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Reproducing Kernel for the Herglotz Functions in $\mathbb{R}^n$ and Solutions of the Helmholtz Equation

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Abstract The purpose of this article is to extend to $\mathbb{R}^n$ known results in dimension 2 concerning the structure of a Hilbert space with reproducing kernel of the space of Herglotz wave functions. These functions are the solutions of Helmholtz equation in $\mathbb{R}^n$ that are the Fourier transform of measures supported in the unit sphere with density in $L^2(\mathbb{S}^{n-1})$. As a natural extension of this, we define Banach spaces of solutions of the Helmholtz equation in $\mathbb{R}^n$ belonging to weighted *Sobolev type* spaces $\mathcal{H}^p$ having in a non local norm that involves radial derivatives and spherical gradients. We calculate the reproducing kernel of the Herglotz wave functions and study in $\mathcal{H}^p$ and in mixed norm spaces, the continuity of the orthogonal projection $\mathcal{P}$ of $\mathcal{H}^2$ onto the Herglotz wave functions.

Keywords Reproducing kernel · Herglotz wave functions · Helmholtz equation · The restriction of Fourier transform

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1 Introduction and Preliminaries

Consider the Fourier extension operator

$$W\phi(x) := \widehat{\phi d\sigma(x)} = (2\pi)^{(1-n)/2} \int_{\mathbb{S}^{n-1}} e^{ix\cdot\xi} \phi(\xi) d\sigma(\xi), \tag{1}$$

where $\phi \in L^2(\mathbb{S}^{n-1})$, $d\sigma$ is the Lebesgue measure in $\mathbb{S}^{n-1}$ and $\widehat{\cdot}$ denotes the Fourier transform in $\mathbb{R}^n$.

We have that $W\phi$ is an entire solution (a solution in $\mathbb{R}^n$) of the Helmholtz equation

$$\Delta u + u = 0. \tag{2}$$

The functions $u = W\phi$ with $\phi \in L^2(\mathbb{S}^{n-1})$ called Herglotz wave functions are relevant in analysis and in particular are extensively used in scattering theory. Hartman and Wilcox in [10] proved the familiar characterization of the Herglotz functions as the entire solutions of the Helmholtz equation satisfying

$$\limsup_{R\to\infty} \frac{1}{R} \int_{|x|<R} |u(x)|^2 \, dx < \infty.$$

The operator W is the transpose of the restriction operator for the Fourier transform, namely the operator $Rf = \widehat{f}_{|\mathbb{S}^{n-1}}$ defined in the Schwartz space.

The restriction problem of Stein–Tomas asks for the values of p and q such that

$$\left\| \widehat{f}_{|\mathbb{S}^{n-1}} \right\|_{L^q(\mathbb{S}^{n-1})} \leq C \, \|f\|_{L^p(\mathbb{R}^n)}, \quad f \in S(\mathbb{R}^n).$$

The best known result for $q = 2$ is given in the Stein–Tomas theorem:

Theorem 1 (Stein–Tomas) *If $f \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \frac{2(n+1)}{n+3}$ then*

$$\left\| \widehat{f}_{|\mathbb{S}^{n-1}} \right\|_{L^2(\mathbb{S}^{n-1})} \leq C_{p,n} \, \|f\|_{L^p(\mathbb{R}^n)}.$$

Or equivalent, if $f \in L^2(\mathbb{S}^{n-1})$

$$\|Wf\|_{L^q(\mathbb{R}^n)} \leq C_{q,n} \, \|f\|_{L^2(\mathbb{S}^{n-1})}$$

for $q \geq \frac{2(n+1)}{n-1}$.

In [2], it was proved that the extension operator is an isomorphism of $L^2(\mathbb{S}^1)$ onto the space $\mathcal{W}^2$ consisting of all entire solutions of Helmholtz equation with radial and angular derivatives satisfying

$$\|u\|^2_{\mathcal{H}^2} = \int_{|x|>1} \left(|u(x)|^2 + \left| \frac{\partial u}{\partial r}(x) \right|^2 + \left| \frac{\partial u}{\partial \theta}(x) \right|^2 \right) \frac{dx}{|x|^3} < \infty.$$

This gave a new characterization of the space $\mathcal{W}^2$ of Herglotz wave functions in $\mathbb{R}^2$ as a Hilbert space with reproducing kernel. Also, for $1 < p < 4/3$, it was proved that the orthogonal projection $\mathcal{P}$ of $\mathcal{H}^2$ onto $\mathcal{W}^2$, can't be extended as a bounded operator on $\mathcal{H}^p$, the p-version of $\mathcal{H}^2$. Then in [4], Barceló, Bennet and Ruiz proved that $\mathcal{P}$ can't be extended as a bounded for any $p > 1$ except for $p \neq 2$. However they obtained a positive result for $4/3 < p < 4$, considering mixed norm spaces $\mathcal{H}^{p,2}$, defined by

$$\|u\|^2_{\mathcal{H}^{p,2}} = \int_0^\infty \left(\int_0^{2\pi} \left(|u(r\theta)|^2 + \left| \frac{\partial u}{\partial \theta}(x) \right|^2 \right) d\theta \right)^{p/2} \frac{r dr}{(1+r^2)^{3/2}}.$$

In this article, we will define Banach spaces $\mathcal{H}^p$ and $\mathcal{W}^p$ in $\mathbb{R}^n$, $n \geq 3$, generalizing the mentioned spaces in [2]. $\mathcal{H}^p$ will consist of all functions belonging to $L^p\left(\mathbb{R}^n, \frac{dx}{\langle x\rangle^3}\right)$ jointly with their radial derivative and the euclidean norm of their spherical gradient. Then $\mathcal{W}^p$ will be the closed subspace of all solutions in $\mathcal{H}^p$ of the Helmholtz equation $\Delta u + u = 0$ on the Euclidean space $\mathbb{R}^n$ for $n \geq 3$. We will construct and study the reproducing kernel for $\mathcal{W}^2$, which as for $n = 2$, turns out to be the space of all Herglotz wave functions and it is characterized as the space of all the entire solutions of the Helmholtz equation satisfying

$$\|u\|_{\mathcal{H}^2} = \left(\|u\|^2_{L^2\left(\mathbb{R}^n, \frac{dx}{\langle x\rangle^3}\right)} + \left\| \frac{\partial u}{\partial r} \right\|^2_{L^2\left(\mathbb{R}^n, \frac{dx}{\langle x\rangle^3}\right)} + \||\nabla_S u|\|^2_{L^2\left(\mathbb{R}^n, \frac{dx}{\langle x\rangle^3}\right)} \right)^{1/2},$$

where ∇_S denotes the spherical gradient.

In Sect. 2 we will study the space $\mathcal{W}^2$. We will show that this is precisely the space of all Herglotz wave functions and we will calculate its reproducing kernel as a subspace of $\mathcal{H}^2$. In Sect. 3 we consider the spaces $\mathcal{H}^p$ and $\mathcal{W}^p$ for exponents $p > 1$. We will prove that these are Banach spaces and we will show that the reproducing kernel of $\mathcal{H}^2$ has also reproducing properties for $\mathcal{W}^p$. Finally, in Sect. 4 we study the continuity properties of the orthogonal projection $\mathcal{P}$ of $\mathcal{H}^2$ onto $\mathcal{W}^2$ in mixed-normed spaces $\mathcal{H}^{p,2}$ extending the results in [4] for $n = 2$. Then we consider the continuity of $\mathcal{P}$ in $\mathcal{H}^p$. As in $n = 2$ this continuity is related to the boundedness in $L^p\left(\mathbb{R}^n, \frac{dx}{\langle x\rangle^3}\right)$ of a singular operator T acting on vector fields and given by

$$TU(x) = C_n \int_{\mathbb{R}^n} |x||y| \frac{J_{n/2+1}(|x-y|)}{(|x-y|)^{n/2+1}} \left((x - \mathbf{P}_y x) \cdot U(y) \right) (y - \mathbf{P}_x y) \frac{dy}{\langle y\rangle^3},$$

where $\mathbf{P}_z$ denotes the orthogonal projection in the direction of z and $J_{n/2+1}$ is the Bessel function of the first kind. Finally we give a non-boundedness result of T in $\mathbb{R}^3$.

Throughout paper we will use the following notations and results: $\mathcal{B}_R \subset \mathbb{R}^n$ denotes the open ball with center at the origin and radius R, $\mathcal{B} = \mathcal{B}_1$, and $\mathbb{S}^{n-1}$ is the $(n-1)-$ dimensional unit sphere with surface area $\sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$. Δ_S denotes the Laplacian on $\mathbb{S}^{n-1}$, that is, the Laplace Beltrami operator on $\mathbb{S}^{n-1}$ and ∇_S will be the spherical gradient. The conjugate exponent of p will be denoted by p'.

Throughout this article c and C will denote generic positive constants that may change in each occurrence.

As usual, if μ is a Borel measure in $\mathbb{R}$, $M_\mu f$ will denote the Hardy–Littlewood maximal function of a locally integrable function f on $\mathbb{R}$:

$$M_\mu f(x) = \sup_{I:x\in I} \frac{1}{\mu(I)} \int_I |f(y)|\, d\,\mu(y),$$

where the supremum is taken over intervals $I \subset \mathbb{R}$. Let w be a weight in $\mathbb{R}$, namely a non-negative function in $L^1_{loc}(\mu)$. By $A_p(\mu)$ we will denote the Muckenhoupt classes. We say that w is an $A_p(\mu)$ weight ($w \in A_p(\mu)$) if

$$\left(\frac{1}{\mu(I)} \int_I w(r) d\,\mu(r)\right) \left(\frac{1}{\mu(I)} \int_I w(r)^{1-p'} d\,\mu(r)\right)^{p-1} \le C,$$

for $1 < p < \infty$ and

$$M_\mu w(r) \le C w(r) \text{ a.e.}$$

when $p = 1$, where C is always independent of I.

We have $A_p(\mu) \subset A_q(\mu)$, $1 \le p < q$, in particular, $A_1(\mu) \subset A_2(\mu)$, see [8].

We denote by J_ν the Bessel functions of the first kind of order ν

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \frac{(-1)^k}{k!\,\Gamma(k+v+1)} \left(\frac{z}{2}\right)^{2k}.$$

The Bessel functions satisfy the following recurrence formulas:

($R1$) $J_{\nu-1}(z) - J_{\nu+1}(z) = 2J_\nu'(z)$.
($R2$) $J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z)$.

Also, we have that

$$\left(\frac{J_\nu(t)}{t^\nu}\right)' = -\frac{J_{\nu+1}(t)}{t^\nu}. \tag{3}$$

We will use the following estimates for Bessel functions.

($D1$) For any $\nu > -1/2$ and $z \in \mathbb{C}$,

$$|J_\nu(z)| \le \frac{\left(\frac{|z|}{2}\right)^\nu}{\Gamma(\nu+1)} e^{|Im\, z|}.$$

For integer $n \ge 0$ we have

$$|J_n(z)| \le \frac{|z|^n}{n! 2^n} e^{\frac{|z|^2}{4}}.$$

$(D2)$ For $\nu \geq 1/2$ and $0 < r \leq 1$,

$$|J_\nu(r)| \leq C\left(\frac{r}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)}.$$

$(D3)$ For $\nu \geq 1/2$, $\alpha_0 > 0$, and $1 \leq \nu \operatorname{sech} \alpha \leq \nu \operatorname{sech} \alpha_0$,

$$|J_\nu(\nu \operatorname{sech} \alpha)| \leq C\frac{e^{-\nu(\alpha - \tanh \alpha)}}{\nu^{1/2}}.$$

$(D4)$ If $z = r \in \mathbb{R}$, then

$$\begin{aligned}
|J_\nu(r)| &\leq C\frac{1}{\nu} \quad 0 \leq r \leq \nu/2,\ \nu \geq 1,\\
|J_\nu(r)| &\leq Cr^{-1/3} \quad r \geq 1,\ \nu \geq 0,\\
|J_n(r)| &\leq C_n r^{-1/2} \quad r > 0,\ n \in \mathbb{Z}.
\end{aligned}$$

A known asymptotic formula for Bessel functions is

$$J_\nu(r) = \sqrt{\frac{2}{\pi r}}\cos(r - \frac{\nu\pi}{2} - \frac{\pi}{4}) + O(r^{-3/2}) \tag{4}$$

as $r \to \infty$. In particular,

$$J_\nu(r) = O(r^{-1/2}) \quad if \quad r \to \infty.$$

The proof of following lemma can be found in [5].

