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Half Sampling on Bipartite Graphs

Robert S. Strichartz[1]

Abstract On a bipartite graph G we consider the half sampling problem of uniquely recovering a function from its values on the even vertices, under the appropriate half bandlimited assumption with respect to a Laplacian on the graph. We discuss both finite and infinite graphs, give the appropriate definition of "half bandlimited" that involves splitting the mid frequency, and give an explicit solution to the problem. We discuss in detail the example of a regular tree. We also consider a related sampling problem on graphs that are generated by edge substitution.

1 Introduction

The classical Sampling Theorem on the line has been extended to many different contexts. We refer the reader to [1,3–8,14–20,25] for just a sampling of such results. There are also some related works on filter banks and wavelets on graphs, for example [12,13,23,24]. In general there is a space X where functions are defined and a sampling set $E \subseteq X$ where the values of a function are given. The problem is to recover the values of a function f on the whole space X in terms of its sample values on E, under

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✉ Robert S. Strichartz
str@math.cornell.edu

[1] Mathematics Department, Cornell University, Malott Hall, Ithaca, NY 14853, USA

Birkhäuser

the assumption that a suitable spectral decomposition of f is entirely contained in a suitable low frequency band. Many of these results are of an "overdetermined" type, meaning that there are implicit "consistency conditions" on the sample values $f|_E$ if f is bandlimited, and the algorithm for recovering the function is not unique and often involves infinite series or iterations. The results in this paper are of a more precise nature, in that there are no restrictions on the sample values, and the reconstruction algorithm is unique (so the sampling set is also an interpolation set) and precisely given.

We consider a locally finite bipartite graph G, whose vertices split $V = V_0 \cup V_1$ (disjoint) into even V_0 and odd V_1 vertices, and all edges $x \sim y$ connect one even vertex to one odd vertex. (It is not necessary to assume that G is connected; on the other hand the disconnected case reduces to the analogues results on each connected component graph.) To define a Laplacian on G we assign positive weights $c(x, y)$ to all edges, set $\mu(x) = \sum_{y \sim x} c(x, y)$ and

$$- \Delta f(x) = \frac{1}{\mu(x)} \sum_{y \sim x} c(x, y)(f(x) - f(y)). \qquad (1.1)$$

For simplicity we discuss first the case when G is finite. Then $-\Delta$ is a nonnegative self-adjoint operator with respect to the measure μ, so there exists an orthonormal basis of eigenfunctions u_k with eigenvalues λ_k,

$$- \Delta u_k = \lambda_k u_k. \qquad (1.2)$$

In fact we have the estimate

$$0 \leqslant \lambda_k \leqslant 2, \qquad (1.3)$$

and both endpoints occur, as 0 is the eigenvalue of the constant function and 2 is the eigenvalue of the function that is 1 on V_0 and -1 on V_1. The midfrequency $\lambda = 1$ may or may not occur as an eigenvalue. If it does not then we easily define a *half bandlimited function* to be a function

$$f(x) = \sum_{\lambda_k < 1} a_k u_k(x) \qquad (1.4)$$

that is expanded as a linear combination of low frequency eigenfunctions. Of course $a_k = \langle f, u_k \rangle$ (the inner product with respect to the measure μ) in (1.4), and (1.4) is equivalent to

$$\langle f, u_k \rangle = 0 \text{ if } \lambda_k > 1. \qquad (1.5)$$

If there are eigenfunctions with $\lambda_k = 1$, we need to split up the 1-eigenspace into low frequency and high frequency parts. The details are given in Sect. 2. The main result proved in Sect. 2 is that a half bandlimited function is uniquely determined by its samples on V_0, with no restrictions on the samples, and the reconstruction is given by an explicit formula involving a low-pass projection operator. We also obtain a relationship between the original spectral decomposition and the spectral

decomposition of a Laplacian defined on V_0, but this requires an additional symmetry hypothesis.

In Sect. 3 we discuss analogous results when G is an infinite graph. Here the spectral resolution of the Laplacian in general involves projection valued measures on a spectrum contained in [0, 2]. We make the additional assumption that it may be written in the form

$$f = \int_0^2 P_\lambda f \, d\nu(\lambda) \tag{1.6}$$

where ν is a finite positive measure supported on the spectrum and $P_\lambda f$ is an eigenfunction,

$$-\Delta P_\lambda f = \lambda P_\lambda f. \tag{1.7}$$

Here (1.6) holds for every $f \in \ell^2(\mu)$, but $P_\lambda f$ is not assumed to be in $\ell^2(\mu)$. This form of the spectral resolution is known to hold for many interesting examples. Then we define f to be half bandlimited if the integral in (1.6) is limited to [0, 1]. Again, if ν has a point mass at $\lambda = 1$ we have to split the 1-eigenspace. Then again a half bandlimited function is uniquely determined from its samples on V_0 via an explicit formula.

While none of the above results are surprising or difficult, they set the stage for the discussion of a nontrivial example in Sect. 4. We denote by T_q the connected $(q + 1)$-regular tree, equipped with its completely symmetric Laplacian. The spectral resolution of this Laplacian in the form (1.6) was given explicitly by Cartier [2]. An exposition may be found in [9], and generalizations to other Laplacians may be found in [10]. A related "Plancherel formula" in the case $q = 2$ is given in [22].

