J Fourier Anal Appl (2016) 22:787–808
DOI 10.1007/s00041-015-9434-x

Stable Signal Recovery from Phaseless Measurements

Bing Gao[1] · **Yang Wang**[2] · **Zhiqiang Xu**[1]

Received: 29 April 2015 / Published online: 20 October 2015

Abstract The aim of this paper is to study the stability of the ℓ_1 minimization for the compressive phase retrieval and to extend the instance-optimality in compressed sensing to the real phase retrieval setting. We first show that $m = \mathcal{O}(k \log(N/k))$ measurements are enough to guarantee the ℓ_1 minimization to recover k-sparse signals stably provided the measurement matrix A satisfies the strong RIP property. We second investigate the phaseless instance-optimality presenting a null space property of the measurement matrix A under which there exists a decoder Δ so that the phaseless instance-optimality holds. We use the result to study the phaseless instance-optimality for the ℓ_1 norm. This builds a parallel for compressive phase retrieval with the classical compressive sensing.

Keywords Phase retrieval · Sparse signals · Compressed sensing

Mathematics Subject Classification 94A12

Communicated by Peter G. Casazza.

✉ Zhiqiang Xu
xuzq@lsec.cc.ac.cn

Bing Gao
gaobing@lsec.cc.ac.cn

Yang Wang
yangwang@ust.hk

1 LSEC, Institute of Computational Mathematics, Academy of Mathematics and System Science, Chinese Academy of Sciences, Beijing 100091, China

2 Department of Mathematics, The Hong Kong University of Science and Technology, Clear Watre Bay, Kowloon, Hong Kong

1 Introduction

In this paper we consider the phase retrieval for sparse signals with noisy measurements, which arises in many different applications. Assume that

$$b_j := |\langle a_j, x_0\rangle| + e_j, \quad j = 1, \ldots, m$$

where $x_0 \in \mathbb{R}^N$, $a_j \in \mathbb{R}^N$ and $e_j \in \mathbb{R}$ is the noise. Our goal is to recover x_0 up to a unimodular scaling constant from $b := (b_1, \ldots, b_m)^\top$ with the assumption of x_0 being approximately k-sparse. This problem is referred to as the *compressive phase retrieval problem* [9].

The paper attempts to address two problems. Firstly we consider the stability of ℓ_1 minimization for the compressive phase retrieval problem where the signal x_0 is approximately k-sparse, which is the ℓ_1 minimization problem defined as follows:

$$\min \|x\|_1 \quad \text{subject to} \quad \big\| |Ax| - |Ax_0| \big\|_2 \leq \epsilon, \tag{1.1}$$

where $A := [a_1, \ldots, a_m]^\top$ and $|Ax_0| := [|\langle a_1, x_0\rangle|, \ldots, |\langle a_m, x_0\rangle|]^\top$. Secondly we investigate instance-optimality in the phase retrieval setting.

Note that in the classical compressive sensing setting the stable recovery of a k-sparse signal $x_0 \in \mathbb{C}^N$ can be done using $m = \mathcal{O}(k\log(N/k))$ measurements for several classes of measurement matrices A. A natural question is whether stable compressive phase retrieval can also be attained with $m = \mathcal{O}(k\log(N/k))$ measurements. This has indeed proved to be the case in [6] if $x_0 \in \mathbb{R}^N$ and A is a random real Gaussian matrix. In [8] a two-stage algorithm for compressive phase retrieval is proposed, which allows for very fast recovery of a sparse signal if the matrix A can be written as a product of a random matrix and another matrix (such as a random matrix) that allows for efficient phase retrieval. The authors proved that stable compressive phase retrieval can be achieved with $m = \mathcal{O}(k\log(N/k))$ measurements for complex signals x_0 as well. In [10], the strong RIP (S-RIP) property is introduced and the authors show that one can use the ℓ_1 minimization to recover sparse signals up to a global sign from the *noiseless* measurements $|Ax_0|$ provided A satisfies S-RIP. Naturally, one is interested in the performance of ℓ_1 minimization for the compressive phase retrieval with noisy measurements. In this paper, we shall show that the ℓ_1 minimization scheme given in (1.1) will recover a k-sparse signal stably from $m = \mathcal{O}(k\log(N/k))$ measurements, provided that the measurement matrix A satisfies the strong RIP (S-RIP) property. This establishes an important parallel for compressive phase retrieval with the classical compressive sensing. Note that in [11] such a parallel in terms of the null space property was already established.

The notion of *instance optimality* was first introduced in [5]. We use $\|x\|_0$ to denote the number of non-zero elements in x. Given a norm $\|\cdot\|_X$ such as the ℓ_1-norm and $x \in \mathbb{R}^N$, the best k-term approximation error is defined as

$$\sigma_k(x)_X := \min_{z\in\Sigma_k} \|x - z\|_X,$$

where

$$\Sigma_k := \{x \in \mathbb{R}^N : \|x\|_0 \le k\}.$$

We use $\Delta : \mathbb{R}^m \mapsto \mathbb{R}^N$ to denote a decoder for reconstructing x. We say the pair (A, Δ) is *instance optimal of order k with constant C_0* if

$$\|x - \Delta(Ax)\|_X \le C_0 \sigma_k(x)_X \tag{1.2}$$

holds for all $x \in \mathbb{R}^N$. In extending it to phase retrieval, our decoder will have the input $b = |Ax|$. A pair (A, Δ) is said to be *phaseless instance optimal of order k with constant C_0* if

$$\min\Big\{\|x - \Delta(|Ax|)\|_X, \|x + \Delta(|Ax|)\|_X\Big\} \le C_0 \sigma_k(x)_X \tag{1.3}$$

holds for all $x \in \mathbb{R}^N$. We are interested in the following problem : *Given $\|\cdot\|_X$ and $k < N$, what is the minimal value of m for which there exists (A, Δ) so that (1.3) holds?*

The null space $\mathcal{N}(A) := \{x \in \mathbb{R}^N : Ax = 0\}$ of A plays an important role in the analysis of the original instance optimality (1.2) (see [5]). Here we present a null space property for $\mathcal{N}(A)$, which is necessary and sufficient, for which there exists a decoder Δ so that (1.3) holds. We apply the result to investigate the instance optimality where X is the ℓ_1 norm. Set

$$\Delta_1(|Ax|) := \underset{z\in\mathbb{R}^N}{\operatorname{argmin}}\Big\{\|z\|_1 : |Ax| = |Az|\Big\}.$$

We show that the pair (A, Δ_1) satisfies (1.3) with X being the ℓ_1-norm provided A satisfies the strong RIP property (see Definition 2.1). As shown in [10], the Gaussian random matrix $A \in \mathbb{R}^{m\times N}$ satisfies the strong RIP of order k for $m = \mathcal{O}(k \log(N/k)$. Hence $m = \mathcal{O}(k \log(N/k))$ measurements suffice to ensure the phaseless instance optimality (1.3) for the ℓ_1-norm exactly as with the traditional instance optimality (1.2).

2 Auxiliary Results

In this section we provide some auxiliary results that will be used in later sections. For $x \in \mathbb{R}^N$ we use $\|x\|_p := \|x\|_{\ell_p}$ to denote the p-norm of x for $0 < p \le \infty$. The measurement matrix is given by $A := [a_1, \ldots, a_m]^T \in \mathbb{R}^{m\times N}$ as before. Given an index set $I \subset \{1, \ldots, m\}$ we shall use A_I to denote the sub-matrix of A where only rows with indices in I are kept, i.e.,

$$A_I := [a_j : j \in I]^\top.$$

The matrix A satisfies the *Restricted Isometry Property* (*RIP*) *of order* k if there exists a constant $\delta_k \in [0, 1)$ such that for all k-sparse vectors $z \in \Sigma_k$ we have

$$(1-\delta_k)\|z\|_2^2 \le \|Az\|_2^2 \le (1+\delta_k)\|z\|_2^2.$$

It was shown in [2] that one can use ℓ_1-minimization to recover k-sparse signals provided that A satisfies the RIP of order tk and $\delta_{tk} < \sqrt{1-\frac{1}{t}}$ where $t > 1$.

