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A Spectral Multiplier Theorem Associated with a Schrödinger Operator

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Abstract We establish a Hörmander type spectral multiplier theorem for a Schrödinger operator $H = -\Delta + V(x)$ in $\mathbb{R}^3$, provided V is contained in a large class of short range potentials. This result does not require the Gaussian heat kernel estimate for the semigroup e^{-tH}, and indeed the operator H may have negative eigenvalues. As an application, we show local well-posedness of a 3d quintic nonlinear Schrödinger equation with a potential.

Keywords Spectral multiplier theorem · Schrödinger operator · Schrödinger equation

Mathematics Subject Classification 42B15 · 42B20 · 35Q55

1 Introduction

1.1 Statement of the Main Theorem

In this paper, we establish a Hörmander type spectral multiplier theorem for a Schrödinger operator $H = -\Delta + V$ in $\mathbb{R}^3$, provided that V is contained in a large class of short range potentials. Precisely, we assume that V is contained in $\mathcal{K}_0 \cap L^{3/2,\infty}$, where $\mathcal{K}_0$ is the norm closure of bounded, compactly supported functions with respect to the *global Kato norm*

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$$\|V\|_{\mathcal{K}} := \sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|V(y)|}{|x-y|}dy, \tag{1.1}$$

and $L^{3/2,\infty}$ is the weak $L^{3/2}$-space. We also assume that H has no eigenvalue or resonance on the positive real-line $[0,+\infty)$. By a *resonance*, we mean a complex number λ such that the equation $\psi+(-\Delta-\lambda\pm i0)^{-1}V\psi=0$ has a slowly decaying solution $\psi\in L^{2,-s}\setminus L^2$ for any $s>\frac{1}{2}$, where $L^{2,s}=\{\langle x\rangle^s f\in L^2\}$.

By the above assumptions, the operator H is self-adjoint on L^2. Moreover, its spectrum $\sigma(H)$ consists of purely absolutely continuous spectrum on the positive real-line $[0,+\infty)$ and at most finitely many negative eigenvalues [2]. Therefore, for a bounded Borel function $m:\sigma(H)\subset\mathbb{R}\to\mathbb{C}$, one can define a spectral multiplier $m(H)$ as a bounded operator on L^2 via functional calculus.

A natural question is then to find a sufficient condition to extend boundedness of the multiplier $m(H)$ to L^p for $p\neq 2$. Such a condition is typically given in terms of regularity of symbols. To measure regularity of a symbol $m:\sigma(H)\to\mathbb{C}$, we define a Sobolev type norm by

$$\|m\|_{\mathcal{H}(s)} := \sum_{\lambda_j:\text{negative eigenvalues}}|m(\lambda_j)| + \sup_{t>0}\|\chi(\lambda)m((t\lambda)^2)\|_{W^{s,2}_\lambda((0,+\infty))}, \tag{1.2}$$

where $\chi\in C_c^\infty(\mathbb{R})$ is a standard dyadic partition of unity function such that χ is supported in $[\frac{1}{2},2]$ and $\sum_{N\in 2^{\mathbb{Z}}}\chi(\frac{\cdot}{N})\equiv 1$ on $(0,+\infty)$, and $W^{s,2}$ is the L^2-Sobolev space of order s.

Our main result is the following.

Theorem 1.1 (Spectral multiplier theorem) *Suppose that $V\in\mathcal{K}_0\cap L^{3/2,\infty}$ and $H=-\Delta+V$ has no eigenvalue or resonance on* $[0,+\infty)$. *We also assume that for $s>2$, the symbol $m:\sigma(H)\to\mathbb{C}$ satisfies $\|m\|_{\mathcal{H}(s)}<\infty$. Then, we have*

$$\|m(H)\|_{L^p\to L^p}\lesssim\|m\|_{\mathcal{H}(s)},\quad \forall 1<p<\infty. \tag{1.3}$$

When $V=0$, Theorem 1.1 is simply the classical Hörmander–Mikhlin multiplier theorem [4].

There are several ways to prove the spectral multiplier theorem for Schrödinger operators. For an operator A, we say that the semigroup e^{-tA} satisfies the Gaussian heat kernel estimate if the kernel of e^{-tA}, denoted by $e^{-tA}(x,y)$, obeys

$$e^{-tA}(x,y)\lesssim t^{-3/2}e^{-\frac{|x-y|^2}{ct}},\quad \forall t>0 \tag{1.4}$$

for some $c>0$. Gaussian upper bounds for the heat kernels have been used successfully to prove spectral multiplier theorems for rather general operators, not necessarily Schrödinger operators (see [4,5,16] and references therein). In the case of the Schrödinger operator $H=-\Delta+V$ in $\mathbb{R}^3$, if $V_+=\max(V,0)$ is in local Kato class, that is,

$$\lim_{r\to 0+}\sup_{x\in\mathbb{R}^3}\int_{|x-y|\le r}\frac{|V_+(y)|}{|x-y|}dy=0, \tag{1.5}$$

and if $V_- = \min(V, 0) \in \mathcal{K}_0$ and $\|V_-\|_{\mathcal{K}} < 4\pi$, then it is known that the semigroup e^{-tH} satisfies the Gaussian heat kernel estimate (1.4) [7,20]. The spectral multiplier theorem for H then follows from [5, Theorem 3.1]. However, for Gaussian upper bounds (1.4), operators need to be positive definite, while the Schrödinger operator in Theorem 1.1 may have negative eigenvalues.