It turns out that the spectrum is the smaller interval $\left[1 - \frac{2\sqrt{q}}{q+1}, 1 + \frac{2\sqrt{q}}{q+1}\right]$, and ν is an explicit absolutely continuous measure. The spectral resolution $P_\lambda f$ is given by a spherical convolution operator

$$P_\lambda f(x) = \sum_y \varphi_\lambda(d(x, y)) f(y) \tag{1.8}$$

for an explicit spherical function φ_λ (here $d(x, y)$ denotes the graph distance). The spherical function just fails to be in $\ell^2(G)$. We give here a simple proof of the spectral resolution based on the Cauchy integral formula. Using the same reasoning we obtain an explicit formula for the low-pass operator that gives the reconstruction formula. It is also a spherical convolution operator with a kernel that is now just barely in $\ell^2(G)$.

In Sect. 5 we discuss a related sampling problem on graphs G_H that are generated by edge substitution. That is, we are given graphs G and H and Laplacians and produce G_H by substituting a copy of H for each edge of G. Under some minimal hypotheses on H, we show how to define low-pass bandlimited functions on G_H (in terms of a natural Laplacian) so that every such function can be reconstructed from its samples on G, and all functions on G arise in this manner. The reconstruction algorithm is given explicitly, but it is not as simple as in the half sampling problem.

The results in this paper are motivated by pure mathematical curiosity. We are not aware of any potential applications, but as usual this does not preclude the possibility of future applications.

2 Finite Graphs

For simplicity we first describe the case of finite bipartite graphs G. We assume we are given a set of positive weights $c(x, y)$ on the edges of G, so that $c(x, y) = c(y, x) > 0$ if $x \sim y$ and $c(x, y) = 0$ otherwise. Define

$$\mu(x) = \sum_j c(x, y_j) \tag{2.1}$$

where $\{y_j\}$ is the set of neighboring vertices of x. We consider μ as a measure on the vertices V of G. Then the Laplacian is defined by

$$-\Delta f(x) = \frac{1}{\mu(x)} \sum_j c(x, y_j)(f(x) - f(y_j))$$

$$= f(x) - \sum_j \frac{c(x, y_j)}{\mu(x)} f(y_j). \tag{2.2}$$

Since $-\Delta$ is a nonnegative self-adjoint operator with respect to the measure μ, there exists an orthonormal basis $\{u_k\}$ of eigenfunctions, with

$$-\Delta u_k = \lambda_k u_k. \tag{2.3}$$

We order the eigenvalues in increasing order, $0 = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N \leqslant 2$, with $N = \#V$. Note that u_1 is constant, and if we assume that G is connected then $\lambda_2 > 0$.

We define the *twist* operator T by

$$Tf(x) = (-1)^{p(x)} f(x) \tag{2.4}$$

where $p(x)$ is the parity function

$$p(x) = \begin{cases} 0 & \text{if } x \in V_0 \\ 1 & \text{if } x \in V_1 \end{cases}.$$

Lemma 1 *If u is an eigenfunction with eigenvalue λ, then Tu is an eigenfunction with eigenvalue $2 - \lambda$.*

Proof If $x \in V_0$ then all $y_j \in V_1$, so

$$-\Delta Tu(x) = u(x) + \sum_j \frac{c(x, y_j)}{\mu(x)} u(y_j)$$

$$= 2u(x) + \Delta u(x)$$
$$= (2 - \lambda)u(x)$$
$$= (2 - \lambda)Tu(x).$$

A similar computation works if $x \in V_1$.   □

We may divide up the spectrum into low frequency eigenvalues with $\lambda_j < 1$ and high frequency eigenvalues with $\lambda_j > 1$, and these are interchanged by the twist operator. Thus

$$\lambda_{N-j+1} = 2 - \lambda_j \tag{2.5}$$

and we may redefine the eigenfunctions so that

$$u_{N-j+1} = T u_j \tag{2.6}$$

for low frequency eigenfunctions. Of course it may happen (and often does) that there are midfrequency eigenvalues with $\lambda_j = 1$. If we denote this midfrequency eigenspace by E_1 then the twist operator maps E_1 to itself.

Lemma 2 $E_1 = E_1^+ \oplus E_1^-$ *where the functions in* E_1^+ *are supported in* V_0 *and the functions in* E_1^- *are supported in* V_1. *Moreover*

$$\dim E_1^+ - \dim E_1^- = \#V_0 - \#V_1. \tag{2.7}$$

Proof Since $T^2 = I$ on E_1, and T is self-adjoint, E_1 splits into an orthonormal sum of eigenspaces of T with eigenvalues ± 1, and these are exactly E_1^+ and E_1^-, since $Tu = u$ if and only if u is supported in V_0, and $Tu = -u$ if and only if u is supported in V_1.

To prove (2.7) we note that

$$\dim E_1^+ + M = \#V_0 \text{ and}$$
$$\dim E_1^- + M = \#V_1$$

where M denotes the number of low frequency eigenvalues, since when restricted to V_0 we have $Tu = u$. $\qquad\square$

We choose the orthonormal basis of E_1 so that the E_1^+ eigenfunctions come before the E_1^- eigenfunctions.

Definition 1 (a) A function is said to be *half-bandlimited* if it is a linear combination of low frequency eigenfunctions and midfrequency eigenfunctions in E_1^+. Specifically,

$$f = \sum_{k=1}^{\#V_0} a_k u_k. \tag{2.8}$$

(b) The *half-sampling operator* Hf is defined to be

$$Hf(x) = \begin{cases} f(x) & \text{for } x \in V_0 \\ 0 & \text{for } x \in V_1 \end{cases}, \tag{2.9}$$

so that $H = \frac{1}{2}(I + T)$. Note that we could also have defined an operator $\widetilde{H}$ from functions on V to functions on V_0 by restriction. Clearly H and $\widetilde{H}$ contain the same information.

