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Fourier Frames for the Cantor-4 Set

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Abstract The measure supported on the Cantor-4 set constructed by Jorgensen–Pedersen is known to have a Fourier basis, i.e. that it possess a sequence of exponentials which form an orthonormal basis. We construct Fourier frames for this measure via a dilation theory type construction. We expand the Cantor-4 set to a two dimensional fractal which admits a representation of a Cuntz algebra. Using the action of this algebra, an orthonormal set is generated on the larger fractal, which is then projected onto the Cantor-4 set to produce a Fourier frame.

Keywords Fourier series · Frames · Fractals · Iterated function system · Cuntz algebra

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Jorgensen and Pedersen [10] demonstrated that there exist singular measures ν which are spectral—that is, they possess a sequence of exponential functions which form an orthonormal basis in $L^2(\nu)$. The canonical example of such a singular and spectral measure is the uniform measure on the Cantor 4-set defined as follows:

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$$C_4 = \left\{ x \in [0,1] : x = \sum_{k=1}^{\infty} \frac{a_k}{4^k},\ a_k \in \{0,2\} \right\}.$$

This is analogous to the standard middle third Cantor set where 4^k replaces 3^k. The set C_4 can also be described as the attractor set of the following iterated function system on $\mathbb{R}$:

$$\tau_0(x) = \frac{x}{4}, \qquad \tau_2(x) = \frac{x+2}{4}.$$

The uniform measure on the set C_4 then is the unique probability measure μ_4 which is invariant under this iterated function system:

$$\int f(x) d\mu_4(x) = \frac{1}{2} \left(\int f(\tau_0(x)) d\mu_4(x) + \int f(\tau_2(x)) d\mu_4(x) \right)$$

for all $f \in C(\mathbb{R})$, see [9] for details. The standard spectrum for μ_4 is $\Gamma_4 = \left\{ \sum_{n=0}^{N} l_n 4^n : l_n \in \{0,1\} \right\}$, though there are many spectra [2,4]. Extension to larger classes of available spectra and other considerations can be found for example in [11,12,14].

Remarkably, Jorgensen and Pedersen prove that the uniform measure μ_3 on the standard middle third Cantor set is not spectral. Indeed, there are no three mutually orthogonal exponentials in $L^2(\mu_3)$. Thus, there has been much attention on whether there exists a Fourier frame for $L^2(\mu_3)$–the problem is still unresolved, but see [5,6] for progress in this regard. In this paper, we will construct Fourier frames for $L^2(\mu_4)$ using a dilation theory type argument. The motivation is whether the construction we demonstrate here for μ_4 will be applicable to μ_3. Fourier frames for μ_4 were constructed in [6] using a duality type construction.

A frame for a Hilbert space H is a sequence $\{x_n\}_{n \in I} \subset H$ such that there exists constants $A, B > 0$ such that for all $v \in H$,

$$A \|v\|^2 \leq \sum_{n \in I} |\langle v, x_n \rangle|^2 \leq B \|v\|^2.$$

The largest A and smallest B which satisfy these inequalities are called the frame bounds. The frame is called a Parseval frame if both frame bounds are 1. The sequence $\{x_n\}_{n \in I}$ is a Bessel sequence if there exists a constant B which satisfies the second inequality, whether or not the first inequality holds; B is called the Bessel bound. A Fourier frame for $L^2(\mu_4)$ is a sequence of frequencies $\{\lambda_n\}_{n \in I} \subset \mathbb{R}$ together with a sequence of "weights" $\{d_n\}_{n \in I} \subset \mathbb{C}$ such that $x_n = d_n e^{2\pi i \lambda_n x}$ is a frame. Fourier frames (unweighted) for Lebesgue measure were introduced by Duffin and Schaffer [3], see also Ortega-Cerda and Seip [13].

It was proven in [8] that a frame for a Hilbert space can be dilated to a Riesz basis for a bigger space, that is to say, that any frame is the image under a projection of a Riesz basis. Moreover, a Parseval frame is the image of an orthonormal basis under a projection. This result is now known to be a consequence of the Naimark dilation theory. This will be our recipe for constructing a Fourier frame: constructing a basis in a bigger space and then projecting onto a subspace. We require the following result along these lines [1]:

Lemma 1 *Let H be a Hilbert space, V, K closed subspaces, and let P_V be the projection onto V. If $\{x_n\}_{n\in I}$ is a frame in K with frame bounds A, B, then:*

1. *$\{P_V x_n\}_{n\in I}$ is a Bessel sequence in V with Bessel bound no greater than B;*
2. *if the projection $P_V : K \to V$ is onto, then $\{P_V x_n\}_{n\in I}$ is a frame in V;*
3. *if $V \subset K$, then then $\{P_V x_n\}_{n\in I}$ is a frame in V with frame bounds between A and B.*

Note that if $V \subset K$ and $\{x_n\}_{n\in I}$ is a Parseval frame for K, then $\{P_V x_n\}_{n\in I}$ is a Parseval frame for V. In the second item above, it is possible that the lower frame bound for $\{P_V x_n\}$ is smaller than A, but the upper frame bound is still no greater than B.

The foundation of our construction is a dilation theory type argument. Our first step, described in Sect. 1, is to consider the fractal like set $C_4 \times [0, 1]$, which we will view in terms of an iterated function system. This IFS will give rise to a representation of the Cuntz algebra $\mathcal{O}_4$ on $L^2(\mu_4 \times \lambda)$ since $\mu_4 \times \lambda$ is the invariant measure under the IFS. Then in Sect. 2, we will generate via the action of $\mathcal{O}_4$ an orthonormal set in $L^2(\mu_4 \times \lambda)$ whose vectors have a particular structure. In Sect. 3, we consider a subspace V of $L^2(\mu_4 \times \lambda)$ which can be naturally identified with $L^2(\mu_4)$, and then project the orthonormal set onto V to, ultimately, obtain a frame. Of paramount importance will be whether the orthonormal set generated by $\mathcal{O}_4$ spans the subspace V so that the projection yields a Parseval frame. Section 4 demonstrates concrete constructions in which this occurs, and identifies all possible Fourier frames that can be constructed using this method.

We note here that there may be Fourier frames for $L^2(\mu_4)$ which cannot be constructed in this manner, but we are unaware of such an example.

1 Dilation of the Cantor-4 Set

We wish to construct a Hilbert space H which contains $L^2(\mu_4)$ as a subspace in a natural way. We will do this by making the fractal C_4 bigger as follows. We begin with an iterated function system on $\mathbb{R}^2$ given by:

$$\Upsilon_0(x, y) = \left(\frac{x}{4}, \frac{y}{2}\right), \Upsilon_1(x, y) = \left(\frac{x+2}{4}, \frac{y}{2}\right),$$
$$\Upsilon_2(x, y) = \left(\frac{x}{4}, \frac{y+1}{2}\right), \Upsilon_3(x, y) = \left(\frac{x+2}{4}, \frac{y+1}{2}\right).$$

As these are contractions on $\mathbb{R}^2$, there exists a compact attractor set, which is readily verified to be $C_4 \times [0, 1]$. Likewise, by Hutchinson [9], there exists an invariant probability measure supported on $C_4 \times [0, 1]$; it is readily verified that this invariant measure is $\mu_4 \times \lambda$, where λ denotes the Lebesgue measure restricted to $[0, 1]$. Thus, for every continuous function $f : \mathbb{R}^2 \to \mathbb{C}$,

$$\int f(x,y)\,d(\mu_4\times\lambda)=\frac{1}{4}\left(\int f\left(\frac{x}{4},\frac{y}{2}\right)d(\mu_4\times\lambda)+\int f\left(\frac{x+2}{4},\frac{y}{2}\right)d(\mu_4\times\lambda)\right.$$
$$\left.+\int f\left(\frac{x}{4},\frac{y+1}{2}\right)d(\mu_4\times\lambda)+\int f\left(\frac{x+2}{4},\frac{y+1}{2}\right)d(\mu_4\times\lambda)\right). \tag{1}$$

The iterated function system Υ_j has a left inverse on $C_4\times[0,1]$, given by

$$R: C_4\times[0,1]\to C_4\times[0,1]:(x,y)\mapsto(4x,2y)\mod 1,$$

so that $R\circ\Upsilon_j(x,y)=(x,y)$ for $j=0,1,2,3$.

