J Fourier Anal Appl (2016) 22:1141–1156
DOI 10.1007/s00041-015-9451-9

Characterization of the SG-Wave Front Set in Terms of the FBI-Transform

René M. Schulz[1]

Received: 3 June 2015 / Published online: 23 December 2015

Abstract The SG-wave front set, which measures microlocally the deviation of a tempered distribution from being rapidly decaying and smooth, is studied using a Fourier–Bros–Iagolnitzer transform. This generalizes the established characterization of the classical Hörmander $\mathscr{C}^\infty$-wave front set. In particular, the transform used is capable of identifying singularities both at finite arguments as well as such arising at infinity.

Keywords FBI-transform · SG-calculus · Wave front set

Mathematics Subject Classification 35A18 · 46F12

1 Introduction

An important aspect of microlocal analysis is the study of distributions by means of localization in position and frequency, meaning in the cotangent space. The fundamental notion in this analysis is the wave front set, which may be obtained through several equivalent approaches—indeed, Bony [1] proved that there is only one sensible notion of wave front set encoding (micro-)local analyticity. Two particular approaches which may be found equivalent in the $\mathscr{C}^\infty$-setting as well are the formulation of the $\mathscr{C}^\infty$-wave front set by means of pseudo-differential analysis, in terms of cut-offs and

Communicated by Hans G. Feichtinger.

✉ René M. Schulz
rschulz@math.uni-hannover.de

1 Institut für Analysis, Leibniz Universität Hannover, Welfenplatz 1, 30167 Hannover, Germany

by means of the FBI transform as originally considered by Bros and Iagolnitzer [2] and the wave package transform [5].

To quickly recall the classical definitions and preliminary results, a distribution $u \in \mathscr{D}'(\mathbb{R}^d)$ is microlocally $\mathscr{C}^\infty$-regular at some $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$, i.e. $(x_0, \xi_0) \notin \mathrm{WF}_{\mathrm{cl}}(u)$, if one of the following equivalent criteria is met:

(1) There exists a pseudo-differential operator A with symbol in the Hörmander class $a \in S^0(\mathbb{R}^d \times \mathbb{R}^d)$ non-characteristic at (x_0, ξ_0) such that $Au \in \mathscr{C}^\infty(\mathbb{R}^d)$.
(2) There exists a cut-off $\phi^{x_0} \in \mathscr{C}_c^\infty(\mathbb{R}^d)$ with $\phi^{x_0} \equiv 1$ in a neighbourhood of x_0 such that $|\mathcal{F}(\phi^{x_0} u)|$ is of rapid decay in a conic neighbourhood Γ of ξ_0, meaning

$$\sup_{\xi \in \Gamma} |\mathcal{F}(\phi^{x_0} u)(\xi)|\,|\xi|^N < \infty \quad \forall N \in \mathbb{N}_0.$$

(3) There exists a neighbourhood $U \times V$ of (x_0, ξ_0) on which $\left\| \mathscr{F}_\lambda(u)|_{U\times V} \right\|_\infty = \mathcal{O}\left(\lambda^{-\infty}\right)$, where $\mathscr{F}_\lambda(u)(x,\xi) = \langle u, e^{-\lambda|x-\cdot|^2} e^{-i\lambda\xi\cdot} \rangle$ is the (classical) FBI-transform.

The equivalence between (1) $\Leftrightarrow$ (2) was proved in [17], the one with (3)—or in terms of wave package transforms—is by now firmly established (see e.g. [13,20,31]) and was extended to several functional settings for which there are adapted wave front sets, such as Gevrey or Denjoy–Carleman classes (see e.g. [3]), Fourier–Lebesgue-spaces (see e.g. [19] and [26]) as well as Sobolev spaces (see e.g. [11,14]). The FBI transform may even be generalized to manifolds, see [30].

This paper serves to make public that the equivalence extends to the global setting, meaning to tempered distributions and their singularities, i.e. microlocal deviations of $u \in \mathscr{S}'(\mathbb{R}^d)$ from being an element of $\mathscr{S}(\mathbb{R}^d)$. Such singularities were originally studied in [4,9,21] in terms of a global wave front set here called the SG-wave front set $\mathrm{WF}_{\mathrm{SG}}$ (as opposed to ZF, $\mathrm{WF}_{\mathscr{S}}$ and ${}^{\mathrm{SC}}\mathrm{WF}$ respectively), which is a generalization of the classical wave front set and may be obtained by replacing the Hörmander symbol classes used for testing of regularity by the so-called SG-classes. In fact, $\mathrm{WF}_{\mathrm{SG}}$ generalizes $\mathrm{WF}_{\mathrm{cl}}$, which appears as one of its three components. The other two components encode singularities "at infinity", see [9]. From a perspective of time-frequency analysis , $\mathrm{WF}_{\mathrm{SG}}$ collects all directions in the time-frequency plane in which the signal is not rapidly decaying. The study of such singularities and of associated classes of global Fourier Integral distributions is subject to active research, with several recent contributions, see e.g. [6–8,10,15].

The proof of the equivalence between the FBI-picture and the pseudo-differential approach in this global setting has first been carried out in [27] and is slightly corrected and repeated here in a self-contained form. Since the classical FBI-transform is not designed for a resolution at large x, a generalized notion $\mathscr{F}_{\lambda,\mu}(u)$ needs to be considered. In fact, the second scaling factor μ is introduced in order to scale x to large arguments and to counteract the sharpening of the window in this regime. Figuratively speaking this provides a partition of the time-frequency plane in which the resolution in time and frequency may be adjusted—at infinity—in any way needed to characterize all components of $\mathrm{WF}_{\mathrm{SG}}$.

The classical equivalence (1) $\Leftrightarrow$ (3) has numerous applications. One of the main advantages of the viewpoint provided by (3) is that one obtains a (parameter-dependent) function—and not a distribution—from which the singularities of u may be read off. This often greatly simplifies proofs and reduction-of-order arguments, see e.g. [5]. It is therefore reasonable to assume that the newly-established characterization of $\mathrm{WF}_{\mathrm{SG}}$ will have similar applications in proofs of results involving global singularities. Its main application in [27] was to facilitate the study of operations on distributions, which may often be expressed in explicit form using the FBI transforms of the involved distributions.

Structure of the paper Section 2 is concerned with recalling the basics of SG-calculus (Sect. 2.1), the definition and properties of the SG-wave front set (Sect. 2.2) and a Littlewood-Paley-like construction of a conic localizer by summing up dilated copies of a single cut-off function (Sect. 2.1.2). In Sect. 3 the main result of this paper is achieved, that is the characterization of $\mathrm{WF}_{\mathrm{SG}}(u)$ in terms of a FBI-transform in two parameters. In Sect. 3.1 the classical results are used to obtain such a characterization of two of the three components of $\mathrm{WF}_{\mathrm{SG}}$. In Sect. 3.2 the remaining component is treated. The FBI transforms of certain model distributions are discussed in an Appendix. They provide a visualization of the action of the FBI transform in the time-frequency plane and of the main result.

2 SG-Type Singularities of Tempered Distributions

Notation The standard notation of $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d)$ for the spaces of rapidly decaying functions and tempered distributions is used. $\mathscr{C}^\infty(X)$ denotes the smooth functions on some open $X \subset \mathbb{R}^d$, $\mathscr{C}_c^\infty(X)$ those of compact support.

The symbol $\mathcal{F}u$ or $\widehat{u}$ is used to denote the Fourier transform of $u \in \mathscr{S}'(\mathbb{R}^d)$, normalized to coincide for $f \in \mathscr{S}(\mathbb{R}^d)$ with $\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi} f(x)\,dx$. The symbol $\langle\cdot,\cdot\rangle$ denotes the real pairing between functions and/or distributions, whereas $(\cdot,\cdot)$ is used for the complex pairing, which if both paired elements are in $L^2(\mathbb{R}^d)$ coincides with the standard scalar product.