To investigate compressive phase retrieval, a stronger notion of RIP is given in [10]:

Definition 2.1 (*S-RIP*) We say the matrix $A = [a_1, \ldots, a_m]^\top \in \mathbb{R}^{m\times N}$ has the *Strong Restricted Isometry Property* of order k with bounds $\theta_-,\ \theta_+ \in (0, 2)$ if

$$\theta_-\|x\|_2^2 \le \min_{I\subseteq[m],|I|\ge m/2} \|A_I x\|_2^2 \le \max_{I\subseteq[m],|I|\ge m/2} \|A_I x\|_2^2 \le \theta_+\|x\|_2^2 \tag{2.1}$$

holds for all k-sparse signals $x \in \mathbb{R}^N$, where $[m] := \{1, \ldots, m\}$. We say A has the *Strong Lower Restricted Isometry Property* of order k with bound θ_- if the lower bound in (2.1) holds. Similarly we say A has the *Strong Upper Restricted Isometry Property* of order k with bound θ_+ if the upper bound in (2.1) holds.

The authors of [10] proved that Gaussian matrices with $m = \mathcal{O}(tk\log(N/k))$ satisfy S-RIP of order tk with high probability.

Theorem 2.1 ([10]) *Suppose that* $t > 1$ *and* $A = (a_{ij}) \in \mathbb{R}^{m\times N}$ *is a random Gaussian matrix with* $m = \mathcal{O}(tk\log(N/k))$ *and* $a_{ij} \sim \mathcal{N}(0, \frac{1}{\sqrt{m}})$. *Then there exist* $\theta_-, \theta_+ \in (0, 2)$ *such that with probability* $1-\exp(-cm/2)$ *the matrix A satisfies the S-RIP of order* tk *with constants* θ_- *and* θ_+, *where* $c > 0$ *is an absolute constant and* θ_-, θ_+ *are independent of* t.

The following is a very useful lemma for this study.

Lemma 2.1 *Let* $x_0 \in \mathbb{R}^N$ *and* $\rho \ge 0$. *Suppose that* $A \in \mathbb{R}^{m\times N}$ *is a measurement matrix satisfying the restricted isometry property with* $\delta_{tk} \le \sqrt{\frac{t-1}{t}}$ *for some* $t > 1$. *Then for any*

$$\hat{x} \in \left\{x \in \mathbb{R}^N : \|x\|_1 \le \|x_0\|_1 + \rho,\ \|Ax - Ax_0\|_2 \le \epsilon\right\}$$

we have

$$\|\hat{x} - x_0\|_2 \le c_1\epsilon + c_2\frac{2\sigma_k(x_0)_1}{\sqrt{k}} + c_2\cdot\frac{\rho}{\sqrt{k}},$$

where $c_1 = \frac{\sqrt{2(1+\delta)}}{1-\sqrt{t/(t-1)}\delta}$, $c_2 = \frac{\sqrt{2}\delta+\sqrt{(\sqrt{t(t-1)}-\delta t)\delta}}{\sqrt{t(t-1)}-\delta t} + 1$.

Remark 2.1 We build the proof of Lemma 2.1 following the ideas of Cai and Zhang [2]. The full proof is given in Appendix for completeness. It is well-known that an effective method to recover approximately-sparse signals x_0 in the traditional compressive sensing is to solve

$$x^{\#} := \underset{x}{\operatorname{argmin}}\{\|x\|_1 : \|Ax - Ax_0\|_2 \le \epsilon\}. \tag{2.2}$$

The definition of $x^{\#}$ shows that

$$\|x^{\#}\|_1 \le \|x_0\|_1, \quad \|Ax^{\#} - Ax_0\|_2 \le \epsilon,$$

which implies that

$$\|x^{\#} - x_0\|_2 \le C_1\epsilon + C_2\frac{\sigma_k(x_0)_1}{\sqrt{k}},$$

provided that A satisfies the RIP condition with $\delta_{tk} \le \sqrt{1 - 1/t}$ for $t > 1$ (see [2]). However, in practice one prefers to design fast algorithms to find an approximation solution of (2.2), say $\hat{x}$. Thus it is possible to have $\|\hat{x}\|_1 > \|x_0\|_1$. Lemma 2.1 gives an estimate of $\|\hat{x} - x_0\|_2$ for the case where $\|\hat{x}\|_1 \le \|x_0\|_1 + \rho$.

Remark 2.2 In [7], Han and Xu extend the definition of S-RIP by replacing the $m/2$ in (2.1) by βm where $0 < \beta < 1$. They also prove that, for any fixed $\beta \in (0, 1)$, the $m \times N$ random Gaussian matrix satisfies S-RIP of order k with high probability provided $m = \mathcal{O}(k \log(N/k))$.

3 Stable Recovery of Real Phase Retrieval Problem

3.1 Stability Results

The following lemma shows that the map $\phi_A(x) := |Ax|$ is stable on Σ_k modulo a unimodular constant provided A satisfies strong lower RIP of order $2k$. Define the equivalent relation $\sim$ on $\mathbb{R}^N$ and $\mathbb{C}^N$ by the following: for any x, y, $x \sim y$ iff $x = cy$ for some unimodular scalar c, where x, y are in $\mathbb{R}^N$ or $\mathbb{C}^N$. For any subset Y of $\mathbb{R}^N$ or $\mathbb{C}^N$ the notation $Y/\sim$ denotes the equivalent classes of elements in Y under the equivalence. Note that there is a natural metric $D_\sim$ on $\mathbb{C}^N/\sim$ given by

$$D_\sim(x, y) = \min_{|c|=1} \|x - cy\|.$$

Our primary focus in this paper will be on $\mathbb{R}^N$, and in this case $D_\sim(x, y) = \min\{\|x - y\|_2, \|x + y\|_2\}$.

Lemma 3.1 *Let $A \in \mathbb{R}^{m\times N}$ satisfy the strong lower RIP of order $2k$ with constant θ_-. Then for any $x, y \in \Sigma_k$ we have*

$$\||Ax| - |Ay|\|_2^2 \ge \theta_- \min(\|x - y\|_2^2, \|x + y\|_2^2).$$

Proof For any $x, y \in \Sigma_k$ we divide $\{1, \dots, m\}$ into two subsets:

$$T = \{j : \operatorname{sign}(\langle a_j, x\rangle) = \operatorname{sign}(\langle a_j, y\rangle)\}$$

and

$$T^c = \{j : \operatorname{sign}(\langle a_j, x\rangle) = -\operatorname{sign}(\langle a_j, y\rangle)\}.$$

Clearly one of T and T^c will have cardinality at least $m/2$. Without loss of generality we assume that T has cardinality no less than $m/2$. Then

$$\begin{aligned} \||Ax| - |Ay|\|_2^2 &= \|A_T x - A_T y\|_2^2 + \|A_{T^c} x + A_{T^c} y\|_2^2 \\ &\geq \|A_T x - A_T y\|_2^2 \\ &\geq \theta_- \|x - y\|_2^2 \\ &\geq \theta_- \min(\|x - y\|_2^2, \|x + y\|_2^2). \end{aligned}$$

□

Remark 3.1 Note that the combination of Lemma 3.1 and Theorem 2.1 shows that for an $m \times N$ Gaussian matrix A with $m = O(k \log(N/k))$ one can guarantee the stability of the map $\phi_A(x) := |Ax|$ on $\Sigma_k / \sim$.

3.2 The Main Theorem

In this part, we will consider how many measurements are needed for the stable sparse phase retrieval by ℓ_1-minimization via solving the following model:

$$\min \|x\|_1 \ \text{ subject to } \ \||Ax| - |Ax_0|\|_2^2 \leq \epsilon^2, \tag{3.1}$$

where A is our measurement matrix and $x_0 \in \mathbb{R}^N$ is a signal we wish to recover. The next theorem tells under what conditions the solution to (3.1) is stable.

Theorem 3.1 *Assume that $A \in \mathbb{R}^{m \times N}$ satisfies the S-RIP of order tk with bounds $\theta_-, \theta_+ \in (0, 2)$ such that*

$$t \geq \max\left\{\frac{1}{2\theta_- - \theta_-^2}, \frac{1}{2\theta_+ - \theta_+^2}\right\}.$$

Then any solution $\hat{x}$ for (3.1) satisfies

$$\min\{\|\hat{x} - x_0\|_2, \|\hat{x} + x_0\|_2\} \leq c_1 \epsilon + c_2 \frac{2\sigma_k(x_0)_1}{\sqrt{k}},$$

where c_1 and c_2 are constants defined in Lemma 2.1.