We will use the iterated function system to define an action of the Cuntz algebra $\mathcal{O}_4$ on $L^2(\mu_4\times\lambda)$. To do so, we choose filters

$$\begin{aligned}m_0(x,y)&=H_0(x,y)\\ m_1(x,y)&=e^{2\pi ix}H_1(x,y)\\ m_2(x,y)&=e^{4\pi ix}H_2(x,y)\\ m_3(x,y)&=e^{6\pi ix}H_3(x,y)\end{aligned}$$

where

$$H_j(x,y)=\sum_{k=0}^{3}a_{jk}\chi_{\Upsilon_k(C_4\times[0,1])}(x,y)$$

for some choice of scalar coefficients a_{jk}. In order to obtain a representation of $\mathcal{O}_4$ on $L^2(\mu_4\times\lambda)$, we require that the above filters satisfy the matrix equation $\mathcal{M}^*(x,y)\mathcal{M}(x,y)=I$ for $\mu_4\times\lambda$ almost every (x,y), where

$$\mathcal{M}(x,y)=\begin{pmatrix}m_0(\Upsilon_0(x,y)) & m_0(\Upsilon_1(x,y)) & m_0(\Upsilon_2(x,y)) & m_0(\Upsilon_3(x,y))\\ m_1(\Upsilon_0(x,y)) & m_1(\Upsilon_1(x,y)) & m_1(\Upsilon_2(x,y)) & m_1(\Upsilon_3(x,y))\\ m_2(\Upsilon_0(x,y)) & m_2(\Upsilon_1(x,y)) & m_2(\Upsilon_2(x,y)) & m_2(\Upsilon_3(x,y))\\ m_3(\Upsilon_0(x,y)) & m_3(\Upsilon_1(x,y)) & m_3(\Upsilon_2(x,y)) & m_3(\Upsilon_3(x,y))\end{pmatrix}$$

For our choice of filters, the matrix $\mathcal{M}$ becomes

$$\mathcal{M}(x,y)=\begin{pmatrix}a_{00} & a_{01} & a_{02} & a_{03}\\ e^{\pi ix/2}a_{10} & -e^{\pi ix/2}a_{11} & e^{\pi ix/2}a_{12} & -e^{\pi ix/2}a_{13}\\ e^{\pi ix}a_{20} & e^{\pi ix}a_{21} & e^{\pi ix}a_{22} & e^{\pi ix}a_{23}\\ e^{3\pi ix/2}a_{30} & -e^{3\pi ix/2}a_{31} & e^{3\pi ix/2}a_{32} & -e^{3\pi ix/2}a_{33}\end{pmatrix},$$

which is unitary if and only if the matrix

$$H=\begin{pmatrix}a_{00} & a_{01} & a_{02} & a_{03}\\ a_{10} & -a_{11} & a_{12} & -a_{13}\\ a_{20} & a_{21} & a_{22} & a_{23}\\ a_{30} & -a_{31} & a_{32} & -a_{33}\end{pmatrix}$$

is unitary. For the remainder of this section, we assume that H is unitary.

Lemma 2 *The operator* $S_j : L^2(\mu_4 \times \lambda) \to L^2(\mu_4 \times \lambda)$ *given by*

$$[S_j f](x, y) = \sqrt{4} m_j(x, y) f(R(x, y))$$

is an isometry.

Proof We calculate:

$$\begin{aligned}
\|S_j f\|^2 &= \int |\sqrt{4} m_j(x, y) f(R(x, y))|^2 \, d(\mu_4 \times \lambda) \\
&= \frac{1}{4} \sum_{k=0}^{3} \int 4 |m_j(\Upsilon_k(x, y)) f(R(\Upsilon_k(x, y)))|^2 \, d(\mu_4 \times \lambda) \\
&= \int \left(\sum_{k=0}^{3} |m_j(\Upsilon_k(x, y))|^2 \right) |f(x, y)|^2 \, d(\mu_4 \times \lambda).
\end{aligned}$$

We used Eq. (1) in the second line. The sum in the integral is the square of the Euclidean norm of the jth row of the matrix $\mathcal{M}$, which is unitary. Hence, the sum is 1, so the integral is $\|f\|^2$, as required. □

Lemma 3 *The adjoint is given by*

$$[S_j^* f](x, y) = \frac{1}{2} \sum_{k=0}^{3} \overline{m_j(\Upsilon_k(x, y))} f(\Upsilon_k(x, y)).$$

Proof Let $f, g \in L^2(\mu_4 \times \lambda)$. We calculate

$$\begin{aligned}
\langle S_j f, g \rangle &= \int \sqrt{4} m_j(x, y) f(R(x, y)) \overline{g(x, y)} \, d(\mu_4 \times \lambda) \\
&= \frac{1}{4} \sum_{k=0}^{3} \int \sqrt{4} m_j(\Upsilon_k(x, y)) f(R(\Upsilon_k(x, y))) \overline{g(\Upsilon_k(x, y))} \, d(\mu_4 \times \lambda) \\
&= \int f(x, y) \overline{\left(\frac{1}{2} \sum_{k=0}^{3} \overline{m_j(\Upsilon_k(x, y))} g(\Upsilon_k(x, y)) \right)} \, d(\mu_4 \times \lambda)
\end{aligned}$$

where we use Eq. (1) and the fact that R is a left inverse of Υ_k. □

Lemma 4 *The isometries* S_j *satisfy the Cuntz relations:*

$$S_j^* S_k = \delta_{jk} I, \qquad \sum_{k=0}^{3} S_k S_k^* = I.$$

Proof We consider the orthogonality relation first. Let $f \in L^2(\mu_4 \times \lambda)$. We calculate:

$$\begin{aligned}
[S_j^* S_k f](x, y) &= \frac{1}{2} \sum_{\ell=0}^{3} \overline{m_j(\Upsilon_\ell(x, y))} [S_k f](\Upsilon_\ell(x, y)) \\
&= \frac{1}{2} \sum_{\ell=0}^{3} \overline{m_j(\Upsilon_\ell(x, y))} \sqrt{4} m_k(\Upsilon_\ell(x, y)) f(R(\Upsilon_\ell(x, y))) \\
&= \left(\sum_{\ell=0}^{3} \overline{m_j(\Upsilon_\ell(x, y))} m_k(\Upsilon_\ell(x, y)) \right) f(x, y).
\end{aligned}$$

Note that the sum is the scalar product of the kth row with the jth row of the matrix $\mathcal{M}$, which is unitary. Hence, the sum is δ_{jk} as required.