(c) We define the low-pass operator L by

$$L\left(\sum_{k=1}^{\#V} a_k u_k\right) = \sum_{k=1}^{\#V_0} a_k u_k. \tag{2.10}$$

Note that L is given explicitly as

$$Lf(x) = \sum_{y\in V} \mathcal{L}(x,y) f(y)\mu(y) \quad \text{for} \tag{2.11}$$

$$\mathcal{L}(x,y) = \sum_{k=1}^{\#V_0} u_k(x)u_k(y). \tag{2.12}$$

Theorem 1 *A half-bandlimited function f is uniquely determined from its half-samples $\widetilde{H}f$, and every function on V_0 arises as $\widetilde{H}f$ for some unique half-bandlimited function f. In other words, V_0 is both a sampling set and an interpolation set for the space of half-bandlimited functions. Explicitly*

$$f(x) = \sum_{y\in V_0} K(x,y) Hf(y)\mu(y) \quad \text{for} \tag{2.13}$$

$$K(x,y) = 2\sum_{k=1}^{M} u_k(x)u_k(y) + \sum_{k=M+1}^{\#V_0} u_k(x)u_k(y). \tag{2.14}$$

Proof Suppose $f = u_k$ for $1 \leqslant k \leqslant \#V_0$. Then $Hu_k = \frac{1}{2}u_k + \frac{1}{2}Tu_k$. If $1 \leqslant k \leqslant M$ so that u_k is low frequency then Tu_k is high frequency so $LTu_k = 0$ and $Lu_k = u_k$. Thus $LHu_k = \frac{1}{2}u_k$. On the other hand, if $M+1 \leqslant k \leqslant \#V_0$ so that u_k is mid-frequency in E_1^+, then $Tu_k = u_k$ and $Lu_k = u_k$. Thus $LHu_k = u_k$. Taking linear combinations we arrive at (2.13) and (2.14).

Conversely, if g is any function on V_0, extend g to $\widetilde{g}$ on V in any way. Then $\widetilde{g}$ is a linear combination of all the eigenfunctions. But the eigenfunctions in E_1^- vanish on V_0 and the high frequency eigenfunctions $u_{N-j+1} = Tu_j$ are equal to the low frequency eigenfunctions u_j when restricted to V_0. So g is equal to a half-bandlimited function restricted to V_0. □

Remark 1 We could reverse the roles of the even and odd vertices, and consider half-sampling on the odd vertices. We would then obtain the same sort of results, but we would have to replace E_1^+ by E_1^- in the definition of half-bandlimited. Only in graphs where E_1 is empty would we have a single class of bandlimited functions that would allow half-sampling on either even or odd vertices.

Next we consider the half graph G_0 whose vertices are V_0 and with edges $x \sim z$ if and only if there exists $y \in V_1$ such that $x \sim y$ and $y \sim z$ are edges in G. Note that such y need not be unique. We consider the question of whether we can define weights $c_0(x,z)$ on the edges of G_0 so that the functions $\widetilde{u}_k$ for $1 \leqslant k \leqslant \#V_0$ give a

complete set of eigenfunctions for the Laplacian $-\Delta_0$ on G_0. It appears that we need to make an additional assumption on the original set of weights of a weak symmetry type in order to do this.

Definition 2 We say that the graph G and its weights $\{c(x, y)\}$ are *weakly symmetric* if there exists a constant A independent of $x \in V_0$ such that

$$\sum_{y \sim x} \frac{c(y, x)^2}{\mu(x)\mu(y)} = A \tag{2.15}$$

for all $x \in V_0$. In that case we define weights $c_0(x, z)$ on G_0 by

$$c_0(x, z) = \sum_{\substack{y \in V_1 \\ x \sim y \sim z}} \frac{c(x, y)c(y, z)}{\mu(y)(1 - A)}. \tag{2.16}$$

Theorem 2 *Assume that G and its weights are weakly symmetric, and consider the Laplacian $-\Delta_0$ on G_0 with weights given by (2.16). Then $\widetilde{H}u_k$ for $1 \leqslant k \leqslant \#V_0$ is an eigenfunction of $-\Delta_0$ with eigenvalue*

$$\widetilde{\lambda}_k = \frac{\lambda_k(2 - \lambda_k)}{1 - A}, \tag{2.17}$$

and these give a complete set of eigenfunctions for $-\Delta_0$.

Proof Fix $x \in V_0$ and let $\{y_j\}$ enumerate the vertices in V_1 that are neighbors of x. Let $\{z_{j\ell}\}$ enumerate the vertices in V_0 that are neighbors of y_j. Note that x is one of these neighbors and we choose the enumeration so $x = z_{j1}$. The complete union of $z_{j\ell}$ with $\ell \geqslant 2$ gives all $z \in V_0$ that are neighbors of x in G_0, but there may be more than one representation of a fixed z. The eigenvalue equation for $-\Delta$ at x yields

$$\sum_j c(x, y_j)u_k(y_j) = \mu(x)(1 - \lambda_k)u_k(x) \tag{2.18}$$

and similarly at y_j we have

$$\sum_\ell \frac{c(y_j, z_{j\ell})u_k(z_{j\ell})}{\mu(y_j)} = (1 - \lambda_k)u_k(y_j). \tag{2.19}$$

