1)(q - \lambda_3)(1 + p - \lambda_3)} \alpha_3, \tag{3.36}$$

$$\alpha_2 = \frac{(\lambda_1 - \lambda_3)(q - \lambda_2)(1 + p - \lambda_2)}{(\lambda_2 - \lambda_1)(q - \lambda_3)(1 + p - \lambda_3)} \alpha_3, \tag{3.37}$$

$$\alpha_3 = \frac{(1 + p - \lambda_3)(q - \lambda_3)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \alpha. \tag{3.38}$$

It is possible to find α_3 from α and λ_3 only. From (3.3) we know that $R(\lambda_i) = \lambda$ for $i \in \{1, 2, 3\}$, which means that

$$(z - \lambda_1)(z - \lambda_2)(z - \lambda_3) = pqR(z) - \lambda. \tag{3.39}$$

By differentiating both sides of (3.39) with respect to z , and then evaluating at $z = \lambda_3$, we obtain

$$(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2) = pqR'(\lambda_3) = 3\lambda_3^2 - 6\lambda_3 + 2 + pq. \tag{3.40}$$

This allows us to replace the denominator in the RHS of (3.38), which leads to

$$\alpha_3 = \frac{(1 + p - \lambda_3)(q - \lambda_3)}{3\lambda_3^2 - 6\lambda_3 + 2 + pq} \alpha. \quad (3.41)$$

This proves the induction from $\alpha_{n,*}$ to $\alpha_{n+1,*}$ in the case where the eigenvalue $\lambda \notin \{0, 2\}$.

If $\lambda \in \{0, 2\}$, then f has two eigenfunction extensions to the next level. The matching condition stated in (3.31) remains the same, but degenerates to a linear system of 2 equations with 2 unknowns. We omit the details.

As a simple corollary, we now show that the weights $\alpha_{*,*}$ are all nonnegative in our convention. Recall that $\alpha_{0,1} = \alpha_{0,2} = \frac{1}{2}$. By the structure of the cubic polynomial $R(z)$ (see Fig. 2), $\lambda_1 \in [0, \min(p, q)]$, $\lambda_2 \in [\max(p, q), \min(1 + p, 1 + q)]$, and $\lambda_3 \in [\max(1 + p, 1 + q), 2]$. So if α is nonnegative, it is direct to verify using (3.36) through (3.38) that α_1 , α_2 , and α_3 are all nonnegative. By induction we deduce that all weights $\alpha_{*,*}$ are nonnegative. \square

Using the aforementioned result, we now define the “approximate delta functions.” Based on Lemma 2.5, and the fact that the lowest branch of $R^{-1}(z)$ is increasing, we can deduce that the lowest $|V_n|$ eigenvalues of Δ_{n+1} are determined recursively by

$$\lambda_{n+1,k} = (\text{The lowest branch of } R^{-1})(\lambda_{n,k}) \quad \text{for } 0 \leq k \leq 3^n. \quad (3.42)$$

Given the level- n delta function $\delta_0^{(n)}$, we define its approximation at level $n_0 < n$ by

$$\delta_0^{(n_0,n)}(x) := \sum_{k=0}^{3^{n_0}} \alpha_{n_0,k} f_{n,k}(x) \quad \text{for } x \in V_n. \quad (3.43)$$

In other words, we consider a truncated series of the spectral representation at level n_0 , fixing the coefficients $\alpha_{n_0,k}$, but taking the eigenfunctions $f_{n,k}$ to level n .

4 Estimates of Eigenvalues and Eigenfunctions

In this section, we use the spectral decimation to derive finer estimates of the eigenvalues and the eigenfunctions, which will be used in Sect. 5. Of particular importance is the constant $\mathbb{C} := R'(0) = \frac{2+pq}{pq}$, the renormalization factor for the eigenvalues $\{\lambda_{n,k}\}_n$. Its significance derives from the following fact.

Proposition 4.1 *For each $k \in \mathbb{N} \cup \{0\}$, the limit $\lim_{n \rightarrow \infty} [R'(0)]^n \lambda_{n,k}$ exists.*

Proof Let φ be the lowest branch of R^{-1} , which we regard as a function on \mathbb{C} . Via a power series expansion, we see that $\varphi(z)$ has an attracting fixed point at $z = 0$, with $\varphi(0) = 0$ and $\varphi'(0) = [R'(0)]^{-1} = \frac{pq}{2+pq} < 1$. By Koenigs' theorem (see for example [46, Sect. 8]), the renormalized iterates $\{z \mapsto [\varphi'(0)]^{-n} \varphi^n(z)\}_n$ converge uniformly on compact subsets of a local neighborhood of 0.

Now given a fixed k , the recurrence relation (3.42) implies that there exists $n_0 = n_0(k)$ such that $\lambda_{n+1,k} = \varphi(\lambda_{n,k})$ for all $n \geq n_0$. Combine this with the foregoing result and we conclude that the limit

$$\lim_{n \rightarrow \infty} R'(0)^n \lambda_{n,k} = [\varphi'(0)]^{-n_0} \lim_{n \rightarrow \infty} [\varphi'(0)]^{-n+n_0} \varphi^{n-n_0}(\lambda_{n_0,k}) \tag{4.1}$$

exists. □

In what follows we denote $\lambda_k := \lim_{n \rightarrow \infty} \mathbb{C}^n \lambda_{n,k}$. Our next result gives an upper and a lower bound on λ_k .

Theorem 4.2 Fix $p \in (0, \frac{1}{2})$, and let k and n_0 be as in the proof of Proposition 4.1. Then

$$\lambda_{n_0,k} \left(1 + \frac{3(pq)}{(2 + pq)^2} \lambda_{n_0,k} \right) \leq \frac{\lambda_k}{\mathbb{C}^{n_0}} \leq \lambda_{n_0,k} \exp \left(\lambda_{n_0,k} \frac{p(2 + q)}{2q(2 - p)} \right). \tag{4.2}$$

Proof As in the previous proof, let φ be the lowest branch of R^{-1} . The lower bound on λ_k will come from the Taylor approximation to φ , while the upper bound will come from a quadratic function which is at least as large as φ .

Lower bound. We compute the Taylor series expansion of φ about 0 to 2nd order in z . It is

$$a(z) = \frac{pq}{2 + pq} z + \frac{3(pq)^2}{(2 + pq)^3} z^2. \tag{4.3}$$

This is explained by the fact that the first derivative of the inverse function $\varphi(z)$ is given by $\varphi' = 1/R'$, and its second derivative is given by $\varphi'' = -R''/(R')^3$. Computing these derivatives at zero gives the quadratic function (4.3).

