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The Bochner–Schoenberg–Eberlein Property for Totally Ordered Semigroup Algebras

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Abstract The concepts of BSE property and BSE algebras were introduced and studied by Takahasi and Hatori in 1990 and later by Kaniuth and Ülger. This abbreviation refers to a famous theorem proved by Bochner and Schoenberg for $L^1(\mathbb{R})$, where $\mathbb{R}$ is the additive group of real numbers, and by Eberlein for $L^1(G)$ of a locally compact abelian group G. In this paper we investigate this property for the Banach algebra $L^p(S, \mu)$ $(1 \leq p < \infty)$ where S is a compact totally ordered semigroup with multiplication $xy = \max\{x, y\}$ and μ is a regular bounded continuous measure on S. As an application, we have shown that $L^1(S, \mu)$ is not an ideal in its second dual.

Keywords BSE algebra · Totally ordered semigroup · Cantor function

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1 Introduction

Let A be a commutative Banach algebra. Denote by $\Delta(A)$ and $\mathcal{M}(A)$ the Gelfand spectrum and the multiplier algebra of A, respectively. A bounded continuous function

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σ on $\Delta(A)$ is called a *BSE-function* if there exists a constant $C > 0$ such that for every finite number of $\varphi_1, \ldots, \varphi_n$ in $\Delta(A)$ and complex numbers $c_1, \ldots, c_n$, the inequality

$$\left|\sum_{j=1}^{n} c_j \sigma(\varphi_j)\right| \leq C \,.\, \left\|\sum_{j=1}^{n} c_j \varphi_j\right\|_{A^*}$$

holds. The BSE-norm of σ ($\|\sigma\|_{BSE}$) is defined to be the infimum of all such C. The set of all BSE-functions is denoted by $C_{BSE}(\Delta(A))$. Takahasi and Hatori [16] showed that under the norm $\|.\|_{BSE}$, $C_{BSE}(\Delta(A))$ is a commutative semisimple Banach algebra.

A bounded linear operator on A is called a *multiplier* if it satisfies $xT(y) = T(xy)$ for all $x, y \in A$. The set $\mathcal{M}(A)$ of all multipliers of A is a unital commutative Banach algebra, called the *multiplier algebra* of A.

For each $T \in \mathcal{M}(A)$ there exists a unique bounded continuous function $\widehat{T}$ on $\Delta(A)$ such that $\widehat{T(a)}(\varphi) = \widehat{T}(\varphi)\hat{a}(\varphi)$ for all $a \in A$ and $\varphi \in \Delta(A)$. See [9] for a proof.

Define

$$\widehat{\mathcal{M}(A)} = \left\{\widehat{T} : T \in \mathcal{M}(A)\right\}.$$

A bounded net $(e_\alpha)_\alpha$ in a Banach algebra A is called a Δ-weak bounded approximate identity for A if $\varphi(e_\alpha) \to 1$ (equivalently, $\varphi(e_\alpha a) \to \varphi(a)$ for every $a \in A$) for all $\varphi \in \Delta(A)$. As is shown in [16], A has a Δ-weak bounded approximate identity if and only if $\widehat{\mathcal{M}(A)} \subseteq C_{BSE}(\Delta(A))$.

A commutative Banach algebra A is called without order if $aA = \{0\}$ implies $a = 0$ $(a \in A)$.

A commutative and without order Banach algebra A is called a BSE-algebra (or has *BSE-property*) if it satisfies the condition

$$C_{BSE}(\Delta(A)) = \widehat{\mathcal{M}(A)}.$$

The abbreviation BSE stands for Bochner–Schoenberg–Eberlein and refers to a famous theorem, proved by Bochner and Schoenberg [2,15] for the additive group of real numbers and in general by Eberlein [5] for a locally compact abelian group G, saying that, in the above terminology, the group algebra $L^1(G)$ is a BSE-algebra (See [12] for a proof).

It is worth to note that the semigroup algebra $l^1(\mathbb{Z}^+)$ (where $\mathbb{Z}^+$ is the additive semigroup of nonnegative integers) is a BSE algebra [17], but for $k \geq 1$, $l^1(\mathbb{N}_k)$ $(\mathbb{N}_k = \{k, k+1, k+2, \ldots\})$ is not a BSE algebra.

In [7], we established affirmatively a question raised by Takahasi and Hatori [16] whether $L^1(\mathbb{R}^+)$ is a BSE-algebra.