Lemma 1 *Let $\nu > 0$, $p \geq 1$ and $a \geq 1$, then there exists a constant C depending only on p and a, such that*

$$\frac{1}{C}\nu^{\frac{1}{3}-\frac{p}{3}}\sum_{j=0}^{K-1} 2^{j(1-\frac{p}{4})} \leq \int_{\frac{\nu}{a}}^{2\nu} |J_\nu(r)|^p dr \leq C\nu^{\frac{1}{3}-\frac{p}{3}}\sum_{j=0}^{K-1} 2^{j(1-\frac{p}{4})}, \tag{5}$$

where $\nu^{\frac{2}{3}} \leq 2^K \leq 2\nu^{\frac{2}{3}}$.

The following lemma [9, p. 675] is useful in this paper.

Lemma 2 *Let $\nu(m) = m + \frac{n-2}{2}$. Then*

$$\int_0^\infty J^2_{\nu(m)}(r)\frac{dr}{r^2} = \frac{1}{\pi}\frac{1}{\nu(m)^2 - 1/4}, \tag{6}$$

for all $m \geq 1$ if $n = 3$ and for all $m \geq 0$ if $n \geq 4$.

The space of all surface spherical harmonics of degree m will be denoted by $\mathcal{Y}_m$. In addition, $\{Y_m^j : m \in \mathbb{N}, j = 1, \ldots, d_m\}$ will always denote a basis of real valued spherical harmonics for $L^2(\mathbb{S}^{n-1})$, where

$$d_m = \begin{cases} 1 & \text{if } m = 0 \\ \frac{(2m+n-2)(m+n-3)!}{m!(n-2)!} & \text{if } m \geq 1. \end{cases}$$

Theorem 2 (Spherical Harmonic Addition Theorem) *Let $\{Y_m^j\}$, $j = 1, \ldots, d_m$ be an orthonormal basis for $\mathcal{Y}_m$. Then*

$$Z_m(\xi, \eta) = \frac{d_m}{\sigma_n} P_m(\xi \cdot \eta), \tag{7}$$

where $Z_m(\xi, \eta) = \sum_{j=1}^{d_m} \overline{Y_m^j(\xi)} Y_m^j(\eta)$ are called zonal harmonics of degree m, P_m is the Legendre polynomial of degree m and σ_n is the total surface area of $\mathbb{S}^{n-1}$.

The following lemma is known as the Addition Theorem of the Bessel functions (see [12, Lemma 2, p. 121]).

Lemma 3 *If $x = r\xi$, $y = s\theta$, we have*

$$\mathcal{J}_0(n; |x - y|) = \sum_{m=0}^{\infty} d_m \mathcal{J}_m(n; r) \mathcal{J}_m(n; s) P_m(\xi \cdot \theta), \tag{8}$$

where

$$\mathcal{J}_m(n; r) = \Gamma\left(\frac{n}{2}\right)\left(\frac{r}{2}\right)^{\frac{2-n}{2}} J_{\nu(m)}(r). \tag{9}$$

We have that

$$\nabla = \frac{\partial}{\partial r}\xi + \frac{1}{r}\nabla_S,$$

that is,

$$\nabla_S u = r\left(\nabla u - \frac{\partial u}{\partial r}\xi\right). \tag{10}$$

An identity that relates the eigenvalues of the spherical Laplacian with the norm $L^2(\mathbb{S}^{n-1})$ of the spherical gradient for some spherical harmonic Y_k of degree k is given by

$$\int_{\mathbb{S}^{n-1}} |\nabla_S Y_k(\xi)|^2 \, d\sigma(\xi) = k(k + n - 2) \int_{\mathbb{S}^{n-1}} |Y_k(\xi)|^2 \, d\sigma(\xi), \tag{11}$$

which implies that the norms $\|(-\Delta_S)^{1/2} u\|_{L^2(\mathbb{S}^{n-1})}$ and $\||\nabla_S u|\|_{L^2(\mathbb{S}^{n-1})}$ are equivalent.

A classical result due to Bakry (see [3]), valid for any Riemannian manifold with non-negative Ricci curvature and in particular for the sphere, is the following.

Theorem 3 (Bakry) *If* $1 < p < \infty$*, there exist constants* c_p *and* C_p *such that*

$$c_p \left\| (-\Delta_S)^{1/2} u \right\|_{L^p(\mathbb{S}^{n-1})} \le \left\| |\nabla_S u| \right\|_{L^p(\mathbb{S}^{n-1})} \le C_p \left\| (-\Delta_S)^{1/2} u \right\|_{L^p(\mathbb{S}^{n-1})} \tag{12}$$

for all $u \in C_c^\infty(\mathbb{S}^{n-1})$.

Definition 1 (i) For $1 \le p < \infty$, we denote by $\mathcal{H}^p$ the space of all $u \in \mathcal{D}'(\mathbb{R}^n)$ such that u, $\frac{\partial u}{\partial r}$ and $|\nabla_S u| \in L^1_{loc}(\mathbb{R}^n)$

$$\|u\|_{\mathcal{H}^p} = \left\{ \int_{\mathbb{R}^n} \left(|u(x)|^p + \left| \frac{\partial u}{\partial r}(x) \right|^p + |\nabla_S u(x)|^p \right) \frac{dx}{\langle x \rangle^3} \right\}^{1/p} \tag{13}$$

$$= \left(\|u\|^p_{L^p(\mathbb{R}^n, \frac{dx}{\langle x \rangle^3})} + \left\| \frac{\partial u}{\partial r} \right\|^p_{L^p(\mathbb{R}^n, \frac{dx}{\langle x \rangle^3})} + \| |\nabla_S u| \|^p_{L^p(\mathbb{R}^n, \frac{dx}{\langle x \rangle^3})} \right)^{1/p}, \tag{14}$$

where $\langle x \rangle := 1/(1+|x|^2)^{1/2}$.

(ii) We denote by $\mathcal{W}^p$ the space of all functions $u \in \mathcal{H}^p$ satisfying the Helmholtz equation (2) in $\mathbb{R}^n$.

Remark 1 (1) $C^\infty(\mathbb{R}^n) \cap \mathcal{H}^p$ is dense in $\mathcal{H}^p$ and the elements of $\mathcal{H}^p$ belong locally to a weighted Sobolev space in $\mathbb{R}^n$.

(2) By Theorem 3, we can define in $\mathcal{H}^p$ the equivalent norm $\|.\|^{\Delta^{\frac{1}{2}}}_{\mathcal{H}^p}$ given by

$$\|u\|^{\Delta^{\frac{1}{2}}}_{\mathcal{H}^p} = \left(\|u\|^p_{L^p\left(\mathbb{R}^n, \frac{dx}{\langle x \rangle^3}\right)} + \left\| \frac{\partial u}{\partial r} \right\|^p_{L^p\left(\mathbb{R}^n, \frac{dx}{\langle x \rangle^3}\right)} + \|(-\Delta_S)^{1/2} u\|^p_{L^p\left(\mathbb{R}^n, \frac{dx}{\langle x \rangle^3}\right)} \right)^{1/p}.$$

Throughout this article will be exchanging these norms as needed.

2 The Fourier Extension Operator in $L^2(\mathbb{S}^{n-1})$ and $\mathcal{W}^2$

In this section we prove that the space $\mathcal{W}^2$ is precisely the space of all Herglotz wave functions.

Lemma 4 *If* Y_m *is a spherical harmonic and* $F_m := WY_m$*, then*

(i) $F_m(x) = (2\pi)^{\frac{1}{2}} i^m r^{-(n-2)/2} J_{\nu(m)}(r) Y_m(\xi)$, $x = r\xi$.

(ii) $\{F_m^j(r\xi) := (2\pi)^{\frac{1}{2}} i^m r^{-(n-2)/2} J_{\nu(m)}(r) Y_m^j(\xi)\}_{m,j}$, $m = 0, 1, \ldots$, $j = 1, 2, \ldots, d_m$ *is an orthogonal family and*

$$\|F_m\|_{\mathcal{H}^2} = \sqrt{2} + O\left(\frac{1}{m^2}\right). \tag{15}$$

(iii) *If* $f = \sum_{m,j} a_{mj} Y_m^j \in L^2(\mathbb{S}^{n-1})$ *and* $u = \sum_{m,j} a_{mj} F_m^j \in \mathcal{W}^2$, *then*

$$\|u\|_{\mathcal{H}^2} \sim \|f\|_{L^2(\mathbb{S}^{n-1})}, \tag{16}$$

and the series of u *converges absolutely and uniformly on compact subsets of* $\mathbb{R}^n$.

Proof (i) Is a direct consequence of the Funk–Hecke's formula (see [11, p. 37]) with $x = r\xi$,

$$\int_{\mathbb{S}^{n-1}} \exp(-ix \cdot w) Y_m(w) d\sigma(w) \\ = (2\pi)^{n/2}(-i)^m r^{-(n-2)/2} J_{\nu(m)}(r) Y_m(\xi). \tag{17}$$

(ii) By the Lemma 2, (11) and the recursion formula $(R1)$ we have that

$$\|F_m\|^2_{\mathcal{H}^2} = \int_{\mathbb{R}^n} \left(|F_m(x)|^2 + \left|\frac{\partial F_m}{\partial r}(x)\right|^2 + |\nabla_S F_m(x)|^2 \right) \frac{dx}{\langle x \rangle^3} \\ = 2 + O\left(\frac{1}{m^2}\right). \tag{18}$$

Then

$$\|F_m\|_{\mathcal{H}^2} = \sqrt{2} + O\left(\frac{1}{m^2}\right).$$

The orthogonality follows from the orthogonality of the spherical harmonics in $L^2(\mathbb{S}^{n-1})$.

(iii) By (ii) it follows that $\|.\|_{\mathcal{H}^2} \sim \|.\|_{L^2(\mathbb{S}^{n-1})}$. Furthermore, using the recurrence formula $(R1)$ for Bessel functions and the estimate (D1), it follows that the series for u converges absolutely and uniformly on compact subsets of $\mathbb{R}^n$ □

Theorem 4 *The operator* W *is a topological isomorphism of* $L^2(\mathbb{S}^{n-1})$ *onto* $\mathcal{W}^2$.