Now for the identity relation, let $f, g \in L^2(\mu_4 \times \lambda)$. We calculate:

$$\begin{aligned}
&\left\langle \sum_{k=0}^{3} S_k S_k^* f, g \right\rangle = \sum_{k=0}^{3} \left\langle S_k^* f, S_k^* g \right\rangle \\
&= \sum_{k=0}^{3} \int \left(\frac{1}{2} \sum_{\ell=0}^{3} \overline{m_k(\Upsilon_\ell(x, y))} f(\Upsilon_\ell(x, y)) \right) \\
&\quad \times \overline{\left(\frac{1}{2} \sum_{n=0}^{3} \overline{m_k(\Upsilon_n(x, y))} g(\Upsilon_n(x, y)) \right)} \, d(\mu_4 \times \lambda) \\
&= \sum_{\ell=0}^{3} \sum_{n=0}^{3} \frac{1}{4} \int \left(\sum_{k=0}^{3} \overline{m_k(\Upsilon_\ell(x, y))} m_k(\Upsilon_n(x, y)) \right) f(\Upsilon_\ell(x, y)) \overline{g(\Upsilon_n(x, y))} \, d(\mu_4 \times \lambda) \\
&= \frac{1}{4} \sum_{n=0}^{3} \int f(\Upsilon_n(x, y)) \overline{g(\Upsilon_n(x, y))} \, d(\mu_4 \times \lambda) \\
&= \int f(x, y) \overline{g(x, y)} \, d(\mu_4 \times \lambda) = \langle f, g \rangle.
\end{aligned}$$

Note that the sum over k in the third line is the scalar product of the ℓth column with the nth column of $\mathcal{M}$, so the sum collapses to $\delta_{\ell n}$. The sum on n in the fourth line collapses by Eq. (1). □

2 Orthonormal Sets in $L^2(\mu_4 \times \lambda)$

Since the isometries S_j satisfy the Cuntz relations, we can use them to generate orthonormal sets in the space $L^2(\mu_4 \times \lambda)$. We do so by having the isometries act on a generating vector. We consider words in the alphabet $\{0, 1, 2, 3\}$; let W_4 denote the set of all such words. For a word $\omega = j_K j_{K-1} \ldots j_1$, we denote by $|\omega| = K$ the length of the word, and define

$$S_\omega f = S_{j_K} S_{j_{K-1}} \ldots S_{j_1} f.$$

Definition 1 Let

$$X_4 = \{\omega \in W_4 : |\omega| = 1\} \cup \{\omega \in W_4 : |\omega| \geq 2,\ \ j_1 \neq 0\}.$$

For convenience, we allow the empty word $\omega_\emptyset$ with length 0, and define $S_{\omega_\emptyset} = I$, the identity.

Lemma 5 *Suppose* $f \in L^2(\mu_4 \times \lambda)$ *with* $\|f\| = 1$, *and that* $S_0 f = f$. *Then,*

$$\{S_\omega f : \omega \in X_4\}$$

is an orthonormal set.

Proof Suppose $\omega, \omega' \in X_4$ with $\omega \neq \omega'$. First consider $|\omega| = |\omega'|$, with $\omega = j_K \dots j_1$ and $\omega' = i_K \dots i_1$. Suppose that ℓ is the largest index such that $j_\ell \neq i_\ell$. Then we have

$$\langle S_\omega f, S_{\omega'} f\rangle = \langle S_{j_\ell} \dots S_{j_1} f, S_{i_\ell} \dots S_{i_1} f\rangle = \langle S_{i_\ell}^* S_{j_\ell} \dots S_{j_1} f, S_{i_{\ell-1}} \dots S_{i_1} f\rangle = 0$$

by the orthogonality condition of the Cuntz relations.

Now, if $K = |\omega| > |\omega'| = M$, with $\omega' = i_M \dots i_1$, we define the word $\rho = i_M \dots i_1 0 \dots 0$ so that $|\rho| = K$. Note that $\rho \notin X_4$ so $\omega \neq \rho$. Note further that $S_{\omega'} f = S_\rho f$. Thus, by a similar argument to that above, we have

$$\langle S_\omega f, S_{\omega'} f\rangle = 0.$$

□

Remark 1 The set $\{S_\omega f : \omega \in X_4\}$ need not be complete. We will provide an example of this in Example 1 in Sect. 4.

Our goal is to project the set $\{S_\omega f : \omega \in X_4\}$ onto some subspace V of $L^2(\mu_4 \times \lambda)$ to obtain a frame. To that end, we need to know when the projection $\{P_V S_\omega f : \omega \in X_4\}$ is a frame, which by Lemma 1 requires the projection $P_V : K \to V$ to be onto, where K is the subspace spanned by $\{S_\omega f : \omega \in X_4\}$. The tool we will use is the following result, which is a minor adaptation of Theorem 3.1 from [7].

Theorem 1 *Let* $\mathcal{H}$ *be a Hilbert space,* $\mathcal{K} \subset \mathcal{H}$ *a closed subspace, and* $(S_i)_{i=0}^{N-1}$ *be a representation of the Cuntz algebra* $\mathcal{O}_N$. *Let* $\mathcal{E}$ *be an orthonormal set in* $\mathcal{H}$ *and* $f : X \to \mathcal{K}$ *a norm continuous function on a topological space* X *with the following properties:*

(i) $\mathcal{E} = \cup_{i=0}^{N-1} S_i \mathcal{E}$ *where the union is disjoint.*
(ii) $\overline{span}\{f(t) : t \in X\} = \mathcal{K}$ *and* $\|f(t)\| = 1$, *for all* $t \in X$.
(iii) *There exist functions* $\mathfrak{m}_i : X \to \mathbb{C}$, $g_i : X \to X$, $i = 0, \dots, N-1$ *such that*

$$S_i^* f(t) = \mathfrak{m}_i(t) f(g_i(t)), \quad t \in X. \tag{2}$$

(iv) *There exist* $c_0 \in X$ *such that* $f(c_0) \in \overline{span}\mathcal{E}$.

(v) *The only function* $h \in \mathcal{C}(X)$ *with* $h \geq 0$, $h(c) = 1$, $\forall\, c \in \{x \in X : f(x) \in \overline{span}\mathcal{E}\}$, *and*

$$h(t) = \sum_{i=0}^{N-1} |\mathfrak{m}_i(t)|^2 h(g_i(t)), \quad t \in X \tag{3}$$

are the constant functions.

Then $\mathcal{K} \subset \overline{span}\mathcal{E}$.

3 The Projection

Recall the definition of the filters $m_j(x, y) = e^{2\pi i j x} H_j(x, y)$ from Sect. 1. We choose the filter coefficients a_{jk} so that the matrix H is unitary. We place the additional constraint that

$$a_{00} = a_{01} = a_{02} = a_{03} = \frac{1}{2},$$

so that $S_0\mathbb{1} = \mathbb{1}$, where $\mathbb{1}$ the function in $L^2(\mu_4 \times \lambda)$ which is identically 1. As $S_0\mathbb{1} = \mathbb{1}$, by Lemma 5, the set $\{S_\omega\mathbb{1} : \omega \in X_4\}$ is orthonormal. Moreover, we place the additional constraint that for every j, $a_{j0} + a_{j2} = a_{j1} + a_{j3}$, which will be required for our calculation of the projection.