The aim of the present paper is to show that for any totally ordered compact semigroup S with multiplication $xy = \max\{x, y\}$ and a regular bounded continuous measure μ on S, $L^p(S, \mu)$ $(1 \leq p < \infty)$ is not a BSE algebra. However, for any compact abelian group G and $1 \leq p < \infty$, the Banach algebra $L^p(G)$ is BSE [18].

As an application, we will show that the Banach $L^1(S,\mu)$ is not an ideal in its second dual. However, for a locally compact group G, the Banach algebra $L^1(G)$ is an ideal in its second dual if and only if G is compact.

Finally, we prove that $C_{BSE(n)}((0,1]) = C_b((0,1])$ for any natural number n.

2 BSE Property of Totally Ordered Semigroup Algebras

The Banach algebra $L^p(S,\mu)$ $(1 \leq p < \infty)$, whenever S is a totally ordered compact space with a regular bounded continuous measure μ on S, was first introduced by Sapounakis [13,14] and then extensively studied more by Baker, Pym and Vasudeva [1].

Recall that a totally ordered locally compact space S is a totally ordered space S which is locally compact in its order topology. This has a natural continuous multiplication $xy = \max\{x, y\}$. For convenience, we adjoin a minimal element 0 to S if it has not already got one. The convolution product $v_1 * v_2$ of two bounded regular measures v_1, v_2 on S is defined (as a linear functional on the space $C_0(S)$ of continuous functions vanishing at infinity on S) by the usual formula

$$\int f dv_1 * v_2 = \int\int f(xy)dv_1(x)dv_2(y) \qquad (f \in C_0(S)).$$

By dividing the range of the inner integral on the right into the sets where $x < y$ and $x \geq y$ we have

$$dv_1 * v_2(x) = v_1[0, x[\ .\ dv_2(x) + v_2[0, x[\ .\ dv_1(x).$$

In particular, if both v_1 and v_2 are absolutely continuous with respect to some positive measure μ, say $dv_1 = f d\mu$ and $dv_2 = g d\mu$, then so is $v_1 * v_2$; and if we put $dv_1 * v_2 = f * g.d\mu$, then

$$f * g(x) = g(x)\int_{[0,x[} f d\mu + f(x)\int_{[0,x]} g d\mu. \qquad (III)$$

By defining the convolution of two measurable functions f and g with respect to μ as in (III), one has the following result.

Proposition 1 *For $p = 1$, $L^p(S,\mu)$ is a Banach algebra. For $1 < p \leq \infty$, $L^p(S,\mu)$ is a Banach algebra if and only if μ is bounded. Moreover, for $1 \leq p \leq \infty$ the algebra $L^p(S,\mu)$ is commutative and semisimple. It has an approximate identity if $1 \leq p < \infty$, which is bounded if and only if $p = 1$.*

Proof See [1]. □

Remark 1 Here we suppose that μ is bounded and it will do no harm to suppose that its total mass is 1. By taking the order completion of S, we may also assume that S is compact, and we lose nothing by taking S to be the support of μ. In [1] it is shown that all L^p-algebras associated with continuous measures of mass 1 with support S are algebra isomorphic to the Banach algebra $L^p([0, 1], \lambda)$, where λ is the Lebesgue measure on the interval [0, 1]. Thus, we need only consider $L^p([0, 1], \lambda)$ and this we shall do (See Theorem 1.1 and page 48 of [1]).Then the Gelfand spectrum $\Delta(L^p(S, \mu))$ of $L^p(S, \mu)$ is equal to the set

$$\{\chi_{[0,x]} : x \in (0, 1]\}$$

which may be identified with the half-open interval (0, 1], and the Gelfand transform $\widehat{f}$ of $f \in L^p(S, \mu)$ is given by

$$\widehat{f}(t) = \int_0^x f d\mu \qquad (t \in [0, 1]),$$

where 0 is the minimal element of S and $\sigma(x) = t$, where σ is a continuous increasing surjection from S to [0, 1] given in the statement of Theorem 1.1 of [1].

Before turning to the next result we need to recall the following definition from [11].

Definition 1 A real-valued function f defined on $[a, b]$ is said to be absolutely continuous on $[a, b]$ if, given $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that

$$\sum_{i=1}^{n} |f(y_i) - f(x_i)| < \epsilon$$

for every finite collection $\{(x_i, y_i)\}$ of non overlapping intervals with

$$\sum_{i=1}^{n} |y_i - x_i| < \delta_\epsilon.$$

A complex-valued function $f = u + iv$ is said to be absolutely continuous if u and v are absolutely continuous.