For two maps $f, g : X \to [0,\infty)$ we write $f \lesssim g$ if there exists $C > 0$ such that $f(x) \leq Cg(x)$ for all $x \in X$, and $f \asymp g$ means that $f \lesssim g$ and $g \lesssim f$.

The Japanese bracket $\langle\cdot\rangle : \mathbb{R}^d \to \mathbb{R}_+$ is given by $\langle x\rangle := \sqrt{1+|x|^2}$, where $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$.

2.1 A Quick Recap of the SG-Calculus

2.1.1 Localization at Infinity

Since we are concerned with singularities "at infinity", we need an efficient way to localize these. For that we pass to a compactification of the base space (see [4,21,22]). In the simple case of $\mathbb{R}^d$ we may consider a directional compactification, namely $\overline{\mathbb{R}^d} := \mathbb{R}^d \sqcup (\mathbb{R}^d \setminus \{0\}/\sim)$ with $x \sim y \Leftrightarrow x = \lambda y$ for $\lambda > 0$, i.e. we attach to $\mathbb{R}^d$ a sphere at infinity. We denote an equivalence class given by some representative x_0 by $x_0\infty$, essentially adopting the notation of [18].

A system of fundamental neighbourhoods on this compactification is given by

- $U (= U \sqcup \emptyset)$, where U is a bounded open ball on $\mathbb{R}^d$,
- $V \sqcup \{x\infty \,|\, x \in V\}$, where V is an open convex cone in $\mathbb{R}^d$.

This amounts to equipping $\mathbb{R}^d$ with a conic structure at infinity. We now wish to localize around a given point $x_0 \in \overline{\mathbb{R}^d}$ by means of cut-offs. A *cut-off* around x_0 is a smooth function $\phi^{x_0} \in \mathscr{C}^\infty(\mathbb{R}^d)$ satisfying

- $|\partial_x^\alpha \phi^{x_0}(x)| \lesssim \langle x \rangle^{-|\alpha|}$ for all $\alpha \in \mathbb{N}_0^d$,
- $\phi^{x_0} \equiv 1$ on the projection to $\mathbb{R}^d$ of some neighbourhood of x_0.

If $x_0 \in \mathbb{R}^d$, ϕ^{x_0} is always assumed to have compact support. If $x_0 \in (\mathbb{R}^d \setminus \{0\}/\sim)$, i.e. $x_0 = y_0\infty$ for some $y_0 \in \mathbb{S}^{d-1}$, then ϕ^{x_0} is called an *asymptotic cut-off*. An asymptotic cut-off around $x_0 = y_0\infty$ may be constructed as follows: take some smooth function $\psi \in \mathscr{C}^\infty(\mathbb{S}^{d-1})$ with $\psi \equiv 1$ in a neighbourhood (in $\mathbb{S}^{d-1}$) of y_0 and vanishing outside a slightly bigger neighbourhood as well as some cut-off $\phi^0 \in \mathscr{C}_c^\infty(\mathbb{R}^d)$ around 0. Then set $\phi^{x_0}(x) := \left(1 - \phi^0\left(\frac{x}{R}\right)\right)\psi\left(\frac{x}{|x|}\right)$ for some $R > 0$. In Sect. 2.1.2 another way to obtain an asymptotic cut-off will be presented.

The space where the microlocal study of singularities takes place is $T^*\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^d$. In the global setting, in order to obtain such representation by means of the SG-calculus, $T^*\mathbb{R}^d$ is compactified in fibre and base space independently, i.e. set

$$\overline{T^*\mathbb{R}^d} := (\mathbb{R}^d \times \mathbb{R}^d) \sqcup \mathrm{W}_{\mathrm{SG}} := \overline{\mathbb{R}^d} \times \overline{\mathbb{R}^d}$$

with

$$\mathrm{W}_{\mathrm{SG}} := \underbrace{\mathbb{R}^d \times ((\mathbb{R}^d \setminus \{0\})/\sim)}_{=:\mathrm{W}_{\mathrm{SG}}^{\psi}} \sqcup \underbrace{((\mathbb{R}^d \setminus \{0\})/\sim) \times \mathbb{R}^d}_{=:\mathrm{W}_{\mathrm{SG}}^{e}} \sqcup \underbrace{((\mathbb{R}^d \setminus \{0\})/\sim) \times ((\mathbb{R}^d \setminus \{0\})/\sim)}_{=:\mathrm{W}_{\mathrm{SG}}^{\psi e}}.$$

Therein, the notation of [12,29] is used, meaning we attach the label "e" to objects associated with large ("exit") behaviour in the (first set of) variables, "ψ" indicates large behaviour in the second factor, the covariable, and "ψe" stands for large behaviour in both independently. We note that these components are labelled 1, 2, 3 in [6–8].

2.1.2 Construction of an Asymptotic Cut-Off by Dilation

A more sophisticated way to obtain an asymptotic cut-off from some cut-off around zero is by means of scaling, more precisely by a construction related to the Littlewood-Paley partition of unity. This will be introduced in the following.

Let $\phi^0 \in \mathscr{C}_c^\infty(\mathbb{R})$ be a positive function supported in $[-1, 1]$ such that for all $x \in [-1, 0]$, $\phi^0(x) + \phi^0(x+1) = 1$ is satisfied.[1] Then, for any given $\gamma > 1$, one may construct $\phi^1(x) = \phi^0(\log_\gamma(x))$ for $x \in (0, \infty)$, which may be trivially extended to all $x \in \mathbb{R}$ since $\phi^1(x) \equiv 0$ for $x < \gamma^{-1}$. Then define, for some $M \in \mathbb{N}_0$ and $k \in \mathbb{Z}$,

$$a_{M,k}^{+\infty}(x) = \sum_{j=k}^{\infty} \gamma^{jM} \phi^1(\gamma^{-j} x).$$

In arbitrary dimensions, one may decompose any $x_0 \in \mathbb{R}^d \setminus \{0\}$ in polar coordinates $x_0 = |x_0| \cdot \frac{x_0}{|x_0|} \in (0, \infty) \times \mathbb{S}^{d-1}$ and take any $\psi \in \mathscr{C}^\infty(\mathbb{S}^{d-1})$ equal to 1 in some neighbourhood of $\frac{x_0}{|x_0|}$ and vanishing outside of a bigger neighbourhood. Then, set

$$a_{M,k}^{x_0}(x) := a_{M,k}^{+\infty}(|x|) \cdot \psi\left(\frac{x}{|x|}\right),$$

smoothly extended by 0 to $x = 0$. If the exact value of k is of no importance, it may be dropped from the notation. One observes that on the support of $\phi^1(\gamma^{-j}|x|)$ one has, for $j \in \mathbb{N}_0$

$$\gamma^{j-1} \leq |x| \leq \langle x \rangle \lesssim |x| \leq \gamma^{j+1}.$$

From this one concludes

(1) $\forall \alpha \in \mathbb{N}_0^d \; \exists C_\alpha > 0$: $|\partial_x^\alpha a_M^{x_0}(x)| \leq C_\alpha \langle x \rangle^{M-|\alpha|}$,
(2) $\exists c > 0, \; R > 0$ such that $|a_M^{x_0}(x)| \geq c\langle x \rangle^M$ for $|x| > R > 0$.

In particular this procedure yields asymptotic cut-offs for $M = 0$.

2.1.3 SG-*Symbols and Pseudo-Differential Operators*

The prototype of a cut-off around any point (x_0, ξ_0) in $\mathrm{W_{SG}}$ is of the form $\phi^{x_0} \otimes \phi^{\xi_0}$. The SG-class of symbols, originally considered by Parenti [25], provides a generalization of such functions. What follows is a minimal introduction to the SG-calculus. For more information, the reader is advised to consider e.g. [12,24,29].