Furthermore we claim that $a(z) < \varphi(z)$ for all $z \in (0, 2)$. It is enough to check that $\varphi'''(z) > 0$. Here we use the identity

$$\frac{d^3 y}{dx^3} = -\frac{d^3 x}{dy^3} \left(\frac{dy}{dx} \right)^4 + 3 \left(\frac{d^2 x}{dy^2} \right)^2 \left(\frac{dy}{dx} \right)^5, \tag{4.4}$$

which in our context reads

$$\varphi'''(z) = -R'''(\varphi(z))[\varphi'(z)]^4 + 3[R''(\varphi(z))]^2[\varphi'(z)]^5. \tag{4.5}$$

Since $\varphi'(z) > 0$, we can factor out $[\varphi'(z)]^5$ from (4.5), and use the identity $\varphi'(z) = [R'(\varphi(z))]^{-1}$ so that we reduce the original sign question to checking the sign of

$$-R'''(\varphi(z))R'(\varphi(z)) + 3[R''(\varphi(z))]^2 \tag{4.6}$$

$$= \left(\frac{1}{pq} \right)^4 \left[-6 \cdot (3[\varphi(z)]^2 - 6\varphi(z) + (2 + pq)) + 3(6\varphi(z) - 6)^2 \right] \tag{4.7}$$

$$= \left(\frac{1}{pq} \right)^4 \cdot 6 \cdot (15[\varphi(z)]^2 - 30\varphi(z) + 16 - pq) \tag{4.8}$$

$$= \left(\frac{1}{pq}\right)^4 \cdot 6 \cdot [15(\varphi(z) - 1)^2 + (1 - pq)], \tag{4.9}$$

which is always positive. This shows that $\varphi'''(z) > 0$, and thus $a(z) < \varphi(z)$ for $z \in (0, 2)$. Combined with the fact that the functions $z \mapsto a(z)$ and $z \mapsto \varphi(z)$ are both monotone increasing on $[0, 2)$, this implies that for each $n \in \mathbb{N}$, $a^n(z) \leq \varphi^n(z)$ for $z \in [0, 2)$.

Fix $k \in \mathbb{N} \cup \{0\}$ and $n_0 = n_0(k)$ as in the proof of Proposition 4.1. Put $z_0 = \lambda_{n_0,k}$, and define the sequence of numbers $\{z_n\}_n$ inductively by $z_{n+1} = \varphi(z_n)$. Then, by the inequalities above, we have

$$\varphi(z_n) \geq a(z_n) = \frac{pq}{2 + pq} z_n + \frac{3(pq)^2}{(2 + pq)^3} z_n^2 \tag{4.10}$$

$$= z_n \left(\frac{pq}{2 + pq}\right) \left(1 + \frac{3(pq)}{(2 + pq)^2} z_n\right) \tag{4.11}$$

$$\geq a(z_{n-1}) \left(\frac{pq}{2 + pq}\right) \left(1 + \frac{3(pq)}{(2 + pq)^2} a(z_{n-1})\right). \tag{4.12}$$

Iterating this process we arrive at the estimate

$$\varphi(z_n) \geq z_{n_0} \left(\frac{pq}{2 + pq}\right)^{n-n_0} \prod_{j=n_0}^n \left(1 + \frac{3(pq)}{(2 + pq)^2} z_j\right). \tag{4.13}$$

Noting that $1 + \frac{3(pq)}{(2+pq)^2} z > 1$, we obtain a slightly crude but still efficient estimate

$$\varphi(z_n) \geq z_{n_0} \left(\frac{pq}{2 + pq}\right)^{n-n_0} \left(1 + \frac{3(pq)}{(2 + pq)^2} z_{n_0}\right), \tag{4.14}$$

which is the claimed lower bound in (4.2).

Upper bound. To bound $\varphi(z)$ from above, we construct a quadratic function $h(z)$ such that $h(0) = 0$, $h(2) = p$ and $h'(0) = \varphi'(0)$. A simple calculation shows that

$$h(z) = \frac{pqz}{2 + pq} \left(1 + \frac{p(2 + q)}{4q} z\right), \tag{4.15}$$

and $h(z) \geq \varphi(z)$ for $z \in [0, 2]$. Using this, along with the fact that $z_{n+1} = \varphi(z_n) \leq \frac{pq}{2+pq} z_n \left(1 + \frac{p(2+q)}{4q} z_n\right)$, and $z_{n+1} \leq \frac{p}{2} z_n$, we get the following estimate:

$$\varphi(z_n) \leq z_{n_0} \left(\frac{pq}{2 + pq}\right)^{n-n_0} \prod_{j=n_0}^n \left(1 + \frac{p(2 + q)}{4q} z_j\right) \tag{4.16}$$

$$\leq \lambda_{n_0,k} \left(\frac{pq}{2 + pq}\right)^{n-n_0} \prod_{j=n_0}^n \left(1 + \frac{p(2 + q)}{4q} \left(\frac{p}{2}\right)^{j-n_0} \lambda_{n_0,k}\right). \tag{4.17}$$

Therefore, using the inequality $1 + x \leq e^x$,

$$\begin{aligned} \lambda_k &= \lim_{n \rightarrow \infty} \left(\frac{2 + pq}{pq}\right)^n \varphi(z_n) \leq \lim_{n \rightarrow \infty} \left(\frac{2 + pq}{pq}\right)^n \left(\frac{pq}{2 + pq}\right)^{n-n_0} \lambda_{n_0,k} \\ &\times \prod_{j=n_0}^n \left(1 + \frac{p(2+q)}{4q} \left(\frac{p}{2}\right)^{j-n_0} \lambda_{n_0,k}\right) \leq \left(\frac{2 + pq}{pq}\right)^{n_0} \lambda_{n_0,k} \\ &\times \exp\left(\frac{p(2+q)}{4q} \frac{1}{1-\frac{p}{2}} \lambda_{n_0,k}\right) = \left(\frac{2 + pq}{pq}\right)^{n_0} \lambda_{n_0,k} \exp\left(\frac{p(2+q)}{2q} \frac{1}{2-p} \lambda_{n_0,k}\right). \end{aligned} \tag{4.18}$$

This gives the claimed upper bound in (4.2). □

For a function $f : V_n \rightarrow \mathbb{R}$, we denote its sup norm by $\|f\|_{n,\infty} = \sup\{|f(x)| : x \in V_n\}$. Likewise, the sup norm of $h : I \rightarrow \mathbb{R}$ is denoted by $\|h\|_\infty = \sup\{|h(x)| : x \in I\}$. Our next result is an estimate on the sup norms of the eigenfunctions of Δ_n .