Theorem 1 (Fundamental Theorem of Lebesgue Integral Calculus). *The following conditions on a real-valued function f on a compact interval $[a, b]$ are equivalent:*

1. *f is absolutely continuous on $[a, b]$.*
2. *There exists a Lebesgue integrable function g on $[a, b]$ such that*

$$f(x) = f(a) + \int_a^x g(t)dt$$

for all $x \in [a, b]$.

Proof See Page 106 of [11]. □

2.1 The Cantor Function

The function $\varphi : [0, 1] \to [0, 1]$ defined by

$$\varphi(x) = \sum_{k=1}^{\infty} \frac{t_{xk}/2}{2^k},$$

whenever the digit 1 does not appear in the ternary expansion of x, and

$$\varphi(x) = \sum_{k=1}^{j_x - 1} \frac{t_{xk}/2}{2^k} + \frac{1}{2^{j_x}},$$

whenever the digit 1 does appear in the ternary expansion of x, and $j_x = \min\{k : t_{xk} = 1\}$, is called the Cantor function.

Proposition 2 *For the Cantor function φ the following statements are valid.*

1. *φ is continuous and of bounded variation.*
2. *φ is not absolutely continuous.*
3. *φ is differentiable almost every where and $\varphi' = 0$ a. e. on* $[0, 1]$.
4. $\int_0^1 \varphi(t)dt = \frac{1}{2}$.

Proof See [3] and [6]. □

Lemma 1 *Let φ be the Cantor function. Then the function g defined by $g(x) = x\varphi(x)$ is not absolutely continuous on* $[0, 1]$.

Proof By Proposition 2, the function g is differentiable almost every where on $[0, 1]$. Therefore

$$g'(x) = x\varphi'(x) + \varphi(x) = \varphi(x) \quad a.e.$$

holds. Assume towards a contradiction that g is absolutely continuous on [0,1]. So by Theorem 1, we get

$$1 = g(1) - g(0) = \int_0^1 g'(t)dt = \int_0^1 \varphi(t)dt = \frac{1}{2}.$$

This contradiction completes the proof. □

Let (X, Σ, μ) be a positive measure space. The space $ba(X, \mu)$ is the Banach space consisting of those bounded and finitely additive signed measures on Σ which vanish on sets of μ-measure zero. The norm of an element in $ba(X, \mu)$ is its total variation. Before stating our next result, we need to recall the following theorem from page 296 of [4].

Theorem 2 *Let (X, Σ, μ) be a σ-finite measure space . There is an isometric isomorphism between $L^\infty(X, \mu)^*$ and $ba(X, \mu)$determined by the identity*

$$F(f) = \int_X f(x)d\mu(x) \qquad (f \in L^\infty(X, \mu)).$$

The following theorem is indeed the main result of this paper.

Theorem 3 *The Banach algebra $L^p(S, \mu)(1 \le p < \infty)$ is not a BSE algebra.*

Proof As in Remark 1, we may assume that $S = ([0, 1], \max)$ and μ is the Lebesgue measure on S.

Case I. $p = 1$. Suppose that σ is a continuous function of bounded variation on $[0, 1]$ and $\sigma(0) = 0$. By Theorem 3.1 of [10], there exists $\nu \in ba(S, \mu)$ such that $\sigma(x) = \nu([0, x])$ for all $x \in [0, 1]$. By Theorem 2, the mapping F given by

$$F(f) = \int_X f(x)d\nu(x) \qquad \big(f \in L^\infty(X, \mu)\big).$$

is in $L^\infty(S, \mu)^*$. In particular,

$$F(\chi_{[0,x]}) = \int_S \chi_{[0,x])}(t)d\nu(t) = \nu([0, x]) = \sigma(x) \qquad (x \in (0, 1]).$$

Therefore $F \in L^\infty(S, \mu)^*(= L^1(S, \mu)^{**})$ and $\sigma|_{(0,1]} = F|_{(0,1]}$. Hence by Theorem 4 (ii) of [16], we have

$$\sigma|_{(0,1]} \in L^1(S, \mu)^{**}|_{\Delta(L^1(S,\mu))(=(0,1])} \bigcap C_b((0, 1]) = C_{BSE}((0, 1]).$$