One can also use the wave operators to show the spectral multiplier theorem. The forward-in-time (backward-in-time, resp) wave operator of the Schrödinger operator $H = -\Delta + V$ is defined by

$$W_+ = \underset{t\to+\infty}{s\text{-}\lim} e^{itH}e^{-it(-\Delta)} \quad \Big(W_- = \underset{t\to-\infty}{s\text{-}\lim} e^{itH}e^{-it(-\Delta)}, \text{resp}\Big). \tag{1.6}$$

An important feature of wave operators is its intertwining property, that is, $P_c f(H) = W_\pm f(-\Delta)(W_\pm)^*$, where P_c is the spectral projection to the continuous spectrum and $(W_\pm)^*$ is the dual of $W_\pm$. In [22], Yajima proved that the wave operators $W_\pm$ are bounded on L^p for all $1 \le p \le \infty$, provided that $|V(x)| \lesssim \langle x\rangle^{-5-\epsilon}$ for $\epsilon > 0$, and zero is not an eigenvalue or a resonance of H. Later, in [1], Beceanu extended this result to a larger space

$$B := \left\{ V : \sum_{k=-\infty}^{\infty} 2^{k/2}\|V(x)\|_{L_x^2(2^k\le|x|<2^{k+1})} < \infty \right\}. \tag{1.7}$$

The spectral multiplier theorem then follows immediately from the intertwining property and boundedness of wave operators and the classical Hörmander–Mikhlin multiplier theorem, since

$$\begin{aligned} \|P_c f(H)\|_{L^p\to L^p} &= \|W_\pm f(-\Delta)(W_\pm)^*\|_{L^p\to L^p} \\ &\lesssim \|f(-\Delta)(W_\pm)^*\|_{L^p\to L^p} \lesssim \|(W_\pm)^*\|_{L^p\to L^p} < \infty \end{aligned} \tag{1.8}$$

and $(I - P_c)f(H)$ is bounded on L^p by Lemma 3.6. Theorem 1.1 improves the spectral multiplier theorem as a consequence boundedness of the wave operator, in that the potential class $\mathcal{K}_0 \cap L^{3/2,\infty}$ is larger than the potential class B. Note that a potential having many singular points, such as $\sum_{k=1}^N 1_{|x-x_j|\le 1}\frac{1}{|x-x_j|^{2-\epsilon}}$ with $x_j \ne x_k$ and $\epsilon > 0$, is contained in $\mathcal{K}_0 \cap L^{3/2,\infty}$, but not in B.

Our proof of the spectral multiplier theorem is perturbative, and it relies heavily on the explicit integral representation of the kernel of the multiplier. We consider the spectral multiplier $m(H)P_c$ as a perturbation of the Fourier multiplier $m(-\Delta)$, and then we show that the difference $(m(H)P_c - m(-\Delta))$ is bounded on L^p. In order to estimate the difference, we first decompose it into its dyadic pieces

$$\sum_{N\in 2^{\mathbb{Z}}} \chi\left(\tfrac{\sqrt{H}}{N}\right)\Big(m(H) - m(-\Delta)\Big), \tag{1.9}$$

where χ is the function given in (1.2). Then, we generate a formal series expansion for each dyadic piece to get explicit integral representations of kernels of terms in the series using the free resolvent formula

$$((-\Delta - z)^{-1} f)(x) = \int_{\mathbb{R}^3} \frac{e^{i\sqrt{z}|x-y|}}{4\pi|x-y|} f(y)dy. \tag{1.10}$$

We estimate these integral kernels. Summing them up, we prove the spectral multiplier theorem.

A key observation is that in spite of the singular integral nature of both $m(H)P_c$ and $m(-\Delta)$ as Calderon–Zygmund operators, the kernel of their difference is less singular than usual Calderon–Zygmund operators. This fact is essential in our analysis, since it allows us to avoid using the delicate classical Calderon–Zygmund theory for the complicated operator $m(H)$ (see Remark 4.4). Instead, we just make use of the fractional integration inequality and Hölder inequality.

1.2 Application to NLS

The choice of the potential class in the main theorem is motivated by the following nonlinear application.

First, we recall the Strichartz estimates for the linear propagator e^{-itH}.

Proposition 1.2 (Strichartz estimates) *If $V \in \mathcal{K}_0$ and H has no eigenvalue or resonance on $[0, +\infty)$, then*

$$\|e^{-itH} P_c f\|_{L_t^q L_x^r} \lesssim \|f\|_{L^2}, \tag{1.11}$$

$$\left\| \int_0^t e^{-i(t-s)H} P_c F(s) ds \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^2 L_x^{6/5}}, \tag{1.12}$$

where $\frac{2}{q} + \frac{3}{r} = \frac{3}{2}$ *and* $2 \le q, r \le \infty$.