Proof Clearly any $\hat{x} \in \mathbb{R}^N$ satisfying (3.1) must have

$$\|\hat{x}\|_1 \leq \|x_0\|_1 \tag{3.2}$$

and

$$\||A\hat{x}| - |Ax_0|\|_2^2 \leq \epsilon^2. \tag{3.3}$$

Now the index set $\{1, 2, \ldots, m\}$ is divisible into two subsets

$$T = \{j : \operatorname{sign}(\langle a_j, \hat{x}\rangle) = \operatorname{sign}(\langle a_j, x_0\rangle)\},$$
$$T^c = \{j : \operatorname{sign}(\langle a_j, \hat{x}\rangle) = -\operatorname{sign}(\langle a_j, x_0\rangle)\}.$$

Then (3.3) implies that

$$\|A_T\hat{x} - A_Tx_0\|_2^2 + \|A_{T^c}\hat{x} + A_{T^c}x_0\|_2^2 \leq \epsilon^2. \tag{3.4}$$

Here either $|T| \geq m/2$ or $|T^c| \geq m/2$. Without loss of generality we assume that $|T| \geq m/2$. We use the fact

$$\|A_T\hat{x} - A_Tx_0\|_2^2 \leq \epsilon^2. \tag{3.5}$$

From (3.2) and (3.5) we obtain

$$\hat{x} \in \left\{x \in \mathbb{R}^N : \|x\|_1 \leq \|x_0\|_1, \|A_Tx - A_Tx_0\|_2 \leq \epsilon\right\}. \tag{3.6}$$

Recall that A satisfies S-RIP of order tk and constants θ_-, θ_+. Here

$$t \geq \max\{\frac{1}{2\theta_- - \theta_-^2}, \frac{1}{2\theta_+ - \theta_+^2}\} > 1. \tag{3.7}$$

The definition of S-RIP implies that A_T satisfies the RIP of order tk in which

$$\delta_{tk} \leq \max\{1 - \theta_-, \theta_+ - 1\} \leq \sqrt{\frac{t-1}{t}} \tag{3.8}$$

where the second inequality follows from (3.7). The combination of (3.6), (3.8) and Lemma 2.1 now implies

$$\|\hat{x} - x_0\|_2 \leq c_1\epsilon + c_2\frac{2\sigma_k(x_0)_1}{\sqrt{k}},$$

where c_1 and c_2 are defined in Lemma 2.1. If $|T^c| \geq \frac{m}{2}$ we get the corresponding result

$$\|\hat{x} + x_0\|_2 \leq c_1\epsilon + c_2\frac{2\sigma_k(x_0)_1}{\sqrt{k}}.$$

The theorem is now proved. □

This theorem demonstrates that, if the measurement matrix has the S-RIP, the real compressive phase retrieval problem can be solved stably by ℓ_1-minimization.

4 Phase Retrieval and Best k-term Approximation

4.1 Instance Optimality from the Linear Measurements

We introduce some definitions and results in [5]. Recall that for a given encoder matrix $A \in \mathbb{R}^{m\times N}$ and a decoder $\Delta : \mathbb{R}^m \mapsto \mathbb{R}^N$, the pair (A, Δ) is said to have instance optimality of order k with constant C_0 with respect to the norm X if

$$\|x - \Delta(Ax)\|_X \le C_0\sigma_k(x)_X \tag{4.1}$$

holds for all $x \in \mathbb{R}^N$. Set $\mathcal{N}(A) := \{\eta \in \mathbb{R}^N : A\eta = 0\}$ to be the null space of A. The following theorem gives conditions under which the (4.1) holds.

Theorem 4.1 ([5]) *Let $A \in \mathbb{R}^{m\times N}$, $1 \le k \le N$ and $\|\cdot\|_X$ be a norm on $\mathbb{R}^N$. Then a sufficient condition for the existence of a decoder Δ satisfying* (4.1) *is*

$$\|\eta\|_X \le \frac{C_0}{2}\sigma_{2k}(\eta)_X, \quad \forall \eta \in \mathcal{N}(A). \tag{4.2}$$

A necessary condition for the existence of a decoder Δ satisfying (4.1) *is*

$$\|\eta\|_X \le C_0\sigma_{2k}(\eta)_X, \quad \forall \eta \in \mathcal{N}(A). \tag{4.3}$$

For the norm $X = \ell_1$ it was established in [5] that instance optimality of order k can indeed be achieved, e.g. for a Gaussian matrix A, with $m = O(k\log(N/k))$. The authors also considered more generally taking different norms on both sides of (4.1). Following [5], we say the pair (A, Δ) has (p, q)-*instance optimality of order k with constant C_0* if

$$\|x - \Delta(Ax)\|_p \le C_0 k^{\frac{1}{q}-\frac{1}{p}}\sigma_k(x)_q, \quad \forall x \in \mathbb{R}^N, \tag{4.4}$$

with $1 \le q \le p \le 2$. It was shown in [5] that the (p, q)-instance optimality of order k can be achieved at the cost of having $m = \mathcal{O}(k(N/k)^{2-2/q})\log(N/k)$ measurements.

4.2 Phaseless Instance Optimality

A natural question here is whether an analogous result to Theorem 4.1 exists for phaseless instance optimality defined in (1.3). We answer the question by presenting such a result in the case of real phase retrieval.

Recall that a pair (A, Δ) is said to be have the phaseless instance optimality of order k with constant C_0 for the norm $\|.\|_X$ if

$$\min\Big\{\|x - \Delta(|Ax|)\|_X, \|x + \Delta(|Ax|)\|_X\Big\} \le C_0\sigma_k(x)_X \tag{4.5}$$

holds for all $x \in \mathbb{R}^N$.

Theorem 4.2 *Let $A \in \mathbb{R}^{m\times N}$, $1 \le k \le N$ and $\|\cdot\|_X$ be a norm. Then a sufficient condition for the existence of a decoder Δ satisfying the phaseless instance optimality (4.5) is: For any $I \subseteq \{1,\ldots,m\}$ and $\eta_1 \in \mathcal{N}(A_I)$, $\eta_2 \in \mathcal{N}(A_{I^c})$ we have*

$$\min\{\|\eta_1\|_X, \|\eta_2\|_X\} \le \frac{C_0}{4}\sigma_k(\eta_1-\eta_2)_X + \frac{C_0}{4}\sigma_k(\eta_1+\eta_2)_X. \tag{4.6}$$

A necessary condition for the existence of a decoder Δ satisfying (4.5) is: For any $I \subseteq \{1,\ldots,m\}$ and $\eta_1 \in \mathcal{N}(A_I)$, $\eta_2 \in \mathcal{N}(A_{I^c})$ we have

$$\min\{\|\eta_1\|_X, \|\eta_2\|_X\} \le \frac{C_0}{2}\sigma_k(\eta_1-\eta_2)_X + \frac{C_0}{2}\sigma_k(\eta_1+\eta_2)_X. \tag{4.7}$$

Proof We first assume (4.6) holds, and show that there exists a decoder Δ satisfying the phaseless instance optimality (4.5). To this end, we define a decoder Δ as follows:

$$\Delta(|Ax_0|) = \operatorname*{argmin}_{|Ax|=|Ax_0|} \sigma_k(x)_X.$$

Suppose $\hat{x} := \Delta(|Ax_0|)$. We have $|A\hat{x}| = |Ax_0|$ and $\sigma_k(\hat{x})_X \le \sigma_k(x_0)_X$. Note that $\langle a_j, \hat{x}\rangle = \pm\langle a_j, x_0\rangle$. Let $I \subseteq \{1,\ldots,m\}$ be defined by

$$I = \Big\{ j : \langle a_j, \hat{x}\rangle = \langle a_j, x_0\rangle \Big\}.$$

Then

$$A_I(x_0 - \hat{x}) = 0, \quad A_{I^c}(x_0 + \hat{x}) = 0.$$

Set

$$\begin{aligned}\eta_1 &:= x_0 - \hat{x} \in \mathcal{N}(A_I),\\ \eta_2 &:= x_0 + \hat{x} \in \mathcal{N}(A_{I^c}).\end{aligned}$$

A simple observation yields

$$\sigma_k(\eta_1-\eta_2)_X = 2\sigma_k(\hat{x})_X \le 2\sigma_k(x_0)_X, \quad \sigma_k(\eta_1+\eta_2)_X = 2\sigma_k(x_0)_X. \tag{4.8}$$

Then (4.6) implies that

$$\begin{aligned}\min\{\|\hat{x}-x_0\|_X, \|\hat{x}+x_0\|_X\} &= \min\{\|\eta_1\|_X, \|\eta_2\|_X\}\\ &\le \frac{C_0}{4}\sigma_k(\eta_1-\eta_2)_X + \frac{C_0}{4}\sigma_k(\eta_1+\eta_2)_X\\ &\le C_0\sigma_k(x_0)_X.\end{aligned}$$

Here the last equality is obtained by (4.8). This proves the sufficient condition.

