J Fourier Anal Appl (2017) 23:656–659
DOI 10.1007/s00041-016-9481-y

Fatou's Interpolation Theorem Implies the Rudin–Carleson Theorem

Arthur A. Danielyan[1]

Received: 21 October 2015 / Revised: 11 May 2016 / Published online: 21 July 2016

Abstract The purpose of this paper is to show that the Rudin–Carleson interpolation theorem is a direct corollary of Fatou's much older interpolation theorem (of 1906).

Keywords Disc algebra · Fatou's interpolation theorem · Rudin–Carleson theorem · Uniform continuity

Mathematics Subject Classification Primary 30H50 · 30J99

1 Introduction

Denote by Δ and T the open unit disk and the unit circle in the complex plane, respectively. Recall that the disk algebra A is the algebra of all continuous functions on the closed unit disk $\overline{\Delta}$ that are analytic on Δ. The following theorem is fundamental; in particular it implies the F. and M. Riesz theorem on analytic measures (cf. [11], pp. 28-31).

Theorem A *(P. Fatou, 1906). Let E be a closed set of Lebesgue measure zero on T. Then there exists a function $\lambda_E(z)$ in the disk algebra A such that $\lambda_E(z) = 1$ on E and $|\lambda_E(z)| < 1$ on $T \setminus E$.*

Communicated by Hans G. Feichtinger.

✉ Arthur A. Danielyan
adaniely@usf.edu

1 Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA

In its original form Fatou's theorem states the existence of an element of A which vanishes precisely on E, but it is equivalent to the above version (cf. [11], p. 30, or [9], pp. 80–81).

The following famous theorem, due to W. Rudin [15] and L. Carleson [4], has been the starting point of many investigations in complex and functional analysis (including several complex variables).

Theorem B *(Rudin - Carleson). Let E be a closed set of Lebesgue measure zero on T and let f be a continuous (complex valued) function on E. Then there exists a function g in the disk algebra A agreeing with f on E.*

It is obvious from Theorem A and Theorem B that for any $\epsilon > 0$ one can choose the extension function g in Theorem B such that it is bounded by $||f||_E + \epsilon$, where $||f||_E$ is the sup norm of f on E. Rudin has shown that one can even choose the function g such that it is bounded by $||f||_E$.

Quite naturally, as mentioned already by Rudin, Theorem B may be regarded as a strengthened form of Theorem A (cf. [15], p. 808).

The present paper shows that Theorem B also is an elementary corollary of Theorem A. To be more specific, we present a brief proof of Theorem B merely using Theorem A and the Heine - Cantor theorem (from Calculus I course); this approach may find further applications. We use a simple argument based on uniform continuity, which has been known (at least since 1930s) in particular to M.A. Lavrentiev [12], M.V. Keldysh, and S.N. Mergelyan, but has not been used for the proof of Theorem B before.

We close the introduction by mentioning some references related to Theorem B. An abstract theorem of E. Bishop [3] generalizes Theorem B to any situation where the F. and M. Riesz theorem is valid. A version of this result is the known peak-interpolation theorem of Bishop (see [16], p. 135); a new approach to this theorem is given in [6]. Further generalizations of Theorem B have been proved by S.Ya. Khavinson [10], and A. Pelczyński [14]. Some other developments and extensions of Theorem B have been given by D. Oberlin [13], and by S. Berhanu and J. Hounie [1,2]. The paper by R. Doss [7] provides elementary proofs for Theorem B and the F. and M. Riesz theorems; note that the argument of [7] (see p. 600) is based on the Weierstrass approximation theorem (cf. the remark below).

2 Proof of Theorem B

Let $\epsilon > 0$ be given. By uniform continuity we cover E by disjoint open intervals $I_k \subset T$ of a finite number n such that $|f(z_1) - f(z_2)| < \epsilon$ for any $z_1, z_2 \in E \cap I_k$ $(k = 1, 2, ..., n)$. Since the intervals I_k are disjoint, the endpoints of I_k are not in E. Thus each $E \cap I_k$ is closed. Denote $E_k = E \cap I_k$ and let $\lambda_{E_k}(z)$ be the function provided by Theorem A. Fix a natural number N so large that $|\lambda_{E_k}(z)|^N < \frac{\epsilon}{n}$ on $T \setminus I_k$ for all k. Fix a point $t_k \in E_k$ for each k and denote $h(z) = \sum_{k=1}^{n} f(t_k)[\lambda_{E_k}(z)]^N$. Obviously the function $h \in A$ is bounded on T by the number $(1+\epsilon)||f||_E$ and $|f(z) - h(z)| < \epsilon(1 + ||f||_E)$ if $z \in E$. Replacing h by $\frac{1}{1+\epsilon}h$ allows to assume that h is bounded on T simply by $||f||_E$ and $|f(z) - h(z)| < \epsilon(1 + 2||f||_E)$ if $z \in E$.