Proof Beceanu–Goldberg [2] proved the dispersive estimate

$$\|e^{-itH} P_c\|_{L^1 \to L^\infty} \lesssim |t|^{-3/2}, \tag{1.13}$$

where P_c is the spectral projection to the continuous spectrum. Strichartz estimates then follow by the argument of Keel–Tao [15]. □

Remark 1.3 The dispersive estimate of the form (1.13) was first proved by Journé–Soffer–Sogge under suitable assumptions on potentials [14]. The assumptions have been relaxed by Rodnianksi–Schlag [17], Goldberg–Schlag [10] and Goldberg [8,9]. Recently, Beceanu–Goldberg established (1.13) for a scaling-critical potential class $\mathcal{K}_0$ [2].

An interesting question is then whether one can use the above Strichartz estimates to show the local well-posedness (LWP), for instance, for a 3d quintic nonlinear Schrödinger equation with a potential

$$iu_t + \Delta u - Vu \pm |u|^4 u = 0;\ u(0) = u_0 \qquad (\text{NLS}_V)$$

assuming that V satisfies the conditions in Proposition 1.2. However, if one tries to show local well-posedness by the standard contraction mapping argument as in [4,21], one will realize that there is a subtle problem, mainly because the linear propagator e^{-itH} does not commute with the differential operators from the Sobolev norms.

We overcome this subtle problem by the two norm estimates lemma, whose proof relies on the spectral multiplier theorem.

Lemma 1.4 (Two norm estimates) *If* $V \in \mathcal{K}_0 \cap L^{3/2,\infty}$ *and* H *has no eigenvalue or resonance on the positive real-line* $[0, +\infty)$*, then*

$$\|H^{\frac{s}{2}} P_c(-\Delta)^{-\frac{s}{2}} f\|_{L^r} \lesssim \|f\|_{L^r}, \tag{1.14}$$

$$\|(-\Delta)^{\frac{s}{2}} H^{-\frac{s}{2}} P_c f\|_{L^r} \lesssim \|f\|_{L^r}. \tag{1.15}$$

for $0 \leq s \leq 2$ *and* $1 < r < \frac{3}{s}$.

Together with Strichartz estimates and the two norm estimates lemma, we prove local well-posedness.

Theorem 1.5 (LWP) *Suppose that* $V \in \mathcal{K}_0 \cap L^{3/2,\infty}$ *and* H *has no eigenvalue or resonance on the positive real-line* $[0, +\infty)$. *Then,* (NLS_V) *is locally well-posed in* $\dot{H}^1$.

Remark 1.6 (i) The range of r in the two norm estimates lemma is sharp. See the counterexample in [19].

(ii) The additional hypothesis $V \in L^{3/2,\infty}$, compared to Strichartz estimates, is from the two norm estimates lemma. In the proof of the two norm estimates lemma, we used this additional assumption.

1.3 Organization of the Paper

The outline of the proof of Theorem 1.1 is given in Sect. 2. We decompose the spectral representation of the difference $(m(H)P_c - m(-\Delta))$ into the low, medium and high frequencies, and then analyze them separately in Sects. 4, 5 and 6. In Sect. 7, we establish LWP of a 3d quintic nonlinear Schrödinger equation with a potential.

1.4 Notations

For an integral operator T, its integral kernel is denoted by $T(x, y)$. We denote by A“ = ”B the formal identity which will be proved later.

2 Reduction to the Key Lemma

Suppose that $V \in \mathcal{K}_0$ and H has no eigenvalue or resonance on $[0, +\infty)$. Then, the spectrum of H, denoted by $\sigma(H)$, consists of purely continuous spectrum on the positive real-line $[0, +\infty)$ and at most finitely many negative eigenvalues. For $z \notin \sigma(H)$, we define the resolvent by $R_V(z) := (H - z)^{-1}$, and denote

$$R_V^{\pm}(\lambda) := \text{s}-\lim_{\epsilon \to 0+} R_V(\lambda \pm i\epsilon). \tag{2.1}$$

Let P_c be the spectral projection on the continuous spectrum. Then, by the Stone's formula, the spectral multiplier operator $m(H)P_c$ is represented by

$$m(H)P_c = \frac{1}{2\pi i}\int_0^\infty m(\lambda)[R_V^+(\lambda) - R_V^-(\lambda)]d\lambda = \frac{1}{\pi}\int_0^\infty m(\lambda)\,\text{Im}\,R_V^+(\lambda)d\lambda. \tag{2.2}$$

Applying the identity

$$\begin{aligned} R_V^+(\lambda) &= R_0^+(\lambda)(I + VR_0^+(\lambda))^{-1} \\ &= R_0^+(\lambda)\Big(I - (I + VR_0^+(\lambda))^{-1}VR_0^+(\lambda)\Big) \\ &= R_0^+(\lambda) - R_0^+(\lambda)(I + VR_0^+(\lambda))^{-1}VR_0^+(\lambda), \end{aligned} \tag{2.3}$$

we split $m(H)P_c$ into the pure and the perturbed parts,

$$\begin{aligned} m(H)P_c &= \frac{1}{\pi}\int_0^\infty m(\lambda)\,\text{Im}\,R_0^+(\lambda)d\lambda \\ &\quad - \frac{1}{\pi}\int_0^\infty m(\lambda)\,\text{Im}[R_0^+(\lambda)(I + VR_0^+(\lambda))^{-1}VR_0^+(\lambda)]d\lambda \\ &=: m(-\Delta) + \text{Pb}, \end{aligned} \tag{2.4}$$

where $m(-\Delta)$ is the Fourier multiplier such that $\widehat{m(-\Delta)f}(\xi) = m(|\xi|^2)\hat{f}(\xi)$. For the pure part $m(-\Delta)$, it follows from the classical Hörmander–Mikhlin multiplier theorem [13] that for $s > \frac{3}{2}$,

$$\|m(-\Delta)\|_{L^p \to L^p} \lesssim \|m\|_{\mathcal{H}(s)}, \quad \forall 1 < p < \infty. \tag{2.5}$$