We next turn to the necessary condition. Let Δ be a decoder for which the phaseless instance optimality (4.5) holds. Let $I \subseteq \{1, \ldots, m\}$. For any $\eta_1 \in \mathcal{N}(A_I)$ and $\eta_2 \in \mathcal{N}(A_{I^c})$ we have

$$|A(\eta_1 + \eta_2)| = |A(\eta_1 - \eta_2)| = |A(\eta_2 - \eta_1)|. \tag{4.9}$$

The instance optimality implies

$$\begin{aligned} &\min\Big\{\|\Delta(|A(\eta_1+\eta_2)|) + \eta_1 + \eta_2\|_X, \|\Delta(|A(\eta_1+\eta_2)|) - (\eta_1+\eta_2)\|_X\Big\} \\ &\quad \le C_0\sigma_k(\eta_1+\eta_2)_X. \end{aligned} \tag{4.10}$$

Without loss of generality we may assume that

$$\|\Delta(|A(\eta_1+\eta_2)|) + \eta_1 + \eta_2\|_X \ \le\ \|\Delta(|A(\eta_1+\eta_2)|) - (\eta_1+\eta_2)\|_X.$$

Then (4.10) implies that

$$\|\Delta(|A(\eta_1+\eta_2)|) + \eta_1 + \eta_2\|_X \le C_0\sigma_k(\eta_1+\eta_2)_X. \tag{4.11}$$

By (4.9), we have

$$\begin{aligned} \|\Delta(|A(\eta_1+\eta_2)|) + \eta_1 + \eta_2\|_X &= \|\Delta(|A(\eta_2-\eta_1)|) - (\eta_2-\eta_1) + 2\eta_2\|_X \\ &\ge 2\|\eta_2\|_X - \|\Delta(|A(\eta_2-\eta_1)|) - (\eta_2-\eta_1)\|_X. \end{aligned} \tag{4.12}$$

Combining (4.11) and (4.12) yields

$$2\|\eta_2\|_X \le C_0\sigma_k(\eta_1+\eta_2)_X + \|\Delta(|A(\eta_2-\eta_1)|) - (\eta_2-\eta_1)\|_X. \tag{4.13}$$

At the same time, (4.9) also implies

$$\begin{aligned} \|\Delta(|A(\eta_1+\eta_2)|) + \eta_1 + \eta_2\|_X &= \|\Delta(|A(\eta_2-\eta_1)|) + (\eta_2-\eta_1) + 2\eta_1\|_X \\ &\ge 2\|\eta_1\|_X - \|\Delta(|A(\eta_2-\eta_1)|) + (\eta_2-\eta_1)\|_X. \end{aligned} \tag{4.14}$$

Putting (4.11) and (4.14) together, we obtain

$$2\|\eta_1\|_X \le C_0\sigma_k(\eta_1+\eta_2)_X + \|\Delta(|A(\eta_2-\eta_1)|) + (\eta_2-\eta_1)\|_X. \tag{4.15}$$

It follows from (4.13) and (4.15) that

$$\begin{aligned} \min\{\|\eta_1\|_X, \|\eta_2\|_X\} \le\ & \frac{C_0}{2}\sigma_k(\eta_1+\eta_2)_X \\ & + \frac{1}{2}\min\{\|\Delta(|A(\eta_2-\eta_1)|) - (\eta_2-\eta_1)\|_X, \|\Delta(|A(\eta_2-\eta_1)|) \\ & + (\eta_2-\eta_1)\|_X\} \le \frac{C_0}{2}\sigma_k(\eta_1+\eta_2)_X + \frac{C_0}{2}\sigma_k(\eta_1-\eta_2)_X. \end{aligned}$$

Here the last inequality is obtained by the instance optimality of (A, Δ). For the case where

$$\|\Delta(|A(\eta_1+\eta_2)|)-(\eta_1+\eta_2)\|_X \;\le\; \|\Delta(|A(\eta_1+\eta_2)|)+\eta_1+\eta_2\|_X,$$

we obtain

$$\min\{\|\eta_1\|_X, \|\eta_2\|_X\} \le \frac{C_0}{2}\sigma_k(\eta_1+\eta_2)_X + \frac{C_0}{2}\sigma_k(\eta_1-\eta_2)_X$$

via the same argument. The theorem is now proved. □

We next present a null space property for phaseless instance optimality, which allows us to establish parallel results for sparse phase retrieval.

Definition 4.1 We say a matrix $A \in \mathbb{R}^{m\times N}$ satisfies the *strong null space property (S-NSP) of order k with constant C* if for any index set $I \subseteq \{1,\dots,m\}$ with $|I| \ge m/2$ and $\eta \in \mathcal{N}(A_I)$ we have

$$\|\eta\|_X \le C\cdot\sigma_k(\eta)_X.$$

Theorem 4.3 *Assume that a matrix $A \in \mathbb{R}^{m\times N}$ has the strong null space property of order $2k$ with constant $C_0/2$. Then there must exist a decoder Δ having the phaseless instance optimality (1.3) with constant C_0. In particular, one such decoder is*

$$\Delta(|Ax_0|) = \underset{|Ax|=|Ax_0|}{argmin}\ \sigma_k(x)_X.$$

Proof Assume that $I \subseteq \{1,\dots,m\}$. For any $\eta_1 \in \mathcal{N}(A_I)$ and $\eta_2 \in \mathcal{N}(A_{I^c})$ we must have either $\|\eta_1\|_X \le \frac{C_0}{2}\sigma_{2k}(\eta_1)_X$ or $\|\eta_2\|_X \le \frac{C_0}{2}\sigma_{2k}(\eta_2)_X$ by the strong null space property. If $\|\eta_1\|_X \le \frac{C_0}{2}\sigma_{2k}(\eta_1)_X$ then

$$\|\eta_1\|_X \le \frac{C_0}{2}\sigma_{2k}(\eta_1)_X \le \frac{C_0}{4}\sigma_k(\eta_1-\eta_2)_X + \frac{C_0}{4}\sigma_k(\eta_1+\eta_2)_X.$$

Similarly if $\|\eta_2\|_X \le \frac{C_0}{2}\sigma_{2k}(\eta_2)_X$ we will have

$$\|\eta_2\|_X \le \frac{C_0}{2}\sigma_{2k}(\eta_2)_X \le \frac{C_0}{4}\sigma_k(\eta_1-\eta_2)_X + \frac{C_0}{4}\sigma_k(\eta_1+\eta_2)_X.$$

It follows that

$$\min\{\|\eta_1\|_X, \|\eta_2\|_X\} \le \frac{C_0}{4}\sigma_k(\eta_1-\eta_2)_X + \frac{C_0}{4}\sigma_k(\eta_1+\eta_2)_X. \tag{4.16}$$

Theorem 4.2 now implies that the required decoder Δ exists. Furthermore, by the proof of the sufficiency part of Theorem 4.2,

$$\Delta(|Ax_0|) = \underset{|Ax|=|Ax_0|}{\operatorname{argmin}} \ \sigma_k(x)_X$$

is one such decoder. □

4.3 The Case $X = \ell_1$

We will now apply Theorem 4.3 to the ℓ_1-norm case. The following lemma establishes a relation between S-RIP and S-NSP for the ℓ_1-norm.

Lemma 4.1 *Let a, b, k be integers. Assume that $A \in \mathbb{R}^{m\times N}$ satisfies the S-RIP of order $(a+b)k$ with constants $\theta_-,\ \theta_+ \in (0,2)$. Then A satisfies the S-NSP of order ak under the ℓ_1-norm with constant*

$$C_0 = 1 + \sqrt{\frac{a(1+\delta)}{b(1-\delta)}},$$

where δ is the restricted isometry constant and $\delta := \max\{1-\theta_-, \theta_+ - 1\} < 1$.