By Proposition 2 and the above argument, $\varphi|_{(0,1]} \in C_{BSE}((0, 1])$, where φ is the Cantor function on $[0, 1]$. We claim that $\varphi|_{(0,1]} \notin \mathcal{M}(\widehat{L^1(S, \mu)})$. Suppose towards a contradiction that $\varphi|_{(0,1]} \in \mathcal{M}(\widehat{L^1(S, \mu)})$. Then for any $f \in L^1(S, \mu)$, there exists $h_f \in L^1(S, \mu))$ such that

$$\varphi(x)\widehat{f}(x) = \widehat{h_f}(x) \qquad x \in (0, 1].$$

Thus

$$\varphi(x) \int_0^x f(t)d\mu(t) = \int_0^x h_f(t)d\mu(t) \qquad x \in (0, 1],$$

which is an absolutely continuous function by Theorem 1. In particular, for the constant function $f(t) = 1$ $(t \in (0, 1])$, we get

$$x\varphi(x) = \int_0^x h_f(t)dt \qquad (x \in (0, 1]).$$

This implies that the function g defined by $g(x) = x\varphi(x)$ $(x \in [0, 1])$ is absolutely continuous. This is a contradiction, by Lemma 1. Therefore $\varphi|_{(0,1]} \notin \mathcal{M}(\widehat{L^1(S, \mu)})$ and consequently, $L^1(S, \mu)$ is not a BSE algebra.

Case II. $1 < p < \infty$. We prove that $L^p(S, \mu)$ has no $\Delta-$weak bounded approximate identity. Assume towards a contradiction that $\{f_\alpha\}_\alpha$ is a $\Delta-$weak bounded approximate identity for $L^p(S, \mu)$ with bound C. By the definition of $\{f_\alpha\}$ we have

$$\lim_\alpha \phi_x(f_\alpha) = 1 \quad (x \in (0, 1]),$$

for all $\phi_x \in \Delta(L^p(S, \mu))$. By Remark 1,

$$\lim_\alpha \int_0^x f_\alpha(t) d\mu(t) = 1 \quad (x \in (0, 1]).$$

By Holder's inequality for every $x \in S$ we have

$$\begin{aligned} \left| \int_0^x f_\alpha(t) d\mu(t) \right| &\leq \int_0^x |f_\alpha(t)| d\mu(t) \\ &\leq \left(\int_0^x |f_\alpha(t)|^p d\mu(t) \right)^{\frac{1}{p}} . \left(\int_0^x 1^q d\mu(t) \right)^{\frac{1}{q}} \\ &\leq C.x^{\frac{1}{q}} \qquad (x \in (0, 1]), \end{aligned}$$

This implies that $1 \leq C.x^{\frac{1}{q}}$, so $x \geq \frac{1}{C^q}$ for all $x \in (0, 1]$. This contradiction completes the proof. □

It is well known that for a locally compact group G, the Banach algebra $L^1(G)$ is an ideal in its second dual if and only if G is compact. As a consequence of the above theorem, in the following result, we prove that this is not the case for the compact totally ordered semigroup S.

Corollary 1 *The Banach algebra $L^1(S, \mu)$ is not an ideal in its second dual.*

Proof By Theorem 3, $L^1(S, \mu)$ is not a BSE algebra, however, it admits a bounded approximate identity. So by Theorem 3.1 of [8], it is not an ideal in its second dual. □

Remark 2 Let σ be a function on $[0, 1]$ such that $\sigma(0) = 0$ and $\sigma|_{(0,1]} \in C_{BSE}((0, 1])$. By Theorem 4 (ii) of [16], there exists $F \in L^1(S, \mu)^{**}(= L^\infty(S, \mu)^*)$ such that $\sigma(x) = F(\chi_{[0,x]})$ $(x \in (0, 1])$. Since $F \in L^\infty(S, \mu)^*$, from Theorem 2, it follows that there exists $\nu \in ba(S, \mu)$ such that

$$F(f) = \int_S f(t) d\nu(t) \qquad \left(f \in L^\infty(S, \mu)\right).$$

In particular,

$$\sigma(x) = F(\chi_{[0,x]}) = \int_S \chi_{[0,x])}(t) d\nu(t) = \nu([0, x]) \qquad (x \in (0, 1]).$$

From page 190 of [10], we conclude that σ is of bounded variation on [0, 1].