Therefore, it suffices to show boundedness of the perturbed part. For the perturbed part Pb, we further decompose it into dyadic pieces. Let χ be the smooth dyadic partition of unity function chosen in (1.2), and decompose

$$\text{Pb} = \sum_{N \in 2^{\mathbb{Z}}} \text{Pb}_N, \tag{2.6}$$

where

$$\text{Pb}_N := -\frac{1}{\pi}\int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda})\,\text{Im}[R_0^+(\lambda)(I + VR_0^+(\lambda))^{-1}VR_0^+(\lambda)]d\lambda. \tag{2.7}$$

For a small dyadic number N_0 and a large dyadic number N_1 to be chosen later, we denote the low (high, resp) frequency part by

$$\mathrm{Pb}_{\leq N_0} := \sum_{N \leq N_0} \mathrm{Pb}_N \left(\mathrm{Pb}_{\geq N_1} := \sum_{N \geq N_1} \mathrm{Pb}_N, \text{resp} \right). \tag{2.8}$$

In the next four sections, we will show the following lemma.

Lemma 2.1 (Key lemma) *Suppose that* $V \in \mathcal{K}_0 \cap L^{3/2,\infty}$ *and* H *has no eigenvalue or resonance on* $[0, +\infty)$. *Let* $s > 2$. *Then, there exists* $p > 1$ *but sufficiently close to 1 such that the following hold.*

(i) (High frequency) There exists $N_1 = N_1(V) \gg 1$ *such that*

$$\|\mathrm{Pb}_{\geq N_1}\|_{L^{p,1} \to L^{p,\infty}} \lesssim \|m\|_{\mathcal{H}(s)}, \tag{2.9}$$

where $L^{p,1}$ *and* $L^{p,\infty}$ *are the Lorentz spaces (see* "Appendix"*).*

(ii) (Low frequency) There exists $N_0 = N_0(V) \ll 1$ *such that*

$$\|\mathrm{Pb}_{\leq N_0}\|_{L^{p,1} \to L^{p,\infty}} \lesssim \|m\|_{\mathcal{H}(s)}. \tag{2.10}$$

(iii) (Medium frequency) For $N_0 < N < N_1$,

$$\|\mathrm{Pb}_N\|_{L^{p,1} \to L^{p,\infty}} \lesssim_{N_0, N_1} \|m\|_{\mathcal{H}(s)}. \tag{2.11}$$

Proof of Theorem 1.1, assuming Lemma 2.1 Let $p > 1$ be sufficiently close to 1 as in Lemma 2.1. Summing the estimates in Lemma 2.1, we prove that Pb is bounded from $L^{p,1}$ to $L^{p,\infty}$. Then, it follows from the classical Hörmander–Mikhlin multiplier theorem that $m(H)P_c = m(-\Delta) + \mathrm{Pb}$ is bounded from $L^{p,1}$ to $L^{p,\infty}$. Moreover, by Lemma 3.6 (see below), $m(H) : L^{p,1} \to L^{p,\infty}$ is bounded.

Recall that by functional calculus, $m(H)$ is bounded on L^2. Thus, by the real interpolation lemma (Corollary 7.8), $m(H)$ is bounded on L^p for all $1 < p \leq 2$. Finally, applying the spectral multiplier theorem to the symbol $\bar{m}$ and the standard duality argument with $m(H) = \bar{m}(H)^*$, we conclude that $m(H)$ is bounded on L^p for $2 < p < \infty$. □

3 Preliminaries

3.1 Resolvent Estimates

Following Beceanu–Goldberg [2], we collect kernel estimates for $VR_0^+(\lambda)$, $V(R_0^+(\lambda) - R_0^+(\lambda_0))$, $(VR_0^+(\lambda))^4$ and $(I + VR_0^+(\lambda))^{-1}$, all of which will play as *building blocks* to analyze the kernel of Pb_N.

Lemma 3.1 (Resolvent estimates) *Suppose that* $V \in \mathcal{K}_0$.