We remark that the above lemma is the analogous to the following lemma providing a relationship between RIP and NSP, which was shown in [5]:

Lemma 4.2 ([5, Lemma 4.1]) *Let $a = l/k$, $b = l'/k$ where $l, l' \geq k$ are integers. Assume that $A \in \mathbb{R}^{m\times N}$ satisfies the RIP of order $(a+b)k$ with $\delta = \delta_{(a+b)k} < 1$. Then A satisfies the null space property under the ℓ_1-norm of order ak with constant $C_0 = 1 + \frac{\sqrt{a(1+\delta)}}{\sqrt{b(1-\delta)}}$.*

Proof By the definition of S-RIP, for any index set $I \subseteq \{1,\dots,m\}$ with $|I| \geq m/2$, the matrix $A_I \in \mathbb{R}^{|I|\times N}$ satisfies the RIP of order $(a+b)k$ with constant $\delta_{(a+b)k} = \delta := \max\{1-\theta_-, \theta_+ - 1\} < 1$. It follows from Lemma 4.2 that

$$\|\eta\|_1 \leq \left(1 + \sqrt{\frac{a(1+\delta)}{b(1-\delta)}}\right)\sigma_{ak}(\eta)_1$$

for all $\eta \in \mathcal{N}(A_I)$. This proves the lemma. □

Set $a = 2$ and $b = 1$ in Lemma 4.1 we infer that if A satisfies the S-RIP of order $3k$ with constants $\theta_-,\ \theta_+ \in (0,2)$, then A satisfies the S-NSP of order $2k$ under the ℓ_1-norm with constant $C_0 = 1 + \sqrt{\frac{2(1+\delta)}{1-\delta}}$. Hence by Theorem 4.3, there must exist a decoder that has the instance optimality under the ℓ_1-norm with constant $2C_0$. According to Theorem 2.1, by taking $m = O(k\log(N/k))$ a Gaussian random matrix A satisfies S-RIP of order $3k$ with high probability. Hence, there exists a decoder Δ so that the pair (A, Δ) has the the ℓ_1-norm phaseless instance optimality at the cost of $m = O(k\log(N/k))$ measurements, as with the traditional instance optimality.

We are now ready to prove the following theorem on phaseless instance optimality under the ℓ_1-norm.

Theorem 4.4 *Let $A \in \mathbb{R}^{m\times N}$ satisfy the S-RIP of order tk with constants $0 < \theta_- < 1 < \theta_+ < 2$, where*

$$t \geq \max\left\{\frac{2}{\theta_-}, \frac{2}{2-\theta_+}\right\} > 2.$$

Let

$$\Delta(|Ax_0|) = \underset{x\in\mathbb{R}^N}{\mathit{argmin}}\left\{\|x\|_1 : |Ax| = |Ax_0|\right\}. \tag{4.17}$$

Then (A, Δ) has the ℓ_1-norm phaseless instance optimality with constant $C = \frac{2C_0}{2-C_0}$, where $C_0 = 1 + \sqrt{\frac{1+\delta}{(t-1)(1-\delta)}}$ and as before

$$\delta := \max\{1-\theta_-, \theta_+ - 1\} \leq 1 - \frac{2}{t}.$$

Proof of Lemma 4.1 Let $x_0 \in \mathbb{R}^N$ and set $\hat{x} = \Delta(|Ax_0|)$. Then by definition

$$\|\hat{x}\|_1 \leq \|x_0\|_1 \quad \text{and} \quad |A\hat{x}| = |Ax_0|.$$

Denote by $I \subseteq \{1, \ldots, m\}$ the set of indices

$$I = \left\{j : \langle a_j, \hat{x}\rangle = \langle a_j, x_0\rangle\right\},$$

and thus $\langle a_j, \hat{x}\rangle = -\langle a_j, x_0\rangle$ for $j \in I^c$. It follows that

$$A_I(\hat{x} - x_0) = 0 \quad \text{and} \quad A_{I^c}(\hat{x} + x_0) = 0.$$

Set

$$\eta := \hat{x} - x_0 \in \mathcal{N}(A_I).$$

We know that A satisfies the S-RIP of order tk with constants $\theta_-,\ \theta_+$ where

$$t \geq \max\left\{\frac{2}{\theta_-}, \frac{2}{2-\theta_+}\right\} > 2.$$

For the case $|I| \geq m/2$, A_I satisfies the RIP of order tk with RIP constant

$$\delta = \delta_{tk} := \max\{1-\theta_-, \theta_+ - 1\} \leq 1 - \frac{2}{t} < 1.$$

Take $a := 1,\ b := t - 1$ in Lemma 4.1. Then A satisfies the ℓ_1-norm S-NSP of order k with constant

$$C_0 = 1 + \sqrt{\frac{1+\delta}{(t-1)(1-\delta)}} < 2.$$

This yields

$$\|\eta\|_1 \le C_0 \|\eta_{T^c}\|_1, \tag{4.18}$$

where T is the index set for the k largest coefficients of x_0 in magnitude. Hence $\|\eta_T\|_1 \le (C_0 - 1)\|\eta_{T^c}\|_1$. Since $\|\hat{x}\|_1 \le \|x_0\|_1$ we have

$$\begin{aligned} \|x_0\|_1 \ge \|\hat{x}\|_1 &= \|x_0 + \eta\|_1 = \|x_{0,T} + x_{0,T^c} + \eta_T + \eta_{T^c}\|_1 \\ &\ge \|x_{0,T}\|_1 - \|x_{0,T^c}\|_1 + \|\eta_{T^c}\|_1 - \|\eta_T\|_1. \end{aligned}$$

It follows that

$$\|\eta_{T^c}\|_1 \le \|\eta_T\|_1 + 2\sigma_k(x_0)_1 \le (C_0 - 1)\|\eta_{T^c}\|_1 + 2\sigma_k(x_0)_1$$

and thus

$$\|\eta_{T^c}\|_1 \le \frac{2}{2 - C_0}\sigma_k(x_0)_1.$$

Now (4.18) yields

$$\|\eta\|_1 \le C_0\|\eta_{T^c}\|_1 \le \frac{2C_0}{2 - C_0}\sigma_k(x_0)_1,$$

which implies

$$\|\hat{x} - x_0\|_1 \le C_0\|\eta_{T^c}\|_1 \le \frac{2C_0}{2 - C_0}\sigma_k(x_0)_1.$$

For the case $|I^c| \ge m/2$ identical argument yields

$$\|\hat{x} + x_0\|_1 \le C_0\|\eta_{T^c}\|_1 \le \frac{2C_0}{2 - C_0}\sigma_k(x_0)_1.$$

The theorem is now proved. □

By Theorem 2.1, an $m \times N$ random Gaussian matrix with $m = \mathcal{O}(tk\log(N/k))$ satisfies the S-RIP of order tk with high probability. We therefore conclude that the ℓ_1-norm phaseless instance optimality of order k can be achieved at the cost of $m = \mathcal{O}(tk\log(N/k))$ measurements.

4.4 Mixed-Norm phaseless Instance Optimality

We now consider *mixed-norm phaseless instance optimality*. Let $1 \le q \le p \le 2$ and $s = 1/q - 1/p$. We seek estimates of the form

$$\min\{\|x - \Delta(|Ax|)\|_p, \|x + \Delta(|Ax|)\|_p\} \le C_0 k^{-s} \sigma_k(x)_q \tag{4.19}$$

for all $x \in \mathbb{R}^N$. We shall prove both necessary and sufficient conditions for mixed-norm phaseless instance optimality.

Theorem 4.5 *Let $A \in \mathbb{R}^{m\times N}$ and $1 \le q \le p \le 2$. Set $s = 1/q - 1/p$. Then a decoder Δ satisfying the mixed norm phaseless instance optimality (4.19) with constant C_0 exists if: for any index set $I \subseteq \{1, \ldots, m\}$ and any $\eta_1 \in \mathcal{N}(A_I)$, $\eta_2 \in \mathcal{N}(A_{I^c})$ we have*

$$\min\{\|\eta_1\|_p, \|\eta_2\|_p\} \le \frac{C_0}{4} k^{-s} \Big(\sigma_k(\eta_1 - \eta_2)_q + \sigma_k(\eta_1 + \eta_2)_q\Big). \tag{4.20}$$

Conversely, assume a decoder Δ satisfying the mixed norm phaseless instance optimality (4.19) exists. Then for any index set $I \subseteq \{1, \ldots, m\}$ and any $\eta_1 \in \mathcal{N}(A_I)$, $\eta_2 \in \mathcal{N}(A_{I^c})$ we have

$$\min\{\|\eta_1\|_p, \|\eta_2\|_p\} \le \frac{C_0}{2} k^{-s} \Big(\sigma_k(\eta_1 - \eta_2)_q + \sigma_k(\eta_1 + \eta_2)_q\Big).$$

Proof of Lemma 4.1 The proof is virtually identical to the proof of Theorem 4.2. We shall omit the details here in the interest of brevity. □

Definition 4.2 (*Mixed-Norm Strong Null Space Property*) We say that A has the mixed strong null space property in norms (ℓ_p, ℓ_q) of order k with constant C if for any index set $I \subseteq \{1, \ldots, m\}$ with $|I| \ge m/2$ the matrix $A_I \in \mathbb{R}^{|I|\times N}$ satisfies

$$\|\eta\|_p \le C k^{-s} \sigma_k(\eta)_q$$

for all $\eta \in \mathcal{N}(A_I)$, where $s = 1/q - 1/p$.