Now if we let $\sigma(x) = x\sin\left(\frac{1}{x}\right)$ $(x \in (0, 1])$ and

$$\sigma_1(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Obviously, $\sigma \in C_b((0, 1])$. The function σ_1 is not of bounded variation on [0, 1] and so as the above argument, $\sigma \notin C_{BSE}((0, 1])$. This implies that $C_{BSE}((0, 1]) \subsetneqq C_b((0, 1])$. However, in the sequel we will prove that for any $n \in \mathbb{N}$, the natural numbers, $C_{BSE(n)}((0, 1]) = C_b((0, 1])$, where $C_{BSE(n)}(\Delta(A))$ denotes the set of all complex valued continuous functions σ on $\Delta(A)$ which satisfy the following condition: there exists a positive real numbers β such that the inequality

$$\left|\sum_{j=1}^{n} c_j \sigma(\varphi_j)\right| \leq \beta\, .\, \left\|\sum_{j=1}^{n} c_j \varphi_j\right\|_{A^*}$$

holds for any choice of complex numbers $c_1, \ldots, c_n$ and $\varphi_1, \ldots, \varphi_n \in \Delta(A)$. For each $\sigma \in C_{BSE(n)}(\Delta(A))$ we denote by $\|\sigma\|_{BSE(n)}$ the infimum of such β. Let $C_{BSE(\infty)}(\Delta(A)) = \bigcap_{n\in\mathbb{N}} C_{BSE(n)}(\Delta(A)$. It is evident that $\|\sigma\|_{BSE} = \sup_{n\in\mathbb{N}} \|\sigma\|_{BSE(n)}$ and

$$C_{BSE}(\Delta(A)) = \{\sigma \in C_{BSE(\infty)}(\Delta(A) : \|\sigma\|_{BSE} < \infty\}.$$

Also we have

$$\begin{aligned} \widehat{A} \subseteq C_{BSE}(\Delta(A)) &\subseteq C_{BSE(\infty)}(\Delta(A)) \\ &\subseteq \cdots \subseteq C_{BSE(2)}(\Delta(A)) \subseteq C_{BSE(1)}(\Delta(A)) = C_b(\Delta(A)). \end{aligned}$$

For more details see [17].

Theorem 4 *For* $S = ([0, 1], \max)$ *and* $n \in \mathbb{N}$, *we have*

$$\widehat{L^1}(S, \mu) \subsetneqq C_{BSE}((0, 1]) \subsetneqq C_{BSE(n)}((0, 1]) = C_b((0, 1]).$$

Proof By Remark 2 and the proof of Theorem 3, we only need to prove the last equality. By Lemma 1 of [17], it is enough to show that for any natural number n, there exists a real number β_n such that for $c_1, \ldots, c_n \in \mathbb{C}$ with $|c_i| \leq 1$ $(1 \leq i \leq n)$ and $x_1, \ldots, x_n \in (0, 1]$, there exists a function $f \in L^1(S, \mu)$ such that $\widehat{f}(x_i) = c_i$ $(1 \leq i \leq n)$ and $\|f\|_1 \leq \beta_n$. For every $x \in [0, 1]$ we define

$$f(x) = \begin{cases} \frac{c_1}{x_1} & x \in [0, x_1] \\ \frac{c_2 - c_1}{x_2 - x_1} & x \in (x_1, x_2] \\ . & . \\ . & . \\ . & . \\ \frac{c_n - c_{n-1}}{x_n - x_{n-1}} & x \in (x_{n-1}, x_n] \\ 0 & x \in (x_n, 1] \end{cases}$$

Now for every $1 \leq i \leq n$, we have

$$\widehat{f}(x_i) = \int_0^{x_i} f d\mu = \int_0^{x_1} f d\mu + \int_{x_1}^{x_2} f d\mu + \cdots + \int_{x_{i-1}}^{x_i} f d\mu = c_i,$$

and

$$\begin{aligned} \|f\|_1 &= \int_0^1 |f| d\mu = \int_0^{x_1} |f| d\mu + \int_{x_1}^{x_2} |f| d\mu + \cdots + \int_{x_n}^1 |f| d\mu \\ &= |c_1| + |c_2 - c_1| + \cdots + |c_n - c_{n-1}| \\ &\leq 2n - 1. \end{aligned}$$

So if we choose $\beta_n = 2n - 1$, then $f \in L^1(S, \mu)$ is the required function. □

Conjecture *It is well known that $L^\infty(S, \mu)$ with the pointwise multiplication is a C^*-algebra and so it is a BSE algebra. We conjecture that $L^\infty(S, \mu)$ with the convolution multiplication is not a BSE algebra.*

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