SG-symbols are obtained as the class of symbols in the Weyl-Hörmander calculus associated with the metric $\frac{|dx|^2}{\langle x \rangle^2} + \frac{|d\xi|^2}{\langle \xi \rangle^2}$. More explicitly, $a \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ is called SG-*symbol* of *order* $(m_e, m_\psi) \in \mathbb{R} \times \mathbb{R}$, denoted by $a \in \mathrm{SG}^{m_e, m_\psi}(\mathbb{R}^d \times \mathbb{R}^d)$, if it fulfils estimates of the form

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \lesssim \langle x \rangle^{m_e - |\alpha|} \langle \xi \rangle^{m_\psi - |\beta|} \quad \forall \alpha, \beta \in \mathbb{N}_0^d. \tag{2.1}$$

[1] Such a function may be constructed as follows: take any smooth positive function f supported in $[-1, 0]$ with $\int |f(x)|\, dx = C$. Then set, for $x \in [-1, 0]$, $\phi^0(x) = C^{-1} \int_{-1}^{x} f(x)\, dx$ and $\phi^0(x) = 1 - \phi^0(x-1)$ for $x \in [0, 1]$.

The essential feature of the SG-calculus used in this document is that their notion of ellipticity respects the bi-conic structure at infinity of $\overline{T^*\mathbb{R}^d}$:

An element $a \in \mathrm{SG}^{m_e,m_\psi}(\mathbb{R}^d \times \mathbb{R}^d)$ is called *SG-elliptic* at a point $(x_0, \xi_0) \in \mathrm{W}_{\mathrm{SG}}$ if it also fulfils, in the projection to $\mathbb{R}^d \times \mathbb{R}^d$ of an open neighbourhood $U \times V \subset \overline{\mathbb{R}^d} \times \overline{\mathbb{R}^d}$ of (x_0, ξ_0)

$$|a(x,\xi)| \gtrsim \langle x\rangle^{m_e}\langle \xi\rangle^{m_\psi} \quad \forall (x,\xi) \in U \times V. \tag{2.2}$$

Consequently, a symbol is called *SG-elliptic* if it is elliptic on all of W_{SG}, which is equivalent to the existence of some $R > 0$ such that

$$|a(x,\xi)| \gtrsim \langle x\rangle^{m_e}\langle \xi\rangle^{m_\psi} \quad |x| + |\xi| \geq R. \tag{2.3}$$

Let $t \in [0, 1]$. The t-operator of such a symbol, which interpolates between the left $(t = 0)$ and the right $(t = 1)$ quantization of $a \in \mathrm{SG}^{m_e,m_\psi}(\mathbb{R}^d \times \mathbb{R}^d)$ is the pseudo-differential operator $a_t(x, D)$, which is given by the oscillatory integral

$$a_t(x, D) f = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(x-y)\xi} a\left((1-t)x + ty, \xi\right) f(y)\, dy d\xi$$

for $f \in \mathscr{S}(\mathbb{R}^d)$, which yields continuous maps $\mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$. Furthermore, these operators have good mapping properties with respect to the weighted Sobolev spaces, also called Sobolev-Kato-spaces, $H^{s_e,s_\psi}(\mathbb{R}^d) := \langle x\rangle^{-s_e} H^{s_\psi}(\mathbb{R}^d)$, for $s_e, s_\psi \in \mathbb{R} \times \mathbb{R}$. Therein, $H^{s_\psi}(\mathbb{R}^d)$ is the usual Sobolev space $\langle D\rangle^{-s_\psi} L^2(\mathbb{R}^d)$. Indeed, $a_t(x, D)$ maps

$$H^{s_e,s_\psi}(\mathbb{R}^d) \to H^{s_e-m_e,s_\psi-m_\psi}(\mathbb{R}^d).$$

Furthermore, elliptic regularity holds, namely if a is elliptic, then

$$a_t(x, D)u \in H^{s_e,s_\psi}(\mathbb{R}^d) \Leftrightarrow u \in H^{s_e+m_e,s_\psi+m_\psi}(\mathbb{R}^d).$$

This is of particular use since

$$\bigcup_{\mathbb{R}\times\mathbb{R}} H^{s_e,s_\psi}(\mathbb{R}^d) = \mathscr{S}'(\mathbb{R}^d) \quad \text{and} \quad \bigcap_{\mathbb{R}\times\mathbb{R}} H^{s_e,s_\psi}(\mathbb{R}^d) = \mathscr{S}(\mathbb{R}^d).$$

2.2 The SG-Wave Front Set of a Distribution

A first set encoding the global singularities of a tempered distribution is the *cone singular support*, see [23]. For $u \in \mathscr{S}'(\mathbb{R}^d)$ define

$$\mathrm{Css}(u) := \overline{\mathbb{R}^d} \setminus \{x_0 \in \overline{\mathbb{R}^d} \mid \exists \phi^{x_0} \text{ s.t. } \phi^{x_0} u \in \mathscr{S}(\mathbb{R}^d)\},$$

where ϕ^{x_0} is a cut-off around x_0 in the sense of the previous sections. Note that the projection of this set to $\mathbb{R}^d$ is the ordinary singular support and that, by a partition of unity argument, $\mathrm{Css}(u) = \emptyset$ if and only if $u \in \mathscr{S}(\mathbb{R}^d)$. $\mathrm{Css}(u)$ is, however, not simply the “asymptotic closure” of singsupp(u), as the example $\mathrm{Css}(\mathbb{1}) = ((\mathbb{R}^d \setminus \{0\})/\sim)$ with singsupp$(\mathbb{1}) = \emptyset$ demonstrates.

A generalization of this notion of singular support is given by the SG-wave front, studied in different approaches by various authors (see [4,9,21]), which is by now an established tool to formulate and study singularities “at infinity” of tempered distributions. In the following, some of its properties will be recalled to lay the foundation for the main result. For a broader presentation, the reader is advised to consider [9,23,27].

Definition 2.1 (SG-wave front set) A distribution $u \in \mathscr{S}'(\mathbb{R}^d)$ is SG-*microlocally* $\mathscr{S}$*-regular* at $(x_0, \xi_0) \in \mathrm{W_{SG}}$ if there exists $a \in \mathrm{SG}^{0,0}(\mathbb{R}^d)$, SG-elliptic at (x_0, ξ_0), such that $a_t(x, D)u \in \mathscr{S}(\mathbb{R}^d)$ for some $t \in [0, 1]$.

The complement $\mathrm{WF_{SG}}(u)$ of all such points in $\mathrm{W_{SG}}$ where u is SG-microlocally $\mathscr{S}$-regular is called the SG-*wave front set* of u. One further calls, for $\bullet \in \{e, \psi, \psi e\}$, the components $\mathrm{WF}^{\bullet}_{\mathrm{SG}}(u) := \mathrm{WF_{SG}} \cap \mathrm{W}^{\bullet}_{\mathrm{SG}}$ the $\bullet$-wave front set.

Remark 1 The independence on t (cf. [6]) stems from the fact that $a_t(x, D) = (e^{i(t-t')D_x D_\xi} a)_{t'}(x, D)$, where the operator $e^{i(t-t')D_x D_\xi}$ preserves ellipticity, see [24].

The notion of $\mathrm{WF_{SG}}$ admits the following properties:

Proposition 2.2 *Let* $u \in \mathscr{S}'(\mathbb{R}^d)$.