The above is an extension of the standard definition of the mixed null space property of order k in norms (ℓ_p, ℓ_q) (see [5]) for a matrix A, which requires

$$\|\eta\|_p \le C k^{-s} \sigma_k(\eta)_q$$

for all $\eta \in \mathcal{N}(A)$. We have the following straightforward generalization of Theorem 4.3.

Theorem 4.6 *Assume that $A \in \mathbb{R}^{m\times N}$ has the mixed strong null space property of order $2k$ in norms (ℓ_p, ℓ_q) with constant $C_0/2$, where $1 \le q \le p \le 2$. Then there exists a decoder Δ such that the mixed-norm phaseless instance optimality (4.19) holds with constant C_0.*

We establish relationships between mixed-norm strong null space property and the S-RIP. First we present the following lemma that was proved in [5].

Lemma 4.3 ([5, Lemma 8.2]) *Let $k \geq 1$ and $\tilde{k} = \lceil k(\frac{N}{k})^{2-2/q} \rceil$. Assume that $A \in \mathbb{R}^{m\times N}$ satisfies the RIP of order $2k + \tilde{k}$ with $\delta := \delta_{2k+\tilde{k}} < 1$. Then A satisfies the mixed null space property in norms (ℓ_p, ℓ_q) of order $2k$ with constant $C_0 = 2^{1/p+1/2}\sqrt{\frac{1+\delta}{1-\delta}} + 2^{1/p-1/q}$.*

Proposition 4.1 *Let $k \geq 1$ and $\tilde{k} = \lceil k(\frac{N}{k})^{2-2/q} \rceil$. Assume that $A \in \mathbb{R}^{m\times N}$ satisfies the S-RIP of order $2k + \tilde{k}$ with constants $0 < \theta_- < 1 < \theta_+ < 2$. Then A satisfies the mixed strong null space property in norms (ℓ_p, ℓ_q) of order $2k$ with constant $C_0 = 2^{1/p+1/2}\sqrt{\frac{1+\delta}{1-\delta}} + 2^{1/p-1/q}$, where δ is the RIP constant and $\delta := \delta_{2k+\tilde{k}} =$* $\max\{1-\theta_-, \theta_+ - 1\}$.

Proof of Lemma 4.1 By definition for any index set $I \subseteq \{1, \ldots, m\}$ with $|I| \geq m/2$, the matrix $A_I \in \mathbb{R}^{|I|\times N}$ satisfies RIP of order $2k + \tilde{k}$ with constant $C_0 = 2^{1/p+1/2}\sqrt{\frac{1+\delta}{1-\delta}} + 2^{1/p-1/q}$, where δ is the RIP constant and $\delta := \delta_{2k+\tilde{k}} = \max\{1 - \theta_-, \theta_+ - 1\}$. By Lemma 4.3, we know that A_I satisfies the mixed null space property in norms (ℓ_p, ℓ_q) of order $2k$ with constant $C_0 = 2^{1/p+1/2}\sqrt{\frac{1+\delta}{1-\delta}} + 2^{1/p-1/q}$, in other words for any $\eta \in \mathcal{N}(A_I)$,

$$\|\eta\|_p \leq Ck^{-s}\sigma_{2k}(\eta)_q.$$

So A satisfies the mixed strong null space property. □

Corollary 4.1 *Let $k \geq 1$ and $\tilde{k} = k(\frac{N}{k})^{2-2/q}$. Assume that $A \in \mathbb{R}^{m\times N}$ satisfies the S-RIP of order $2k + \tilde{k}$ with constants $0 < \theta_- < 1 < \theta_+ < 2$. Let $\delta := \delta_{2k+\tilde{k}} =$* $\max\{1-\theta_-, \theta_+ - 1\} < 1$. *Define the decoder Δ for A by*

$$\Delta(|Ax_0|) = \underset{|Ax|=|Ax_0|}{argmin}\ \sigma_k(x)_q. \tag{4.21}$$

Then (4.19) holds with constant $2C_0$, where $C_0 = 2^{1/p+1/2}\sqrt{\frac{1+\delta}{1-\delta}} + 2^{1/p-1/q}$.

Proof of Lemma 4.1 By the Proposition 4.1, the matrix A satisfies the mixed strong null space property in (ℓ_p, ℓ_q) of order $2k$ with constant $C_0 = 2^{1/p+1/2}\sqrt{\frac{1+\delta}{1-\delta}} + 2^{1/p-1/q}$. The corollary now follows immediately from Theorem 4.6. □

Remark 4.1 Combining Theorem 2.1 and Corollary 4.1, the mixed phaseless instance optimality of order k in norms (ℓ_p, ℓ_q) can be achieved for the price of $\mathcal{O}(k(N/k)^{2-2/q} \log(N/k))$ measurements, just as with the traditional mixed (ℓ_p, ℓ_q)-norm instance optimality. Theorem 3.1 implies that the ℓ_1 decoder satisfies the $(p, q) = (2, 1)$ mixed-norm phaseless instance optimality at the price of $\mathcal{O}(k \log(N/k))$ measurements.

Appendix: Proof of Lemma 2.1

We will first need the following two Lemmas to prove Lemma 2.1.

Lemma 5.1 (Sparse Representation of a Polytope [2,12]) *Let* $s \geq 1$ *and* $\alpha > 0$. *Set*

$$T(\alpha, s) := \Big\{u \in \mathbb{R}^n : \|u\|_\infty \leq \alpha,\ \|u\|_1 \leq s\alpha\Big\}.$$

For any $v \in \mathbb{R}^n$ *let*

$$U(\alpha, s, v) := \Big\{u \in \mathbb{R}^n : supp(u) \subseteq supp(v), \|u\|_0 \leq s, \|u\|_1 = \|v\|_1, \|u\|_\infty \leq \alpha\Big\}.$$

Then $v \in T(\alpha, s)$ *if and only if* v *is in the convex hull of* $U(\alpha, s, v)$*, i.e.* v *can be expressed as a convex combination of some* $u_1, \dots, u_N$ *in* $U(\alpha, s, v)$.

Lemma 5.2 ([1, Lemma 5.3]) *Assume that* $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$. *Let* $r \leq m$ *and* $\lambda \geq 0$ *such that* $\sum_{i=1}^r a_i + \lambda \geq \sum_{i=r+1}^m a_i$. *Then for all* $\alpha \geq 1$ *we have*

$$\sum_{j=r+1}^m a_j^\alpha \leq r\left(\sqrt[\alpha]{\frac{\sum_{i=1}^r a_i^\alpha}{r}} + \frac{\lambda}{r}\right)^\alpha. \tag{5.1}$$

In particular for $\lambda = 0$ *we have*

$$\sum_{j=r+1}^m a_j^\alpha \leq \sum_{i=1}^r a_i^\alpha.$$

We are now ready to prove Lemma 2.1.