(i) For $\lambda \geq 0$,

$$\|VR_0^+(\lambda)f\|_{L^1} \leq \frac{\|V\|_{\mathcal{K}}}{4\pi}\|f\|_{L^1}. \tag{3.1}$$

(ii) Define the difference operator by

$$B_{\lambda,\lambda_0} := V(R_0^+(\lambda) - R_0^+(\lambda_0)). \tag{3.2}$$

For $\epsilon > 0$*, there exist* $\delta > 0$ *and an integral operator* $B : L^1 \to L^1$ *such that for* $|\lambda - \lambda_0| \leq \delta$ *and* $\lambda, \lambda_0 \geq 0$,

$$|B_{\lambda,\lambda_0}(x, y)| \leq B(x, y), \text{ and} \|B(x, y)\|_{L_y^\infty L_x^1} \leq \epsilon. \tag{3.3}$$

(iii) For $\epsilon > 0$*, there exist* $N_1 \gg 1$ *and an integral operator* $D = D_\epsilon : L^1 \to L^1$ *such that for* $\lambda \geq N_1$,

$$|(VR_0^+(\lambda))^4(x, y)| \leq D(x, y), \text{ and} \|D(x, y)\|_{L_y^\infty L_x^1} \leq \epsilon. \tag{3.4}$$

Proof (i) By the free resolvent formula $R_0^+(\lambda)(x, y) = \frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|}$, the Minkowski inequality and the definition of the global Kato norm (1.1), we have

$$\|VR_0^+(\lambda)f\|_{L^1} \leq \int_{\mathbb{R}^3} \left\| \frac{|V(x)|}{4\pi|x - y|} \right\|_{L_x^1} |f(y)|dy \leq \frac{\|V\|_{\mathcal{K}}}{4\pi}\|f\|_{L^1}. \tag{3.5}$$

(ii) For $\epsilon > 0$, decompose $V = V_1 + V_2$ such that V_1 is bounded and compactly supported and $\|V_2\|_{\mathcal{K}} \leq \epsilon$. We choose $\delta > 0$ such that $|\sqrt{\lambda} - \sqrt{\lambda_0}| \leq \epsilon\|V_1\|_{L^1}^{-1}$ for all $\lambda, \lambda_0 \geq 0$ with $|\lambda - \lambda_0| \leq \delta$. By the mean-value theorem,

$$\begin{aligned} |B_{\lambda,\lambda_0}(x, y)| &\leq \left| \frac{V_1(x)(e^{i\sqrt{\lambda}|x-y|} - e^{i\sqrt{\lambda_0}|x-y|})}{4\pi|x - y|} \right| + \left| \frac{V_2(x)(e^{i\sqrt{\lambda}|x-y|} - e^{i\sqrt{\lambda_0}|x-y|})}{4\pi|x - y|} \right| \\ &\leq \frac{|V_1(x)||\sqrt{\lambda} - \sqrt{\lambda_0}|}{4\pi} + \frac{|V_2(x)|}{2\pi|x - y|} \\ &\leq \frac{\epsilon|V_1(x)|}{4\pi\|V_1\|_{L^1}} + \frac{|V_2(x)|}{2\pi|x - y|} =: B_\epsilon(x, y). \end{aligned} \tag{3.6}$$

Then, we have

$$\|B_\epsilon(x, y)\|_{L_y^\infty L_x^1} \leq \frac{\epsilon}{4\pi} + \frac{\|V_2\|_{\mathcal{K}}}{2\pi} \leq \epsilon. \tag{3.7}$$

(iii) Similarly, for $\epsilon > 0$, decompose $V = V_1 + V_2$ such that V_1 is bounded and compactly supported and $\|V_2\|_{\mathcal{K}} \leq \epsilon\|V\|_{\mathcal{K}}^{-3}$. We then write

$$|(VR_0^+(\lambda))^4(x, y)| \leq |(V_1R_0^+(\lambda))^4(x, y)| + |(VR_0^+(\lambda))^4(x, y) - (V_1R_0^+(\lambda))^4(x, y)|. \tag{3.8}$$

For the first term, by the fractional integration inequalities, the Hölder inequalities in the Lorentz spaces (Lemma 7.5) and the free resolvent estimate $\|R_0^+(\lambda)\|_{L^{4/3}\to L^4} \lesssim \langle\lambda\rangle^{-1/4}$ [11, Lemma 2.1], we get

$$
\begin{aligned}
&\|R_0^+(\lambda)(V_1R_0^+(\lambda))^3 f\|_{L^\infty}\\
&\lesssim \|(V_1R_0^+(\lambda))^3 f\|_{L^{3/2,1}} \le \|V_1\|_{L^{3,1}}\|R_0^+(\lambda)(V_1R_0^+(\lambda))^2 f\|_{L^{3,\infty}}\\
&\lesssim \|(V_1R_0^+(\lambda))^2 f\|_{L^1} \le \|V_1\|_{L^{4/3}}\|R_0^+(\lambda)V_1R_0^+(\lambda)f\|_{L^4}\\
&\lesssim \langle\lambda\rangle^{-1/4}\|V_1R_0^+(\lambda)f\|_{L^{4/3}} \lesssim \langle\lambda\rangle^{-\frac14}\|V_1\|_{L^\infty}\|R_0^+(\lambda)f\|_{L^{4/3}_{x\in \operatorname{supp} V_1}}\\
&\lesssim \langle\lambda\rangle^{-\frac14}\int_{\mathbb{R}^3}\left\|\frac{1}{|x-y|}\right\|_{L^{4/3}_{x\in \operatorname{supp} V_1}}|f(y)|dy \lesssim \langle\lambda\rangle^{-\frac14}\|f\|_{L^1}.
\end{aligned}
\tag{3.9}
$$