(1) $\mathrm{WF_{SG}}(u)$ *is a closed subset of* $\mathrm{W_{SG}}$.
(2) $\mathrm{pr}_1(\mathrm{WF_{SG}}(u)) = \mathrm{Css}(u)$, *where* pr_1 *is the projection on the first set of variables. In particular* $u \in \mathscr{S}(\mathbb{R}^d) \Leftrightarrow \mathrm{WF_{SG}}(u) = \emptyset$.
(3) $\mathrm{WF}^{\psi}_{\mathrm{SG}}(u)$ *coincides with the Hörmander* $\mathscr{C}^\infty$*-wave front set* $\mathrm{WF_{cl}}(u)$.
(4) *For any* $(x_0, \xi_0) \in \mathrm{W_{SG}}$ *it holds that*

$$(x_0, \xi_0) \in \mathrm{WF_{SG}}(u) \Leftrightarrow (\xi_0, -x_0) \in \mathrm{WF_{SG}}(\mathcal{F}u). \tag{2.4}$$

It was noticed in [9] that, similarly to $\mathrm{WF_{cl}}(u)$, the SG-wave front set may as well be characterized by suitable (asymptotic) cut-offs:

Proposition 2.3 *Let* $u \in \mathscr{S}'(\mathbb{R}^d)$, $(x_0, \xi_0) \in \mathrm{W_{SG}}$. *Then* $(x_0, \xi_0) \notin \mathrm{WF_{SG}}(u)$ *if and only if there exist two (asymptotic) cut-offs* ϕ^{x_0} *and* ϕ^{ξ_0} *around* x_0 *and* ξ_0 *respectively such that* $\phi^{\xi_0}(D)(\phi^{x_0}u) \in \mathscr{S}(\mathbb{R}^d)$. *To be precise:*

- *If* $(x_0, \xi_0) \in \mathrm{W}^e_{\mathrm{SG}}$, *then* ϕ^{x_0} *is an asymptotic cut-off and* ϕ^{ξ_0} *a cut-off,*
- *If* $(x_0, \xi_0) \in \mathrm{W}^{\psi}_{\mathrm{SG}}$, *then* ϕ^{x_0} *is a cut-off and* ϕ^{ξ_0} *an asymptotic cut-off,*
- *If* $(x_0, \xi_0) \in \mathrm{W}^{\psi e}_{\mathrm{SG}}$, *then* ϕ^{x_0} *and* ϕ^{ξ_0} *are both asymptotic cut-offs.*

Remark 2 By a change of quantization, the order of the cut-offs may also be exchanged.

3 Characterization of the SG-Wave Front Set by FBI Fransforms

3.1 Characterization of $\mathrm{WF}^{e}_{\mathrm{SG}}$ and $\mathrm{WF}^{\psi}_{\mathrm{SG}}$

In the following, a characterization of this wave front set in terms of a generalized FBI-transform, parametrized by two parameters λ and μ, is established. This FBI transform may be easily formulated in terms of the L^2-isometries (extended to $\mathscr{S}'(\mathbb{R}^d)$ by duality) defined for $f \in \mathscr{S}(\mathbb{R}^d)$ (wherein $y \in \mathbb{R}^d$):

Translation by $x_0 \in \mathbb{R}^d$	$\mathcal{T}_{x_0} f(y)$	$= f(y - x_0)$,
Modulation by $\xi_0 \in \mathbb{R}^d$	$\mathcal{M}_{\xi_0} f(y)$	$= e^{iy\xi_0} f(y)$,
Dilation by $\lambda > 0$	$\mathcal{D}_\lambda f(y)$	$= \lambda^{d/2} f(\lambda y)$.

Definition 3.1 Let $\psi_0 = \pi^{-d/4} e^{-\frac{|x|^2}{2}} \in \mathscr{S}(\mathbb{R}^d)$. The *FBI*-transform $\mathscr{F}_{\lambda,\mu}(u)$ of $u \in \mathscr{S}'(\mathbb{R}^d)$ is the map from $\mathbb{R}^d \times \mathbb{R}^d \times [1,\infty)^2$ to $\mathbb{C}$ given by

$$(x, \xi, \lambda, \mu) \mapsto \mathscr{F}_{\lambda,\mu}(u)(x, \xi) := (2\pi)^{-d/2} \mathcal{D}_{\sqrt{\lambda\mu}} \left(u, \mathcal{D}_{\sqrt{\lambda/\mu}} \mathcal{T}_x \mathcal{M}_\xi \psi_0 \right).$$

Remark 3 Note that in the definition of the FBI-transform, the "inner" dilation $\mathcal{D}_{\sqrt{\lambda/\mu}}$ is with respect to the argument of ψ_0 (denoted y below), whereas the "outer" one, $\mathcal{D}_{\sqrt{\lambda\mu}}$, is the $2d$-dimensional dilation with respect to (x, ξ).

For $u \in \mathscr{S}(\mathbb{R}^d)$, this FBI-transform may thus be expressed as the integral

$$\mathscr{F}_{\lambda,\mu} u(x, \xi) = (2\pi^{3/2})^{-d/2} \mu^{d/4} \lambda^{3d/4} \int u(y)\, e^{-\frac{\lambda}{2\mu}|y-\mu x|^2} e^{-i\lambda(y-\mu x)\xi}\, dy. \quad (3.1)$$

$\mathscr{F}_{\lambda,\mu}(u)$ coincides with the convolution $\left(\mathcal{D}_\mu \left(u * k_{\lambda,\mu,\xi}\right)\right)(x)$ where $k_{\lambda,\mu,\xi} \in \mathscr{S}(\mathbb{R}^d)$ given by

$$\begin{aligned} k_{\lambda,\mu,\xi}(y) &:= (2\pi)^{-d/2} \mu^{d/4} \lambda^{3d/4} \psi_0 \left(\sqrt{\lambda/\mu}\, y \right) e^{i\lambda y \xi} \\ &= (2\pi)^{-d/2} \mu^{d/2} \lambda^{d/2} \mathcal{M}_{\lambda\xi} \mathcal{D}_{\sqrt{\lambda/\mu}} \psi_0 (y). \end{aligned} \quad (3.2)$$

From continuity of $u \in \mathscr{S}'(\mathbb{R}^d)$ one immediately gathers the following:

Lemma 3.2 *Let $u \in \mathscr{S}'(\mathbb{R}^d)$. Then $\mathscr{F}_{\lambda,\mu} u$ is a continuous map in all arguments and is locally—in (x_0, ξ_0)—bounded by $\lambda^N \mu^M$ for some $N, M \in \mathbb{N}_0^2$, i.e. that for all (x_0, ξ_0) there exists some open neighbourhood $U \times V$ of (x_0, ξ_0) such that $\left|\mathscr{F}_{\lambda,\mu}(u)|_{U\times V}\right| = \mathcal{O}\left(\lambda^N \mu^M\right)$ for $\lambda \geq 1$, $\mu \geq 1$.*

Remark 4 This FBI-transform in two parameters is closely related to the one used in [20], denoted $T_\mu u(x, \xi; h)$. In fact it is simply a dilated version

$$\mu^{-d/2} (T_{\mu^{-1}} u)(\mu x, \xi; \lambda^{-1}) = (\mathscr{F}_{\lambda,\mu} u)(x, \xi). \quad (3.3)$$

For $\mu = \lambda$ one further obtains (up to normalization) the h-dependent Short-time Fourier transform of [28] used to characterize the so-called Gabor-wave front set, which may also be introduced in terms of the Shubin symbol classes, see [27].

Using $\mathrm{WF}^{\psi}_{\mathrm{SG}}(u) = \mathrm{WF}_{\mathrm{cl}}(u)$, one may state the by now well-known characterization of the wave front set in terms of the FBI transform (see e .g. [11,14,16,19,20]) for the ψ-component of $\mathrm{WF}^{\psi}_{\mathrm{SG}}(u)$ in the following form, wherein the behaviour in μ is due to same reasoning as in [20, Proposition 3.2.5], using (3.3).