Lemma 4.3 Fix $p \in (0, \frac{1}{2})$. Let $f_{n,k}$ be the eigenfunction corresponding to the $(k + 1)$ th lowest eigenvalue $\lambda_{n,k}$ of Δ_n . Then for every $m > n \geq n_0$,

$$\|f_{m,k}\|_{m,\infty} \leq \|f_{n,k}\|_{n,\infty} \prod_{j=n+1}^m \left(1 + \left(\frac{p}{2}\right)^{j-n} \frac{\lambda_{n,k}}{q - \lambda_{n+1,k}}\right). \tag{4.19}$$

In particular, if $f_k = \lim_{i \rightarrow \infty} f_{n_0+i,k}$ per Theorem 3.3, then

$$\|f_k\|_\infty \leq \|f_{n,k}\|_{n,\infty} \exp\left(\frac{\lambda_{n,k}}{q - \lambda_{n+1,k}} \frac{p}{2-p}\right). \tag{4.20}$$

Proof Let us introduce the function

$$F(A, B, z) = \frac{q(1-z)A + pqB}{(q-z)(1+p-z)}, \tag{4.21}$$

which is derived from the eigenfunction extension algorithm (3.1) and (3.2). Note that $z \in [0, p)$ because the extension uses the lowest branch of R^{-1} starting from level n_0 . First we would like to control the linear growth of $z \mapsto F(A, B, z) - F(A, B, 0)$:

$$\begin{aligned} &|F(A, B, z) - F(A, B, 0)| \\ &\leq |A| \left| \frac{q(1-z)}{(q-z)(1+p-z)} - \frac{q}{1-p^2} \right| \\ &\quad + |B| \left| \frac{pq}{(q-z)(1+p-z)} - \frac{pq}{1-p^2} \right| \end{aligned}$$

$$\begin{aligned}
&\leq \max(|A|, |B|)q \left(\left| \frac{(1-z)(1-p^2) - 1 + 2z + p^2 - z^2}{(q-z)(1+p-z)(1-p^2)} \right| \right. \\
&\quad \left. + p \left| \frac{2z - z^2}{(q-z)(1+p-z)(1-p^2)} \right| \right) \\
&= \max(|A|, |B|)q|z| \left(\left| \frac{1 + p^2 - z}{(q-z)(1+p-z)(1-p^2)} \right| \right. \\
&\quad \left. + p \left| \frac{2-z}{(q-z)(1+p-z)(1-p^2)} \right| \right). \tag{4.22}
\end{aligned}$$

Since $0 < z < p < \frac{1}{2}$, the absolute value terms in the RHS of (4.22) are positive, so we can drop the absolute value signs and add the two terms in the bracket to get

$$\begin{aligned}
&|F(A, B, z) - F(A, B, 0)| \\
&\leq \max(|A|, |B|)q|z| \left(\frac{1 + 2p + p^2 - z - pz}{(q-z)(1+p-z)(1-p^2)} \right) \\
&= \max(|A|, |B|)q|z|(1+p) \left(\frac{1+p-z}{(q-z)(1+p-z)(1-p^2)} \right) \\
&= \max(|A|, |B|) \frac{|z|}{(q-z)}, \tag{4.23}
\end{aligned}$$

which implies that

$$|F(A, B, z)| \leq \max(|A|, |B|)|z| \frac{1}{(q-z)} + |F(A, B, 0)|. \tag{4.24}$$

We use (4.24) to estimate the sup norms of the eigenfunctions: for all $n \geq n_0$,

$$\begin{aligned}
\|f_{n+1,k}\|_{n+1,\infty} &\leq \|f_{n,k}\|_{n,\infty} |F(1, 1, z_{n+1})| \\
&\leq \|f_{n,k}\|_{n,\infty} \left(\frac{|z_{n+1}|}{q - z_{n+1}} + |F(1, 1, 0)| \right) \\
&\leq \|f_{n,k}\|_{n,\infty} \left(\frac{z_{n+1}}{q - z_{n+1}} + 1 \right), \tag{4.25}
\end{aligned}$$

Iterating the inequality (4.25) and using the fact that $z_{n+m} \leq \left(\frac{p}{2}\right)^m z_n$ (3.14) gives

$$\begin{aligned}
\|f_{m,k}\|_{m,\infty} &\leq \|f_{n,k}\|_{n,\infty} \prod_{j=n+1}^m \left(1 + \frac{z_j}{q - z_j} \right) \\
&\leq \|f_{n,k}\|_{n,\infty} \prod_{j=n+1}^m \left(1 + \left(\frac{p}{2}\right)^{j-n} \frac{z_n}{q - z_{n+1}} \right) \\
&= \|f_{n,k}\|_{n,\infty} \prod_{j=n+1}^m \left(1 + \left(\frac{p}{2}\right)^{j-n} \frac{\lambda_{n,k}}{q - \lambda_{n+1,k}} \right) \tag{4.26}
\end{aligned}$$

for all $m > n \geq n_0$. This shows (4.19).

Next, using the triangle inequality and taking the supremum, we have

$$\left| \sup_{x \in I} |f_k(x)| - \sup_{x \in I} |f_{m,k}(x)| \right| \leq \sup_{x \in I} |f_k(x) - f_{m,k}(x)|. \tag{4.27}$$

Recall from Theorem 3.3 that the limit $\lim_{i \rightarrow \infty} f_{n_0+i,k}$ is uniform on I . This along with the bound (4.27) implies that

$$\|f_k\|_\infty = \lim_{m \rightarrow \infty} \|f_{m,k}\|_{m,\infty} \tag{4.28}$$

So by taking the limit $m \rightarrow \infty$ on both sides of (4.3), we arrive at the estimate

$$\begin{aligned} \|f_k\|_\infty &\leq \|f_{n,k}\|_{n,\infty} \lim_{m \rightarrow \infty} \prod_{j=n+1}^m \left(1 + \left(\frac{p}{2}\right)^{j-n} \frac{\lambda_{n,k}}{q - \lambda_{n+1,k}} \right) \\ &\leq \|f_{n,k}\|_{n,\infty} \exp \left(\sum_{j=n+1}^\infty \left(\frac{p}{2}\right)^{j-n} \frac{\lambda_{n,k}}{q - \lambda_{n+1,k}} \right) \\ &= \|f_{n,k}\|_{n,\infty} \exp \left(\frac{\lambda_{n,k}}{q - \lambda_{n+1,k}} \frac{p}{2 - p} \right), \end{aligned} \tag{4.29}$$

where in the second line we used the inequality $1 + x \leq e^x$. This proves (4.20). \square

The following result provides a quantitative estimate of the convergence of $f_{n,k}$ to f_k in sup norm. As in the previous section, we harmonically extend $f_{n,k}$ from V_n to I , and abuse notation by calling the extension $f_{n,k}$ still. Then $\Delta_\mu f_{n,k}(x) = 0$ for all $x \in I \setminus V_n$.