Taking $f\to\delta(\cdot-y)$, we obtain that $|R_0^+(\lambda)(V_1R_0^+(\lambda))^3(x,y)|\to 0$ as $\lambda\to+\infty$. Thus, there exists $N_1 = N_1(\epsilon, V_1)\gg 1$ such that if $\lambda\ge N_1$, then

$$
|(V_1R_0^+(\lambda))^4)(x,y)| \le \frac{\epsilon|V_1(x)|}{2\|V_1\|_{L^1}} =: D_1(x,y). \tag{3.10}
$$

Then, it is obvious that $\|D_1(x,y)\|_{L_y^\infty L_x^1} \le \frac{\epsilon}{2}$. For the second term, we split

$$
\begin{aligned}
&(VR_0^+(\lambda))^4(x,y) - (V_1R_0^+(\lambda))^4(x,y)\\
&= (V_2R_0^+(\lambda)(VR_0^+(\lambda))^3)(x,y) + (V_1R_0^+(\lambda)V_2R_0^+(\lambda)(VR_0^+(\lambda))^2)(x,y)\\
&\quad + ((V_1R_0^+(\lambda))^2V_2R_0^+(\lambda)VR_0^+(\lambda))(x,y) + ((V_1R_0^+(\lambda))^3V_2R_0^+(\lambda))(x,y).
\end{aligned}
\tag{3.11}
$$

Since the kernel of $R_0^+(\lambda)$ is bounded by the kernel of $(-\Delta)^{-1}$, we have

$$
\begin{aligned}
&|(VR_0^+(\lambda))^4(x,y) - (V_1R_0^+(\lambda))^4(x,y)|\\
&\le (|V_2|(-\Delta)^{-1}(|V|(-\Delta)^{-1})^3)(x,y)\\
&\quad + (|V_1|(-\Delta)^{-1}|V_2|(-\Delta)^{-1}(|V|(-\Delta)^{-1})^2)(x,y)\\
&\quad + ((|V_1|(-\Delta)^{-1})^2|V_2|(-\Delta)^{-1}|V|(-\Delta)^{-1})(x,y)\\
&\quad + ((|V_1|(-\Delta)^{-1})^3|V_2|(-\Delta)^{-1})(x,y)\\
&=: D_2(x,y).
\end{aligned}
\tag{3.12}
$$

Then,

$$\begin{aligned}\|D_2(x,y)\|_{L^\infty_y L^1_x} &= \|D_2\|_{L^1\to L^1}\\ &\le \||V_2|(-\Delta)^{-1}\|_{L^1\to L^1}\||V|(-\Delta)^{-1}\|^3_{L^1\to L^1}\\ &\quad +\||V_1|(-\Delta)^{-1}\|_{L^1\to L^1}\||V_2|(-\Delta)^{-1}\|_{L^1\to L^1}\||V|(-\Delta)^{-1}\|^2_{L^1\to L^1}\\ &\quad +\||V_1|(-\Delta)^{-1}\|^2_{L^1\to L^1}\||V_2|(-\Delta)^{-1}\|_{L^1\to L^1}\||V|(-\Delta)^{-1}\|_{L^1\to L^1}\\ &\quad +\||V_1|(-\Delta)^{-1}\|^3_{L^1\to L^1}\||V_2|(-\Delta)^{-1}\|_{L^1\to L^1}\\ &\le 4\Big(\frac{\|V\|_{\mathcal{K}}+\|V_2\|_{\mathcal{K}}}{4\pi}\Big)^3\frac{\|V_2\|_{\mathcal{K}}}{4\pi}\le\frac{\epsilon}{2},\end{aligned}\tag{3.13}$$

where D_2 is an integral operator with kernel $D_2(x,y)$. Therefore, we conclude that

$$|(VR_0^+(\lambda))^4(x,y)|\le D(x,y):=D_1(x,y)+D_2(x,y)\tag{3.14}$$

and $\|D(x,y)\|_{L^\infty_y L^1_x}\le\epsilon$. □

By algebra, the resolvent $R_V^+(\lambda)$ can be written as

$$R_V^+(\lambda)“=”R_0^+(\lambda)(I+VR_0^+(\lambda))^{-1}.\tag{3.15}$$

Let $\mathcal{L}(L^1)$ be the space of bounded operators on L^1. The following lemmas say that $(I+VR_0^+(\lambda))$ is invertible in $\mathcal{L}(L^1)$ for $\lambda\ge 0$, its inverse $(I+VR_0^+(\lambda))^{-1}$ is uniformly bounded in $\mathcal{L}(L^1)$, and is the sum of the identity map and an integral operator.