Proposition 3.3 *Let $u \in \mathscr{S}'(\mathbb{R}^d)$ and $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$. Then $(x_0, \xi_0\infty) \notin \mathrm{WF}^{\psi}_{\mathrm{SG}}(u)$ if and only if there exists a (bounded) open neighbourhood $U \subset \mathbb{R}^d \times \mathbb{R}^d$ of $(\mu^{-1}x_0, \xi_0)$ and $R > 0$ such that $\left|\mathscr{F}_{\lambda,\mu}(u)|_U(x, \xi)\right| = \mathcal{O}(\lambda^{-\infty})$ for $\lambda > R$ and fixed $\mu > 0$.*

In order to characterize $\mathrm{WF}^{e}_{\mathrm{SG}}(u)$, we use (2.4). For that we note that by Parseval's identity,

$$\mathscr{F}_{\lambda,\mu}(u)(x, \xi) = (2\pi)^{d/2} e^{i\lambda\mu x\xi} \mathscr{F}_{\mu,\lambda}(\widehat{u})(\xi, -x). \tag{3.4}$$

Proposition 3.3 then yields the following characterization of $\mathrm{WF}^{e}_{\mathrm{SG}}(u)$:

Proposition 3.4 *Let $u \in \mathscr{S}'(\mathbb{R}^d)$ and $(x_0, \xi_0) \in (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^d$. Then $(x_0\infty, \xi_0) \notin \mathrm{WF}^{e}_{\mathrm{SG}}(u)$ if and only if there exists a (bounded) open neighbourhood $U \subset \mathbb{R}^d \times \mathbb{R}^d$ of $(x_0, \lambda^{-1}\xi_0)$ and $R > 0$ such that $\left|\mathscr{F}_{\lambda,\mu}(u)|_U(x, \xi)\right| = \mathcal{O}(\mu^{-\infty})$ for $\mu > R$ and fixed $\lambda > 0$.*

3.2 Characterization of the Corner Component

This section is concerned with the main result of this paper, meaning an analogous characterization of the corner component of $\mathrm{WF}_{\mathrm{SG}}$, that is $\mathrm{WF}^{\psi e}_{\mathrm{SG}}$. As one may expect from Propositions 3.3 and 3.4, it is obtained in the region where μ and λ are both large. The main result of this section will be:

Theorem 3.5 *Let $u \in \mathscr{S}'(\mathbb{R}^d)$ and $(x_0, \xi_0) \in (\mathbb{R}^d \setminus \{0\}) \times (\mathbb{R}^d \setminus \{0\})$. Then $(x_0\infty, \xi_0\infty) \notin \mathrm{WF}^{\psi e}_{\mathrm{SG}}(u)$ if and only if there exists a (bounded) open neighbourhood $U \times V \subset \mathbb{R}^d \times \mathbb{R}^d$ of (x_0, ξ_0) such that for all $M, N \in \mathbb{N}_0$ and some $R > 0$ fixed we have*

$$\left|\mathscr{F}_{\lambda,\mu}(u)|_{U\times V}\right| = \mathcal{O}\left(\lambda^{-N}\mu^{-M}\right), \qquad \lambda > R, \quad \mu > R. \tag{3.5}$$

Remark 5 Notice that (3.5) may be rephrased as

$$\left|\mathscr{F}_{\lambda,\mu}(u)|_{U\times V}\right| = \mathcal{O}\left(\lambda^{-\infty}\right)\mathcal{O}\left(\mu^{-\infty}\right), \qquad \lambda > R, \quad \mu > R.$$

Theorem 3.5 will be proved in several steps. First, one handles the cases where μ and λ are large in case u is locally $\mathscr{S}$-regular. The first lemma corresponds to the fact that $u \in \mathscr{S}(\mathbb{R}^d) \Rightarrow \mathrm{WF}^{\psi e}_{\mathrm{SG}} = \emptyset$.

Lemma 3.6 *Let $u \in \mathscr{S}(\mathbb{R}^d)$. Then for all $(x_0, \xi_0) \in (\mathbb{R}^d \setminus \{0\}) \times (\mathbb{R}^d \setminus \{0\})$ there exists an open neighbourhood $U \times V$ of (x_0, ξ_0) such that for all $N, M \in \mathbb{N}_0$ and some $R > 1$, we have*

$$\left|\mathscr{F}_{\lambda,\mu}(u)|_{U\times V}\right| = \mathcal{O}\left(\lambda^{-N}\mu^{-M}\right) \qquad \lambda > R \quad \mu > R. \tag{3.6}$$

Proof Because of Fourier symmetry, (3.4), we may assume $\mu \geq \lambda$.

For arbitrary $N \in \mathbb{N}_0$ we may estimate in a neighbourhood of $x_0 \neq 0$

$$\begin{aligned}
\mu^N |\mathscr{F}_{\lambda,\mu} u(x,\xi)| &\lesssim \mu^{N+d} \int |u(y)|\, e^{-\frac{\lambda}{2\mu}|y-\mu x|^2}\, dy \\
&\lesssim \mu^{N+d} \int |u(y)|\, e^{-\frac{1}{2}|\sqrt{\mu}^{-1} y - \sqrt{\mu} x|^2}\, dy \\
&\lesssim \mu^{N+d} \int |u(y)|\, \langle \sqrt{\mu}^{-1} y - \sqrt{\mu} x\rangle^{-2N-2d}\, dy \\
&\lesssim \mu^{N+d} \int |u(y)|\, \langle \sqrt{\mu}^{-1} y\rangle^{2N+2d} \langle \sqrt{\mu} x\rangle^{-2N-2d}\, dy \\
&\lesssim \int |u(y)|\, \langle y\rangle^{2N+2d}\, dy < \infty,
\end{aligned}$$

using the rapid decay of the Gaussian and Peetre's inequality $\langle x+y\rangle^z \lesssim \langle x\rangle^z \langle y\rangle^{|z|}$. □

The second lemma corresponds to the refined statement that if some $x_0 \notin \mathrm{Css}(u)$, then for all ξ_0 then $(x_0\infty, \xi_0\infty) \notin \mathrm{WF}^{\psi e}_{\mathrm{SG}}$.

Lemma 3.7 *Let $u \in \mathscr{S}'(\mathbb{R}^d)$. If for $x_0 \in \mathbb{R}^d \setminus \{0\}$ it holds that $x_0\infty \notin \mathrm{Css}(u)$, then for any $\xi_0 \in \mathbb{R}^d \setminus \{0\}$, there exists an open neighbourhood $U \times V$ of (x_0, ξ_0) such that for all $N, M \in \mathbb{N}_0$ and some $R > 1$, we have*

$$\left|\mathscr{F}_{\lambda,\mu}(u)|_{U\times V}\right| = \mathcal{O}\left(\lambda^{-N}\mu^{-M}\right) \qquad \lambda > R \quad \mu > R. \tag{3.7}$$

Sketch of proof. One may reduce to the case in which u vanishes in (the projection to $\mathbb{R}^d$ of) some open neighbourhood of $x_0\infty$ by writing $u = (1-\phi^{x_0\infty})u + \phi^{x_0\infty}u$ for some asymptotic cut-off around $x_0\infty$ and by applying Lemma 3.6.

We may thus assume that u vanishes around suitably large arguments in an open conic neighbourhood of x_0. Continuity of u yields that for some $N \in \mathbb{N}_0$

$$|\langle u,\ f\rangle| \lesssim \sup_{y\in \mathrm{supp}(u)} \sum_{|\alpha|+|\beta|\leq N} \langle y\rangle^{|\alpha|} |\partial_y^\beta f(y)|. \tag{3.8}$$

This may now be used to estimate $\left|\mathscr{F}_{\lambda,\mu}(u)|_{U\times V}\right|$, i.e. $\left|\mathcal{D}_{\sqrt{\lambda\mu}}\left(u, \mathcal{D}_{\sqrt{\lambda/\mu}} \mathcal{T}_x \mathcal{M}_\xi \psi_0\right)\right|$. In fact, for x_0 as in the statement and x in a suitably small neighbourhood of x_0, μx

will be in a conic neighbourhood of x_0 separated from supp(u) outside some compact set and one may deduce from this, by scaling, the estimate

$$|y-\mu x|^2 \gtrsim (|y|^2+|\mu x|^2)$$

for y in the support of u, and $\mu > R$ for some $R > 0$. One may now estimate the term arising from (3.8) with $|\alpha|+|\beta| = 0$ by (cf. (3.1))

$$\sup_{y\in\mathrm{supp}(u)} |e^{-\frac{\lambda}{2\mu}|y-\mu x|^2}| \lesssim \sup_{y\in\mathrm{supp}(u)} e^{-\epsilon\frac{\lambda}{2\mu}(|y|^2+\mu^2|x|^2)} \lesssim e^{-\epsilon'\mu\lambda|x_0|^2}.$$

The remaining terms involving finite derivatives and products with $\langle x\rangle$ are estimated similarly, using Peetre's inequality and rapid decay of the Gaussian. □

We are now in the situation to prove Theorem 3.5:

Proof of Theorem 3.5 Let $(x_0,\xi_0) \in (\mathbb{R}^d\setminus\{0\})\times(\mathbb{R}^d\setminus\{0\})$ such that $(x_0\infty,\xi_0\infty) \notin \mathrm{WF}_{\mathrm{SG}}(u)$. By the characterization of $\mathrm{WF}_{\mathrm{SG}}$ in Proposition 2.3, one may find two asymptotic cut-offs ϕ^{x_0}, ϕ^{ξ_0} such that

$$\phi^{\xi_0}(D)\left(\phi^{x_0}u\right) \in \mathscr{S}(\mathbb{R}^d). \tag{3.9}$$

and $\phi^{x_0}\otimes\phi^{\xi_0}$ is SG-elliptic at $(x_0\infty,\xi_0\infty)$. We first write $\mathscr{F}_{\lambda,\mu}(u) = \mathscr{F}_{\lambda,\mu}((1-\phi^{x_0})u) + \mathscr{F}_{\lambda,\mu}(\phi^{x_0}u)$. Since $x_0 \notin \mathrm{Css}\big((1-\phi^{x_0})u\big)$, one may reduce the analysis to $\phi^{x_0}u$, by Lemma 3.7. In this case, (3.9) may be rewritten as $\xi_0 \notin \mathrm{Css}(\mathcal{F}\phi^{x_0}u)$ and the assertion follows from Lemma 3.7 as well as (3.4). This establishes the first implication.

The second half of the proof is inspired by that in [11] , see also [14]. We construct a pseudo-differential operator acting on and regularizing u by suitably summing up of cut-offs as in Sect. 2.1.2. By assumption, there is an open neighbourhood $U\times V \subset \mathbb{R}^d\times\mathbb{R}^d$ of (x_0,ξ_0) such that for all $N, M \in \mathbb{N}_0^2$ and some $R > 0$ fixed (3.5) holds. Possibly reducing to a smaller set, assume that U and V are conic segments in the sense that they may be written (in polar coordinates) as $U = (\gamma^k,\gamma^{k+1})\cdot\tilde{U}$, $V = (\gamma^j,\gamma^{j+1})\cdot\tilde{V}$ with $\tilde{U},\tilde{V} \subset \mathbb{S}^{d-1}$ for some $\gamma > 1$, $k, j \in \mathbb{Z}$.

One may treat $\mathscr{F}_{\lambda,\mu}(u)$ as an element in $\mathscr{C}^\infty(\mathbb{R}^{2d})\cap\mathscr{S}'(\mathbb{R}^{2d})$ and calculate, for arbitrary $N, M \in \mathbb{N}_0^2$, the following pairing with two arbitrary cut-offs in $\mathscr{C}_c^\infty$, ϕ^{x_0} and ϕ^{ξ_0}, supported in U and V respectively,

$$\begin{aligned}\left\langle\mathscr{F}_{\lambda,\mu}(u)(x,\cdot)\,,\,\lambda^N\mu^M\phi^{x_0}(x)\phi^{\xi_0}\right\rangle_\xi &:= \lambda^N\mu^M\int\mathscr{F}_{\lambda,\mu}(u)(x,\xi)\phi^{\xi_0}(\xi)\phi^{x_0}(x)d\xi\\ &= \lambda^N\mu^M\int\mathcal{F}^{-1}_{\eta\to x}\left\{\mathcal{F}_{y\to\eta}\mathscr{F}_{\lambda,\mu}(u)(y,\xi)\right\}\phi^{\xi_0}(\xi)\phi^{x_0}(x)d\xi\end{aligned}$$

The integral converges since ϕ^{ξ_0} is compactly supported. Writing $\mathscr{F}_{\lambda,\mu}(u)$ as the convolution with $k_{\lambda,\mu,\xi}$ we may express this as

$$\begin{aligned}
&\left\langle \mathscr{F}_{\lambda,\mu}(u)(x,\cdot)\,,\ \lambda^N\mu^M\phi^{x_0}(x)\phi^{\xi_0}\right\rangle_\xi \\
&\quad= \lambda^N\mu^M \int \mathcal{F}^{-1}_{\eta\to x}\left(\mathcal{D}_{\mu^{-1}}\left(\widehat{u}(\eta)\,\widehat{k_{\lambda,\mu,\xi}}(\eta)\right)\right)\phi^{\xi_0}(\xi)\phi^{x_0}(x)d\xi \\
&\quad= \lambda^N\mu^M \int \mathcal{D}_\mu\left(\phi^{x_0}(\mu^{-1}x)\mathcal{F}^{-1}_{\eta\to x}\left(\widehat{u}(\eta)\,\widehat{k_{\lambda,\mu,\xi}}(\eta)\right)\right)\phi^{\xi_0}(\xi)d\xi \\
&\quad=: \mathcal{D}_\mu\left(\mathcal{F}^{-1}_{\eta\to x}\left(a^{N,M}_{\lambda,\mu}(x,\eta)\widehat{u}(\eta)\right)\right). \qquad (3.10)
\end{aligned}$$

Note that therein, the dilation $\mathcal{D}_\mu$ is with respect to the x-variable. The latter expression is the dilation of a pseudo-differential operator applied to u with the operator given by the symbol (see (3.2))

$$\begin{aligned}
a^{N,M}_{\lambda,\mu}(x,\eta) &= \lambda^N\mu^M \int \phi^{x_0}\left(\mu^{-1}x\right)\widehat{k_{\lambda,\mu,\xi}}(\eta)\phi^{\xi_0}(\xi)d\xi \\
&= (2\pi)^{-d/2}\lambda^N\mu^M \int \phi^{x_0}\left(\mu^{-1}x\right)\mathcal{F}\left(\lambda^{d/2}\mathcal{M}_{\lambda\xi}\mathcal{D}_{\left(\frac{\lambda}{\mu}\right)^{1/2}}\psi_0\right)(\eta)\phi^{\xi_0}(\xi)d\xi \\
&= \lambda^{N-3d/4}\mu^{M+d/4}\int \phi^{x_0}\left(\mu^{-1}x\right)\psi_0\left(\left(\frac{\mu}{\lambda}\right)^{1/2}(\eta+\xi)\right)\phi^{\xi_0}(\lambda^{-1}\xi)d\xi \qquad (3.11)
\end{aligned}$$

Specifying $\lambda = \gamma^n$ and $\mu = \gamma^m$, by (3.5) one may estimate

$$\begin{aligned}
\infty &> \sum_{m=k}^\infty\sum_{n=j}^\infty\left(\int\left|\left\langle \mathscr{F}_{\gamma^n,\gamma^m}(u)(x,\cdot),\,\gamma^{nN}\gamma^{mM}\phi^{x_0}(x)\phi^{\xi_0}\right\rangle_\xi\right|^2dx\right)^{1/2} \\
&\overset{(3.10)}{=} \sum_{m=k}^\infty\sum_{n=j}^\infty\left(\int\left|\mathcal{D}_{\gamma^m}\left(\mathcal{F}^{-1}_{\eta\to x}\left(a^{N,M}_{\gamma^n,\gamma^m}(x,\eta)\widehat{u}(\eta)\right)\right)\right|^2dx\right)^{1/2} \\
&= \sum_{m=k}^\infty\sum_{n=j}^\infty\left(\int\left|\mathcal{F}^{-1}_{\eta\to x}\left(a^{N,M}_{\gamma^n,\gamma^m}(x,\eta)\widehat{u}(\eta)\right)\right|^2dx\right)^{1/2} \\
&\geq \left(\int\left|\sum_{m=k}^\infty\sum_{n=j}^\infty\mathcal{F}^{-1}_{\eta\to x}\left(a^{N,M}_{\gamma^n,\gamma^m}(x,\eta)\widehat{u}(\eta)\right)\right|^2dx\right)^{1/2}. \qquad (3.12)
\end{aligned}$$