Theorem 4.4 Fix $p \in (0, \frac{1}{2})$. Then

$$\|f_k - f_{n,k}\|_\infty \leq \mathbb{C}^{-n} \lambda_k \|f_{n,k}\|_{n,\infty} \|g\|_\infty \exp \left(\frac{\lambda_{n,k}}{q - \lambda_{n+1,k}} \frac{p}{2 - p} \right), \tag{4.30}$$

where $g : I \times I \rightarrow \mathbb{R}_+$ is the Green’s function associated with Δ_μ , and

$$\|g\|_\infty := \sup_{(x,y) \in I \times I} g(x, y). \tag{4.31}$$

Proof Since f_k and $f_{n,k}$ agree on V_n , it is enough to estimate their difference on $I \setminus V_n$. Based on the construction described in Sect. 2, $I \setminus V_n$ is the disjoint union of $\{F_w\{(0, 1)\} : |w| = n\}$, where $w = w_1 w_2 \cdots w_n$ is a word of length n with $w_i \in \{1, 2, 3\}$ for $1 \leq i \leq n$, and $F_w := F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_n}$. So our task is to show that for every word w of length n , the function $|(f_k - f_{n,k}) \circ F_w|$ on $(0, 1)$ has a uniform upper bound.

Our strategy is to exploit the self-similarity of the fractal Laplacian Δ_μ (Proposition 2.4), as well as properties of the corresponding Green’s function. We remind the reader

that $G_\mu = \Delta_\mu^{-1}$ is the Green's operator associated to Δ_μ . It admits an integral kernel $g(\cdot, \cdot)$ called the Green's function, defined by

$$(G_\mu u)(y) = \int_I g(y, y')u(y') d\mu(y'). \quad (4.32)$$

The existence of G_μ and g follows from the theory of Kigami [37, Sects. 3.5–3.6]. In particular, $G_\mu : L^1(I, \mu) \rightarrow C(I)$, and g is a nonnegative continuous function on $I \times I$.

To begin the proof, we start with the self-similarity of Δ_μ (Proposition 2.4):

$$\Delta_\mu((f_k - f_{n,k}) \circ F_w) = \mathbb{C}^{-|w|} \Delta_\mu(f_k - f_{n,k}) \circ F_w \quad \text{on } I. \quad (4.33)$$

Since $f_{n,k}$ is harmonic with respect to Δ_μ on $F_w\{(0, 1)\}$ for every word w of length n , it follows that

$$\Delta_\mu(f_k - f_{n,k}) \circ F_w = \Delta_\mu f_k \circ F_w = \lambda_k f_k \circ F_w \quad \text{on } (0, 1). \quad (4.34)$$

Combine (4.33) and (4.34) and we get

$$\Delta_\mu((f_k - f_{n,k}) \circ F_w) = \mathbb{C}^{-n} \lambda_k f_k \circ F_w \quad \text{on } (0, 1). \quad (4.35)$$

Now apply the Green's operator G_μ on both sides of (4.35) to get

$$(f_k - f_{n,k}) \circ F_w = \mathbb{C}^{-n} \lambda_k G_\mu(f_k \circ F_w) \quad \text{on } (0, 1). \quad (4.36)$$

Using the representation (4.32) we obtain the estimate

$$\begin{aligned} |(f_k - f_{n,k}) \circ F_w| &\leq \mathbb{C}^{-n} \lambda_k \|G_\mu(f_k \circ F_w)\|_\infty \\ &\leq \mathbb{C}^{-n} \lambda_k \|g\|_\infty \|(f_k \circ F_w)\|_\infty \\ &\leq \mathbb{C}^{-n} \lambda_k \|g\|_\infty \|f_k\|_\infty \quad \text{on } (0, 1). \end{aligned} \quad (4.37)$$

This proves that

$$\|f_k - f_{n,k}\|_\infty \leq \mathbb{C}_0^{-n} \lambda_k \|g\|_\infty \|f_k\|_\infty. \quad (4.38)$$

In order to turn the RHS of (4.38) into a more useful estimate, we use (4.20) to replace $\|f_k\|_\infty$ by $\|f_{n,k}\|_{n,\infty} \exp\left(\frac{\lambda_{n,k}}{q-\lambda_{n+1,k}} \frac{p}{2-p}\right)$, which yields (4.30). \square

5 Estimates on the Solution of the Wave Equation

We now apply the results of Sects. 3 and 4 to estimate the solution of the wave equation on the interval I endowed with the fractal measure μ .

Numerically, we can only compute the eigensolutions of the fractal Laplacian up to a finite level, so in practice we solve the “approximate” wave equation

$$\begin{cases} \partial_{tt}u_n = -\Delta_n u_n & \text{on } V_n \times [0, T], \\ u_n(\cdot, 0) = \delta_0^{(n_0, n)} & \text{on } V_n, \\ \partial_t u_n(\cdot, 0) = 0 & \text{on } V_n, \end{cases} \tag{5.1}$$

where $\delta_0^{(n, n_0)} = \sum_{k=0}^{3^{n_0}} \alpha_k f_{n,k}$ is the approximate δ -function built up from the first $|V_{n_0}| = (3^{n_0} + 1)$ eigenfunctions of Δ_n (with $\Delta_n f_{n,k} = \lambda_{n,k} f_{n,k}$), and $\alpha_k := \alpha_{n_0, k} \geq 0$ are the coefficients found in Sect. 3.3. Throughout the section n_0 will be fixed, and we will not mention n_0 explicitly unless the context demands it.

Following the exact same argument, the solution to (5.1) has the series representation

$$u_n(x, t) = \sum_{k=0}^{3^{n_0}} \alpha_k f_{n,k}(x) \cos\left(t\sqrt{\lambda_{n,k}}\right) \quad \text{for all } x \in V_n \text{ and } t \in [0, T]. \tag{5.2}$$

For each t , we harmonically extend the function $x \mapsto u_n(x, t)$ from V_n to I . This procedure allows us to compare $u_n(x, t)$ with

$$\tilde{u}(x, t) = \sum_{k=0}^{3^{n_0}} \alpha_k f_k(x) \cos\left(t\sqrt{\lambda_k}\right) \quad \text{for all } x \in I \text{ and } t \in [0, T], \tag{5.3}$$

the solution of the wave equation on (I, μ) whose initial condition is the truncated series representation of the δ -impulse. We note that \tilde{u} is differentiable in t and continuous in x because the eigenfunctions functions f_k are continuous. However it is highly localized function at $t = 0$, and therefore it mimics wave propagation from a delta function initial values.

5.1 Convergence of Approximate Solutions of Wave Equation

In this subsection we establish an upper bound on

$$\left| u_n\left(x, \frac{t\pi}{\sqrt{\lambda_{n,1}}}\right) - \tilde{u}\left(x, \frac{t\pi}{\sqrt{\lambda_1}}\right) \right| \tag{5.4}$$

for all n and t , uniform in x . This would then give us the convergence of u_n to \tilde{u} at fixed t and uniformly in x . Note that we are normalizing t in such a way that the orthogonal projection of the wave onto the lowest eigenfunction (corresponding to eigenvalue $\lambda_{*,0}$) propagates at speed 1.