Definition 2 We define the subspace $V = \{f \in L^2(\mu_4 \times \lambda) : f(x, y) = g(x)\chi_{[0,1]}(y),\ g \in L^2(\mu_4)\}$. Note that the subspace V can be identified with $L^2(\mu_4)$ via the isometric isomorphism $g \mapsto g(x)\chi_{[0,1]}(y)$. We will suppress the y variable in the future.

Definition 3 We define a function $c : X_4 \to \mathbb{N}_0$ as follows: for a word $\omega = j_K j_{K-1} \cdots j_1$,

$$c(\omega) = \sum_{k=1}^{K} j_k 4^{K-k}.$$

Here $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. It is readily verified that c is a bijection.

Lemma 6 *For a word* $\omega = j_K j_{K-1} \ldots j_1$,

$$S_\omega\mathbb{1} = e^{2\pi i c(\omega) x}\left(\prod_{k=1}^{K} 2H_{j_k}(R^{K-k}(x, y))\right).$$

Proof We proceed by induction on the length of the word ω. The equality is readily verified for $|\omega| = 1$. Let $\omega_0 = j_{K-1} j_{n-2} \ldots j_1$. We have

$$\begin{aligned} S_\omega\mathbb{1} &= S_{j_K} S_{\omega_0}\mathbb{1} \\ &= S_{j_K}\left[e^{2\pi i c(\omega_0) x}\left(\prod_{k=1}^{K-1} 2H_{j_k}(R^{K-1-k}(x, y))\right)\right] \end{aligned}$$

$$= 2e^{2\pi i j_K x} H_{j_K}(x, y) e^{2\pi i c(\omega_0)\cdot 4x} \left(\prod_{k=1}^{K-1} H_{j_k}(R^{K-k}(x, y)) \right)$$
$$= 2e^{2\pi i (j_K + 4c(\omega_0))x} H_{j_K}(R^{K-K}(x, y)) \left(\prod_{k=1}^{K-1} 2H_{j_k}(R^{K-k}(x, y)) \right)$$
$$= 2e^{2\pi i c(\omega) x} \left(\prod_{k=1}^{K-1} 2H_{j_k}(R^{K-k}(x, y)) \right).$$

The last line above is justified by the following calculation:

$$\begin{aligned} j_K + 4c(\omega_0) &= j_K + 4\left(\sum_{k=1}^{K-1} j_k 4^{K-1-k} \right) \\ &= j_K 4^{K-K} + \sum_{k=1}^{K-1} j_k 4^{K-k} \\ &= \sum_{k=1}^{K} j_k 4^{K-k} \\ &= c(\omega). \end{aligned}$$

□

We wish to project the vectors $S_\omega \mathbb{1}$ onto the subspace V. The following lemma calculates that projection, where P_V denotes the projection onto the subspace V.

Lemma 7 *If $f(x, y) = g(x)h(x, y)$ with $g \in L^2(\mu_4)$ and $h \in L^\infty(\mu_4 \times \lambda)$, then*

$$[P_V f](x, y) = g(x)G(x)$$

where $G(x) = \int_{[0,1]} h(x, y) d\lambda(y)$.

Proof We verify that for every $F(x) \in L^2(\mu_4)$, $f(x, y) - g(x)G(x)$ is orthogonal to $F(x)$. We calculate utilizing Fubini's theorem:

$$\begin{aligned} \langle f - gG, F \rangle &= \int\int g(x)h(x, y)\overline{F(x)}\, d(\mu_4 \times \lambda) - \int\int g(x)G(x)\overline{F(x)}\, d(\mu_4 \times \lambda) \\ &= \int_{C_4} g(x)\overline{F(x)} \left(\int_{[0,1]} h(x, y) - G(x)\, d\lambda(y) \right) d\mu_4(x) \\ &= \int_{C_4} g(x)\overline{F(x)}\, (G(x) - G(x))\, d\mu_4(x) \\ &= 0. \end{aligned}$$

□

For the purposes of the following lemma, αx and βy are understood to be modulo 1.

Lemma 8 *For any word* $\omega = j_K j_{K-1} \dots j_1$,

$$\int \prod_{k=1}^{K} 2H_{j_k}(R^{k-1}(x,y))\, d\lambda(y) = \prod_{k=1}^{K} 2 \int H_{j_k}(4^{k-1}x, y)\, d\lambda(y).$$

Proof Let $F_m(x,y) = \prod_{k=m}^{K} 2H_{j_k}(4^{k-1}x, 2^{k-m}y)$. Note that

$$\begin{aligned} F_m\left(x, \frac{y}{2}\right) &= 2H_{j_m}\left(4^{m-1}x, \frac{y}{2}\right)\left(\prod_{k=m+1}^{K} 2H_{j_k}(4^{k-1}x, 2^{k-(m+1)}y)\right) \\ &= 2H_{j_m}\left(4^{m-1}x, \frac{y}{2}\right) F_{m+1}(x,y). \end{aligned}$$

Likewise for $F_m\left(x, \frac{y+1}{2}\right)$.

Since λ is the invariant measure for the iterated function system $y \mapsto \frac{y}{2}$, $y \mapsto \frac{y+1}{2}$, we calculate:

$$\begin{aligned} \int_0^1 F_m(x,y)\, d\lambda(y) &= \frac{1}{2}\left[\int_0^1 F_m\left(x, \frac{y}{2}\right) d\lambda(y) + \int_0^1 F_m\left(x, \frac{y+1}{2}\right) d\lambda(y)\right] \\ &= \frac{1}{2}\left[\int_0^1 2H_{j_m}\left(4^{m-1}x, \frac{y}{2}\right) F_{m+1}(x,y)\right. \\ &\left. +2H_{j_m}\left(4^{m-1}x, \frac{y+1}{2}\right) F_{m+1}(x,y)\, d\lambda(y)\right] \\ &= \frac{1}{2}\left[\int_0^1 2a_{j_m,q} F_{m+1}(x,y) + 2a_{j_m,q+2} F_{m+1}(x,y)\, d\lambda(y)\right] \\ &= \frac{1}{2}\left[2a_{j_m,q} + 2a_{j_m,q+2}\right] \cdot \left[\int_0^1 F_{m+1}(x,y)\, d\lambda(y)\right] \\ &= \left[\int_0^1 2H_{j_m}(4^{m-1}x, y)\, d\lambda(y)\right] \cdot \left[\int_0^1 F_{m+1}(x,y)\, d\lambda(y)\right] \end{aligned}$$

where $q = 0$ if $0 \le 4^{m-1}x < \frac{1}{2}$, and $q = 1$ if $\frac{1}{2} \le 4^{m-1}x < 1$.

The result now follows by a standard induction argument. □

Proposition 1 *Suppose the filters* $m_j(x,y)$ *are chosen so that*

(i) *the matrix* H *is unitary,*
(ii) $a_{00} = a_{01} = a_{02} = a_{03} = \frac{1}{2}$, *and*
(iii) *for* $j = 0, 1, 2, 3$, $a_{j0} + a_{j2} = a_{j1} + a_{j3}$.