Proof By Lemma 4, to prove that $\|Wf\|_{\mathcal{H}^2} \sim \|f\|_{L^2(\mathbb{S}^{n-1})}$ it suffices to show that $Wf = \sum_{m,j} a_{mj} F_m^j$ for any $f = \sum_{m,j} a_{mj} Y_m^j \in L^2(\mathbb{S}^{n-1})$. Notice that if f_n converges to f in $L^2(\mathbb{S}^{n-1})$ then Wf_n converges uniformly to Wf uniformly on compact sets of $\mathbb{R}^n$. Let L_0^2 be the linear span of $\{Y_m^j\}$ and $W' = W \mid_{L_0^2}$. If ϕ is a finite sum $\sum_{m,j} a_{mj} Y_m^j \in L_0^2$ then $W\phi = \sum_{m,j} a_{mj} F_m^j$ and by Lemma 4(iii) we have that $\|W\phi\|_{\mathcal{H}^2} \sim \|\phi\|_{L^2(\mathbb{S}^{n-1})}$. Moreover, W' can be extended to a continuous operator from $L^2(\mathbb{S}^{n-1})$ into $\mathcal{W}^2$ so that $W'\big(\sum_{m,j} a_{mj} Y_m^j\big) = \sum_{m,j} a_{mj} F_m^j$ converges uniformly on compact subsets of $\mathbb{R}^n$.

Now let $f = \sum_{m,j} a_{mj} Y_m^j \in L^2(\mathbb{S}^{n-1})$ and $\phi_m = \sum_{k,j,k \le m} a_{kj} Y_k^j$. Then $W(\phi_n) = W'(\phi_n) \to Wf$ uniformly on compact subsets and $Wf = W'f$. Thus, $\|Wf\|_{\mathcal{H}^2} \sim \|f\|_{L^2(\mathbb{S}^{n-1})}$.

It remains to prove that $\mathcal{W}$ is onto.

Let $u \in \mathcal{W}^2$, we have that $u \in C^\infty(\mathbb{R}^n)$, so, for r fixed, consider the Fourier series in spherical harmonics of $u(r\xi)$, that is,

$$u(r\xi) = (2\pi)^{\frac{1}{2}} r^{-(n-2)/2} \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} i^m A_{mj}(r) Y_m^j(\xi),$$

with

$$(2\pi)^{\frac{1}{2}} r^{-(n-2)/2} i^m A_{mj}(r) = \int_{\mathbb{S}^{n-1}} u(r\eta) \overline{Y_m^j(\eta)} d\sigma(\eta).$$

Thus, we can apply term by term the Helmholtz operator in polar coordinates

$$\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_S + 1$$

to the representation of u. We obtain

$$\sum_{m=0}^{\infty} \sum_{j=1}^{d_m} \left(A''_{mj}(r) + \frac{1}{r} A'_{mj}(r) + \left(1 - \frac{\nu(m)^2}{r^2}\right) A_{mj}(r) \right) Y_m^j(\xi) = 0,$$

and using the orthogonality of the spherical harmonics we have that

$$A''_{mj}(r) + \frac{1}{r} A'_{mj}(r) + \left(1 - \frac{\nu(m)^2}{r^2}\right) A_{mj}(r) = 0,$$

for each $m \in \mathbb{N} \cup \{0\}$, $j = 1, \ldots, d_m$, that is, the function $A_{mj}(r)$ satisfies the Bessel equation of order $\nu(m)$. Then, A_{mj} can be written as a linear combination,

$$A_{mj}(r) = a_{mj} J_{\nu(m)}(r) + b_{mj} N_{\nu(m)}(r),$$

where $N_{\nu(m)}(r)$ is the Neumann function of order $\nu(m)$. Since $N_{\nu(m)}(r)$ has a singularity at $r = 0$ and $A_{mj}(r)$ is bounded, it follows that $b_{mj} = 0$ for all m, j; therefore, $A_{mj}(r) = a_{mj} J_{\nu(m)}(r)$.

We see that $\sum_{m,j} \left|a_{mj}\right|^2 \leq C\|u\|_{\mathcal{H}^2}$, so taking $\phi = \sum_{m,j} a_{mj} Y_m^j$, we conclude that $\phi \in L^2(\mathbb{S}^{n-1})$ and $u = W\phi$. □

Now we will construct the reproducing kernel for $\mathcal{W}^2$ as a subspace of the Hilbert space $\mathcal{H}^2$. Before, we observe that family $\{\beta_m^{-1} F_m^j\}$ is an orthonormal basis for $\mathcal{W}^2$, where $\beta_m = \left\| F_m^j \right\|_{\mathcal{H}^2}$.

Let

$$\mathcal{K}(x,y)=\sum_{m=0}^{\infty}\sum_{j=1}^{d_m}\frac{F_m^j(x)\overline{F_m^j(y)}}{\beta_m^2}$$
$$=2\pi(rs)^{-(n-2)/2}\sum_{m=0}^{\infty}\sum_{j=1}^{d_m}\frac{J_{\nu(m)}(r)J_{\nu(m)}(s)}{\beta_m^2}Y_m^j(\xi)\overline{Y_m^j(\theta)},$$

where $x=r\xi$, $y=s\theta$. Using directly the Addition Theorem 2 we have

$$\mathcal{K}(x,y)=2\pi(rs)^{-(n-2)/2}\sum_{m=0}^{\infty}\frac{J_{\nu(m)}(r)J_{\nu(m)}(s)}{\beta_m^2}Z_m(\xi,\theta) \tag{19}$$
$$=2\pi(rs)^{-(n-2)/2}\sum_{m=0}^{\infty}\frac{J_{\nu(m)}(r)J_{\nu(m)}(s)}{\beta_m^2}\frac{d_m}{\sigma_n}P_m(\xi\cdot\theta). \tag{20}$$

By the estimate (D1) for Bessel functions we can prove that the series that define $\mathcal{K}(x,y)$ converges absolutely and uniformly on compacts subsets of $\mathbb{R}^n\times\mathbb{R}^n$. Since $Z_m(\xi,\theta)$ is real then $\mathcal{K}(x,y)$ is symmetric.

The orthogonal projection of $\mathcal{H}^2$ onto $\mathcal{W}^2$ is given by

$$\mathcal{P}u=\sum_{m=0}^{\infty}\sum_{j=1}^{d_m}\langle u,\beta_m^{-1}F_m^j\rangle_{\mathcal{H}^2}\beta_m^{-1}F_m^j,$$

with convergence in $\mathcal{W}^2$ and also pointwise.

For $x\in\mathbb{R}^n$ fixed, we have

$$\mathcal{P}u(x)=\langle u,\overline{\mathcal{K}(x,\cdot)}\rangle_{\mathcal{H}^2}$$
$$=\int_{\mathbb{R}^n}\left(\mathcal{K}(x,y)u(y)+\frac{\partial\mathcal{K}}{\partial s}(x,y)\frac{\partial u}{\partial s}(y)+\nabla_{S_\theta}\mathcal{K}(x,y)\cdot\nabla_{S_\theta}u(y)\right)\frac{dy}{\langle y\rangle^3}. \tag{21}$$

The function $\mathcal{K}(x,y)$ is the reproducing kernel for the space $\mathcal{W}^2$.

The following lemma shows that after a topological isomorphism of $\mathcal{W}^2$, the kernel $\mathcal{K}(x,y)$ has a closed form.

We call $\mathcal{M}$ a multiplier on the sphere $\mathbb{S}^{n-1}$ defined by a complex sequence $\{\mu_m\}$ to the operator

$$\mathcal{M}\left(\sum_{m,j}a_{mj}Y_m^j(\xi)\right)=\sum_{m,j}\mu_m a_{mj}Y_m^j(\xi)$$

for any finite sum $\sum_{m,j}a_{mj}Y_m^j(\xi)$.

Lemma 5 *Let $\mathcal{M}$ be the multiplier on the sphere $\mathbb{S}^{n-1}$ defined by the sequence $\{\beta_m^2\}$. Then, $\mathcal{M}$ is a topological isomorphism of $\mathcal{W}^2$ onto itself, where here*

$$\mathcal{M}\left(\sum_{m,j} a_{mj} F_m^j(\xi)\right) = \sum_{m,j} \beta_m^2 a_{mj} F_m^j(\xi).$$

Moreover, the kernel function of the composition $\mathcal{M} \circ \mathcal{P}$ is

$$\widetilde{\mathcal{K}}(x, y) = (2\pi\,|x-y|)^{-(n-2)/2} J_{\frac{n-2}{2}}(|x-y|), \tag{22}$$

namely $(\mathcal{M} \circ \mathcal{P})u(x) = \langle u, \widetilde{\mathcal{K}}(x, \cdot)\rangle_{\mathcal{H}^2}$.

Proof Since $c \le \beta_m^2 \le C$ for some constants $c, C > 0$, then it is clear that $\mathcal{M}$ is a topological isomorphism of $\mathcal{W}^2$ onto itself. In particular, by (19), (8) and (9), we have that

$$\begin{aligned}
\widetilde{\mathcal{K}}(x, y) &:= \mathcal{M}\mathcal{K}(x, y) \\
&= \frac{2\pi}{\sigma_n} \sum_{m=0}^{\infty} d_m r^{-(n-2)/2} J_{m+\frac{n-2}{2}}(r) s^{-(n-2)/2} J_{m+\frac{n-2}{2}}(s) P_m(\xi \cdot \theta) \\
&= (2\pi\,|x-y|)^{-(n-2)/2} J_{\frac{n-2}{2}}(|x-y|),
\end{aligned}$$

where $\mathcal{M}$ may be thought of as acting on ξ or on θ.
In $\mathcal{H}^2$, the kernel $\widetilde{\mathcal{K}}(x, y)$ defines a continuous operator $\widetilde{\mathcal{P}}$ on $\mathcal{H}^2$ given by

$$\widetilde{\mathcal{P}}u(x) = \int_{\mathbb{R}^n} \left(\widetilde{\mathcal{K}}(x, y)u(y) + \frac{\partial \widetilde{\mathcal{K}}}{\partial s}(x, y)\frac{\partial u}{\partial s}(y) + \nabla_{S_\theta}\widetilde{\mathcal{K}}(x, y) \cdot \nabla_{S_\theta} u(y) \right) \frac{dy}{\langle y\rangle^3}.$$

Let $\mathcal{H}_0$ be the linear span of the set $\{A(r)Y_m^j(\xi) : A \in C_c^\infty(0, \infty)\}_{m,j}$. We can prove that $\widetilde{\mathcal{P}} = \mathcal{M} \circ \mathcal{P}$ in $\mathcal{H}_0$. Since $\mathcal{H}_0$ is dense in $\mathcal{H}^2$, we conclude that $\widetilde{\mathcal{P}} = \mathcal{M} \circ \mathcal{P}$. □

Below we will need to study the continuity of the multiplier $\mathcal{M}$ in $L^p(\mathbb{S}^{n-1})$. For this we will use the next two results by Strichartz and Bonami–Clerc proved in [13] and [6], respectively.

Theorem 5 *Let $m(x)$ be a function of a real variable satisfying*

$$|x^k m^{(k)}(x)| \le A \quad for \quad k = 0, \ldots, a.$$

If $m_j = m(j)$ then $\{m_j\} \in \mathcal{M}_p(\mathbb{S}^{n-1})$ for

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{a}{n-1}, \; p \neq 1, \; \infty,$$

where $\mathcal{M}_p(\mathbb{S}^{n-1})$ denotes the space of all L^p−multipliers on the sphere $\mathbb{S}^{n-1}$.

Theorem 6 *Let $N = [\frac{n-1}{2}]$ and $\{\mu_k\}_{k\geq 0}$ be a sequence of complex numbers such that*

(A_0) $|\mu_k| \leq C$,
(A_N) $\sup_{j\geq 0} 2^{j(N-1)} \sum_{k=2^j}^{2^{j+1}} |\Delta^N \mu_k| \leq C$.

Then $\{\mu_k\} \in \mathcal{M}_p(\mathbb{S}^{n-1})$ for $1 < p < \infty$. Here Δ denotes the forward difference operator given by $\Delta\mu_k = \mu_{k+1} - \mu_k$.