Now we multiply (2.19) by $c(x, y_j)$ and sum over j to obtain

$$\sum_j \sum_\ell \frac{c(x, y_j)c(y_j, z_{j\ell})u_k(z_{j\ell})}{\mu(y_j)} = (1 - \lambda_k) \sum_j c(x, y_j)u_k(y_j)$$

$$= \mu(x)(1 - \lambda_k)^2 u_k(x). \tag{2.20}$$

Note that the terms corresponding to $\ell = 1$ on the left side of (2.20) sum to exactly $\mu(x)Au_k(x)$ while the other terms sum to exactly $(1-A)\sum_z c_0(x,z)u_k(z)$ where the sum is over all z that are neighbors of x in G_0. Dividing by $(1-A)\mu(x)$, we obtain

$$\sum_z \frac{c_0(x,z)}{\mu(x)} u_k(z) = \left(\frac{(1-\lambda_k)^2}{1-A} - \frac{A}{1-A}\right) u_k(x)$$

$$= (1-\widetilde{\lambda}_k)u_k(x),$$

which is equivalent to $-\Delta_0 \widetilde{H}u_k = \widetilde{\lambda}_k \widetilde{H}u_k$. Note that choosing $k = 1$ so u_1 is constant verifies that $\mu(x) = \sum_k c_0(y,z)$ as required. □

3 Infinite Graphs

If G is infinite we assume that there is an upper bound to the order of each vertex. Then the operator $-\Delta$ on $\ell^2(G)$ (with respect to the measure given by the weights $\mu(x)$) is a bounded, nonnegative self-adjoint operator with spectrum contained in $[0, 2]$. By the spectral theorem it has a spectral resolution. For simplicity we assume this resolution can be given in the form

$$f = \int_0^2 P_\lambda f \, d\nu(\lambda) \tag{3.1}$$

where ν is a finite positive measure on the spectrum, and $P_\lambda f$ is an eigenfunction

$$-\Delta P_\lambda f = \lambda P_\lambda f. \tag{3.2}$$

Note that $P_\lambda f$ is not assumed to be in $\ell^2(G)$ (usually it is not). The usual spectral projection onto an interval $[a, b]$ is then $\int_a^b P_\lambda f \, d\nu(\lambda)$. Note that P_λ and ν are not unique, since you could multiply P_λ by $\varphi(x)$ and ν by $\frac{1}{\varphi(x)}$. The general case may be treated by similar arguments, but the notation is more awkward.

For simplicity we consider first the case when the measure ν does not have an atom at $\lambda = 1$. We define the twist operator T as before, and note that $-\Delta T P_\lambda f = (2-\lambda)P_\lambda f$. We may assume then that $P_{2-\lambda}f = P_\lambda T f$. We define the lowpass operator L by

$$Lf = \int_0^1 P_\lambda f \, d\nu(\lambda) \tag{3.3}$$

and define f to be *half-bandlimited* if $Lf = f$. The half-sampling operators Hf and $\widetilde{H}f$ are defined as before. Then the analogue of Theorem 1 holds with

$$f = 2LHf. \tag{3.4}$$

The proof is essentially the same.

In the case that ν has an atom at $\lambda = 1$, we define the space E_1 to be the set of 1-eigenfunctions in $\ell^2(G)$. If $\{\varphi_j\}$ is an orthonormal basis for E_1 then we define the projection onto E_1 by

$$P_{E_1} f(x) = \sum_j \sum_y \varphi_j(x) \varphi_j(y) f(y) \mu(y). \tag{3.5}$$

We then split E_1 into $E_1^+ \oplus E_1^-$ into eigenfunctions supported on V_0 and V_1, and obtain similar projections $P_{E_1^+}$ and $P_{E_1^-}$. We define P_λ as before for $\lambda \neq 1$, and denote by ν_0 the measure ν with the atom at 1 removed. Then the analogue of (3.1) is

$$f = \int_0^2 P_\lambda f \, d\nu_0 + P_{E_1^+} f + P_{E_1^-} f, \tag{3.6}$$

and the analogue of (3.3) is

$$Lf = \int_0^1 P_\lambda f \, d\nu_0(\lambda) + P_{E_1^+} f. \tag{3.7}$$

We say that f is half-bandlimited if $Lf = f$, in which case we can recover f from Hf by

$$\begin{aligned} f &= 2 \int_0^1 P_\lambda Hf \, d\nu_0(\lambda) + P_{E_1^+} Hf \\ &= 2LHf - P_{E_1^+} Hf. \end{aligned} \tag{3.8}$$

The survey paper [11] discusses many examples of infinite graphs where the spectral resolution (3.1) may be given explicitly.

4 Regular Trees

We follow the standard convention and denote by T_q the connected $(q+1)$-regular tree. All trees are bipartite: choose one vertex x and define V_0 to consist of all vertices that are an even distance to x. Every vertex in T_q has exactly $q + 1$ neighbors, and we place weights $\frac{1}{q+1}$ on each edge so that $\mu(x) = 1$. Note that $T_1 = \mathbb{Z}$, so we are mainly interested in the case $q \geqslant 2$. We note that T_q may be realized as the Cayley graph of a group in several different ways. For example, take the group with $q + 1$ generators $a_1, a_1, \ldots, a_q$ and relations $a_0^2 = a_1^2 = \cdots = a_q^2 = \mathrm{id}$. For some reason, the group theory plays no role in the spectral resolution of the Laplacian on T_q. The theory that we outline is due to Cartier [2] and is described in detail in [9]. We will also provide a proof of the main theorem here.