Then for any word $\omega = j_K \dots j_1$,

$$P_V S_\omega \mathbb{1} = d_\omega e^{2\pi i c(\omega)x},$$

where

$$d_\omega = \prod_{k=1}^{K} \left(a_{j_k0} + a_{j_k2}\right). \tag{4}$$

Proof We apply the previous three Lemmas to obtain

$$\begin{aligned}[P_V S_\omega \mathbb{1}](x, y) &= e^{2\pi i c(\omega)x} \int \prod_{k=1}^{K} 2H_{j_k}(R^{k-1}(x, y))d\lambda(y) \\ &= e^{2\pi i c(\omega)x} \prod_{k=1}^{K} 2 \int H_{j_k}(4^{k-1}x, y)d\lambda(y)\end{aligned}$$

By assumption (iii), the integral $\int H_{j_k}(4^{k-1}x, y)d\lambda(y)$ is independent of x, and the value of the integral is $\frac{a_{j0}}{2} + \frac{a_{j2}}{2}$. Eq. (4) now follows. □

4 Concrete Constructions

We now turn to concrete constructions of Fourier frames for μ_4. The hypotheses of Lemma 5 and Proposition 1 require H to be unitary and requires the matrix

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}$$

to have the first row be identically $\frac{1}{2}$ and to have the vector $\begin{pmatrix}1 & -1 & 1 & -1\end{pmatrix}^T$ in the kernel.

We can use Hadamard matrices to construct examples of such a matrix A. Every 4×4 Hadamard matrix is a permutation of the following matrix:

$$U_\rho = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & \rho & -\rho \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -\rho & \rho \end{pmatrix}$$

where ρ is any complex number of modulus 1.

If we set $H = U_\rho$, we obtain

$$A = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & \rho & \rho \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -\rho & -\rho \end{pmatrix} \tag{5}$$

which has the requisite properties to apply Lemma 5 and Proposition 1.

We define for $k = 1, 2, 3$, $l_k : \mathbb{N}_0 \to \mathbb{N}_0$ by $l_k(n)$ is the number of digits equal to k in the base 4 expansion of n. Note that $l_k(0) = 0$, and we follow the convention that $0^0 = 1$.

Theorem 2 *For the choice A as in Eq.* (5) *with* $\rho \neq -1$, *the sequence*

$$\left\{ \left(\frac{1+\rho}{2}\right)^{l_1(n)} 0^{l_2(n)} \left(\frac{1-\rho}{2}\right)^{l_3(n)} e^{2\pi i n x} : n \in \mathbb{N}_0 \right\} \tag{6}$$

is a Parseval frame in $L^2(\mu_4)$.

Proof By Lemma 5, we have that $\{S_\omega \mathbb{1} : \omega \in X_4\}$ is an orthonormal set. For a word $\omega = j_K j_{K-1} \ldots j_1$, Proposition 1 yields that

$$P_V S_\omega \mathbb{1} = e^{2\pi i c(\omega) x} \prod_{k=1}^{K} \left(a_{j_k 0} + a_{j_k 2}\right).$$

Then, setting $n = c(\omega)$, we obtain

$$P_V S_\omega \mathbb{1} = e^{2\pi i n x} \left(a_{00} + a_{02}\right)^{K - l_1(n) - l_2(n) - l_3(n)} \prod_{j=1}^{3} \left(a_{j0} + a_{j2}\right)^{l_j(n)}.$$

Since

$$a_{00} + a_{02} = 1, \quad a_{10} + a_{12} = \frac{1+\rho}{2}, \quad a_{20} + a_{22} = 0, \quad a_{30} + a_{32} = \frac{1-\rho}{2},$$

it follows that

$$P_V S_\omega \mathbb{1} = \left(\frac{1+\rho}{2}\right)^{l_1(n)} 0^{l_2(n)} \left(\frac{1-\rho}{2}\right)^{l_3(n)} e^{2\pi i n x}.$$

Since c is a bijection, the set $\{P_V S_\omega \mathbb{1} : \omega \in X_4\}$ coincides with the set in (6).

In order to establish that the set (6) is a Parseval frame, we wish to apply Lemma 1, which requires that the subspace V is contained in the closed span of $\{S_\omega \mathbb{1} : \omega \in X_4\}$. Denote the closed span by $\mathcal{K}$. We will proceed in a manner nearly identical to the proof of Theorem 1 and its inspiration [7, Theorem 3.1]. Define the function $f : \mathbb{R} \to V$ by $f(t) = e_t$ where $e_t(x, y) = e^{2\pi i x t}$. Note that $f(0) = \mathbb{1} \in \mathcal{K}$. Likewise, define a function $h_X : \mathbb{R} \to \mathbb{R}$ by

$$h_X(t) = \sum_{\omega \in X_4} |\langle f(t), S_\omega \mathbb{1} \rangle|^2 = \| P_{\mathcal{K}} f(t) \|^2.$$

□

Claim 1 We have $h_X \equiv 1$.

Assuming for the moment that the claim holds, we deduce that $f(t) \in \mathcal{K}$ for every $t \in \mathbb{R}$. Since $\{f(\gamma) : \gamma \in \Gamma_4\}$ is an orthonormal basis for V, it follows that the closed span of $\{f(t) : t \in \mathbb{R}\}$ is all of V. We conclude that $V \subset \mathcal{K}$, and so Lemma 1 implies that $\{P_V S_\omega \mathbb{1} : \omega \in X_4\}$ is a Parseval frame for V, from which the Theorem follows.

Thus, we turn to the proof of Claim 1. First, we require $\{S_\omega \mathbb{1} : \omega \in X_4\} = \cup_{j=0}^{3} \{S_j S_\omega \mathbb{1} : \omega \in X_4\}$, where the union is disjoint. Clearly, the RHS is a subset of the LHS, and the union is disjoint. Consider an element of the LHS: $S_\omega \mathbb{1}$. If $|\omega| \geq 2$, we write $S_\omega \mathbb{1} = S_j S_{\omega_0} \mathbb{1}$ for some j and some $\omega_0 \in X_4$, whence $S_\omega \mathbb{1}$ is in the RHS. If $|\omega| = 1$, then we write $S_\omega \mathbb{1} = S_j \mathbb{1} = S_j S_0 \mathbb{1}$, which is again an element of the RHS. Equality now follows.

As a consequence,

$$\begin{aligned} h_X(t) &= \sum_{\omega \in X_4} |\langle f(t), S_\omega \mathbb{1} \rangle|^2 \\ &= \sum_{j=0}^{3} \sum_{\omega \in X_4} |\langle f(t), S_j S_\omega \mathbb{1} \rangle|^2 \\ &= \sum_{j=0}^{3} \sum_{\omega \in X_4} |\langle S_j^* f(t), S_\omega \mathbb{1} \rangle|^2. \end{aligned}$$