Theorem 7 *For $1 < p < \infty$, the operators $\mathcal{M}$ and $\mathcal{M}^{-1}$ are continuous on $L^p(\mathbb{S}^{n-1})$. That is, the sequences $\left\{\beta_m^2\right\}$ and $\left\{\beta_m^{-2}\right\}$ define bounded multipliers on $L^p(\mathbb{S}^{n-1})$.*

Proof By (6) and (18) we obtain that for all $m \geq 2$,

$$\beta_m^2 = 2 + R(m)$$

with $R(m) = \frac{P(m)}{Q(m)}$ for some polynomials P y Q of degree 4 and 6, respectively. Thus, to prove the continuity of $\mathcal{M}$ it is enough to show that the sequence $\{R(m)\}$ defines a bounded multiplier on $L^p(\mathbb{S}^{n-1})$. We have that $R^{(k)}(x) \sim \frac{1}{|x|^{k+2}}$ for $|x|$ large. Hence,

$$|x^k R^{(k)}(x)| \leq A,$$

for all $k \in \mathbb{N} \cup \{0\}$. Then by Theorem 5, the above inequality implies that $\{R(m)\}$ defines a bounded multiplier on $L^p(\mathbb{S}^{n-1})$ for $1 < p < \infty$. To prove the continuity of $\mathcal{M}^{-1}$, it suffices to prove the continuity of the multiplier defined by the sequence $\{\gamma_m\}$ given by

$$\gamma_m = \frac{1}{1 + \frac{R(m)}{2}}.$$

For m large and $L \in \mathbb{N}$ fixed, there exists a sequence $\{r_m\}$ such that

$$\gamma_m = 1 - \frac{R(m)}{2} + \frac{R(m)^2}{2^2} - \cdots + \frac{R(m)^{L-1}}{2^{L-1}} + r(m),$$

$|r(m)| \sim O(\frac{1}{m^{2L}})$. Using Strichartz's Theorem we see that each $\{R(m)^k\}$ defines a bounded multiplier in $L^p(\mathbb{S}^{n-1})$ for $1 < p < \infty$ and $k = 0, 1, \ldots, L-1$. Thus, to end the proof we will show that if we choose L large enough, $\{r(m)\}$ defines a bounded multiplier in $L^p(\mathbb{S}^{n-1})$. Let $N = \left[\frac{n-1}{2}\right]$, then for m large,

$$\sum_{m=2^j}^{2^{j+1}} \left|\Delta^N r_m\right| = \sum_{m=2^j}^{2^{j+1}} \left|\sum_{i=0}^{N} (-1)^i \binom{N}{i} r_{m-i}\right|$$
$$\leq \frac{C_{N,L}}{2^{2jL}}$$

for all $j \geq 2$. Therefore,

$$2^{j(N-1)} \sum_{m=2^j}^{2^{j+1}} \left|\Delta^N r_m\right| \leq C_L 2^{j(N-2L)} = O(1)$$

if we choose any $L > N/2$. By Theorem 6, we conclude that $\{r_m\}$ defines a bounded multiplier on $L^p(\mathbb{S}^{n-1})$. □

Remark 2 By (18), we have that

$$\|u\|_{\mathcal{H}^2} \sim \left(\int_{\mathbb{R}^n} \left(|u(x)|^2 + |\nabla_S u(x)|^2\right) \frac{dx}{\langle x\rangle^3}\right)^{1/2}$$

for $u \in \mathcal{H}^2$.

Hence we may replace $\mathcal{H}^2$ by the Hilbert space ${\mathcal{H}'}^2$ with the norm

$$\|u\|_{\mathcal{H}'^2} = \left(\int_{\mathbb{R}^n} \left(|u(x)|^2 + |\nabla_S u(x)|^2\right) \frac{dx}{\langle x\rangle^3}\right)^{1/2}, \tag{23}$$

to define the kernel

$$\mathcal{K}'(x, y) = \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} \frac{F_m^j(x)\overline{F_m^j(y)}}{\gamma_m^2},$$

where $\gamma_m = \|F_m\|_{\mathcal{H}'^2} \sim \sqrt{2} + O\left(\frac{1}{m^2}\right)$. In this case, the orthogonal projection $\mathcal{P}'$ on ${\mathcal{H}'}^2$ is given by

$$\mathcal{P}'u(x) = \int_{\mathbb{R}^n} \left(\mathcal{K}'(x, y)u(y) + (-\Delta_{S_\theta})^{1/2}\mathcal{K}'(x, y)(-\Delta_{S_\theta})^{1/2}u(y)\right) \frac{dy}{\langle y\rangle^3}. \tag{24}$$

3 Structure and Properties of $\mathcal{W}^p$

Now we give estimates of the kernel $\widetilde{\mathcal{K}}(x, y)$.

Lemma 6 *Consider* $\widetilde{\mathcal{K}}(x, y) = \frac{J_{\frac{n-2}{2}}(|x-y|)}{(2\pi|x-y|)^{(n-2)/2}}$, $y = s\theta$ *in the polar form. Then we have the following pointwise estimates:*

$$\left|\widetilde{\mathcal{K}}(x, y)\right| \leq \frac{C}{(1 + |x-y|)^{\frac{n-1}{2}}}, \tag{25}$$

$$\left|\frac{\partial}{\partial s}\widetilde{\mathcal{K}}(x, y)\right| \leq \frac{C}{(1 + |x-y|)^{\frac{n-1}{2}}}, \tag{26}$$

$$\left|\nabla_{S_\theta}\widetilde{\mathcal{K}}(x, y)\right| \leq \frac{C|x|\,|y|}{(1 + |x-y|)^{\frac{n+1}{2}}}. \tag{27}$$

Proof The inequality (25) follows from (4) and the fact that the function $J_{\frac{n-2}{2}}(r)$ has a zero of order $(n-2)/2$ at $r=0$. Similarly, we can obtain (26).

To prove (27) we estimate any directional derivative D_ν of $\widetilde{\mathcal{K}}$ in the direction of a unit vector ν tangent to $\mathbb{S}^{n-1}$. Using (3), we have that

$$\begin{aligned}|D_\nu\widetilde{\mathcal{K}}(x,s\theta)| &= s|\nabla_y\widetilde{\mathcal{K}}(x,y)\cdot\nu|\\ &= C|y|\left|\frac{J_{\frac{n}{2}}(|x-y|)}{|x-y|^{n/2}}(x-y)\cdot\nu\right|\\ &= C|y|\left|\frac{J_{\frac{n}{2}}(|x-y|)}{|x-y|^{\frac{n}{2}}}x\cdot\nu\right|\\ &\le C|y|\left|\frac{J_{\frac{n}{2}}(|x-y|)}{|x-y|^{\frac{n}{2}}}\right||x|.\end{aligned}$$

Thus in particular we obtain (27). □

Proposition 1 *Let*

$$\alpha_n = \begin{cases} 1 & \text{if } n=2,3,4,5\\ \frac{2(n-3)}{n-1} & \text{if } n>5.\end{cases}$$

If $p>\alpha_n$ then $\widetilde{\mathcal{K}}(x,.)$, $\frac{\partial}{\partial s}\widetilde{\mathcal{K}}(x,.)$ and $\nabla_{S_\theta}\widetilde{\mathcal{K}}(x,.)$ belong to $L^p(\frac{dy}{\langle y\rangle^3})$ for each $x\in\mathbb{R}^n$.

Proof In fact, using the estimates given in the Lemma 6 and Peetre's inequality $(1+|x-y))^{-1}\le C(1+|x|)/(1+|y|)$, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^n}\left|\widetilde{\mathcal{K}}(x,y)\right|^p\frac{dy}{\langle y\rangle^3}\right)^{1/p} &\le C\left(\int_{\mathbb{R}^n}\frac{1}{(1+|x-y|)^{\frac{n-1}{2}p}}\frac{dy}{\langle y\rangle^3}\right)^{1/p}\\ &\le C(1+|x|)^{\frac{n-1}{2}}\left(\int_{\mathbb{R}^n}\frac{1}{(1+|y|)^{\frac{n-1}{2}p}}\frac{dy}{\langle y\rangle^3}\right)^{1/p}\\ &\le C(x)<\infty.\end{aligned}$$

Similarly, $\frac{\partial}{\partial s}\widetilde{\mathcal{K}}(x,.)\in L^p\left(\frac{dy}{\langle y\rangle^3}\right)$.

Finally,

$$\begin{aligned}\left(\int_{\mathbb{R}^n}\left|\nabla_{S_\theta}\widetilde{\mathcal{K}}(x,y)\right|^p\frac{dy}{\langle y\rangle^3}\right)^{1/p} &\le C|x|\left(\int_{\mathbb{R}^n}\frac{|y|^p}{(1+|x-y|)^{\frac{n+1}{2}p}}\frac{dy}{\langle y\rangle^3}\right)^{1/p}\\ &\le C|x|(1+|x|)^{\frac{n+1}{2}}\left(\int_{\mathbb{R}^n}\frac{|y|^p}{(1+|y|)^{\frac{n+1}{2}p}}\frac{dy}{\langle y\rangle^3}\right)^{1/p}\\ &\le C|x|(1+|x|)^{\frac{n+1}{2}}\left(\int_{\mathbb{R}^n}\frac{1}{(1+|y|)^{\frac{n-1}{2}p}}\frac{dy}{\langle y\rangle^3}\right)^{1/p}\end{aligned}$$

$$\leq C(x) < \infty.$$

□

Proposition 2 *If $p > \alpha_n$ then $F_m^j \in \mathcal{W}^p$ for any m, j. Moreover, $\mathcal{W}^p \neq \{\mathbf{0}\}$ if and only if $p > \alpha_n$.*

Proof We know that F_m^j is an entire solution of the Helmholtz equation and if $p > \alpha_n$, $F_m^j \in L^p\left(\frac{dx}{\langle x\rangle^3}\right)$. In fact, by (4)

$$F_m^j \in L^p(\langle x\rangle^{-3}dx) \iff \int_0^\infty \left|\frac{J_{\nu(m)}(r)}{r^{(n-2)/2}}\right|^p \frac{r^{n-1}dr}{(1+r^2)^{3/2}} < \infty$$
$$\iff \int_0^\infty r^{(n-1)-\frac{p}{2}(n-1)+3}dr < \infty$$

whenever $p > \alpha_n$. Thus, $F_m^j \in \mathcal{W}^p$.