Lemma 3.2 (Invertibility of $(I+VR_0^+(\lambda))$) *If $V\in\mathcal{K}_0$ and H has no eigenvalue or resonance on* $[0,+\infty)$*, then $(I+VR_0^+(\lambda))$ is invertible in $\mathcal{L}(L^1)$ for $\lambda\ge 0$.*

Proof If it is not invertible, there exists $\varphi\in L^1$, $\varphi\neq 0$, such that $(I+VR_0^+(\lambda))\varphi=0$. Then, $\psi:=R_0^+(\lambda)\varphi$ solves the eigenvalue equation $(-\Delta+V)\psi=(\lambda+i0)\psi\iff\psi+R_0^+(\lambda)V\psi=0$. Moreover, by the resolvent formula $R_0^+(\lambda)(x,y)=\frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|}$, if $s>\frac{1}{2}$, then

$$\|\langle x\rangle^{-s}\psi\|_{L^2}=\|\langle x\rangle^{-s}R_0^+(\lambda)\varphi\|_{L^2}\le\int_{\mathbb{R}^3}\Big\|\frac{1}{\langle x\rangle^s 4\pi|x-y|}\Big\|_{L^2_x}|\varphi(y)|dy\lesssim\|\varphi\|_{L^1}.\tag{3.16}$$

Hence, λ is an eigenvalue or a resonance (contradiction!). □

Lemma 3.3 (Uniform bound for $(I+VR_0^+(\lambda))^{-1}$) *If $V\in\mathcal{K}_0$ and H has no eigenvalue or resonance on* $[0,+\infty)$*, then $S_\lambda:=(I+VR_0^+(\lambda))^{-1}:[0,+\infty)\to\mathcal{L}(L^1)$ is uniformly bounded.*

Proof Iterating the resolvent identity, we get the formal identity

$$(I+VR_0^+(\lambda))^{-1}\text{``}=\text{''}(I-VR_0^+(\lambda)+(VR_0^+(\lambda))^2-(VR_0^+(\lambda))^3)\sum_{n=0}^{\infty}(VR_0^+(\lambda))^{4n}. \tag{3.17}$$

Indeed, by Lemma 3.1 (iii), $\|(VR_0^+(\lambda))^4\|_{L^1\to L^1}<\frac{1}{2}$ for all sufficiently large λ. Hence, the formal identity (3.16) makes sense, and $(I+VR_0^+(\lambda))^{-1}$ is uniformly bounded for all sufficiently large λ. Thus, it suffices to show that $(I+VR_0^+(\lambda))^{-1}$ is continuous. To see this, we fix $\lambda_0\geq 0$ and write

$$\begin{aligned}&(I+VR_0^+(\lambda))^{-1}-(I+VR_0^+(\lambda_0))^{-1}=(I+VR_0^+(\lambda_0)+B_{\lambda,\lambda_0})^{-1}-S_{\lambda_0}\\&=[(I+VR_0^+(\lambda_0)(I+S_{\lambda_0}B_{\lambda,\lambda_0})]^{-1}-S_{\lambda_0}=(I+S_{\lambda_0}B_{\lambda,\lambda_0})^{-1}S_{\lambda_0}-S_{\lambda_0}\\&\text{``}=\text{''}\sum_{n=0}^{\infty}(-S_{\lambda_0}B_{\lambda,\lambda_0})^nS_{\lambda_0}-S_{\lambda_0}\text{``}=\text{''}\sum_{n=1}^{\infty}(-S_{\lambda_0}B_{\lambda,\lambda_0})^nS_{\lambda_0}.\end{aligned} \tag{3.18}$$

Then, by Lemma 3.1 (ii), we have

$$\begin{aligned}&\|(I+VR_0^+(\lambda))^{-1}-(I+VR_0^+(\lambda_0))^{-1}\|_{L^1\to L^1}\leq\sum_{n=1}^{\infty}\|S_{\lambda_0}\|_{L^1\to L^1}^{n+1}\|B_{\lambda,\lambda_0}\|_{L^1\to L^1}^n\\&=\frac{\|S_{\lambda_0}\|_{L^1\to L^1}^2\|B_{\lambda,\lambda_0}\|_{L^1\to L^1}}{1-\|S_{\lambda_0}\|_{L^1\to L^1}\|B_{\lambda,\lambda_0}\|_{L^1\to L^1}}\to 0\text{ as }\lambda\to\lambda_0.\end{aligned} \tag{3.19}$$

Therefore, the formal identity (3.17) makes sense, and $(I+VR_0^+(\lambda))^{-1}$ is continuous. □

Lemma 3.4 *If $V\in\mathcal{K}_0$ and H has no eigenvalue or resonance on $[0,+\infty)$, then $\tilde{S}_\lambda:=(S_\lambda-I)=(I+VR_0^+(\lambda))^{-1}-I:[0,+\infty)\to\mathcal{L}(L^1)$ is not only uniformly bounded but also an integral operator with kernel $\tilde{S}_\lambda(x,y)$:*

$$\tilde{S}:=\sup_{\lambda\geq 0}\|\tilde{S}_\lambda\|_{L^1\to L^1}=\sup_{\lambda\geq 0}\|\tilde{S}_\lambda(x,y)\|_{L_y^\infty L_x^1}<\infty. \tag{3.20}$$

Proof By algebra, we have

$$\tilde{S}_\lambda=(I+VR_0^+(\lambda))^{-1}-I=-(I+VR_0^+(\lambda))^{-1}VR_0^+(\lambda)=-S_\lambda VR_0^+(\lambda). \tag{3.21}$$