We calculate:

$$\begin{aligned} [S_j^* f(t)](x, y) &= \frac{1}{2} \sum_{k=0}^{3} \overline{m_j(\Upsilon_k(x, y))} e_t(\Upsilon_k(x, y)) \\ &= \frac{1}{2} \Bigg[\overline{a_{j0}} e^{-2\pi i j x/4} e_t\left(\frac{x}{4}, \frac{y}{2}\right) + e^{-\pi i j} \overline{a_{j1}} e^{-2\pi i j x/4} e_t\left(\frac{x+2}{4}, \frac{y}{2}\right) \\ &\qquad + \overline{a_{j2}} e^{-2\pi i j x/4} e_t\left(\frac{x}{4}, \frac{y+1}{2}\right) \\ &\qquad + e^{-\pi i j} \overline{a_{j3}} e^{-2\pi i j x/4} e_t\left(\frac{x+2}{4}, \frac{y+1}{2}\right) \Bigg] \\ &= \frac{1}{2} \Bigg[\overline{a_{j0}} e^{-2\pi i j x/4} e_t\left(\frac{x}{4}, \frac{y}{2}\right) + e^{-\pi i j} \overline{a_{j1}} e^{-2\pi i j x/4} e^{\pi i t} e_t\left(\frac{x}{4}, \frac{y}{2}\right) \\ &\qquad + \overline{a_{j2}} e^{-2\pi i j x/4} e_t\left(\frac{x}{4}, \frac{y}{2}\right) + e^{-\pi i j} \overline{a_{j3}} e^{-2\pi i j x/4} e^{\pi i t} e_t\left(\frac{x}{4}, \frac{y}{2}\right) \Bigg] \\ &= \frac{1}{2} \left[\overline{a_{j0}} + e^{-\pi i j} \overline{a_{j1}} e^{\pi i t} + \overline{a_{j2}} + e^{-\pi i j} \overline{a_{j3}} e^{\pi i t} \right] e^{-2\pi i j x/4} e_t\left(\frac{x}{4}, \frac{y}{2}\right) \\ &= \frac{1}{2} \left[\overline{a_{j0}} + e^{-\pi i j} \overline{a_{j1}} e^{\pi i t} + \overline{a_{j2}} + e^{-\pi i j} \overline{a_{j3}} e^{\pi i t} \right] e^{2\pi i (t \frac{x}{4} - j \frac{x}{4})} \\ &= \frac{1}{2} \left[\overline{a_{j0}} + e^{-\pi i j} \overline{a_{j1}} e^{\pi i t} + \overline{a_{j2}} + e^{-\pi i j} \overline{a_{j3}} e^{\pi i t} \right] e^{2\pi i \left(\frac{t-j}{4} x\right)} \\ &= \frac{1}{2} \left[\overline{a_{j0}} + e^{-\pi i j} \overline{a_{j1}} e^{\pi i t} + \overline{a_{j2}} + e^{-\pi i j} \overline{a_{j3}} e^{\pi i t} \right] e_{\frac{t-j}{4}}(x, y). \end{aligned}$$

Thus, we define

$$\mathfrak{m}_j(t) = \frac{1}{2} \left(\overline{a_{j0}} + \overline{a_{j2}} \right) + \frac{e^{-\pi i j}}{2} \left(\overline{a_{j1}} + \overline{a_{j3}} \right) e^{\pi i t},$$

and

$$g_j(t) = \frac{t-j}{4}.$$

As a consequence, we obtain

$$\begin{aligned} h_X(t) &= \sum_{j=0}^{3} \sum_{\omega \in X_4} |\langle S_j^* f(t), S_\omega \mathbb{1} \rangle|^2 \\ &= \sum_{j=0}^{3} \sum_{\omega \in X_4} |\langle \mathfrak{m}_j(t) f(g_j(t)), S_\omega \mathbb{1} \rangle|^2 \\ &= \sum_{j=0}^{3} |\mathfrak{m}_j(t)|^2 h_X(g_j(t)). \end{aligned} \tag{7}$$

Because of our choice of coefficients in the matrix A, which has the vector $\begin{pmatrix}1 & -1 & 1 & -1\end{pmatrix}^T$ in the kernel, we have for every j: $a_{j0} + a_{j2} = a_{j1} + a_{j3}$. Thus, if we let $b_j = \overline{a_{j0}} + \overline{a_{j2}}$, the functions $\mathfrak{m}_j$ simplify to

$$\mathfrak{m}_j(t) = \frac{b_j}{2} e^{\pi i \frac{t}{2}} \cos\left(\pi \frac{t}{2}\right)$$

for $j = 0, 2$, and

$$\mathfrak{m}_j(t) = -\frac{i b_j}{2} e^{\pi i \frac{t}{2}} \sin\left(\pi \frac{t}{2}\right)$$

for $j = 1, 3$. Substituting these into Eq. (7),

$$\begin{aligned} h_X(t) = \cos^2\left(\frac{\pi t}{2}\right) h_X\left(\frac{t}{4}\right) + \sin^2\left(\frac{\pi t}{2}\right) \frac{|1+\overline{\rho}|^2}{4} h_X\left(\frac{t-1}{4}\right) \\ + \sin^2\left(\frac{\pi t}{2}\right) \frac{|1-\overline{\rho}|^2}{4} h_X\left(\frac{t-3}{4}\right). \end{aligned} \tag{8}$$

Claim 2 The function h_X can be extended to an entire function.

Assume for the moment that Claim 2 holds, we finish the proof of Claim 1. If $h_X(t) = 1$ for $t \in [-1, 0]$, then $h_X(z) = 1$ for all $z \in \mathbb{C}$, and Claim 1 holds.

Now, assume to the contrary that $h_X(t)$ is not identically 1 on $[-1, 0]$. Since $0 \leq h_X(t) \leq 1$ for t real, then $\beta = \min\{h_X(t) : t \in [-1, 0]\} < 1$. Because constant functions satisfy (8), $h_1 := h_X - \beta$ also satisfies Eq. (8). There exists t_0 such that $h_1(t_0) = 0$ and $t_0 \neq 0$ as $h_X(0) = 1$. Since $h_1 \geq 0$ each of the terms in (8) must vanish :

$$\cos^2\left(\frac{\pi t_0}{2}\right) h_1\left(\frac{t_0}{4}\right) = 0 \tag{9}$$

$$\sin^2\left(\frac{\pi t_0}{2}\right)\frac{|1+\overline{\rho}|^2}{4}h_1\left(\frac{t_0-1}{4}\right)=0 \tag{10}$$

$$\sin^2\left(\frac{\pi t_0}{2}\right)\frac{|1-\overline{\rho}|^2}{4}h_1\left(\frac{t_0-3}{4}\right)=0 \tag{11}$$

Our hypothesis is that $\rho \neq -1$, so in Eq. (10), the coefficient $\frac{|1+\overline{\rho}|^2}{4} \neq 0$.

Case 1 If $t_0 \neq -1$ then Eq. (9) implies $h_1(t_0/4) = 0 = h_1(g_0(t_0))$. Let $t_1 := g_0(t_0) \in (-1, 0)$; iterating the previous argument implies that $h_1(g_0(t_1)) = 0$. Thus, we obtain an infinite sequence of zeroes of h_1.

Case 2 If $t_0 = -1$, then the previous argument does not hold. However, we can construct another zero of h_1, $t_0' \in (-1, 0)$ to which the previous argument will hold. Indeed, if $t_0 = -1$, Eq. (10) implies $h_1((t_0 - 1)/4) = h_1(-1/2) = 0$. Let $t_0' = -1/2$ and continue as in Case 1.

In either case, h_1 vanishes on a (countable) set with an accumulation point, and since h_1 is analytic it follows that $h_1 \equiv 0$, a contradiction, and Claim 1 holds.