Everything is described in terms of what might be called *spherical convolution* operators $f \mapsto \sum f(y) G(\mathrm{dist}(x, y))$ for a function G defined on $\mathbb{Z}^+$. We begin by defining the *c*-function on $\mathbb{C}$ by

$$c(z) = \left(\frac{1}{q+1} \right) \frac{q^{1-z} - q^{z-1}}{q^{-z} - q^{z-1}} \tag{4.1}$$

and the spherical functions

$$\varphi_z(n) = c(z)q^{-nz} + c(1-z)q^{-n(1-z)}. \tag{4.2}$$

A direct computation shows that for any fixed vertex x_0, the function $\varphi_z(d(x, x_0))$ is an eigenfunction of $-\Delta$ with eigenvlue

$$\lambda = 1 - \frac{q^z + q^{1-z}}{q+1}. \tag{4.3}$$

Indeed, if $x \neq x_0$ then x has q neighbors y with $d(y, x_0) = d(x, x_0) + 1$ and one neighbor with $d(y, x_0) = d(x, x_0) - 1$, so $-\Delta q^{-d(x,x_0)z} = \lambda q^{-d(x,x_0)z}$ and similarly for z replaced by $1 - z$. In other words, the eigenvalue equation holds separately for each of the functions q^{-nz} and $q^{-n(1-z)}$ that form $\varphi_z(n)$ by linear combination. But x_0 has $q + 1$ neighbors y with $d(x_0, y) = 1$ so $-\Delta\varphi_z(d(x, x_0))|_{x=x_0} = \varphi_z(0) - \varphi_z(1)$, and the specific linear combinations in (4.2) yields $\varphi_z(0) - \varphi_z(1) = \lambda\varphi_z(0)$ as required.

We will actually only be interested in the choice $z = \frac{1}{2} + it$ for $0 \leqslant t \leqslant \frac{\pi}{\log q}$. This yields

$$\lambda = 1 - \frac{2\sqrt{q}}{q+1}\cos(t\log q), \tag{4.4}$$

and we will see that $\left[1 - \frac{2\sqrt{q}}{q+1}, 1 + \frac{2\sqrt{q}}{q+1}\right]$ is the spectrum of $-\Delta$. Note that we may write

$$c\left(\frac{1}{2} + it\right) = \left(\frac{1}{q+1}\right)\left(\frac{q e^{it\log q} - e^{it\log q}}{e^{-it\log q} - e^{it\log q}}\right). \tag{4.5}$$

Now define

$$P_\lambda f(x) = \sum_y \varphi_{\frac{1}{2}+it}(d(x, y)) f(y) \tag{4.6}$$

where λ and t are related by (4.4). We may initially assume that f has finite support, since such functions are dense in $\ell^2(T_q)$. This eliminates all questions of convergence of sums, and we see from the above computation that

$$-\Delta P_\lambda f = \lambda P_\lambda f. \tag{4.7}$$

We will take P_λ to be the spectral projection. It turns out that the associated measure is

$$d\nu(\lambda) = \frac{q\log q}{2\pi i(q+1)}\left|c\left(\frac{1}{2} + it\right)\right|^{-2} dt = \frac{4q(q+1)\log q \sin^2(t\log q)}{2\pi((q-1)^2 + 4q\sin^2(t\log q))} dt. \tag{4.8}$$

Theorem 3 *(Cartier)The spectral resolution of* $-\Delta$ *is*

$$f = \int_{1-\frac{2\sqrt{q}}{q+1}}^{1+\frac{2\sqrt{q}}{q+1}} P_\lambda f \, d\nu(\lambda). \tag{4.9}$$

Proof We need to verify (4.9), which follows easily from

$$\int_0^{\pi/\log q} \varphi_{\frac{1}{2}+it}(n) \frac{q \log q}{2\pi(q+1)} \left| c\left(\frac{1}{2}+it\right) \right|^{-2} dt = \delta_{n0}. \qquad (4.10)$$

Since $c(\frac{1}{2} - it) = \overline{c(\frac{1}{2} + it)}$, the integrand in (4.10) is

$$\left(\frac{q \log q}{2\pi(q+1)} \right) q^{-n/2} \left(\frac{e^{-int \log q}}{c(\frac{1}{2} - it)} + \frac{e^{in+\log q}}{c(\frac{1}{2} + it)} \right).$$

Thus the integral in (4.10) is

$$\left(\frac{q \log q}{2\pi(q+1)} \right) q^{-n/2} \int_{-\frac{\pi}{\log q}}^{\frac{\pi}{\log q}} \frac{e^{int \log q}}{c(\frac{1}{2} + it)} dt. \qquad (4.11)$$

We introduce the variable $w = e^{it \log q}$. Note that $\frac{1}{c(\frac{1}{2}+it)} = (q+1)\left(\frac{1-w^2}{q-w^2}\right)$ and $dt = \frac{e^{-it \log q}}{i \log q} dw$, so (4.11) becomes the contour integral over the unit circle

$$q^{1-\frac{\pi}{2}} \frac{1}{2\pi i} \oint \frac{w^{n-1}(1-w^2)}{q-w^2} dw. \qquad (4.12)$$