Proof Set $h := \hat{x} - x_0$. Let T_0 denote the set of the largest k coefficients of x_0 in magnitude. Then

$$\begin{aligned}\|x_0\|_1 + \rho \geq \|\hat{x}\|_1 &= \|x_0 + h\|_1 \\ &= \|x_{0,T_0} + h_{T_0} + x_{0,T_0^c} + h_{T_0^c}\|_1 \\ &\geq \|x_{0,T_0}\|_1 - \|h_{T_0}\|_1 - \|x_{0,T_0^c}\|_1 + \|h_{T_0^c}\|_1.\end{aligned}$$

It follows that

$$\begin{aligned}\|h_{T_0^c}\|_1 &\leq \|h_{T_0}\|_1 + 2\|x_{0,T_0^c}\|_1 + \rho \\ &= \|h_{T_0}\|_1 + 2\sigma_k(x_0)_1 + \rho.\end{aligned}$$

Suppose that S_0 is the index set of the k largest entries in absolute value of h. Then we can get

$$\|h_{S_0^c}\|_1 \leq \|h_{T_0^c}\|_1 \leq \|h_{T_0}\|_1 + 2\sigma_k(x_0)_1 + \rho$$
$$\leq \|h_{S_0}\|_1 + 2\sigma_k(x_0)_1 + \rho.$$

Set

$$\alpha := \frac{\|h_{S_0}\|_1 + 2\sigma_k(x_0)_1 + \rho}{k}.$$

We divide $h_{S_0^c}$ into two parts $h_{S_0^c} = h^{(1)} + h^{(2)}$, where

$$h^{(1)} := h_{S_0^c} \cdot I_{\{i:\, |h_{S_0^c}(i)| > \alpha/(t-1)\}}, \quad h^{(2)} := h_{S_0^c} \cdot I_{\{i:\, |h_{S_0^c}(i)| \leq \alpha/(t-1)\}}.$$

A simple observation is that $\|h^{(1)}\|_1 \leq \|h_{S_0^c}\|_1 \leq \alpha k$. Set

$$\ell := |\text{supp}(h^{(1)})| = \|h^{(1)}\|_0.$$

Since all non-zero entries of $h^{(1)}$ have magnitude larger than $\alpha/(t-1)$, we have

$$\alpha k \geq \|h^{(1)}\|_1 = \sum_{i \in \text{supp}(h^{(1)})} |h^{(1)}(i)| \geq \sum_{i \in \text{supp}(h^{(1)})} \frac{\alpha}{t-1} = \frac{\alpha \ell}{t-1},$$

which implies $\ell \leq (t-1)k$. Thus we have:

$$\left\langle A(h_{S_0} + h^{(1)}), Ah \right\rangle \leq \|A(h_{S_0} + h^{(1)})\|_2 \cdot \|Ah\|_2 \leq \sqrt{1+\delta} \cdot \|h_{S_0} + h^{(1)}\|_2 \cdot \epsilon. \tag{5.2}$$

Here we apply the facts that $\|h_{S_0} + h^{(1)}\|_0 = \ell + k \leq tk$ and A satisfies the RIP of order tk with $\delta := \delta_{tk}^A$. We shall assume at first that tk as an integer. Note that $\|h^{(2)}\|_\infty \leq \frac{\alpha}{t-1}$ and

$$\|h^{(2)}\|_1 = \|h_{S_0^c}\|_1 - \|h^{(1)}\|_1 \leq k\alpha - \frac{\alpha \ell}{t-1} = (k(t-1) - \ell)\frac{\alpha}{t-1}. \tag{5.3}$$

We take $s := k(t-1) - \ell$ in Lemma 5.1 and obtain that $h^{(2)}$ is a weighted mean

$$h^{(2)} = \sum_{i=1}^{N} \lambda_i u_i, \qquad 0 \leq \lambda_i \leq 1, \quad \sum_{i=1}^{N} \lambda_i = 1$$

where $\|u_i\|_0 \leq k(t-1) - \ell$, $\|u_i\|_1 = \|h^{(2)}\|_1$, $\|u_i\|_\infty \leq \alpha/(t-1)$ and $\text{supp}(u_i) \subseteq \text{supp}(h^{(2)})$. Hence

$$\begin{aligned}\|u_i\|_2 \le \sqrt{\|u_i\|_0} \cdot \|u_i\|_\infty &= \sqrt{k(t-1)-\ell} \cdot \|u_i\|_\infty \\ &\le \sqrt{k(t-1)} \cdot \|u_i\|_\infty \\ &\le \alpha\sqrt{k/(t-1)}.\end{aligned}$$

Now for $0 \le \mu \le 1$ and $d \ge 0$, which will be chosen later, set

$$\beta_j := h_{S_0} + h^{(1)} + \mu \cdot u_j, \quad j = 1, \ldots, N.$$

Then for fixed $i \in [1, N]$

$$\begin{aligned}\sum_{j=1}^N \lambda_j \beta_j - d\beta_i &= h_{S_0} + h^{(1)} + \mu \cdot h^{(2)} - d\beta_i \\ &= (1-\mu-d)(h_{S_0} + h^{(1)}) - d\mu u_i + \mu h.\end{aligned}$$

Recall that $\alpha = \frac{\|h_{S_0}\|_1 + 2\sigma_k(x_0)_1 + \rho}{k}$. Thus

$$\begin{aligned}\|u_i\|_2 &\le \sqrt{k/(t-1)}\alpha \\ &\le \frac{\|h_{S_0}\|_2}{\sqrt{t-1}} + \frac{2\sigma_k(x_0)_1 + \rho}{\sqrt{k(t-1)}} \\ &\le \frac{\|h_{S_0} + h^{(1)}\|_2}{\sqrt{t-1}} + \frac{2\sigma_k(x_0)_1 + \rho}{\sqrt{k(t-1)}} \\ &= \frac{z+R}{\sqrt{t-1}},\end{aligned} \tag{5.4}$$

where $z := \|h_{S_0} + h^{(1)}\|_2$ and $R := \frac{2\sigma_k(x_0)_1 + \rho}{\sqrt{k}}$. It is easy to check the following identity:

$$\begin{aligned}&(2d-1) \sum_{1\le i<j\le N} \lambda_i \lambda_j \|A(\beta_i - \beta_j)\|_2^2 \\ &\quad = \sum_{i=1}^N \lambda_i \Big\| A(\sum_{j=1}^N \lambda_j \beta_j - d\beta_i) \Big\|_2^2 - \sum_{i=1}^N \lambda_i (1-d)^2 \|A\beta_i\|_2^2,\end{aligned} \tag{5.5}$$

provided that $\sum_{i=1}^N \lambda_i = 1$. Choose $d = 1/2$ in (5.5) we then have

$$\sum_{i=1}^N \lambda_i \Big\| A\Big((\frac{1}{2} - \mu)(h_{S_0} + h^{(1)}) - \frac{\mu}{2} u_i + \mu h\Big) \Big\|_2^2 - \sum_{i=1}^N \frac{\lambda_i}{4} \|A\beta_i\|_2^2 = 0.$$

Note that for $d = 1/2$,

$$\begin{aligned}&\Big\|A\Big((\frac{1}{2}-\mu)(h_{S_0}+h^{(1)})-\frac{\mu}{2}u_i+\mu h\Big)\Big\|_2^2\\&\quad=\Big\|A\Big((\frac{1}{2}-\mu)(h_{S_0}+h^{(1)})-\frac{\mu}{2}u_i\Big)\Big\|_2^2\\&\qquad+2\Big\langle A\Big((\frac{1}{2}-\mu)(h_{S_0}+h^{(1)})-\frac{\mu}{2}u_i\Big),\mu Ah\Big\rangle+\mu^2\|Ah\|_2^2.\end{aligned}$$

It follows from $\sum_{i=1}^N \lambda_i = 1$ and $h^{(2)} = \sum_{i=1}^N \lambda_i u_i$ that

$$\begin{aligned}&\sum_{i=1}^N\lambda_i\Big\|A\Big((\frac{1}{2}-\mu)(h_{S_0}+h^{(1)})-\frac{\mu}{2}u_i+\mu h\Big)\Big\|_2^2\\&=\sum_i\lambda_i\Big\|A\Big((\frac{1}{2}-\mu)(h_{S_0}+h^{(1)})-\frac{\mu}{2}u_i\Big)\Big\|_2^2\\&\quad+2\Big\langle A\Big((\frac{1}{2}-\mu)(h_{S_0}+h^{(1)})-\frac{\mu}{2}h^{(2)}\Big),\mu Ah\Big\rangle+\mu^2\|Ah\|_2^2\\&=\sum_i\lambda_i\Big\|A\Big((\frac{1}{2}-\mu)(h_{S_0}+h^{(1)})-\frac{\mu}{2}u_i\Big)\Big\|_2^2\\&\quad+\mu(1-\mu)\Big\langle A(h_{S_0}+h^{(1)}),Ah\Big\rangle-\sum_{i=1}^N\frac{\lambda_i}{4}\|A\beta_i\|_2^2.\end{aligned}\tag{5.6}$$