Theorem 5.1 Fix $p \in (0, \frac{1}{2})$. Let u_n and \tilde{u} be respectively defined as in (5.2) and (5.3). Then there exists a positive constant $C = C(n_0, p)$ such that for each $t \in [0, T]$ and $n > n_0$,

$$\sup_{x \in I} \left| u_n \left(x, \frac{t\pi}{\sqrt{\lambda_{n,1}}} \right) - \tilde{u} \left(x, \frac{t\pi}{\sqrt{\lambda_1}} \right) \right| \leq C (t \vee 1) \mathbb{C}^{-n}. \tag{5.5}$$

Proof Using the series representations (5.2) and (5.3) and the triangle inequality, we find that (5.4) is bounded from above by

$$\sum_{k=0}^{3^{n_0}} \alpha_k \left| f_{n,k}(x) \cos \left(t\pi \frac{\sqrt{\lambda_{n,k}}}{\sqrt{\lambda_{n,1}}} \right) - f_k(x) \cos \left(t\pi \frac{\sqrt{\lambda_k}}{\sqrt{\lambda_1}} \right) \right|, \tag{5.6}$$

which, by a simple manipulation using the sum-to-product trigonometric rules, is equal to

$$\begin{aligned} \sum_{k=0}^{3^{n_0}} \alpha_k & \left| [f_{n,k}(x) - f_k(x)] \cos \left(t\pi \Lambda_{n,k}^+ \right) \cos \left(t\pi \Lambda_{n,k}^- \right) \right. \\ & \left. - [f_{n,k}(x) + f_k(x)] \sin \left(t\pi \Lambda_{n,k}^+ \right) \sin \left(t\pi \Lambda_{n,k}^- \right) \right|, \end{aligned} \tag{5.7}$$

where $\Lambda_{n,k}^\pm := \frac{1}{2} \left(\sqrt{\frac{\lambda_{n,k}}{\lambda_{n,1}}} \pm \sqrt{\frac{\lambda_k}{\lambda_1}} \right)$. Using again the triangle inequality, we can estimate (5.7) from above by $I_1 + I_2$, where

$$I_1 := \sum_{k=0}^{3^{n_0}} \alpha_k |f_{n,k}(x) - f_k(x)| \cdot \left| \cos \left(t\pi \Lambda_{n,k}^+ \right) \right| \left| \cos \left(t\pi \Lambda_{n,k}^- \right) \right|, \tag{5.8}$$

$$I_2 := \sum_{k=0}^{3^{n_0}} \alpha_k |f_{n,k}(x) + f_k(x)| \cdot \left| \sin \left(t\pi \Lambda_{n,k}^+ \right) \right| \left| \sin \left(t\pi \Lambda_{n,k}^- \right) \right|. \tag{5.9}$$

The key term to control in I_1 is $|f_{n,k}(x) - f_k(x)|$, while in I_2 it is $\left| \sin \left(t\pi \Lambda_{n,k}^- \right) \right|$. For the former we invoke Theorem 4.4, while for the latter we apply the Taylor expansion

$$\left| \sin \left(t\pi \Lambda_{n,k}^- \right) \right| \leq t\pi \left| \Lambda_{n,k}^- \right| + O \left((t\Lambda_{n,k}^-)^3 \right). \tag{5.10}$$

For terms other than these two, we apply the simple minded estimates $\left| \cos \left(t\pi \Lambda_{n,k}^\pm \right) \right| \leq 1$, $\left| \sin \left(t\pi \Lambda_{n,k}^+ \right) \right| \leq 1$, and

$$\begin{aligned} |f_{n,k}(x) + f_k(x)| & \leq 2|f_{n,k}(x)| + \|f_{n,k} - f_k\|_\infty \\ & \leq 2\|f_{n,k}\|_\infty + O(\mathbb{C}^{-n}). \end{aligned} \tag{5.11}$$

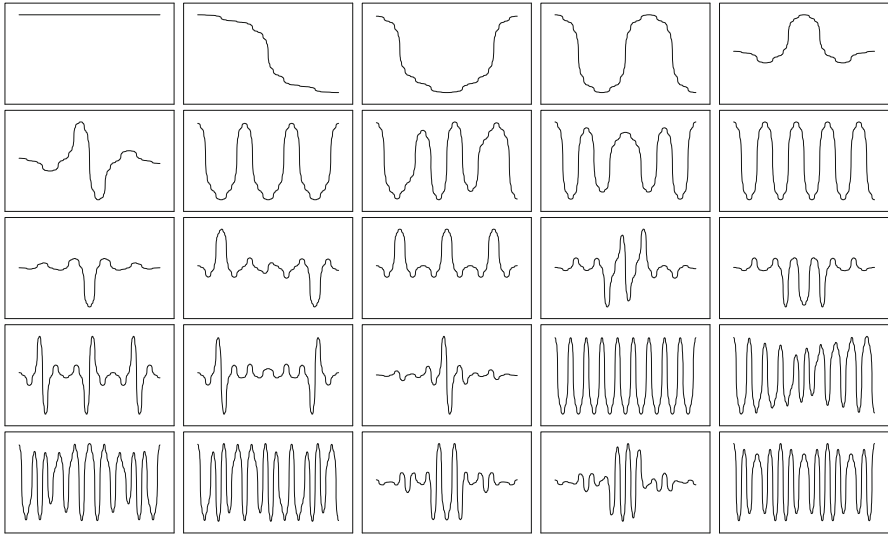


Fig. 4 The first 25 eigenfunctions of the fractal Laplacian with $p = \frac{1}{5}$

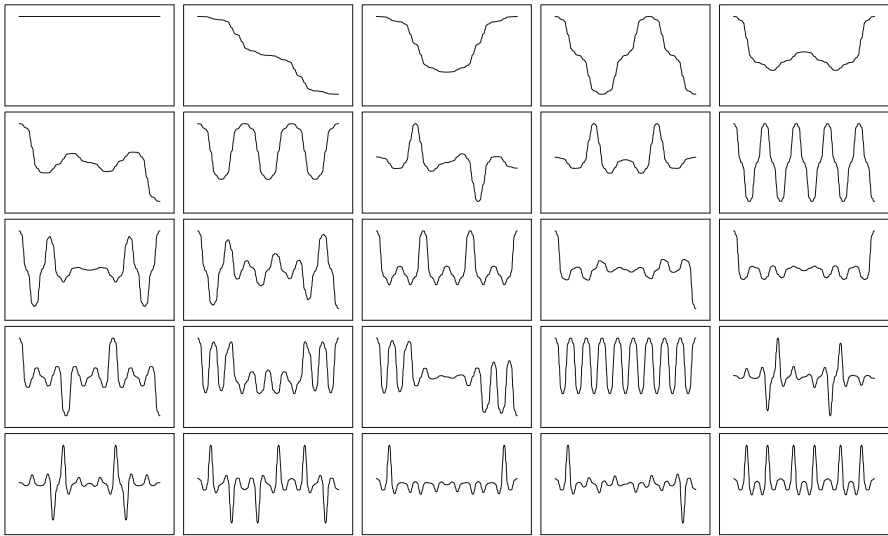


Fig. 5 The first 25 eigenfunctions of the fractal Laplacian with $p = \frac{4}{5}$