When $n \geqslant 1$ the integrand is holomorphic inside the unit disk so the integral is zero. When $n = 0$ we obtain

$$q \frac{1}{2\pi i} \oint \frac{1-w^2}{q-w^2} \frac{dw}{w} = 1$$

by Cauchy's integral formula. □

It is clear that $[1 - \frac{2\sqrt{q}}{q+1}, 1]$ is the lower half of the spectrum, so that the lowpass operator is

$$Lf = \int_{1-\frac{2\sqrt{q}}{q+1}}^1 P_\lambda f \, d\nu(\lambda), \qquad (4.13)$$

and a function is half-bandlimited if $Lf = f$. If f is half-bandlimited it can be reconstructed from its half-samples Hf via $f = 2LHf$. We can get an explicit expression for Lf by following the reasoning in the proof of Theorem 3 to write

$$Lf(x) = \sum_y \mathcal{L}(d(x,y)) f(y) \qquad (4.14)$$

with

$$\mathcal{L}(n) = q^{1-\frac{n}{2}} \frac{1}{2\pi i} \int_{-i}^i \frac{w^{n-1}(1-w^2)}{q-w^2} dw \qquad (4.15)$$

since the low frequencies correspond to $-\frac{\pi}{2\log q} \leqslant t \leqslant \frac{\pi}{2\log q}$. The integral in (4.15) is over the contour that is the right half of the unit circle, but for $n \geqslant 1$ the integral is holomorphic so the integral is independent of the contour. When n is even it follows by symmetry that (4.15) is half of (4.12), so $2LHf(x) = f(x)$ for $x \in V_0$ as required.

When $n = 2m + 1$ is odd we define

$$\alpha_m = \frac{1}{2\pi i} \int_{-i}^{i} \frac{w^{2m}}{q - w^2}\, dw \tag{4.16}$$

so that

$$\mathscr{L}(2m + 1) = q^{\frac{1}{2}-m}(\alpha_m - \alpha_{m+1}). \tag{4.17}$$

We evaluate α_m by standard methods of calculus by writing $\frac{1}{q-w^2} = \frac{1}{2\sqrt{q}}$ $\left(\frac{1}{\sqrt{q}+w} + \frac{1}{\sqrt{q}-w}\right)$ and $w^{2m} = ((\sqrt{q} \pm w) - \sqrt{q})^{2m} = \sum_{k=0}^{2m} \binom{2m}{k}(-1)^k q^{m+\frac{k}{2}}(\sqrt{q} \pm w)^k$ so that the integrand in (4.16) is

$$\frac{1}{2\sqrt{q}} \sum_{k=0}^{2m} \binom{2m}{k}(-1)^k q^{m+\frac{k}{2}}\left((\sqrt{q} + w)^{k-1} + (\sqrt{q} - w)^{k-1}\right).$$

When we integrate the $k = 0$ term we obtain

$$\frac{1}{2\pi i} \int_{-i}^{i} \left(\frac{1}{\sqrt{q} + w} + \frac{1}{\sqrt{q} - w}\right) dw = \frac{1}{2\pi i} \log \frac{\sqrt{q}+i}{\sqrt{q}-i} = \frac{2}{\pi} \tan^{-1}\left(\frac{1}{\sqrt{q}}\right),$$

while for $k \geqslant 1$, we obtain

$$\frac{1}{2\pi i} \int_{-i}^{i} \left((\sqrt{q} + w)^{k-1} + (\sqrt{q} - w)^{k-1}\right) dw = \frac{1}{\pi i k}\left((\sqrt{q} + i)^k - (\sqrt{q} - i)^k\right)$$

$$= \frac{2}{\pi k} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2j+1}(-1)^j q^{\frac{k}{2}-j-\frac{1}{2}}.$$

This leads to

$$\alpha_m = \frac{q^{n-\frac{1}{2}}}{\pi} \tan^{-1}\left(\frac{1}{\sqrt{q}}\right) + \sum_{k=1}^{2m} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{1}{\pi k}\binom{2m}{k}\binom{k}{2j+1}(-1)^{j+k} q^{m+k-j-1}. \tag{4.18}$$

Although (4.17) and (4.18) give an explicit formula for $\mathscr{L}(2m + 1)$, it is not clear how to obtain size estimates from this. However, we may rewrite (4.15) as

$$\mathscr{L}(n) = q^{1-\frac{n}{2}} \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{in\theta}\left(\frac{1 - e^{2\pi i\theta}}{q - e^{i\theta}}\right) d\theta. \tag{4.19}$$

The integral gives the Fourier coefficients of a function with jump discontinuities at $\theta = \pm\frac{\pi}{2}$, so

$$\mathscr{L}(n) = O\left(\frac{q^{-n/2}}{n}\right) \tag{4.20}$$

and this estimate is sharp. This shows that for fixed y, $f(x) = \mathscr{L}(d(x, y))$ is in $\ell^2(G)$ uniformly in y, since there are $(q + 1)q^{n-1}$ points x with $d(x, y) = n$. However, this is essentially best possible, as $f(x) \notin L^p$ for any $p < 2$. This is in contrast to the sinc function on $T_1 = \mathbb{Z}$ which belongs to every L^p with $p > 1$.

We conclude this section by showing how the ideas in Theorem 2 lead to the explicit spectral resolution for the symmetric Laplacian on the graph G_0. Note that G_0 is not a tree, but it is homogeneous and $q(q + 1)$ regular, with each vector lying on the intersection of $q + 1$ complete-$(q + 1)$ graphs. Let us fix a pair of distinct vertices x and y with $d_0(x, y) = n$ (we denote the distance on G_0 by d_0, and of course $2d_0(x, y) = d(x, y)$ regarding x and y as vertices of G). Then y has one neighbor z with $d_0(x, z) = n - 1$, $q - 1$ neighbors z with $d_0(x, z) = n$ and q^2 neighbors z with $d_0(x, z) = n + 1$. Thus if $f(y) = q^{-2(\frac{1}{2}+it)d_0(x,y)}$, an easy computation shows