Now suppose that $\mathcal{W}^p \neq \{\mathbf{0}\}$. Let $u \in \mathcal{W}^p$, $u \neq 0$. Then $u = \sum_{m,j} a_{mj} F_{mj}$ with some $a_{mj} \neq 0$. We have that $u(r\xi)Y_k^l(\xi) \in L^p\left(\frac{dx}{\langle x\rangle^3}\right)$. If φ is a radial function such that $\varphi(|x|) \in L^{p'}\left(\frac{dx}{\langle x\rangle^3}\right)$ and $\|\varphi\|_{L^{p'}(\frac{r^{n-1}}{\langle r\rangle^3})} \leq 1$, then by Hölder's inequality

$$\int_{\mathbb{R}^n} \left|u(x)Y_k^l(\xi)\varphi(|x|)\right| \frac{dx}{\langle x\rangle^3} \leq C,$$

which implies that

$$\int_0^\infty \left|\frac{J_{\nu(k)}(r)}{r^{(n-2)/2}}\varphi(r)\right| \frac{r^{n-1}}{\langle r\rangle^3}dr \leq C.$$

Consequently, by duality

$$\int_0^\infty \left|\frac{J_{\nu(k)}(r)}{r^{(n-2)/2}}\right|^p \frac{r^{n-1}dr}{(1+r^2)^{3/2}} < \infty,$$

and this implies that $p > \alpha_n$. □

Theorem 8 *For $1 < p < \infty$, $\mathcal{W}^p$ is a Banach space.*

Proof Let v any entire solution of the Helmholtz equation and let $\Phi(x, y)$ be the fundamental solution of the Helmholtz equation in $\mathbb{R}^n$ [1, p. 42], given as

$$\Phi(x, y) = \frac{i}{4}(2\pi\,|x-y|)^{-(n-2)/2} H^1_{\frac{n-2}{2}}(|x-y|).$$

Let $x \in \mathcal{B}_R$ fixed with $R > 1$. Using a Green's identity for the functions v and $\Phi(x, \cdot)$ we have (see [7, p. 68–69]) for $\rho > R$,

$$v(x) = \rho^{n-1} \int_{\mathbb{S}^{n-1}} \left(\frac{\partial v}{\partial s}(\rho\omega)\Phi(x, \rho\omega) - \frac{\partial \Phi}{\partial s}(x, \rho\omega)v(\rho\omega) \right) d\sigma(\omega).$$

Next, integrating both sides above with respect to $\frac{d\rho}{(1+\rho^2)^{3/2}}$ on the interval $[2R, 3R]$, we have the integral representation of v for points of $\mathcal{B}_R$,

$$v(x) = C_R \int_{2R \le |y| \le 3R} \left(\frac{\partial v}{\partial s}(y)\Phi(x, y) - \frac{\partial \Phi}{\partial s}(x, y)v(y) \right) \frac{dy}{\langle y \rangle^3}. \tag{28}$$

Now we prove that $\mathcal{W}^p$ is closed in $\mathcal{H}^p$. Differentiating under the integral in (28) and using Hölder's inequality we have that on any compact set K, any partial derivative

$$\left| \frac{\partial^\alpha u}{\partial x^\alpha}(x) \right| \le C_{K,\alpha} \|u\|_{\mathcal{H}^p}, \quad u \in \mathcal{W}^p, \ x \in K.$$

Let $\{u_n\}$ be a sequence in $\mathcal{W}^p$ converging to $u \in \mathcal{H}^p$. Taking a subsequence if necessary, assume that the convergence is also almost everywhere. The relation (28) implies that $\{u_n\}$ (and all their derivatives) is a Cauchy sequence uniformly in compact subsets of $\mathbb{R}^n$, converging to a limit $\tilde{u}$, that satisfies the Helmholtz equation. Then $u = \tilde{u}$ and $u \in \mathcal{W}^p$. □

Remark 3 Using the integral representation (28) we can see that the evaluation functional $\mathcal{W}^p \longrightarrow \mathbb{C}$, $v \longmapsto v(x)$ is continuous for every $x \in \mathbb{R}^n$.

Given $f(\xi) = \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} a_{mj} Y_m^j(\xi) \in L^p(\mathbb{S}^{n-1})$, the Riesz means R_N^δ of f of order δ is defined by

$$R_N^\delta f(\xi) = \sum_{k=0}^{N} \sum_{j=1}^{d_k} \left(1 - \frac{k}{N+1}\right)^\delta a_{kj} Y_k^j(\xi).$$

We will need the following theorem (see [6]) about the convergence of Riesz means to study the density of the linear span of $\{F_m^j\}$ in $\mathcal{W}^p$.

Theorem 9 *Let $1 \le p \le \infty$. If $\delta > (n-2)/2$, then for $f \in L^p(\mathbb{S}^{n-1})$,*

$$R_N^\delta f \to f \quad in\ L^p(\mathbb{S}^{n-1}),$$

moreover, the Riesz means are uniformly bounded on $L^p(\mathbb{S}^{n-1})$, that is, there exists a uniform constant $C_{p,\delta}$ such that

$$\| R_N^\delta f \|_{L^p(\mathbb{S}^{n-1})} \le C_{p,\delta} \|f\|_{L^p(\mathbb{S}^{n-1})}$$

for all N.

Theorem 10 *Let $p > \alpha_n$ and $\mathcal{W}_0^p$ the linear span of $\{F_m^j\}_{m,j}$. Then $\mathcal{W}_0^p$ is dense in $\mathcal{W}^p$.*

Proof Given $u \in \mathcal{W}^p$, the proof of the surjectivity in Theorem 4 shows that there exists $a_{mj} \in \mathbb{C}$ such that

$$u(r\xi) = \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} a_{mj} F_m^j(r\xi),$$

where the convergence is absolute and uniform in compact subsets of $\mathbb{R}^n$. Let r fixed and $\delta > (n-2)/2$, and we consider the Riesz means R_N^δ of u of order δ. By Proposition 2, $R_N^\delta u \in \mathcal{W}^p$ for $p > \alpha_n$.

Let $\Lambda_N^p(r)$ the integral given by

$$\Lambda_N^p(r) = \int_{\mathbb{S}^{n-1}} \left(\left|(R_N^\delta u - u)(r\xi)\right|^p + \left|\frac{\partial}{\partial r}(R_N^\delta u - u)(r\xi)\right|^p \right.$$
$$\left. + \left|(-\Delta_S)^{1/2}(R_N^\delta u - u)(r\xi)\right|^p \right) d\sigma(\xi).$$

By the Theorem 9 we have that $R_N^\delta u \longrightarrow u$ and $\frac{\partial}{\partial r} R_N^\delta u \longrightarrow \frac{\partial u}{\partial r}$ in $L^p(\mathbb{S}^{n-1})$ as $N \to \infty$. Since $(-\Delta_S)^{1/2}(R_N^\delta u) = R_N^\delta((-\Delta_S)^{1/2} u)$ we deduce that $(-\Delta_S)^{1/2} R_N^\delta u$ converges to $(-\Delta_S)^{1/2} u$ in $L^p(\mathbb{S}^{n-1})$. Hence

$$\lim_{N\to\infty} \Lambda_N^p(r) = 0.$$

Also, using the uniform boundedness of the Riesz means (Theorem 9) we obtain

$$\Lambda_N^p(r) \le C \int_{\mathbb{S}^{n-1}} \left(|u(r\xi)|^p + \left|\frac{\partial}{\partial r} u(r\xi)\right|^p + \left|(-\Delta_S)^{1/2} u(r\xi)\right|^p \right) d\sigma(\xi),$$

that is, $\Lambda_N^p(r) \le C g(r)$ with $g \in L^1\big(\mathbb{R}^+, \frac{r^{n-1} dr}{(1+r^2)^{3/2}}\big)$. Then applying the Lebesgue's Dominated Convergence Theorem we have

$$0 = \int_0^\infty \lim_{N\to\infty} \Lambda_N^p(r) r^{n-1} \frac{dr}{(1+r^2)^{3/2}} = \lim_{N\to\infty} \int_0^\infty \Lambda_N^p(r) r^{n-1} \frac{dr}{(1+r^2)^{3/2}}.$$

Therefore, $R_N^\delta u$ converges to u in $\mathcal{H}^p$. So, we conclude that the linear span of $\{F_m^j\}_{m,j}$ is dense in $\mathcal{W}^p$. □

Remark 4 By Theorems 7 and 10, we have that $\mathcal{M}$ and $\mathcal{M}^{-1}$ are continuous in $\mathcal{W}^p$ for any $p > \alpha_n$.

Now we will prove a reproducing property of the orthogonal projection $\mathcal{P}$ for the space $\mathcal{W}^p$.

Theorem 11 *Let $\alpha_n < p < \alpha_n'$. Given $u \in \mathcal{H}^p$, then $u \in \mathcal{W}^p$ if and only if $\mathcal{P}u = u$.*

Proof Let $u \in \mathcal{W}^p$ and $\alpha_n < p < \alpha_n'$. By Theorem 10, there exists a sequence $\{u_n\} \subseteq \mathcal{W}_0^p \subseteq \mathcal{W}^2$ such that $u_n \to u$ in $\mathcal{H}^p$ for $p > \alpha_n$. Also, since $\mathcal{P}$ is continuous in $\mathcal{W}^2$, then $\mathcal{P}u_n = u_n$. On the other hand, by Remark 3, we have that $u_n(x) \to u(x)$ for every $x \in \mathbb{R}^n$. So, to end the proof it is enough to see that $\mathcal{P}u_n(x) \to \mathcal{P}u(x)$ for all $x \in \mathbb{R}^n$. In effect,

$$\begin{aligned}|\mathcal{P}u_n(x) - \mathcal{P}u(x)| \le &\int_{\mathbb{R}^n} |\mathcal{K}(x, y)|\,|(u_n - u)(y)| \frac{dy}{\langle y\rangle^3}\\ &+ \int_{\mathbb{R}^n} \left|\frac{\partial \mathcal{K}}{\partial s}(x, y)\right| \left|\frac{\partial}{\partial s}(u_n - u)(y)\right| \frac{dy}{\langle y\rangle^3}\\ &+ \int_{\mathbb{R}^n} \left|\nabla_{S_\theta}\mathcal{K}(x, y)\right| \left|\nabla_{S_\theta}(u_n - u)(y)\right| \frac{dy}{\langle y\rangle^3}.\end{aligned}$$

Since by Proposition 1, $\widetilde{\mathcal{K}}(x, .)$, $\frac{\partial \widetilde{\mathcal{K}}}{\partial s}(x, .)$ and $|\nabla_{S_\theta}\widetilde{\mathcal{K}}(x, .)| \in L^{p'}\left(\frac{dy}{\langle y\rangle^3}\right)$, applying the Hölder's inequality we have that

$$|u_n(x) - \mathcal{P}u(x)| = |\mathcal{P}u_n(x) - \mathcal{P}u(x)| \le C(x)\|u_n - u\|_{\mathcal{H}^p}^p \longrightarrow 0.$$

Since we also have that $u_n(x) \longrightarrow u(x)$ we conclude that $Pu(x) = u(x)$.

To prove the converse, let $u \in \mathcal{H}^p$ and suppose $u = Pu$, then

$$\begin{aligned}&(\Delta + 1)_x u(x)\\ &= \int_{\mathbb{R}^n} (\Delta + 1)_x \left(\mathcal{K}(x, y)u(y) + \frac{\partial \mathcal{K}}{\partial s}(x, y)\frac{\partial u}{\partial s}(y) + \nabla_{S_\theta}\mathcal{K}(x, y) \cdot \nabla_{S_\theta}u(y)\right) \frac{dy}{\langle y\rangle^3}\\ &= 0,\end{aligned}$$

since $\mathcal{K}(., y)$ satisfies the Helmholtz equation in $\mathbb{R}^n$ for each $y \in \mathbb{R}^n$. Therefore, $u \in \mathcal{W}^p$. □

4 Continuity of $\mathcal{P}'$ in Mixed-Normed Spaces

In this section we prove a positive result about the continuity of $\mathcal{P}$ on mixed-normed spaces, generalizing the results in [4] for $n > 2$.

Definition 2 Let $1 \le p < \infty$, the mixed-normed space $L^p(\mathbb{R}^+; d\mu)(L^2(\mathbb{S}^{n-1}))$ consisting of all the measurable functions $f(r\xi)$ such that

$$\|f\|_{L^{p,2}}^p := \int_0^\infty \left(\int_{\mathbb{S}^{n-1}} |f(r\xi)|^2 d\sigma(\xi)\right)^{\frac{p}{2}} d\mu(r) < \infty,$$

where $d\mu(r) := r^{n-1}/(1 + r^2)^{3/2}dr$.

From now on we will write $L^p(\mathbb{R}^+; d\mu)(L^2(\mathbb{S}^{n-1}))$ as $L^{p,2}$.