One is thus led to study $\sum_{m=k}^\infty\sum_{n=j}^\infty\mathcal{F}^{-1}_{\eta\to x}\left(a^{N,M}_{\gamma^n,\gamma^m}(x,\eta)\widehat{u}(\eta)\right)$. This is a pseudo-differential operator applied to u, with symbol which by (3.11) has the symbol

$$a^{N,M}(x,\eta) := \sum_{m=k}^{\infty}\sum_{n=j}^{\infty}\gamma^{n(N-3d/4)+m(M+d/4)}\int \phi^{x_0}\left(\gamma^{-m}x\right)\psi_0\left(\gamma^{\frac{m-n}{2}}(\eta+\xi)\right)\phi^{\xi_0}\left(\gamma^{-n}\xi\right)d\xi \tag{3.13}$$

If we now specify ϕ^{x_0} and ϕ^{ξ_0} as in the construction of Sect. 2.1.2, we have on the support of $\phi^{x_0}\left(\frac{x}{2^m}\right)\phi^{\xi_0}\left(\frac{\xi}{2^l}\right)$ the bounds $\gamma^m \asymp \langle x\rangle$ and $\gamma^n \asymp \langle\xi\rangle$ jointly over all (n,m). It is thus possible to find asymptotic cut-offs $\phi^{x_0\infty}$ and $\phi^{\xi_0\infty}$ such that all $\phi^{x_0}\left(\frac{x}{2^m}\right)\phi^{\xi_0}\left(\frac{\xi}{2^l}\right)$ are supported in their respective cone supports. We may estimate (3.13) from above and below (for some $c>0$) by

$$\tilde{a}^{N,M}(x,\eta) := \int \langle x\rangle^{M+d/4}\langle\xi\rangle^{N-3d/4}\phi^{x_0\infty}(x)\,\psi_0\left(c\left(\frac{\langle x\rangle}{\langle\xi\rangle}\right)^{1/2}(\eta+\xi)\right)\phi^{\xi_0\infty}(\xi)\,d\xi \tag{3.14}$$

for some cut-offs $\phi^{x_0\infty}$ and $\phi^{\xi_0\infty}$.

(1) One first obtains that $|a^{N,M}(x,\xi)| \lesssim \langle x\rangle^{M+d/4}\langle\eta\rangle^{N-3d/4}$, using decay properties of the Gaussian and Peetre's in equality in (3.14).

(2) Similarly one estimates that $a^{N,M}(x,\eta)\in \mathrm{SG}^{M,N}(\mathbb{R}^d\times\mathbb{R}^d)$. For that, observe that the derivatives of $a^{N,M}$ lead, by partial integration, to expressions of the form

$$|\partial_x^\alpha\partial_\eta^\beta a^{N,M}(x,\eta)| \lesssim \sum_{m=R}^{\infty}\sum_{l=R}^{\infty}(-1)^{|\alpha|}\gamma^{n(N-3d/4-|\beta|)+m(M+d/4-|\alpha|)} \times\left|\int(\partial^\alpha\phi^{x_0})\left(\gamma^{-m}x\right)\psi_0\left(\gamma^{\frac{m-l}{2}}(\eta+\xi)\right)(\partial^\beta\phi^{\xi_0})\left(\gamma^{-n}\xi\right)d\xi\right|.$$

These may be estimated in a similar fashion to (3.13).

(3) Finally, $a^{N,M}$ is elliptic at $(x_0\infty,\xi_0\infty)$. In fact, since the integrand is positive, the integral is bounded from below by integration over any region. Consequently, the integral is bounded from below by $\langle x\rangle^{M+d/4}\langle\eta\rangle^{N-3d/4}$ if η is in a small enough conic neighbourhood of ξ_0 such that $\phi^{x_0\infty}(x)\phi^{\xi_0\infty}(\xi-\eta)\equiv 1$.

By (3.12) we have that $\left(a^{N,M}\right)(x,D)u\in L^2(\mathbb{R}^d)$. That means that we have found a family of symbols such that the associated pseudo-differential operators $A_{N,M}$ map $A_{N,M}u\in L^2(\mathbb{R}^d)$, and such that these $A_{N,M}$ are all elliptic in a fixed neighbourhood of (x_0,ξ_0). Since M and N are arbitrary, one may conclude that, for suitable cut-offs on their joint set of ellipticity, we have for all $M,N\in\mathbb{N}_0$:

$$\phi^{x_0}(x)\phi^{\xi_0}(D)u\in H^{-M-d/4,-N+3d/4}(\mathbb{R}^d),$$

implying $\phi^{x_0}(x)\phi^{\xi_0}(D)u\in\mathscr{S}(\mathbb{R}^d)$ and thus $(x_0\infty,\xi_0\infty)\notin \mathrm{WF}_{\mathrm{SG}}(u)$, which concludes the proof. □

Acknowledgments The author would like to thank Patrik Wahlberg, as well as the anonymous reviewers, for their remarks and corrections which led to numerous improvements of the manuscript. The present article elaborates on certain pieces of the author's thesis [27], which was completed under supervision of Dorothea Bahns at the University of Göttingen. During this time, the author was supported by the Studienstiftung des deutschen Volkes. The author would further like to thank the Leibniz Universität Hannover as well as the Georg-August Universität Göttingen for institutional support.

4 Appendix: The FBI Transform Visualized

Explicitly computable FBI-transforms of certain model distributions may serve as a visualization for Theorem 3.5 as well as the results building up to it. We have (cf. [9])

$$\begin{aligned}
\mathrm{WF}_{\mathrm{SG}}(\delta_{x_0}) &= \mathrm{WF}^{\psi}_{\mathrm{SG}}(\delta_{x_0}) = \{x_0\} \times \{\pm\infty\} && \text{(Dirac delta)},\\
\mathrm{WF}_{\mathrm{SG}}(e^{i\xi_0 x}) &= \mathrm{WF}^{e}_{\mathrm{SG}}(e^{i\xi_0 x}) = \{\pm\infty\} \times \{\xi_0\} && \text{(plane wave)},\\
\mathrm{WF}_{\mathrm{SG}}\big(e^{-\frac{ix^2}{2}}\big) &= \mathrm{WF}^{\psi e}_{\mathrm{SG}}\big(e^{-\frac{ix^2}{2}}\big)\\
&= (\{-\infty\} \times \{-\infty\}) \cup (\{\infty\} \times \{\infty\}) && \text{(chirp)},\\
\mathrm{WF}_{\mathrm{SG}}\big(\mathcal{M}_{x_0}\mathcal{T}_{\xi_0} e^{-\frac{|x|^2}{2}}\big) &= \emptyset && \text{(Gaussian)}.
\end{aligned}$$

The first two identities may be directly read off from

$$\left|\mathscr{F}_{\lambda,\mu}\delta_{x_0}(x,\xi)\right| = (2\pi^{3/2})^{-d/2}\mu^{d/4}\lambda^{3d/4}e^{-\frac{\lambda}{2\mu}|x_0-\mu x|^2}$$