First let us estimate $\Lambda_{n,k}^-$, which amounts to controlling the ratio $\frac{\lambda_{n,k}}{\lambda_{n,1}}$. By Theorem 4.2,

$$\frac{\left(1 + \frac{3pq}{(2+pq)^2} \lambda_{n,1}\right)}{\exp\left(\lambda_{n,k} \frac{p(2+q)}{2q(2-p)}\right)} \leq \frac{\lambda_{n,k}}{\lambda_{n,1}} \cdot \frac{\lambda_1}{\lambda_k} \leq \frac{\exp\left(\lambda_{n,1} \frac{p(2+q)}{2q(2-p)}\right)}{\left(1 + \frac{3pq}{(2+pq)^2} \lambda_{n,k}\right)}. \tag{5.12}$$

We note that in this case we have $0 \leq k \leq 3^{n_0} < n$ and so Theorem 4.2 is applicable.

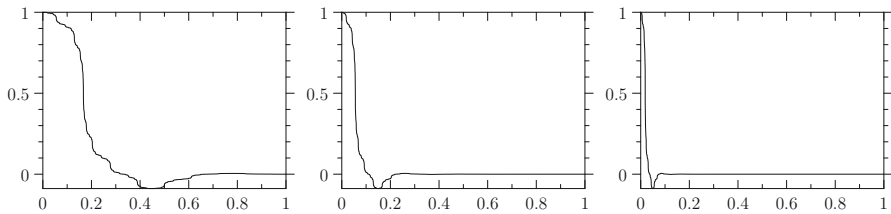


Fig. 6 Fourier approximations for delta function, $p = \frac{1}{5}$. Left to right $n_0 = 2, 3, 4$

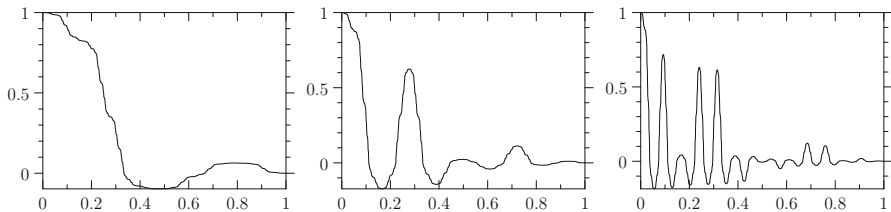


Fig. 7 Fourier approximations for delta function, $p = \frac{4}{5}$. Left to right $n_0 = 2, 3, 4$

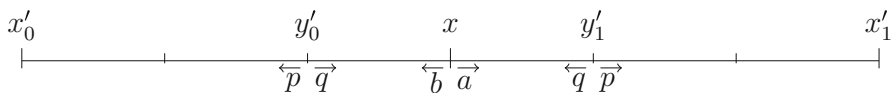


Fig. 8 A diagram of two adjacent $(n + 1)$ -cells used in the proof of Theorem 3.1

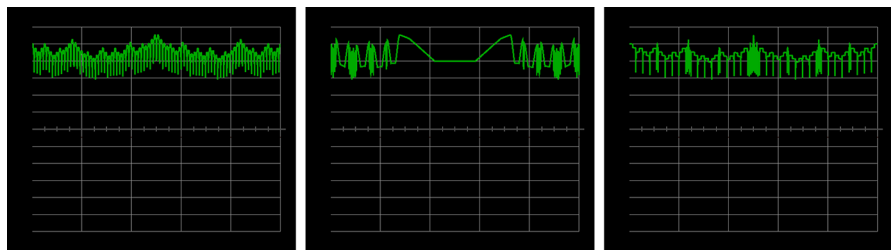


Fig. 9 A graph of $y(x) = \sum_{k=0}^{3n_0} \alpha_k |f_{n,k}(x)|$ and in order: uniform spacing of points, uniform resistance between points, and uniform measure

In what follows we denote $\mathcal{D} := \frac{p(2+q)}{2q(2-p)}$ and $\mathbf{D}_1 := \frac{3pq}{(2+pq)^2}$. Observe from the discussion in the proof of Theorem 4.2 that $\mathcal{D} > \mathbf{D}_1$ whenever $p \in (0, 1]$.

From Proposition 4.1, we know that for each fixed k , $\lambda_{n,k} = O(\mathbb{C}^{-n})$ as $n \rightarrow \infty$. Thus upon expanding the LHS and the RHS of (5.12) up to the $O(\mathbb{C}^{-n})$ terms, we get

$$\begin{aligned}
 1 + \frac{1}{2} (\mathbf{D}_1 \lambda_{n,1} - \mathcal{D} \lambda_{n,k}) + o(\mathbb{C}^{-n}) &\leq \sqrt{\frac{\lambda_{n,k}}{\lambda_{n,1}} \cdot \frac{\lambda_1}{\lambda_k}} \\
 &\leq 1 + \frac{1}{2} (\mathcal{D} \lambda_{n,1} - \mathbf{D}_1 \lambda_{n,k}) + o(\mathbb{C}^{-n}). \quad (5.13)
 \end{aligned}$$

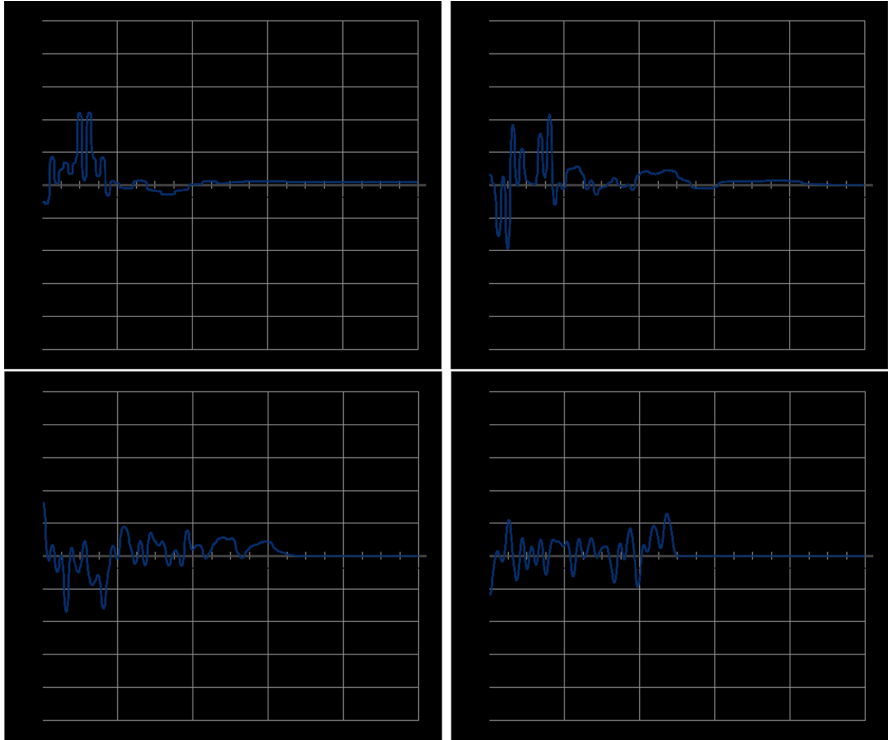


Fig. 10 Uniform spacing of points at time $t = 0.4$. From *left to right* $p = 0.1, 0.2, 0.3, 0.4$