Now, to prove Claim 2, we follow the proof of Lemma 4.2 of [10]. For a fixed $\omega \in X_4$, define $f_\omega : \mathbb{C} \to \mathbb{C}$ by

$$f_\omega(z) = \langle e_z, S_\omega \mathbb{1}\rangle = \int e^{2\pi i z x}\overline{[S_\omega \mathbb{1}](x, y)}\, d(\mu_4 \times \lambda).$$

Since the distribution $\overline{[S_\omega \mathbb{1}](x, y)}\, d(\mu_4 \times \lambda)$ is compactly supported, a standard convergence argument demonstrates that f_ω is entire. Likewise, $f_\omega^*(z) = \overline{f_\omega(\overline{z})}$ is entire, and for t real,

$$f_\omega(t) f_\omega^*(t) = (\langle e_t, S_\omega \mathbb{1}\rangle)\left(\overline{\langle e_t, S_\omega \mathbb{1}\rangle}\right) = |\langle e_t, S_\omega \mathbb{1}\rangle|^2.$$

Thus,

$$h_X(t) = \sum_{\omega \in X_4} f_\omega(t) f_\omega^*(t).$$

For $n \in \mathbb{N}$, let $h_n(z) = \sum_{|\omega| \leq n} f_\omega(z) f_\omega^*(z)$, which is entire. By Hölder's inequality,

$$\sum_{\omega \in X_4} |f_\omega(z) f_\omega^*(z)| \leq \left(\sum_{\omega \in X_4} |\langle e_z, S_\omega \mathbb{1}\rangle|^2\right)^{1/2}\left(\sum_{\omega \in X_4} |\langle e_{\overline{z}}, S_\omega \mathbb{1}\rangle|^2\right)^{1/2}$$
$$\leq \|e_z\|\|e_{\overline{z}}\| \leq e^{K Im(z)}$$

for some constant K. Thus, the sequence $h_n(z)$ converges pointwise to a function $h(z)$, and are uniformly bounded on strips $Im(z) \leq C$. By the theorems of Montel and Vitali, the limit function h is entire, which coincides with h_X for real t, and Claim 2 is proved.

Example 1 As mentioned in Sect. 2, in general, $\{S_\omega \mathbb{1}\}$ need not be complete, and the exceptional point $\rho = -1$ in Theorem 2 provides the example. In the case $\rho = -1$, the set (6) becomes

$$\{d_n e^{2\pi i n x} : n \in \mathbb{N}_0\}$$

where the coefficients $d_n = 1$ if $n \in \Gamma_3$ and 0 otherwise. Here,

$$\Gamma_3 = \left\{ \sum_{n=0}^{N} l_n 4^n : l_n \in \{0, 3\} \right\}$$

and it is known [4] that the sequence $\{e^{2\pi i n x} : n \in \Gamma_3\}$ is incomplete in $L^2(\mu_4)$. Thus, $\{P_V S_\omega \mathbb{1}\}$ is incomplete in V, so $\{S_\omega \mathbb{1}\}$ is incomplete in $L^2(\mu_4 \times \lambda)$.

We can generalize the construction of Theorem 2 as follows. We want to choose a matrix

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ h_{10} & h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & h_{33} \end{pmatrix}$$

such that $\begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix}^T$ is in the kernel of H and the matrix

$$H = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ h_{10} & -h_{11} & h_{12} & -h_{13} \\ h_{20} & h_{21} & h_{22} & h_{23} \\ h_{30} & -h_{31} & h_{32} & -h_{33} \end{pmatrix}$$

is unitary. We obtain a system of nonlinear equations in the 12 unknowns. To parametrize all solutions, we consider the following row vectors:

$$\vec{v}_0 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \qquad \vec{w}_0 = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix} \tag{12}$$

$$\vec{v}_1 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix} \qquad \vec{w}_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix} \tag{13}$$

$$\vec{v}_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix} \qquad \vec{w}_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix} \tag{14}$$

If we construct the matrix A so that the rows are linear combinations of $\{\vec{v}_0, \vec{v}_1, \vec{v}_2\}$, then A will satisfy the desired condition on the kernel. Note that if the jth row of A is $\alpha_{j0}\vec{v}_0 + \alpha_{j1}\vec{v}_1 + \alpha_{j2}\vec{v}_2$ for $j = 1, 3$, then the jth row of H is $\alpha_{j0}\vec{w}_0 + \alpha_{j1}\vec{w}_1 + \alpha_{j2}\vec{w}_2$, whereas if $j = 0, 2$, then the jth row of H is equal to the jth row of A.

Thus, we want to choose coefficients α_{jk}, $j = 0, 1, 2, 3$, $k = 1, 2, 3$ so that the matrix

$$H = \begin{pmatrix} \alpha_{00}\vec{v}_0 + \alpha_{01}\vec{v}_1 + \alpha_{02}\vec{v}_2 \\ \alpha_{10}\vec{w}_0 + \alpha_{11}\vec{w}_1 + \alpha_{12}\vec{w}_2 \\ \alpha_{20}\vec{v}_0 + \alpha_{21}\vec{v}_1 + \alpha_{22}\vec{v}_2 \\ \alpha_{30}\vec{w}_0 + \alpha_{31}\vec{w}_1 + \alpha_{32}\vec{w}_2 \end{pmatrix} \tag{15}$$

is unitary. To satisfy the requirement on the first row, we choose $\alpha_{00} = 1$ and $\alpha_{01} = \alpha_{02} = 0$. Calculating the inner products of the rows of H, we obtain the following necessary and sufficient conditions:

$$|\alpha_{j0}|^2 + |\alpha_{j1}|^2 + |\alpha_{j2}|^2 = 1 \tag{16}$$
$$\alpha_{00}\overline{\alpha_{20}} = 0 \tag{17}$$
$$\alpha_{11}\overline{\alpha_{22}} + \alpha_{12}\overline{\alpha_{21}} = 0 \tag{18}$$
$$\alpha_{10}\overline{\alpha_{30}} + \alpha_{11}\overline{\alpha_{31}} + \alpha_{12}\overline{\alpha_{32}} = 0 \tag{19}$$
$$\alpha_{21}\overline{\alpha_{32}} + \alpha_{22}\overline{\alpha_{31}} = 0 \tag{20}$$

Proposition 2 *Fix* $\alpha_{00} = 1$. *There exists a solution to the Eqs.* (16–20) *if and only if* $\alpha_{10}, \alpha_{30} \in \mathbb{C}$ *with*

$$|\alpha_{10}|^2 + |\alpha_{30}|^2 = 1. \tag{21}$$

Proof ($\Leftarrow$) If $|\alpha_{10}|^2 = 1$, then we choose $\alpha_{21} = \alpha_{31} = 1$ and all other coefficients to be 0 to obtain a solution to Eqs. (16–20). Likewise, if $|\alpha_{10}|^2 = 0$, then choose $\alpha_{11} = \alpha_{21} = 1$ and all other coefficients to be 0.

Now suppose that $0 < |\alpha_{10}| < 1$, and we choose $\lambda = \dfrac{-\overline{\alpha_{10}}\alpha_{30}}{1 - |\alpha_{10}|^2}$. Then choose α_{11} and α_{12} such that $|\alpha_{11}|^2 + |\alpha_{12}|^2 = 1 - |\alpha_{10}|^2$. Now let $\alpha_{31} = \lambda\alpha_{11}$ and $\alpha_{32} = \lambda\alpha_{12}$. We have

$$\begin{aligned}\alpha_{10}\overline{\alpha_{30}} + \alpha_{11}\overline{\alpha_{31}} + \alpha_{12}\overline{\alpha_{32}} &= \alpha_{10}\overline{\alpha_{30}} + \overline{\lambda}|\alpha_{11}|^2 + \overline{\lambda}|\alpha_{12}|^2 \\ &= \alpha_{10}\overline{\alpha_{30}} + \overline{\lambda}(1 - |\alpha_{10}|^2) \\ &= 0,\end{aligned} \tag{22}$$

so Eq. (19) is satisfied.