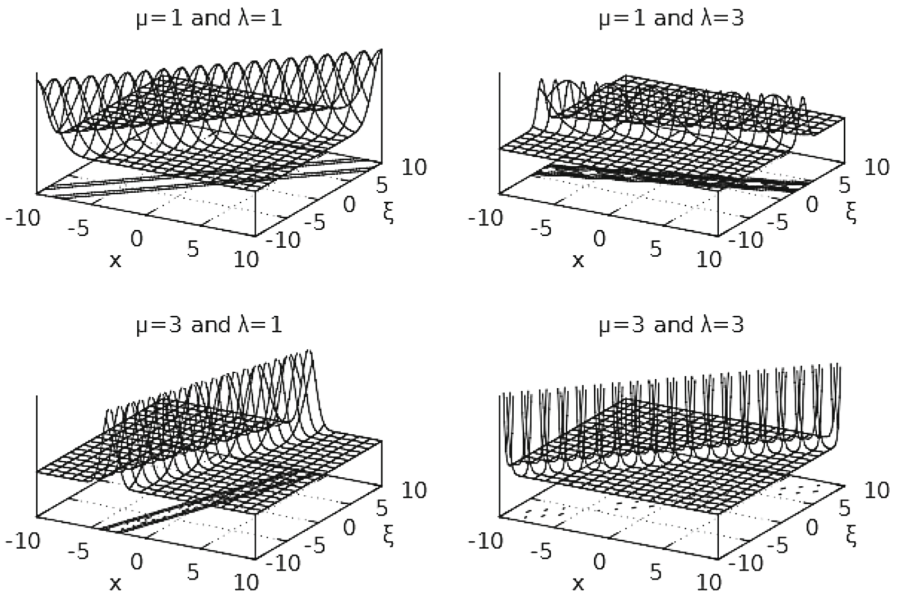

Fig. 1 The transformed chirp for various parameters λ and μ

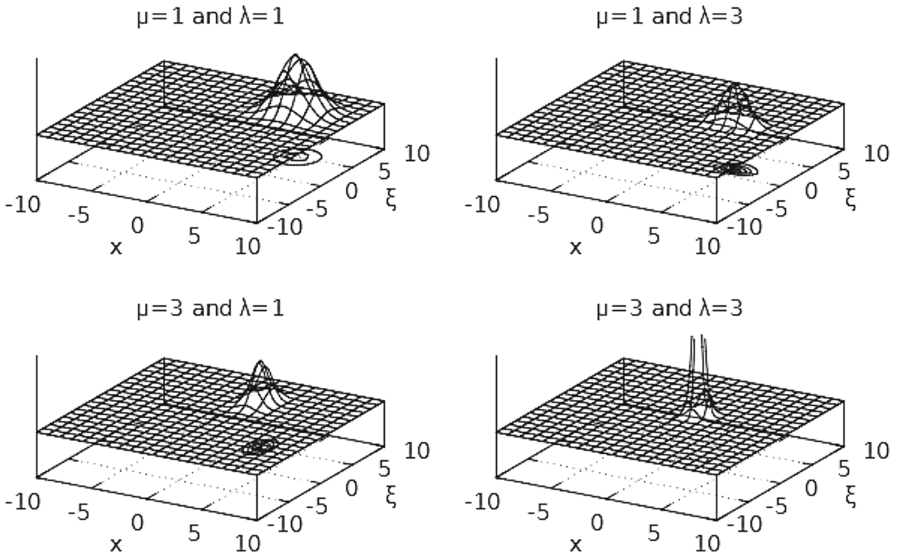

Fig. 2 The transformed Gaussian peak for various parameters λ and μ

and Fourier symmetry (3.4). The third and fourth FBI-transforms may be calculated explicitly as well, and the graphs of absolute value of their transforms are depicted in Figs. 1 and 2 (including level set lines and suitably scaled). These underline how the time-frequency plane is shifted under the action of the generalized FBI transform for various parameters.

References

1. Bony, J.M.: Equivalence des diverses notions de spectre singulier analytique. Séminaire Goulaouic-Schwartz exp. n. 3:1–12 (1976–1977)
2. Bros, J., Iagolnitzer, D.: Support essentiel et structure analytique des distributions. Séminaire Goulaouic-Meyer-Schwartz exp. 18 (1975–1976)
3. Chung, S.Y., Kim, D.: A quasianalytic singular spectrum with res pect to the Denjoy-Carleman class. Nagoya Math. J. **148**, 137–149 (1997)
4. Cordes, H.O.: The Technique of Pseudodifferential Operators. Cambridge University Press, Cambridge (1995)
5. Córdoba, A., Fefferman, C.: Wave packets and Fourier integral operators. Commun. Partial Differ. Equ. **3**(11), 979–1005 (1978)
6. Coriasco, S., Johansson, K., Toft, J.: Global wave-front sets of Banach, Fréchet and modulation space types, and pseudo-differential operators. J. Differ. Equ. **254**(8), 3228–3258 (2013). doi:10.1016/j.jde.2013.01.014
7. Coriasco, S., Johansson, K., Toft, J.: Global wave-front sets of intersection and union type. Fourier Analysis, Trends in Mathematics 2014, vol. 9, pp. 1–106. Springer International Publishing (2014)
8. Coriasco, S., Johansson, K., Toft, J.: Global wave-front properties for Fourier integral operators and hyperbolic problems. J. Fourier. Anal. Appl. (2015). doi:10.1007/s00041-015-9422-1
9. Coriasco, S., Maniccia, L.: Wave front set at infinity and hyperbolic linear operators with multiple characteristics. Ann. Global Anal. Geom. **24**, 375–400 (2003)

10. Coriasco, S., Schulz, R.: The global wave front set of tempered oscillatory integrals with inhomogeneous phase functions. J. Fourier Ana l. Appl. **19**(5), 1093–1121 (2013). doi:10.1007/s00041-013-9283-4
11. Delort, J.-M.: FBI Transformation: Second Microlocalization and SeMilinear Caustics, vol. 1522. Springer, Berlin (1992)
12. Egorov, Y.V., Schulze, B.-W.: Pseudo-Differential Operators, Singularities, Applications. Birkhäuser, Basel (1997)
13. Folland, G.B.: Harmonic Analysis in Phase Space. Princeton University Press, Princeton (1989)
14. Gérard, P.: Moyennisation et régularité deux-microlocale. Ann. scient. Ec. Norm. Sup. 4ème série **23**, 89–121 (1990)
15. Hassel, A., Wunsch, J.: The semiclassical resolvent and the prop agator for non-trapping scattering metrics. Adv. Math. **217**(2), 586–682 (2008)
16. Hörmander, L.: The Analysis of Linear Partial Differential Operators, vol. I. Springer, Berlin (1990)
17. Hörmander, L.: Fourier Integral operators I. Acta Math. **127**(1), 79–183 (1971)
18. Kaneko, A.: On the global existence of real analytic solutions of linear partial differential equations on unbounded domain. J. Fac. Sci. Univ. Tokyo Sec. IA **32**, 319–372 (1985)
19. Kato, K., Kobayashi, M., Ito, S.: Remark on characterization of wave front set by wave packet transform, preprint. arXiv:1408.1370v1 (2014)
20. Martinez, A.: An Introduction to Semiclassical and Microlocal Analysis. Universitext. Springer, New York (2002)
21. Melrose, R.: Spectral and scattering theory for the Laplacian on a symptotically Euclidian spaces. In: Spectral and Scattering Theory", Sanda 1992. Lecture Notes in Pure and Appl. Math., vol. 161, pp. 85–130. Dekker, New York (1994)
22. Melrose, R.: Geometric Scattering Theory, Stanford Lectures. Cambridge University Press, Cambridge (1995)
23. Melrose, R.: Lecture Notes for Graduate Analysis, 2004, available online at http://www-math.mit.edu/~rbm/18.155-F04-notes/Lecture-notes.pdf, last download 17/11/2015
24. Nicola, F., Rodino, L.: Global Pseudo-differential Calculus on Euclidean Spaces. Birkhäuser, Basel (2010)
25. Parenti, C.: Operatori pseudo-differentiali in $\mathbb{R}^n$ e applicazioni. Annali Mat. Pura Appl. **93**, 359–389 (1972)
26. Pilipovic, S., Teofanov, N., Toft, J.: Micro-local analysis with Fourier Lebesgue spaces, Part I. J. Fourier Anal. Appl. **17**:374 -407
27. Schulz, R.: Microlocal Analysis of Tempered Distributions. Diss. Niedersächsische Staats-und Universitätsbibliothek Göttingen (2014)
28. Schulz, R., Wahlberg, P.: The equality of the homogeneous and the Gabor wave front set, preprint. arXiv:1304.7608 (2013)
29. Schulze, B.-W.: Boundary Value Problems and Singular Pseudo-Differential Operators. J. Wiley, Chichester (1998)
30. Wunsch, J., Zworski, M.: The FBI transform on compact C^∞-manifolds. Trans. Am. Math. Soc. **353**(3), 1151–1167 (2001)
31. Zworski, M.: Semiclassical Analysis, Graduate Studies in Mathemat ics **138**. AMS (2012)