It then follows that

$$\frac{1}{4} (\mathbf{D}_1 \lambda_{n,1} - \mathcal{D} \lambda_{n,k}) + o(\mathbb{C}^{-n}) \leq \Lambda_{n,k}^- \cdot \sqrt{\frac{\lambda_1}{\lambda_k}} \leq \frac{1}{4} (\mathcal{D} \lambda_{n,1} - \mathbf{D}_1 \lambda_{n,k}) + o(\mathbb{C}^{-n}). \tag{5.14}$$

Plugging this into (5.10) yields

$$\begin{aligned} \left| \sin \left(t\pi \Lambda_{n,k}^- \right) \right| &\leq \frac{t\pi}{4} \sqrt{\frac{\lambda_k}{\lambda_1}} \max \left(|\mathcal{D} \lambda_{n,1} - \mathbf{D}_1 \lambda_{n,k}|, |\mathcal{D} \lambda_{n,k} - \mathbf{D}_1 \lambda_{n,1}| \right) + o \left(t\mathbb{C}^{-n} \right) \\ &= \frac{t\pi}{4} \sqrt{\frac{\lambda_k}{\lambda_1}} |\mathcal{D} \lambda_{n,k} - \mathbf{D}_1 \lambda_{n,1}| + o \left(t\mathbb{C}^{-n} \right). \end{aligned} \tag{5.15}$$

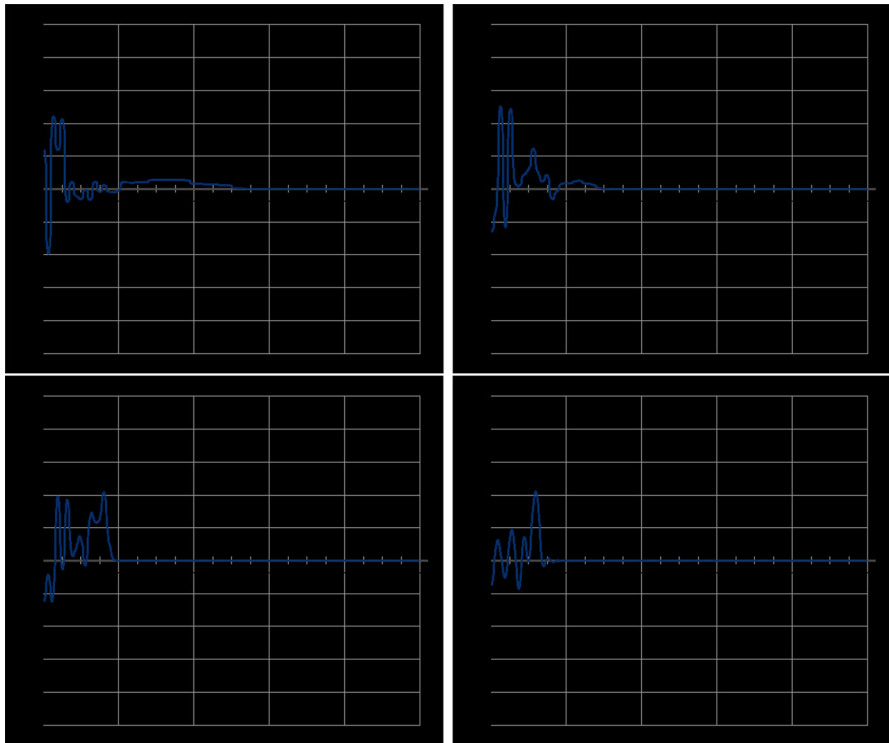


Fig. 11 Uniform spacing of points at time $t = 0.1$. From left to right $p = 0.1, 0.2, 0.3, 0.4$

Putting all the estimates into (5.8) and (5.9) gives

$$\begin{aligned}
 I_1 &\leq \mathbb{C}^{-n} \|g\|_\infty \sum_{k=0}^{3^{n_0}} \alpha_k \lambda_k \|f_{n,k}\|_{n,\infty} \exp\left(\frac{\lambda_{n,k}}{q - \lambda_{n+1,k}} \frac{2}{2-p}\right) + o(\mathbb{C}^{-n}). \\
 I_2 &\leq \frac{t\pi}{2} \sum_{k=0}^{3^{n_0}} \alpha_k \|f_{n,k}\|_\infty \sqrt{\frac{\lambda_k}{\lambda_1}} |\mathcal{D}\lambda_{n,k} - \mathbf{D}_1\lambda_{n,1}| + o(t\mathbb{C}^{-n}).
 \end{aligned}
 \tag{5.16}$$

This means that (5.4) is bounded above by

$$\begin{aligned}
 &\sum_{k=0}^{3^{n_0}} \alpha_k \|f_{n,k}\|_\infty \left[\lambda_k \mathbb{C}^{-n} \|g\|_\infty \exp\left(\frac{\lambda_{n,k}}{q - \lambda_{n+1,k}} \frac{2}{2-p}\right) \right. \\
 &\quad \left. + \frac{t\pi}{2} \sqrt{\frac{\lambda_k}{\lambda_1}} |\mathcal{D}\lambda_{n,k} - \mathbf{D}_1\lambda_{n,1}| \right] + o((t \vee 1)\mathbb{C}^{-n}).
 \end{aligned}
 \tag{5.17}$$