Equation (17) forces $\alpha_{20} = 0$; choose α_{21} and α_{22} such that $|\alpha_{21}|^2 + |\alpha_{22}|^2 = 1$ and $\alpha_{11}\overline{\alpha_{21}} + \alpha_{12}\overline{\alpha_{22}} = 0$. Thus, Eqs. (18) and (20) are satisfied. Finally, regarding Eq. (16), it is satisfied for $j = 0, 1, 2$ by construction. For $j = 3$, we calculate:

$$\begin{aligned}|\alpha_{30}|^2 + |\alpha_{31}|^2 + |\alpha_{32}|^2 &= |\alpha_{30}|^2 + |\lambda|^2\left(|\alpha_{11}|^2 + |\alpha_{12}|^2\right) \\ &= |\alpha_{30}|^2 + \frac{|\alpha_{10}|^2|\alpha_{30}|^2}{(1 - |\alpha_{10}|^2)^2}\left(1 - |\alpha_{10}|^2\right) \\ &= |\alpha_{30}|^2\left(1 + \frac{|\alpha_{10}|^2}{1 - |\alpha_{10}|^2}\right) \\ &= \frac{|\alpha_{30}|^2}{1 - |\alpha_{10}|^2} \\ &= 1\end{aligned} \tag{23}$$

as required.

($\Rightarrow$) Suppose that we have a solution to Eqs. (16–20). If $|\alpha_{10}| = 1$, then we must have $\alpha_{11} = \alpha_{12} = 0$, and thus Eq. (19) requires $\alpha_{30} = 0$, so Eq. (21) holds.

Now suppose $|\alpha_{10}| < 1$. Since $\alpha_{20} = 0$, we must have that $|\alpha_{21}|^2 + |\alpha_{22}|^2 = 1$. Combining this with Eqs. (18) and (20) imply that the matrix

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{31} & \alpha_{32} \end{pmatrix}$$

is singular. Thus, there exists a λ such that $\alpha_{31} = \lambda\alpha_{11}$ and $\alpha_{32} = \lambda\alpha_{12}$. Using the same computation as in Eq. (22), we conclude that $\lambda = \frac{-\overline{\alpha_{10}}\alpha_{30}}{1 - |\alpha_{10}|^2}$; then Eq. (23) implies (21). □

The coefficient matrix we obtain from this construction is

$$H = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ \alpha_{10} + \alpha_{11} + \alpha_{12} & \alpha_{10} - \alpha_{11} + \alpha_{12} & \alpha_{10} - \alpha_{11} - \alpha_{12} & \alpha_{10} + \alpha_{11} - \alpha_{12} \\ \alpha_{21} + \alpha_{22} & -\alpha_{21} + \alpha_{22} & -\alpha_{21} - \alpha_{22} & \alpha_{21} - \alpha_{22} \\ \alpha_{30} + \lambda\alpha_{11} + \lambda\alpha_{12} & \alpha_{30} - \lambda\alpha_{11} + \lambda\alpha_{12} & \alpha_{30} - \lambda\alpha_{11} - \lambda\alpha_{12} & \alpha_{30} + \lambda\alpha_{11} - \lambda\alpha_{12} \end{pmatrix}$$

where we are allowed to choose α_{11}, α_{12}, α_{21} and α_{22} subject to the normalization condition in Eq. (16). However, those choices do not affect the construction, since if we apply Proposition 1 and the calculation from Theorem 2, we obtain

$$P_V S_\omega \mathbb{1} = (\alpha_{10})^{\ell_1(n)} \cdot (0)^{\ell_2(n))} \cdot (\alpha_{30})^{\ell_3(n)} e^{2\pi i n x}. \tag{24}$$

This will in fact be a Parseval frame for $L^2(\mu_4)$, provided $V \subset \mathcal{K}$, as in the proof of Theorem 2.

Theorem 3 *Suppose $p, q \in \mathbb{C}$ with $|p|^2 + |q|^2 = 1$. Then*

$$\{p^{\ell_1(n)} \cdot 0^{\ell_2(n)} \cdot q^{\ell_3(n)} e^{2\pi i n x} : n \in \mathbb{N}_0\}$$

is a Parseval frame for $L^2(\mu_4)$, provided $p \neq 0$.

Proof Substitute $\alpha_{10} = p$ and $\alpha_{30} = q$ in Proposition 2 and Eq. (24). As noted, we only need to verify $V \subset \mathcal{K}$. We proceed as in the proof of Theorem 2; indeed, define f, h_X, $\mathfrak{m}_j$ and g_j as previously. We obtain $b_0 = 1$, $b_1 = \overline{p}$, $b_2 = 0$, and $b_3 = \overline{q}$, so Eq. (8) becomes

$$h_X(t) = \cos^2\left(\frac{\pi t}{2}\right) h_X\left(\frac{t}{4}\right) + |\overline{p}|^2 \sin^2\left(\frac{\pi t}{2}\right) h_X\left(\frac{t-1}{4}\right) + |\overline{q}|^2 \sin^2\left(\frac{\pi t}{2}\right) h_X\left(\frac{t-3}{4}\right).$$

From here, the same argument shows that $h_X \equiv 1$, and $V \subset \mathcal{K}$. □

5 Concluding Remarks

We remark here that the constructions given above for μ_4 does not work for μ_3. Indeed, we have the following no-go result. To obtain the measure $\mu_3 \times \lambda$, we consider the iterated function system:

$$\Upsilon_0(x,y)=\left(\frac{x}{3},\frac{y}{2}\right),\ \Upsilon_1(x,y)=\left(\frac{x+2}{3},\frac{y}{2}\right),$$
$$\Upsilon_2(x,y)=\left(\frac{x}{3},\frac{y+1}{2}\right),\Upsilon_3(x,y)=\left(\frac{x+2}{3},\frac{y+1}{2}\right).$$

Using the same choice of filters, the matrix $\mathcal{M}(x,y)$ reduces to

$$H=\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & e^{4\pi i/3}a_{11} & a_{12} & e^{4\pi i/3}a_{13} \\ a_{20} & e^{2\pi i/3}a_{21} & a_{22} & e^{2\pi i/3}a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}$$

which we require to be unitary. Additionally, we require the same conditions as for μ_4, namely, the first row of H must have all entries $\frac{1}{2}$, and $a_{j0}+a_{j2}=a_{j1}+a_{j3}$. The inner product of the first two rows must be 0. Hence,

$$\frac{1}{2}\left(a_{10}+e^{4\pi i/3}a_{11}+a_{12}+e^{4\pi i/3}a_{13}\right)=\frac{1}{2}\left(a_{10}+a_{12}\right)\left(1+e^{4\pi i/3}\right)=0.$$

Consequently, $a_{10}+a_{12}=0$. Likewise, $a_{20}+a_{22}=a_{30}+a_{32}=0$. As a result,

$$H\begin{pmatrix}1\\0\\1\\0\end{pmatrix}=\begin{pmatrix}a_{00}+a_{02}\\a_{10}+a_{12}\\a_{20}+a_{22}\\a_{30}+a_{32}\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

and so H cannot be unitary.

It may be possible to extend the construction for μ_4 to μ_3 by considering a representation of $\mathcal{O}_n$ for some sufficiently large n, or by considering $\mu_3\times\rho$ for some other fractal measure ρ rather than λ.

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