Definition 3 For $1 \le p < \infty$, we denote by $\mathcal{H}^{p,2}$ the closure of $C_c^\infty(\mathbb{R}^n)$ with respect to the mixed-norm

$$\|u\|^p_{\mathcal{H}^{p,2}} := \int_0^\infty \left(\int_{\mathbb{S}^{n-1}} (|u(r\xi)|^2 + |\nabla_S u(r\xi)|^2) d\sigma(\xi) \right)^{\frac{p}{2}} d\mu(r)$$
$$\sim \int_0^\infty \left(\int_{\mathbb{S}^{n-1}} (|u(r\xi)|^2 + |(-\Delta_S)^{1/2} u(r\xi)|^2) d\sigma(\xi) \right)^{\frac{p}{2}} d\mu(r),$$

and denote by $\mathcal{W}^{p,2}$ the space of all functions $u \in \mathcal{H}^{p,2}$ satisfying the Helmholtz $\Delta u + u = 0$ in $\mathbb{R}^n$.

To study the continuity of $\mathcal{P}'$ in $\mathcal{H}^{p,2}$, we introduce the operator T defined by

$$Tu(r\xi) = (-\Delta_{S_\xi})^{1/2} \int_{\mathbb{R}^n} (-\Delta_{S_\theta})^{1/2} \mathcal{K}'(x, y) u(y) \frac{dy}{\langle y \rangle^3}. \tag{29}$$

T is well defined when $p < \alpha_n'$. In fact, for $u \in L^{p,2}$, by Hölder's inequality, Theorem 3 and Proposition 1, we have

$$\left| \int_{\mathbb{R}^n} (-\Delta_{S_\theta})^{1/2} \mathcal{K}'(x, y) u(y) \frac{dy}{\langle y \rangle^3} \right|$$
$$\le \left\| (-\Delta_{S_\theta})^{1/2} \mathcal{K}'(x, \cdot) \right\|_{L^{p',2}\left(\mathbb{R}^n, \frac{dy}{\langle y \rangle^3}\right)} \|u\|_{L^{p,2}\left(\mathbb{R}^n, \frac{dy}{\langle y \rangle^3}\right)}$$
$$\le C \left\| \nabla_S \mathcal{K}'(x, \cdot) \right\|_{L^{p',2}\left(\mathbb{R}^n, \frac{dy}{\langle y \rangle^3}\right)} \|u\|_{L^{p,2}\left(\mathbb{R}^n, \frac{dy}{\langle y \rangle^3}\right)}$$
$$< \infty.$$

Lemma 7 *Let $w(r)$ be a non-negative function such that $w^\beta \in A_2(d\tilde{\mu}(r))$ for some $\beta > 2$. Then*

$$n^4 \int_0^\infty |J_n(r)|^2 w(r) d\tilde{\mu}(r) \int_0^\infty |J_n(r)|^2 w^{-1}(r) d\tilde{\mu}(r) \le C,$$

where C independent of n.

The proof of this lemma can be found in [4] and we have the following version.

Lemma 8 *Let $w(r)$ be a non-negative function and suppose there exists $\beta > 2$ such that $w^\beta \in A_2(d\tilde{\mu}(r))$ and $-a = (n-2)(1-\frac{2}{p}) < 2 - \frac{1}{\beta}$. Then*

$$m^4 \int_0^\infty |J_{\nu(m)}(r)|^2 r^a w(r) d\tilde{\mu}(r) \int_0^\infty |J_{\nu(m)}(r)|^2 r^{-a} w^{-1}(r) d\tilde{\mu}(r) \le C, \tag{30}$$

where C is independent of m.

Proof Let I^1 and I^2 be the integrals given by

$$I^1 = \int_0^\infty |J_{\nu(m)}(r)|^2 r^a w(r) d\tilde{\mu}(r)$$

and

$$I^2 = \int_0^\infty |J_{\nu(m)}(r)|^2 r^{-a} w^{-1}(r) d\tilde{\mu}(r),$$

respectively.

We split these integrals as

$$I^1 = \int_0^1 + \int_1^{\nu(m)\text{sech}\alpha_0} + \int_{\nu(m)\text{sech}\alpha_0}^{2\nu(m)} + \int_{2\nu(m)}^\infty = \sum_{i=1}^4 I_i^1$$

and

$$I^2 = \int_0^1 + \int_1^{\nu(m)\text{sech}\alpha_0} + \int_{\nu(m)\text{sech}\alpha_0}^{2\nu(m)} + \int_{2\nu(m)}^\infty = \sum_{j=1}^4 I_j^2.$$

We proceed as in the proof of Lemma 7. We will prove that

$$m^4 I_i^1 I_j^2 \le C; \quad i, j \in \{1, 2, 3, 4\}.$$

Suppose $m \ge 1$, then by Hölder's inequality and the estimates of Bessel functions $(D1)$–$(D4)$ we have

$$\begin{aligned} I_1^1 &\le \tilde{\mu}([0,1])^{1/\beta} \left(\int_0^1 |J_{\nu(m)}(r)|^{2\beta'} r^{a\beta'} d\tilde{\mu}(r) \right)^{1/\beta'} \left(\frac{1}{\tilde{\mu}([0,1])} \int_0^1 w^\beta(r) d\tilde{\mu}(r) \right)^{1/\beta} \\ &\le \frac{C}{(2^m m!)^2} \left(\int_0^1 r^{(n-2)\beta' + a\beta'} d\tilde{\mu}(r) \right)^{1/\beta'} \left(\frac{1}{\tilde{\mu}([0,1])} \int_0^1 w^\beta(r) d\tilde{\mu}(r) \right)^{1/\beta} \\ &\le \frac{C}{(2^m m!)^2} \left(\frac{1}{\tilde{\mu}([0,1])} \int_0^1 w^\beta(r) d\tilde{\mu}(r) \right)^{1/\beta}, \end{aligned}$$

$$\begin{aligned} I_4^1 &\le \frac{C}{\nu(m)^{1/\beta}} \left(\int_{2\nu(m)}^\infty r^{-\beta' + a\beta'} d\tilde{\mu}(r) \right)^{1/\beta'} \left(\frac{1}{\tilde{\mu}([2\nu(m),\infty])} \int_{2\nu(m)}^\infty w^\beta(r) d\tilde{\mu}(r) \right)^{1/\beta} \\ &\le \frac{C\nu(m)^a}{m^2} \left(\frac{1}{\tilde{\mu}([2\nu(m),\infty])} \int_{2\nu(m)}^\infty w^\beta(r) d\tilde{\mu}(r) \right)^{1/\beta} \end{aligned}$$

and

$$I_2^1 \le C\left(\int_1^{\nu(m)c} e^{-2\nu(m)\beta'\phi(r)}\,dr\right)^{1/\beta'}\left(\frac{1}{\tilde{\mu}([1,\nu(m)c])}\int_1^{\nu(m)c} w^{\beta}(r)d\tilde{\mu}(r)\right)^{1/\beta}$$
$$\le \frac{C}{e^{2m\beta_0}}\left(\frac{1}{\tilde{\mu}([1,\nu(m)c])}\int_1^{\nu(m)c} w^{\beta}(r)d\tilde{\mu}(r)\right)^{1/\beta},$$

where $c = \operatorname{sech}\alpha_0$ for some $\alpha_0 > 0$, $\phi(r) = \alpha(r) - \tanh\alpha(r)$, $\beta_0 = \phi(\nu(m)c) = \alpha_0 - \tanh\alpha_0 > 0$ and the function $\alpha(r)$ is defined by the equation $\nu(m)\sinh\alpha(r) = r$.

In addition, by Lemma 1 we see that

$$I_3^1 \le \frac{C\nu(m)^a}{m^2}\left(\frac{1}{\tilde{\mu}([\nu(m)c, 2\nu(m)])}\int_{\nu(m)c}^{2\nu(m)} w^{\beta}(r)d\tilde{\mu}(r)\right)^{1/\beta}.$$

Similarly, we have that

$$I_1^2 \le \frac{C}{(2^m m!)^2}\left(\frac{1}{\tilde{\mu}([0,1])}\int_0^1 w^{-\beta}(r)d\tilde{\mu}(r)\right)^{1/\beta},$$
$$I_2^2 \le \frac{C\nu(m)^{-a}}{e^{2m\beta_0}}\left(\frac{1}{\tilde{\mu}([1,2\nu(m)c])}\int_1^{2\nu(m)c} w^{-\beta}(r)d\tilde{\mu}(r)\right)^{1/\beta},$$
$$I_3^2 \le \frac{C\nu(m)^{-a}}{m^2}\left(\frac{1}{\tilde{\mu}([\nu(m)c, 2\nu(m)])}\int_{\nu(m)c}^{2\nu(m)} w^{-\beta}(r)d\tilde{\mu}(r)\right)^{1/\beta}.$$

Furthermore, using that $a > \frac{1}{\beta} - 2$ it follows that

$$I_4^2 \le \frac{C\nu(m)^{-a}}{m^2}\left(\frac{1}{\tilde{\mu}([2\nu(m),\infty])}\int_{2\nu(m)}^{\infty} w^{-\beta}(r)d\tilde{\mu}(r)\right)^{1/\beta},$$

Consequently, since $w^{\beta} \in A_2(d\tilde{\mu}(r))$,

$$m^4 I_i^1 I_j^2 \le C; \quad i, j \in \{1, 2, 3, 4\}.$$

□

Proposition 3 *Let $\beta_n \in (1,\infty)$ such that*

$$\beta_n' = \begin{cases} \infty & \text{if } n = 2, 3 \\ 2 + \frac{4}{n-3} & \text{if } n > 3. \end{cases}$$

If $p \in (\beta_n, \beta_n') \cap (4/3, 4)$ then T is a bounded operator on $L^{p,2}$. Moreover, if $p \notin (4/3, 4)$ then T cannot be extended to a bounded operator on $L^{p,2}$.

Proof First we note that $p \in (\beta_n, \beta'_n) \subset (\alpha_n, \alpha'_n)$ and then p satisfies $(n-1)(1-\frac{2}{p}) < 2$.

It suffices to prove the proposition for (β_n, β'_n) and $p \geq 2$, since T is self adjoint with respect to the duality $(f, g) \to \int_{\mathbb{R}^n} fg \frac{dx}{\langle x\rangle^3}$ of $L^{p,2}$ and $L^{p',2}$.

Next, expanding u in spherical harmonics, that is,

$$u(r\xi) = \sum_{m,j} u_{mj}(r) Y_m^j(\xi),$$

and using the Fourier expansion of the kernel $\mathcal{K}'$ we have

$$Tu(r\xi) = \sum_{m,j} T_{mj} u_{mj}(r) Y_m^j(\xi), \tag{31}$$

where

$$T_{mj} f_{mj}(r) = Cm(m+n-2) J_{\nu(m)}(r) r^{-(n-2)/2} \int_0^\infty J_{\nu(m)}(s) s^{-(n-2)/2} f_{mj}(s)\, d\mu(s). \tag{32}$$

Showing that T is bounded on $L^{p,2}$ is equivalent to prove the vector-valued inequality,

$$\left(\int_0^\infty \left(\sum_{m,j} |T_{mj} u_{mj}(r)|^2\right)^{\frac{p}{2}} d\mu(r)\right)^{\frac{1}{p}} \leq C \left(\int_0^\infty \left(\sum_{m,j} |u_{mj}(r)|^2\right)^{\frac{p}{2}} d\mu(r)\right)^{\frac{1}{p}}, \tag{33}$$

with C independent of m.