Letting $\epsilon = \frac{1}{m}$ provides a sequence $\{h_m\}$, $h_m \in A$, which is uniformly bounded on T by $||f||_E$ and uniformly converges to f on E.[1]

To complete the proof, we use the following known steps (cf. e.g. [5]). Let $\eta > 0$ be given and let $\eta_p > 0$ be such $\sum \eta_p < \eta$. We can find $H_1 = h_{m_1} \in A$ such that $|H_1(z)| \leq ||f||_E$ on T and $|f(z) - H_1(z)| < \eta_1$ on E. Letting $f_1 = f - H_1$ on E, the same reasoning yields $H_2 \in A$ with $|H_2(z)| \leq ||f_1||_E < \eta_1$ on T and $|f_1(z) - H_2(z)| < \eta_2$ on E. Similarly we find $H_p \in A$ for $p = 3, 4, \ldots$, with appropriate properties. The convergence of the series $||f||_E + \eta_1 + \eta_2 + \ldots$ implies that the series $\sum H_p(z)$ converges uniformly on $\overline{\Delta}$ to a function $g \in A$, which is bounded by $||f||_E + \eta$. On E holds $|f - g| = |(f - H_1) - H_2 - \ldots - H_p - \ldots| = |(f_1 - H_2) - H_3 - \ldots - H_p - \ldots| = \ldots = |(f_{p-1} - H_p) - \ldots| \leq \eta_p + \sum_{k=p}^{\infty} \eta_k$. Since $\lim_{p\to\infty}(\eta_p + \sum_{k=p}^{\infty} \eta_k) = 0$, it follows that $g = f$ on E, which completes the proof.

Remark The known proofs of Theorem B use Theorem A and a polynomial approximation theorem (cf. [8], p. 125; or [9], pp. 81–82). The latter is needed to approximate f on E by the elements of the disc algebra A. The above proof uses just Theorem A to provide such approximation of f on E by the elements of A, which in addition are bounded by $||f||_E$ on T.

Acknowledgements The author wishes to thank the referees for the careful reading of the manuscript and useful suggestions.

References

1. Berhanu, S., Hounie, J.: A Rudin–Carleson theorem for planar vector fields. Math. Ann. **347**, 95–110 (2010)
2. Berhanu, S., Hounie, J.: Boundary behavior of generalized analytic functions. J. Funct. Anal. **266**, 4121–4149 (2014)
3. Bishop, E.: A general Rudin–Carleson theorem. Proc. Am. Math. Soc. **13**, 140–143 (1962)
4. Carleson, L.: Representations of continuous functions. Math. Z. **66**, 447–451 (1957)
5. Danielyan, A.A.: On a polynomial approximation problem. J. Approx. Theory **162**, 717–722 (2010)
6. Danielyan, A.A.: The Peak-Interpolation Theorem of Bishop, Complex Analysis and Dynamical Systems IV. Part 1, 27–30. Contemporary Mathematics 553. American Mathematical Society, Providence, RI (2011)
7. Doss, R.: Elementary proofs of the Rudin–Carleson and the F. and M. Riesz theorems. Proc. Am. Math. Soc. **82**, 599–602 (1981)
8. Garnett, J.B.: Bounded Analytic Functions. Academic Press, New York (1981)
9. Hoffman, K.: Banach Spaces of Analytic Functions. Prentice Hall, Englewood Cliffs, NJ (1962)
10. Ya, S.: Khavinson, On the Rudin–Carleson theorem. Dokl. Akad. Nauk. SSSR **165**, 497–499 (1965)
11. Koosis, P.: Introduction to H^p Spaces. Cambridge University Press, Cambridge (1980)
12. Lavrentieff, M.: Sur les fonctions d'une variable complexe représntables par des séries de polynômes. Actualités scientifiques et industrielles, vol. 441. Hermann, Paris (1936)
13. Oberlin, D.: A Rudin–Carleson theorem for uniformly convergent Taylor series. Mich. Math. J. **27**, 309–313 (1980)

[1] Note that using the constructed sequence $\{h_m\}$ one can easily derive the lemma on which the proofs of Doss [7] are based. Indeed, the functions $h_m(z)[\lambda_E(z)]^m \in A$, being uniformly bounded on T by $||f||_E$, obviously uniformly converge to f on E and converge to zero uniformly inside the complementary intervals of E (which implies the lemma used in [7]).

14. Pelczyński, A.: On simultaneous extension of continuous functions. Studia Math. **24**, 285–303 (1964)
15. Rudin, W.: Boundary values of continuous analytic functions. Proc. Am. Math. Soc. **7**, 808–811 (1956)
16. Rudin, W.: Function Theory in Polydiscs. W.A. Benjamin Inc, New York (1969)