$$-\Delta_0 f(y) = \widetilde{\lambda} f(y) \quad \text{for } y \neq x \text{ with} \tag{4.21}$$

$$\widetilde{\lambda} = 1 - \frac{q-1}{q(q+1)} - \frac{q^{2it} + q^{-2it}}{q+1} = \frac{q^2+1}{q^2+q} - \frac{2q\cos(2t\log q)}{q^2+q}, \tag{4.22}$$

and this agrees with (2.17) for λ given by (4.4). Again we can verify directly that $\varphi_{\frac{1}{2}+it}(d(x, y))$ satisfies the eigenvalue equation (4.21) also at $y = x$. Thus we may define

$$P_{\widetilde{\lambda}}^0 f(x) = \sum_{y \in G_0} \varphi_{\frac{1}{2}+it}(2d_0(x, y)) f(y) \tag{4.23}$$

as our spectral projection onto the $\widetilde{\lambda}$-eigenspace, where t and $\widetilde{\lambda}$ are related by (4.22). Note that this will give us a spectrum

$$\left[\frac{(q-1)^2}{q^2+q}, 1+\frac{1}{q}\right] \tag{4.24}$$

for $-\Delta_0$.

Theorem 4 *The spectral resolution of $-\Delta_0$ is*

$$f = \int_{\frac{(q-1)^2}{q^2-q}}^{1+\frac{1}{q}} P_{\widetilde{\lambda}}^0 f \, d\nu_0(\widetilde{\lambda}) \tag{4.25}$$

where

$$d\nu_0(\widetilde{\lambda}) = 2d\nu(\lambda). \tag{4.26}$$

Proof Let g denote the extension of f to V obtained by setting $g = 0$ on V_1. Then $g = \int_{1-\frac{2\sqrt{q}}{q+1}}^{1+\frac{2\sqrt{q}}{q+1}} P_\lambda g \, d\nu(\lambda)$ on V by (4.9). Since $P_\lambda g = P_{2-\lambda} g$ on V_0 we obtain $f = g = 2 \int_{1-\frac{2\sqrt{q}}{q+1}}^{1} P_\lambda g \, d\nu(\lambda)$ on V_0. But $P_\lambda g = P_\lambda^0 f$ on V_0. □

5 Edge Substitution Graphs

In this section we consider edge substitution graphs as discussed in [21]. We consider two graphs G and H, and define a graph G_H obtained from G by substituting a copy of H for each edge in G. We assume that H is finite (with $\#H \geqslant 3$) and has two distinguished boundary points q_0 and q_1. For simplicity we assume that H has a symmetry that permutes q_0 and q_1. The vertices of G_H consist of V_{old}, the *old vertices*, that are just vertices of G, and V_{new}, the *new vertices*, denoted $v(x, y, h)$, where $x \sim y$ is an edge in G and $h \in V_H \setminus \{q_0, q_1\}$. The edges of G_H are exactly the edges E_H of H in each copy of H, so

$$\begin{cases} v(x, y, h) \sim v(x, y, h') & \text{if } h \sim h' \text{ in } H \\ v(x, y, h) \sim x & \text{if } h \sim q_0 \text{ in } H, \text{ and} \\ v(x, y, h) \sim y & \text{if } h \sim q_1 \text{ in } H. \end{cases} \quad (5.1)$$

Because of the symmetry assumption it doesn't matter which order we take for $x \sim y$.

In this section we are not assuming that any of the graphs are bipartite. The sampling problem we consider is to recover a function on G_H from its values on V_{old}, under an appropriate bandlimited assumption. To do this we need to consider Laplacians Δ_G, Δ_H and Δ_{G_H} on the three graphs. We assume we are given a set of positive conductances $c(x, y)$ on the edges of G, with $\mu(x) = \sum_{y \sim x} c(x, y)$ and positive conductances $c_H(h, h')$ on the edges of H, with $\nu(h) = \sum_{h' \sim h} c_H(h, h')$, and we assume that the H conductances are invariant under the symmetry that permutes q_0 and q_1. Then

$$-\Delta_G f(x) = f(x) - \sum_{y \sim x} \frac{c(x, y)}{\mu(x)} f(y) \quad \text{and} \quad (5.2)$$

$$-\Delta_H f(h) = f(h) - \sum_{h' \sim h} \frac{c(h, h')}{\nu(h)} f(h). \quad (5.3)$$

For the graph G_H we assign weights multiplicatively, so

$$\begin{cases} c(v(x, y, h), v(x, y, h')) &= c(x, y) c_H(h, h') \\ c(v(x, y, h), x) &= c(x, y) c_H(q_0, h) \\ c(v(x, y, h), y) &= c(x, y) c_H(q_1, h). \end{cases} \quad (5.4)$$

We may assume that $\nu(q_0) = \nu(q_1) = 1$ by multiplying all the H conductances by the appropriate constant. That means that the weights $\mu(x)$ at the old vertices are unchanged, and $\nu((x, y, h)) = c(x, y)\nu(h)$. Thus, at new vertices

$$-\Delta_{G_H} f(v(x, y, h)) = f(v(x, y, h)) - \sum_{h' \sim h} \frac{c_H(h, h')}{\nu(h)} f(v(x, y, h'))$$
$$= -\Delta_H f(v(x, y, h)), \tag{5.5}$$

while at old vertices

$$-\Delta_{G_H} f(x) = f(x) - \sum_{y \sim x} \sum_{h \sim q_0} \frac{c(x, y)}{\mu(x)} c(q_0, h) f(v(x, y, h)). \tag{5.6}$$

We want to compare the λ'-eigenfuntion equation on G_H with the λ-eigenfunction equation on G. Note that (5.5) implies that λ'-eigenfunctions on G_H, when restricted to the new points $v(x, y, h)$, for fixed edge $x \sim y$, will be λ'-eigenfunctions on the interior of H.