Let r be the dual exponent of $p/2$. By duality, there exists $h \in L^r(d\mu)$ with $\|h\|_{L^r(d\mu)} = 1$ such that

$$\left(\int_0^\infty \left(\sum_{m,j} |T_{mj} u_{mj}(s)|^2\right)^{\frac{p}{2}} d\mu(s)\right)^{\frac{2}{p}} = \int_0^\infty \sum_{m,j} |T_{mj} u_{mj}(s)|^2 h(s) d\mu(s).$$

Let $g(s) = s^{\frac{n-2}{r}} h(s)$ and $\tilde{\mu}$ the measure given by $d\tilde{\mu}(r) = \frac{r\,dr}{(1+r^2)^{3/2}}$. Notice that since $p < 4$ we have that $r > 2$, so we can choose γ such that $2 < \gamma \leq r$, then $g^\gamma \in L^1_{loc}(d\tilde{\mu})$, $g^\gamma \leq M_{\tilde{\mu}}(g^\gamma)$ a.e. and

$$\left(\int_0^\infty \left(\sum_{m,j} |T_{mj} u_{mj}(s)|^2\right)^{\frac{p}{2}} d\mu(s)\right)^{\frac{2}{p}}$$

$$= \int_0^\infty \sum_{m,j} |T_{mj} u_{mj}(s)|^2 s^{-(n-2)/r} g(s) d\mu(s)$$
$$= \int_0^\infty \sum_{m,j} |T_{mj} u_{mj}(s)|^2 s^{(n-2)(1-1/r)} g(s) d\tilde{\mu}(s)$$
$$\leq \sum_{m,j} \int_0^\infty |T_{mj} u_{mj}(s)|^2 s^{2(n-2)/p} (M_{\tilde{\mu}}[g^\gamma](s))^{\frac{1}{\gamma}} d\tilde{\mu}(s)$$
$$\leq C \sum_{m,j} m^4 \int_0^\infty |J_{\nu(m)}(s) s^{-(n-2)/2}|^2 s^{2(n-2)/p} w(s) d\tilde{\mu}(s)$$
$$\times \int_0^\infty |J_{\nu(m)}(s) s^{-(n-2)/2}|^2 s^{2(n-2)/q} w^{-1}(s) d\tilde{\mu}(s)$$
$$\int_0^\infty |u_{mj}(s)|^2 s^{2(n-2)/p} w(s) d\tilde{\mu}(s)$$
$$\leq C \sum_{m,j} \int_0^\infty |u_{mj}(s)|^2 s^{2(n-2)/p} w(s) d\tilde{\mu}(s)$$
$$\times m^4 \int_0^\infty |J_{\nu(m)}(s)|^2 s^{(n-2)(\frac{2}{p}-1)} w(s) d\tilde{\mu}(s)$$
$$\int_0^\infty |J_{\nu(m)}(s)|^2 s^{-(n-2)(\frac{2}{p}-1)} w^{-1}(s) d\tilde{\mu}(s),$$

where $w(s) = (M_{\tilde{\mu}}[g^\gamma](s))^{\frac{1}{\gamma}}$. Furthermore, since $(n-1)\big(1-\frac{2}{p}\big) < 2$, we have that $(n-2)\big(\frac{2}{p}-1\big) - \frac{1}{r} > -2$. Then we can choose γ close enough to r so that for some $2 < \beta < \gamma$ we have $(n-2)\big(\frac{2}{p}-1\big) - \frac{1}{\beta} > -2$. We know (see [8, Theorem 7.7(1)]) that $M_{\tilde{\mu}}(g^\gamma)^{\frac{\beta}{\gamma}} \in A_1(\tilde{\mu})$. Then since $M_{\tilde{\mu}}$ is bounded on $L^s(\tilde{\mu})$ for $s > 1$, by Lemma 8 and Hölder's inequality, we have

$$\left(\int_0^\infty \left(\sum_{m,j} |T_{mj} u_{mj}(s)|^2\right)^{\frac{p}{2}} d\mu(s)\right)^{\frac{2}{p}}$$
$$\leq C \int_0^\infty \sum_{m,j} |u_{mj}(s)|^2 s^{2(n-2)/p} (M_{\tilde{\mu}}[g^\gamma](s))^{\frac{1}{\gamma}} d\tilde{\mu}(s)$$
$$\leq C \left(\int_0^\infty \left(\sum_{m,j} |u_{mj}(s)|^2 s^{2(n-2)/p}\right)^{\frac{p}{2}} d\tilde{\mu}(s)\right)^{\frac{2}{p}} \left(\int_0^\infty (M_{\tilde{\mu}}[g^\gamma](s))^{\frac{r}{\gamma}} d\tilde{\mu}(s)\right)^{\frac{1}{r}}$$
$$\leq C \left(\int_0^\infty \left(\sum_{m,j} |u_{mj}(s)|^2\right)^{\frac{p}{2}} d\mu(s)\right)^{\frac{2}{p}}.$$

Now we prove that T is not continuous on $L^{p,2}$ for $p \notin (4/3, 4)$.

Let $u(r\xi) = \sum_{m,j} u_{mj}(r) Y_m^j(\xi)$, where

$$u_{mj}(r\xi) = r^{\alpha} |J_{\nu(m)}(r)|^{p'-1} sgn(J_{\nu(m)}(r)) \chi_{[\nu(m), 2\nu(m)]} Y_m^j(\xi) \tag{34}$$

with $\alpha = -\frac{(n-2)}{2}\frac{1}{p-1}$ (see in [4], the sequence $\{f_n\}$ in the proof of Theorem 4). Writing $Tu(r\xi) = \sum_{m,j} T_{mj} u_{mj}(r) Y_m^j(\xi)$ as in (31), we have that

$$\|u_{mj}\|_{p,2} = \left(\int_{\nu(m)}^{2\nu(m)} |J_{\nu(m)}(r)|^{p'} r^{-(n-2)p'/2} d\mu(r)\right)^{1/p}$$

and

$$\|T_{mj}u_{mj}\|_{p,2} \geq Cm(m+n-2)\left(\int_{\nu(m)}^{2\nu(m)} |J_{\nu(m)}(r)|^{p} r^{-(n-2)p/2} d\mu(r)\right)^{1/p} \times \|u_{mj}\|_{p,2}^{p}.$$

Therefore,

$$\frac{\|T_{mj}u_{mj}\|_{p,2}}{\|u_{mj}\|_{p,2}} \geq C\left(\int_{\nu(m)}^{2\nu(m)} |J_{\nu(m)}(r)|^{p} dr\right)^{1/p} \left(\int_{\nu(m)}^{2\nu(m)} |J_{\nu(m)}(r)|^{p'} dr\right)^{1/p'},$$

and using the Lemma 1 we see that this last expression is not bounded if $p \notin (4/3, 4)$. □

Now, we are ready to demonstrate the main theorem of this section.

Theorem 12 *If $p \in (\beta_n, \beta_n') \cap (4/3, 4)$ then $\mathcal{P}'$ can be extended to a bounded operator on $\mathcal{H}^{p,2}$. Moreover, if $p \notin (4/3, 4)$ then $\mathcal{P}'$ cannot be extended to a bounded operator on $\mathcal{H}^{p,2}$. In particular, for n =2, 3, 4, 5, $\mathcal{P}'$ is continuous on $\mathcal{H}^{p,2}$ if and only if $p \in (4/3, 4)$.*

Proof Let $p \in (\beta_n, \beta_n') \cap (4/3, 4)$. To prove the $L^{p,2}$ boundedness of $\mathcal{P}'$, it suffices to prove that the operators T_1, T_2, T_3 with kernels

$$\mathcal{K}'(x, y),\ (-\Delta_{S_\xi})^{1/2}\mathcal{K}'(x, y),\ (-\Delta_{S_\xi})^{1/2}(-\Delta_{S_\theta})^{1/2}\mathcal{K}'(x, y),$$

are bounded on $L^{p,2}$. By Proposition 3, we know that T_3 is continuous on $L^{p,2}$. To prove the continuity of T_1 and T_2 notice that

$$\mathcal{K}'(x, y) = \mathcal{M}_1(-\Delta_{S_\xi})^{1/2}(-\Delta_{S_\theta})^{1/2}\mathcal{K}'(x, y)$$

and

$$(-\Delta_{S_\xi})^{1/2}\mathcal{K}'(x, y) = \mathcal{M}_2(-\Delta_{S_\xi})^{1/2}(-\Delta_{S_\theta})^{1/2}\mathcal{K}'(x, y),$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are the multipliers in $\mathbb{S}^{n-1}$ corresponding to the sequences $\frac{1}{m(m+n-2)}$ and $\frac{1}{\sqrt{m(m+n-2)}}$ respectively. Then proceeding as in Theorem 3 we see that the required vector valued inequalities for T_1 and T_2 are less demanding than (33).

Now we show that $\mathcal{P}'$ is not continuous in $\mathcal{H}^{p,2}$ for $p \notin (4/3, 4)$.

If $\mathcal{P}'$ is continuous in $\mathcal{H}^{p,2}$ then since $(-\Delta_{S_\xi})^{-1/2} : \mathcal{H}^{p,2} \to \mathcal{H}^{p,2}$ is bounded (due to the fact that $(-\Delta_{S_\xi})^{-1/2}$ is bounded in $L^2(\mathbb{S}^{n-1})$), we have that

$$\mathcal{L} = (-\Delta_{S_\xi})^{1/2} \circ \mathcal{P}' \circ (-\Delta_{S_\xi})^{-1/2} \tag{35}$$

is continuous in $L^{p,2}$.

But

$$\mathcal{L}u(x) = \int_{\mathbb{R}^n} \mathcal{K}'(x, y)u(y)\frac{dy}{\langle y\rangle^3} + Tu(x),$$

hence, in the notation of Proposition 3,

$$\mathcal{L}u(x) = \sum_{m,j} \left(\frac{1}{m(m+n-2)} + 1\right) T_{mj} u_{mj}(r) Y_m^j(\xi)$$

and it follows proceeding as in Proposition 3, that $\mathcal{L}$ is not bounded in $L^{p,2}$ for $p \notin (4/3, 4)$. □

Now we will obtain a negative result relative to the continuity of projection $\mathcal{P}$. Notice that by Remark 4 the operators $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ have the same continuity properties on $\mathcal{H}^p$. This motivates the study of the continuity of the integral operator $\mathcal{T}$ given by

$$\mathcal{T}u(x) = \nabla_{S_\xi} \int_{\mathbb{R}^n} \nabla_{S_\theta} \widetilde{\mathcal{K}}(x, y) \cdot u(y) \frac{dy}{\langle y\rangle^3}, \;\; x = r\xi, \;\; y = s\theta, \tag{36}$$

since the most singular part of $\widetilde{\mathcal{P}}$ is precisely $\mathcal{T}(\nabla_{S_\theta} u)$.

Using (10), we can split the operator in the sum $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$, where

$$\mathcal{T}_1 u(x) = C_n \int_{\mathbb{R}^n} |x||y| F_{n/2}(|x-y|) \left(\mathbf{A}(u, y) - \mathbf{A}(u, y) \cdot \frac{x}{|x|}\frac{x}{|x|}\right) \frac{dy}{\langle y\rangle^3}, \tag{37}$$

$$\mathcal{T}_2 u(x) = C_n \int_{\mathbb{R}^n} |x||y| F_{n/2+1}(|x-y|)(x - \mathbf{P}_y x) \cdot \nabla_{S_\theta} u(y)(y - \mathbf{P}_x y) \frac{dy}{\langle y\rangle^3}, \tag{38}$$

where $F_\alpha(t) = \frac{J_\alpha(t)}{t^\alpha}$, $\mathbf{A}(u, y) = u(y) - u(y) \cdot \frac{y}{|y|}\frac{y}{|y|}$ and $\mathbf{P}_a b = \frac{a \cdot b}{|a|}\frac{a}{|a|}$ is the orthogonal projection of b in the direction of a.