Set $\mu = \sqrt{t(t-1)} - (t-1)$. We next estimate the three terms in (5.6). Noting that $\|h_{S_0}\|_0 \le k$, $\|h^{(1)}\|_0 \le \ell$ and $\|u_i\|_0 \le s = k(t-1) - \ell$, we obtain

$$\|\beta_i\|_0 \le \|h_{S_0}\|_0 + \|h^{(1)}\|_0 + \|u_i\|_0 \le t \cdot k$$

and $\|(\frac{1}{2}-\mu)(h_{S_0}+h^{(1)}) - \frac{\mu}{2}u_i\|_0 \le t \cdot k$. Since A satisfies the RIP of order $t \cdot k$ with δ, we have

$$\begin{aligned}&\Big\|A\Big((\frac{1}{2}-\mu)(h_{S_0}+h^{(1)})-\frac{\mu}{2}u_i\Big)\Big\|_2^2\\&\quad\le(1+\delta)\|(\frac{1}{2}-\mu)(h_{S_0}+h^{(1)})-\frac{\mu}{2}u_i\|_2^2\\&\quad=(1+\delta)\Big((\frac{1}{2}-\mu)^2\|(h_{S_0}+h^{(1)})\|_2^2+\frac{\mu^2}{4}\|u_i\|_2^2\Big)\\&\quad=(1+\delta)\Big((\frac{1}{2}-\mu)^2z^2+\frac{\mu^2}{4}\|u_i\|_2^2\Big)\end{aligned}$$

and

$$\begin{aligned}\|A\beta_i\|_2^2\ge(1-\delta)\|\beta_i\|_2^2&=(1-\delta)(\|h_{S_0}+h^{(1)}\|_2^2+\mu^2\cdot\|u_i\|_2^2)\\&=(1-\delta)(z^2+\mu^2\cdot\|u_i\|_2^2).\end{aligned}$$

Combining the result above with (5.2) and (5.4) we get

$$
\begin{aligned}
0 \leq\ & (1+\delta)\sum_{i=1}^{N}\lambda_i\Big((\frac{1}{2}-\mu)^2 z^2+\frac{\mu^2}{4}\|u_i\|_2^2\Big)+\mu(1-\mu)\sqrt{1+\delta}\cdot z\cdot\epsilon\\
&-(1-\delta)\sum_{i=1}^{N}\frac{\lambda_i}{4}(z^2+\mu^2\|u_i\|_2^2)\\
=\ & \sum_{i=1}^{N}\lambda_i\Big(\Big((1+\delta)(\frac{1}{2}-\mu)^2-\frac{1-\delta}{4}\Big)z^2+\frac{\delta}{2}\mu^2\|u_i\|_2^2\Big)+\mu(1-\mu)\sqrt{1+\delta}\cdot z\cdot\epsilon\\
\leq\ & \sum_{i=1}^{N}\lambda_i\Big(\Big((1+\delta)(\frac{1}{2}-\mu)^2-\frac{1-\delta}{4}\Big)z^2+\frac{\delta}{2}\mu^2\frac{(z+R)^2}{t-1}\Big)\\
&+\mu(1-\mu)\sqrt{1+\delta}\cdot z\cdot\epsilon\\
=\ & \Big((\mu^2-\mu)+\delta\Big(\frac{1}{2}-\mu+(1+\frac{1}{2(t-1)})\mu^2\Big)\Big)z^2\\
&+\Big(\mu(1-\mu)\sqrt{1+\delta}\cdot\epsilon+\frac{\delta\mu^2 R}{t-1}\Big)z+\frac{\delta\mu^2R^2}{2(t-1)}\\
=\ & -t\Big((2t-1)-2\sqrt{t(t-1)}\Big)(\sqrt{\frac{t-1}{t}}-\delta)z^2\\
&+\Big(\mu^2\sqrt{\frac{t}{t-1}}\sqrt{1+\delta}\cdot\epsilon+\frac{\delta\mu^2R}{t-1}\Big)z+\frac{\delta\mu^2R^2}{2(t-1)}\\
=\ & \frac{\mu^2}{t-1}\Big(-t(\sqrt{\frac{t-1}{t}}-\delta)z^2+(\sqrt{t(t-1)(1+\delta)}\epsilon+\delta R)z+\frac{\delta R^2}{2}\Big),
\end{aligned}
$$

which is a quadratic inequality for z. We know $\delta < \sqrt{(t-1)/t}$. So by solving the above inequality we get

$$
\begin{aligned}
z \leq\ & \frac{(\sqrt{t(t-1)(1+\delta)}\epsilon+\delta R)+\Big((\sqrt{t(t-1)(1+\delta)}\epsilon+\delta R)^2+2t(\sqrt{(t-1)/t}-\delta)\delta R^2\Big)^{1/2}}{2t(\sqrt{(t-1/t)}-\delta)}\\
\leq\ & \frac{\sqrt{t(t-1)(1+\delta)}}{t(\sqrt{(t-1)/t}-\delta)}\epsilon+\frac{2\delta+\sqrt{2t(\sqrt{(t-1)/t}-\delta)\delta}}{2t(\sqrt{(t-1)/t}-\delta)}R.
\end{aligned}
$$

Finally, noting that $\|h_{S_0^c}\|_1 \leq \|h_{S_0}\|_1 + R\sqrt{k}$, in the Lemma 5.2, if we set $m = N$, $r = k$, $\lambda = R\sqrt{k} \geq 0$ and $\alpha = 2$ then $\|h_{S_0^c}\|_2 \leq \|h_{S_0}\|_2 + R$. Hence

$$
\begin{aligned}
\|h\|_2 &= \sqrt{\|h_{S_0}\|_2^2+\|h_{S_0^c}\|_2^2}\\
&\leq \sqrt{\|h_{S_0}\|_2^2+(\|h_{S_0}\|_2+R)^2}
\end{aligned}
$$

$$\leq \sqrt{2\|h_{S_0}\|_2^2} + R \leq \sqrt{2}z + R$$
$$\leq \frac{\sqrt{2(1+\delta)}}{1-\sqrt{t/(t-1)}\delta}\epsilon + \left(\frac{\sqrt{2}\delta + \sqrt{t(\sqrt{(t-1)/t}-\delta)\delta}}{t(\sqrt{(t-1)/t}-\delta)} + 1\right)R.$$

Substitute R into this inequality and the conclusion follows.

For the case where $t \cdot k$ is not an integer, we set $t^* := \lceil tk \rceil / k$, then $t^* > t$ and $\delta_{t^*k} = \delta_{tk} < \sqrt{\frac{t-1}{t}} < \sqrt{\frac{t^*-1}{t^*}}$. We can then prove the result by working on δ_{t^*k}. □

Acknowledgments Yang Wang was supported in part by the AFOSR grant FA9550-12-1-0455 and NSF grant IIS-1302285. Zhiqiang Xu was supported by NSFC grant (11171336, 11422113, 11021101, 11331012) and by National Basic Research Program of China (973 Program 2015CB856000).

References

1. Cai, T.T., Zhang, A.: Sharp RIP bound for sparse signal and low-rank matrix recovery. Appl. Comput. Harmon. Anal. **35**(1), 74–93 (2013)
2. Cai, T.T., Zhang, A.: Sparse representation of a polytope and recovery of sparse signals and low-rank matrices. IEEE Trans. Inf. Theory **60**(1), 122–132 (2014)
3. Candès, E.J., Tao, T.: Decoding by linear programming. IEEE Trans. Inf. Theory **51**(12), 4203–4215 (2005)
4. Candès, E.J., Romberg, J.K., Tao, T.: Stable signal recovery from incomplete and inaccurate measurements. Commun. Pure Appl. Math. **59**(8), 1207–1223 (2006)
5. Cohen, A., Dahmen, W., Devore, R.: Compressed sensing and best k-term approximation. J. Am. Math. Soc. **22**(1), 211–231 (2009)
6. Eldar, Y.C., Mendelson, S.: Phase retrieval: stability and recovery guarantees. Appl. Comput. Harmon. Anal. **36**(3), 473–494 (2014)
7. Han, B., Xu, Z. Q.: Robustness properties of dimensionality reduction with Gaussian random matrices. arXiv:1501.01695 (2015)
8. Iwen, M., Viswanathan, A., Wang, Y.: Robust sparse phase retrieval made easy. arXiv:1410.5295 (2015)
9. Moravec, M.L., Romberg, J.K., Baraniuk, R.G.: Compressive phase retrieval. In: SPIE Proceedings, vol. 6701 (2007)
10. Voroninski, V., Xu, Z.Q.: A strong restricted isometry property, with an application to phaseless compressed sensing. Appl. Comput. Harmon. Anal. (2015). doi:10.1016/j.acha.2015.06.004
11. Wang, Y., Xu, Z.Q.: Phase retrieval for sparse signals. Appl. Comput. Harmon. Anal. **37**(3), 531–544 (2014)
12. Xu, G.W., Xu, Z.Q.: On the ℓ_1-sparse decomposition of signals. J. Oper. Res. Soc. China **1**(4), 537–541 (2013)