We first discuss the case when G is finite. Assume that λ' is not a Dirichlet eigenvalue of $-\Delta_H$. Then interior values are determined by boundary values, so

$$f(v(x, y, h)) = a_0(\lambda', h) f(x) + a_1(\lambda', h) f(y) \tag{5.7}$$

for certain rational functions $a_i(\lambda', h)$ of λ' with a denominator a polynomial of degree $N = \#H - 2$ and numerator polynomial of degree $N - 1$ (see [21] p. 2052). The zeros of the denominator are the Dirichlet eigenvalues. We then define

$$A_i(\lambda') = \sum_{h \sim q_0} c_H(q_0, h) a_i(\lambda', h), \quad i = 0, 1 \tag{5.8}$$

so that

$$-\Delta_{G_H} f(x) = (1 - A_0(\lambda')) f(x) - A_(\lambda')(f(x) - \Delta_G f(x)). \tag{5.9}$$

In other words, f is a λ'-eigenfunction of $-\Delta_{G_H}$ if and only if its restriction to V_{old} is a λ-eigenfunction of $-\Delta_G$ where the eigenvalues are related by

$$A_1(\lambda')(1 - \lambda) = 1 - \lambda' - A_0(\lambda'). \tag{5.10}$$

It is clear from (5.10) that each λ' determines a unique value of λ, but there are multiple solutions for λ' in terms of λ. Note that if f is constant then $\lambda = \lambda' = 0$.

Note that the relationship (5.10) depends on the graph H and its Laplacian, not on G. Suppose λ' is a Neumann eigenvalue of $-\Delta_H$ corresponding to an odd eigenfunction g with $g(q_0) = 1$, $g(q_1) = -1$. (Recall that a Neumann eigenfunction means that if we glue together two copies of H at q_0 and q_1 and extend g symmetrically then the eigenvalue equation continues to hold at q_0 and q_1.) It is clear that such eigenfunctions exist. If G is bipartite then we can take the $-\Delta_G$ eigenfunction $f|_{V_0} = 1$ and $f|_{V_1} = -1$ with $\lambda = 2$, extend it to G_H by interpolating g in every copy of H, and obtain an eigenfunction of $-\Delta_{G_H}$ with eigenvalue λ'. This shows that (5.10) has a solution λ' for $\lambda = 2$.

Denote by λ'_1 the smallest solution of (5.10) with $\lambda = 2$. Then by continuity there is a solution λ' to (5.10) in the interval $[0, \lambda'_1]$ for every $\lambda \in [0, 2]$. We make the

additional assumption that λ is an increasing function of λ' in the interval $[0, \lambda'_1]$. This implies that the solution of (5.10) in this interval is unique. We denote it $\lambda'(\lambda)$.

We may now define the low frequency spectrum on G_H to correspond, roughly speaking, to the eigenfunctions of $-\Delta_{G_H}$ with eigenvalues in $[0, \lambda'_1]$. Each such eigenfunction restricts to a λ-eigenfunction of $-\Delta_G$, and all such eigenfunctions arise in this way. The only exception is that we have to exclude eigenfunctions that restrict to zero on G. These, of course, correspond to Dirichlet eigenvalues on H. For each Dirichlet eigenvalue λ'_D in $[0, \lambda'_1]$, we must consider all such eigenfunctions as high frequency. Note that it may happen that there still remains some low frequency eigenfunctions with eigenvalues λ'_D, and in fact this happens exactly when λ'_D has multiplicity greater than one in the spectrum of $-\Delta_H$. In that case we are simply fine-tuning the distinction between low and high frequency at the endpoint of the band.

Now define a low bandlimited function on G_H to be one whose spectral expansion in eigenfunctions of $-\Delta_{G_H}$ contains only low frequency spectrum

$$f = \sum b_k u_k, \tag{5.11}$$

where the sum extends over all low frequency eigenfunctions. Then $-\Delta_{G_H} u_k = \lambda'_k u_k$, but also the restriction of u_k to G satisfies $-\Delta_G u_k = \lambda_k u_k$ for $\lambda'_k = \lambda'(\lambda_k)$. Then the restriction of (5.11) to G just gives the spectral expansion of $f|_{V_{\text{old}}}$ for $-\Delta_G$. In this manner we obtain all functions on G as restrictions of low bandlimited functions to G. We may then uniquely solve the sampling problem as follows. Given the sample values $F = f|_{V_{\text{old}}}$, expand F in the $-\Delta_G$ eigenfunctions

$$F = \sum c_k \varphi_k \quad \text{with} \tag{5.12}$$

$$-\Delta_G \varphi_k = \lambda_k \varphi_k. \tag{5.13}$$

Solve for $\lambda'_k = \lambda'(\lambda_k)$, and then the restriction of u_k to V_{old} will be a multiple of φ_k, and so (5.12) yields (5.11). (In the case that the eigenvalue λ_k has nontrivial multiplicity it may be necessary to reshuffle the eigenspaces to line up u_k and φ_k.) The extension from φ_k to u_k is given explicitly by (5.7). The coefficients in the expansions (5.11) and (5.12) are given by different inner products. The relationship between the inner products is discussed in detail in [21] p. 2055-2057. The situation when G is an infinite graph is similar to the discussion for half sampling in Sect. 3. We leave the details to the interested reader.

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