We will assume that $n = 3$ and we will prove that $\mathcal{T}$ cannot be extended in general to a bounded operator on $L^p(\langle x\rangle^{-3} dx)$. Let $m \in \mathbb{N}$ and B_m be the unit ball of center $(0, 0, m)$ and fixed radius $\epsilon < 1$. Define $u_m = \chi_{B_m} \mathbf{e_1}$.

We consider the region R of the upper half-space between two cones $c_1^2\big(x_1^2 + x_2^2\big) \le x_3^2 \le c_2^2\big(x_1^2 + x_2^2\big)$ and such that $|x_1| > |x_2|$. Now, for fixed $\lambda > 0$ and $k > \lambda m$, let A_k be the annulus between the spheres centered in $(0, 0, m)$ and radii $\alpha(k)$ and $\alpha(k) + l$, with $\alpha(k) = 2\pi k + C$ and where C and $l > 0$ are chosen so that $\cos\big(t - (\frac{n}{2} + 1)\frac{\pi}{2} - \frac{\pi}{4}\big) \ge 1/2$ for $t \in [\alpha(k), \alpha(k) + l]$.

Lemma 9 *There exists positive constant λ such that, if $k > \lambda m$, then $|R \cap A_k| \sim k^2$ uniformly for large m.*

Proof Clearly $|R \cap A_k| = O(k^2)$. Now consider spherical coordinates $\{(r, \theta, \varphi) : r > 0, \theta \in [0, 2\pi], \varphi \in [0, \pi]\}$ centered at the point (cartesian) $(0, 0, m)$. Notice that as a subset of $\mathbb{R}^2$, every vertical section $R \cap A_k \cap \{(r, \theta_0, \varphi) : r > 0, \varphi \in [0, \pi]\}$ is independent of $\theta_0 \in [0, \pi/4]$. This subset of $\mathbb{R}^2$ contains the region in S_k described as follows.

Let P_1 be the intersection of $(\alpha(k) + l)\mathbb{S}^1$ and the line $s = c_1^{-1}t$ in the plane (s, t) and P_2 the intersection of $\alpha(k)\mathbb{S}^1$ and the line $s = c_2^{-1}t$ in the plane (s, t) both with $t > m$.

Then define S_k as the intersection of the annulus $\alpha_k < |x - (0, m)| < \alpha_k + l$ and the region in the first quadrant between the line l_1 through $(0, m)$ and P_1 and the line l_2 passing through $(0, m)$ and P_2. Let φ_i be such that $\tan(\pi/2 - \varphi_i)$ is the slope of the line l_i for $i = 1, 2$.

It follows that if $A_k' \subset R \cap A_k$ in spherical coordinates centered on $(0, 0, m)$ is given by the inequalities $\alpha(k) \le r \le \alpha(k) + l$, $0 \le \theta \le \frac{\pi}{4}$, $\varphi_2 \le \varphi \le \varphi_1$, then we have

$$\big|A_k'\big| = \int_0^{\frac{\pi}{4}} \int_{\varphi_2}^{\varphi_1} \int_{\alpha(k)}^{\alpha(k)+l} r^2 \sin\varphi dr\, d\varphi\, d\theta \ge Ck^2(\cos\varphi_2 - \cos\varphi_1).$$

Hence, to complete the proof of the lemma, it suffices to show that there exists $c > 0$ such that

$$\cos\phi_2 - \cos\phi_1 \ge c. \tag{39}$$

Denoting $\alpha(k)$ just by α, we observe that $P_2 = (c_2^{-1}t_2, t_2)$ with

$$\frac{t_2}{\alpha} = \frac{m + \sqrt{m^2 + (\alpha^2 - m^2)(c_2^{-2} + 1)}}{(c_2^{-2} + 1)\alpha}.$$

Let $\lambda > 0$ and $\alpha > \lambda m$. Then $1 - \frac{1}{\lambda^2} < 1 - \frac{m^2}{\alpha^2}$, and

$$\frac{t_2}{\alpha} \ge \frac{\sqrt{c_2^{-2} + 1}\sqrt{\alpha^2 - m^2}}{(c_2^{-2} + 1)\alpha} \ge \frac{1}{\sqrt{c_2^{-2} + 1}}\sqrt{1 - \frac{1}{\lambda^2}}. \tag{40}$$

Similarly, we have that $P_1 = (c_1^{-1} t_1, t_1)$ and

$$\frac{t_1}{\alpha + l} = \frac{m + \sqrt{m^2 + [(\alpha + l)^2 - m^2](c_1^{-2} + 1)}}{(c_1^{-2} + 1)(\alpha + l)}.$$

Since $\alpha > \lambda m$ then $\frac{m}{\alpha + l} < \frac{1}{\lambda}$, hence

$$\frac{t_1}{\alpha + l} \leq \frac{1}{(c_1^{-2} + 1)\lambda} + \frac{\sqrt{\frac{1}{\lambda^2} + (c_1^{-2} + 1)}}{c_1^{-2} + 1}. \tag{41}$$

By (40) and (41), we have that

$$\frac{t_2}{\alpha} - \frac{t_1}{\alpha + l} \geq \frac{1}{\sqrt{c_2^{-2} + 1}} \sqrt{1 - \frac{1}{\lambda^2}} - \frac{1}{(c_1^{-2} + 1)\lambda} - \frac{\sqrt{\frac{1}{\lambda^2} + (c_1^{-2} + 1)}}{c_1^{-2} + 1}.$$

Since the limit of the right side is positive as $\lambda \to \infty$, we conclude that choosing λ large enough $t_2/\alpha - t_1/(\alpha + \lambda) \geq \epsilon$, for some $\epsilon > 0$.

Finally for such λ, if $\alpha > \lambda m$ we have that

$$\begin{aligned} \cos \varphi_2 - \cos \varphi_1 &= \frac{t_2 - m}{\alpha} - \frac{t_1 - m}{\alpha + \lambda} \\ &= \left(\frac{t_2}{\alpha} - \frac{t_1}{\alpha + \lambda} \right) + h, \end{aligned}$$

where $|h| \sim O\left(\frac{1}{m}\right)$. Therefore, since $t_2/\alpha - t_1/(\alpha + \lambda) \geq c$ then (39) holds for large m and the proof is complete. □

Theorem 13 *$\mathcal{T}$ cannot be extended to a bounded operator on $L^p(\langle x \rangle^{-3} dx)$ for $p \in (1, 3/2)$.*

Proof Let $y \in B_m$, then we can write to $y = m\mathbf{e_3} + y'$ with $|y'| < \epsilon$, so that

$$(\mathbf{P}_x y)_3 = (\mathbf{P}_x m\mathbf{e_3})_3 + (\mathbf{P}_x y')_3 < Cm + \epsilon.$$

Therefore,

$$(y - \mathbf{P}_x y)_3 \geq (m - \epsilon) - (Cm + \epsilon) = (1 - C)m - 2\epsilon \geq Cm \geq C|y| \tag{42}$$

for all ϵ sufficiently small, m sufficiently large and choosing $C < 1$.

On the other hand, we have that

$$(x - \mathbf{P}_y x) \cdot u_m(y) = x_1 - \frac{y \cdot x}{|y|} \frac{y}{|y|} \cdot \mathbf{e_1},$$

estimating above the right hand side we have

$$\left|\frac{y\cdot x}{|y|}\frac{y}{|y|}\cdot \mathbf{e_1}\right| \leq \frac{\epsilon}{m^2}\left(|x_1y_1|+|x_2y_2|+|x_3y_3|\right) \leq C|x_1|\epsilon/m.$$

Hence,

$$(x-\mathbf{P}_y x)\cdot u_m(y) = x_1 - O(|x_1|\epsilon/m) > C|x_1| > C\,|x|\,. \tag{43}$$

Let $x\in A_k$. For (42), (43) and (4), we deduce that

$$|\mathcal{T}_2 u_m(x)| \geq C\int_{B_m}|x|m\frac{1}{k^3}|x|m\frac{dy}{\langle y\rangle^3} \geq \frac{C|x|^2}{k^3m}.$$

By Lemma 9,

$$\begin{aligned}\|\mathcal{T}_2u_m\|^p_{L^p(\bigcup_{k\geq Cm}R\cap A_k)} &= \int_{\bigcup_{k\geq Cm}R\cap A_k}|\mathcal{T}_2u_m(x)|^p\frac{dx}{\langle x\rangle^3}\geq C\sum_{k\geq Cm}\int_{A_k}\left(\frac{k^2}{k^3m}\right)^p\frac{dx}{\langle x\rangle^3}\\ &\geq C\sum_{k\geq Cm}\left(\frac{1}{km}\right)^p\frac{1}{k^3}\,|R\cap A_k|\geq\frac{C}{m^p}\sum_{k\geq Cm}\frac{1}{k^{p+1}}\geq\frac{C}{m^{2p}},\end{aligned}$$

and so

$$\|\mathcal{T}_2u_m\|_{L^p(\bigcup_{k\geq Cm}R\cap A_k)} \geq \frac{C}{m^2}. \tag{44}$$

Furthermore,

$$|\mathcal{T}_1u_m(x)| \leq C\int_{B_m}|x|\frac{m}{k^2}\frac{dy}{\langle y\rangle^3} \leq \frac{C|x|}{m^2k^2}.$$

Then,

$$\begin{aligned}\|\mathcal{T}_1u_m\|^p_{L^p(\bigcup_{k\geq Cm}R\cap A_k)} &= \int_{\bigcup_{k\geq Cm}R\cap A_k}|\mathcal{T}_1u_m(x)|^p\frac{dx}{\langle x\rangle^3}\\ &\leq C\int_{\bigcup_{k\geq Cm}R\cap A_k}\left(\frac{|x|}{m^2k^2}\right)^p\frac{dx}{\langle x\rangle^3}\\ &\leq C\sum_{k\geq Cm}\int_{R\cap A_k}\left(\frac{k}{m^2k^2}\right)^p\frac{dx}{\langle x\rangle^3}\\ &\leq\frac{C}{m^{2p}}\sum_{k\geq Cm}\int_{R\cap A_k}\frac{1}{k^{p+3}}\,|R\cap A_k|\\ &\leq\frac{C}{m^{2p}}\sum_{k\geq Cm}\frac{1}{k^{p+1}}\leq\frac{C}{m^{3p}}.\end{aligned}$$

Consequently,

$$\|\mathcal{T}_1 u_m\|_{L^p(\bigcup_{k\geq Cm} R\cap A_k)} \leq \frac{C}{m^3}. \tag{45}$$

Finally, by (44) and (45)

$$\begin{aligned}\|\mathcal{T} u_m\|_p &= \|(\mathcal{T}_2 u_m - (-\mathcal{T}_1 u_m)\|_p \\ &\geq \|\mathcal{T}_2 u_m\|_{L^p(\bigcup_{k\geq Cm} R\cap A_k)} - \|\mathcal{T}_1 u_m\|_{L^p(\bigcup_{k\geq Cm} R\cap A_k)} \\ &\geq C\left(\frac{1}{m^2} - \frac{1}{m^3}\right) \geq \frac{C}{m^2},\end{aligned}$$

then, since $\|u_m\|_p \sim m^{-3/p}$,

$$\frac{\|\mathcal{T} u_m\|_p}{\|u_m\|_p} \geq Cm^{3/p-2}. \tag{46}$$

Hence $\mathcal{T}$ is not bounded if $p \in (1, 3/2)$. □

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