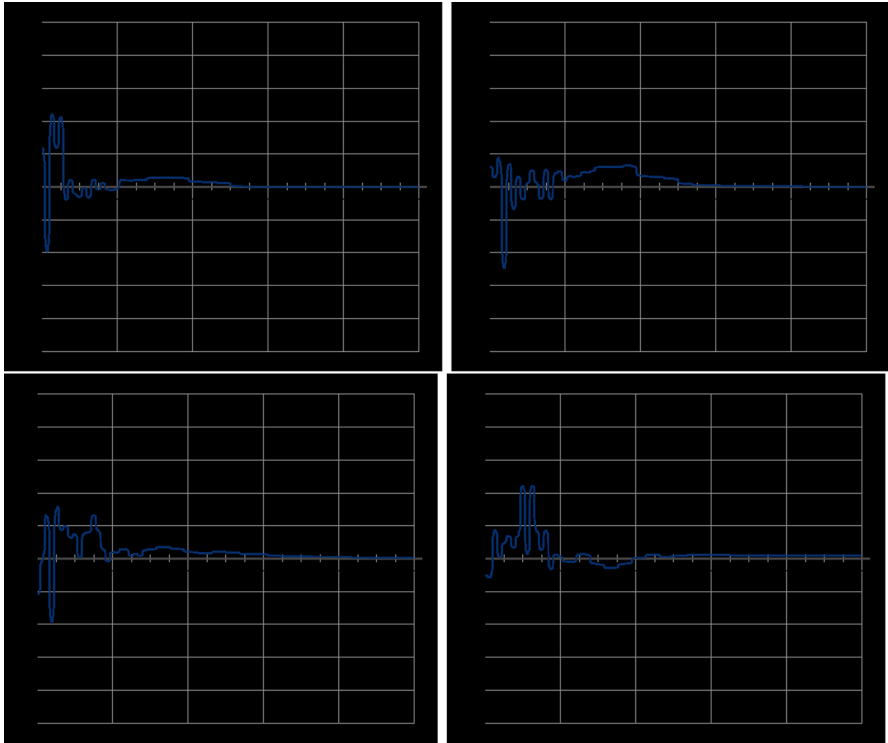


Fig. 12 Uniform spacing of points and, from *left to right* time $t = 0.1, 0.2, 0.3, 0.4$. with $p = 0.1$

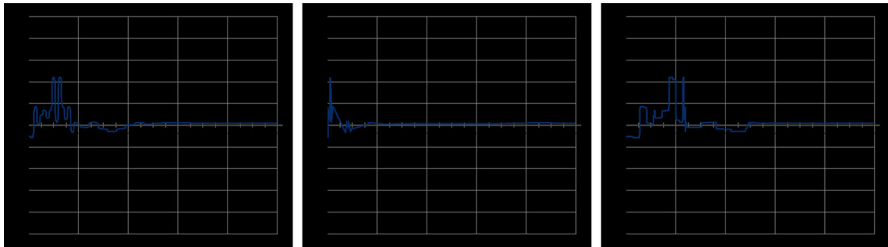


Fig. 13 Time $t = 0.4$ and $p = 0.1$ and from *left to right* uniform spacing of points, uniform resistance between points, and uniform measure

It remains to explain how (5.5) follows from (5.17). For $0 \leq k \leq 3^{n_0}$, $\lambda_k \leq \lambda_{3^{n_0}}$. Also by Proposition 4.1, there exists a positive constant C independent of n such that

$$|\mathcal{D}\lambda_{n,k} - \mathbf{D}_1\lambda_{n,1}| \leq C\mathbb{C}^{-n}|\mathcal{D}\lambda_k - \mathbf{D}_1\lambda_1|. \tag{5.18}$$

By an argument of Kigami [37], the Green’s function corresponding to Δ_μ on I can be constructed independently of the measure μ , whence independently of the value p . In particular when $p = \frac{1}{2}$, we recover the classical Green’s function on I with Lebesgue measure, $g(x, y) = (x \wedge y)((1 - x) \wedge (1 - y))$. Thus $\|g\|_\infty = \frac{1}{4}$.

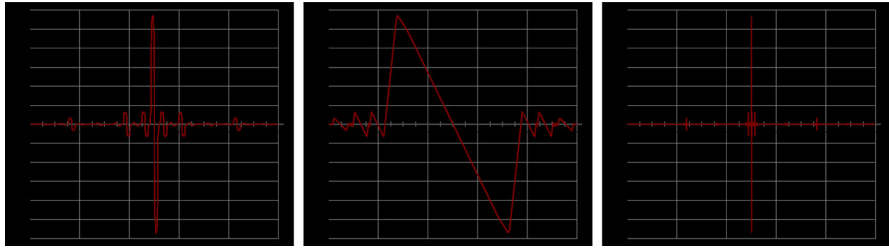


Fig. 14 Eigenfunction 53, with $p = 0.1$ and in order: uniform spacing of points, uniform resistance between points, and uniform measure

For the exponential in the first term, note that since $p \in (0, \frac{1}{2})$,

$$\frac{\lambda_{n,k}}{q - \lambda_{n+1,k}} \frac{2}{2-p} < \frac{\lambda_{n,k}}{\frac{1}{2} - \lambda_{n+1,k}} \frac{2}{2 - \frac{1}{2}} = \frac{4}{3} \frac{\lambda_{n,k}}{\frac{1}{2} - \lambda_{n+1,k}}. \quad (5.19)$$

By Proposition 4.1, the RHS of (5.19) is $O(\mathbb{C}^{-n})$ as $n \rightarrow \infty$. Thus the exponential is $\exp(O(\mathbb{C}^{-n})) = 1 + O(\mathbb{C}^{-n})$.

Finally, we claim that there exists a constant C such that

$$\sum_{k=0}^{3^{n_0}} \alpha_k \|f_{n,k}\|_{\infty} \leq C$$

for all $n > n_0$. To explain this, note that $\sum_{k=0}^{3^{n_0}} \alpha_k = 1$, by virtue of our choice of the initial eigenfunctions ($f_{0,1}$ and $f_{0,2}$) and initial weights $\alpha_{0,1} = \alpha_{0,2} = \frac{1}{2}$, and the matching condition (3.33). Then the rest of the proof follows from Sect. 8 of Rogers' paper [51]. Numerically, C is slightly above 1, see Fig. 9. \square

6 Numerical Computation of Eigenfunctions and Solutions of the Wave Equation

We present some of the numerical results obtained by our spectral decimation method. The spectral decimation is an iterative method, and the code repeats the calculation done in Sect. 3. This code, which is used to produce pictures and to perform the experiments, and a graphical user interface to recreate the results can be found at <http://homepages.uconn.edu/fractals/fractalwave/>. Here we give a representative variety of figures detailing some of the numerical calculations that have been performed. Figures 4, and 5 show the first 25 eigenfunctions, and, in particular, the ways in which the symmetry is broken for values of p and $1 - p$. Figures 6, and 7 show the quality of the approximation for the delta function for various values of n_0 . In particular one can see that for small values of p (in our case $p = .2$) the approximation is significantly better than for the corresponding values of $1 - p$ (in our case $q = .8$). This shows why our efforts focused on the cases where $p < .5$ (Figs. 8, 9). The next set of Figs. 10, 11, 12, and 13 heuristically suggest that the visible portion of the wave propagates

at a speed proportional to $t^{\frac{d_g}{2}}$. But, further investigation will be needed to show this more precisely. Figure 14 shows three different parametrizations of a representative eigenfunction.

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