Since $\tilde{S}_\lambda:L^1\to L^1$ is bounded, sending $f_\epsilon\to\delta(\cdot-y_0)$ as $\epsilon\to 0$, we get

$$(\tilde{S}_\lambda f_\epsilon)(x)=(-S_\lambda VR_0^+(\lambda)f_\epsilon)(x)\to -S_\lambda\left(\frac{V(\cdot)e^{i\sqrt{\lambda}|\cdot-y_0|}}{4\pi|\cdot-y_0|}\right)(x)=:\tilde{S}_\lambda(x,y_0). \tag{3.22}$$

Consider $F_V(x;y,\lambda):=V(x)\frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|}$ as a function of x with parameters $y\in\mathbb{R}^3$ and $\lambda\in\mathbb{R}$. Then, $F_V(x;y,\lambda)$ is bounded in L_x^1 uniformly in y and λ. Therefore, by

Lemma 3.3, we conclude that $\tilde{S}_\lambda(x, y) = -S_\lambda\left(\frac{V(\cdot)e^{i\sqrt{\lambda}|\cdot-y|}}{4\pi|\cdot-y|}\right)(x)$ is also bounded in L^1_x uniformly in λ and y. □

3.2 Spectral Projections and Eigenfunctions

Let χ be the dyadic partition of unity function chosen in (1.2), and let $\tilde{\chi}_N(\lambda) \in C_c^\infty(\mathbb{R})$ such that $\tilde{\chi}_N(\lambda) = \chi(\frac{\sqrt{\lambda}}{N})$ if $\lambda \geq 0$; $\tilde{\chi}_N(\lambda) = 0$ if $\lambda < 0$. By functional calculus, we define the Littlewood-Paley projections by $P_N = \tilde{\chi}_N(H)$, $P_{\leq N_0} = \sum_{N<N_0} P_N$, $P_{N_0<\cdot<N_1} = \sum_{N_0<N<N_1} P_N$ and $P_{\geq N_1} = \sum_{N\geq N_1} P_N$.

Lemma 3.5 *Suppose that $V \in \mathcal{K}_0 \cap L^{3/2,\infty}$ and H has no eigenvalue or resonance on $[0, +\infty)$. Let $\mathfrak{S} := \{f \in L^1 \cap L^\infty : P_c f = P_{N_0<\cdot<N_1} f$ for some $N_0, N_1 > 0\}$. For $1 < r < \infty$, $\mathfrak{S}$ is dense in L^r.*

Proof $L^1 \cap L^\infty$ is dense in L^r. Fix $f \in L^1 \cap L^\infty$. We claim that $\lim_{N_0\to 0} \|P_{<N_0} f\|_{L^r} = 0$. By the spectral theory, $\lim_{N_0\to 0} \|P_{<N_0} f\|_{L^2} = 0$. On the other hand, replacing $\tilde{\chi}_N$ by $\sum_{N<N_0} \tilde{\chi}_N$ in the proof of [12, Corollary 1.6], one can show that $\|P_{<N_0} f\|_{L^1}$ and $\|P_{<N_0} f\|_{L^\infty}$ are bounded uniformly in N_0. Hence the claim follows from interpolation. By the same argument, one can show that $\lim_{N_1\to\infty} \|P_{>N_1} f\|_{L^r} = 0$. Thus, $\mathfrak{S}$ is dense in L^r. □

Lemma 3.6 (Boundedness of eigenfunctions) *Suppose that $V \in \mathcal{K}_0 \cap L^{3/2,\infty}$ and H has no eigenvalue or resonance on $[0, +\infty)$. Let ψ_j be an eigenfunction corresponding to the negative eigenvalue λ_j.*

(i) For all $1 \leq p < \infty$, $\psi_j \in L^p$ and P_{λ_j} is bounded on L^p, where P_{λ_j} is the spectral projection onto the point $\{\lambda_j\}$.
(ii) $\nabla\psi_j \in L^r$ for $1 < r < 3$.

Proof (i) We prove the lemma following the argument in [1]. We decompose $V = V_1 + V_2$ such that V_1 is compactly supported and bounded, and $\|V_2\|_{\mathcal{K}} \leq 1$. Then,

$$\begin{aligned}
&\psi_j + R_0(\lambda_j)V\psi_j = \psi_j + R_0(\lambda_j)(V_1 + V_2)\psi_j = 0 \\
&\Rightarrow \psi_j = -(I + R_0(\lambda_j)V_2)^{-1}R_0(\lambda_j)V_1\psi_j = -\sum_{n=0}^{\infty}(-R_0(\lambda_j)V_2)^n R_0(\lambda_j)V_1\psi_j.
\end{aligned} \tag{3.23}$$

Observe that, since V_1 is compactly supported, and $\lambda_j < 0$, $R_0(\lambda_j)V_1\psi_j$ is exponentially decaying. To see this, we choose sufficiently small $\epsilon > 0$ such that $\epsilon < \sqrt{-\lambda_j}$ for any negative eigenvalue λ_j. Indeed, there exists such ϵ, since by the assumptions, there are at most finitely many negative eigenvalues (see [2]). Then, by the fractional integration inequality and the Hölder inequality in the Lorentz spaces (Lemma 7.5), we get

$$
\begin{aligned}
|e^{\epsilon|x|}(R_0(\lambda_j)V_1 f)(x)| &\le e^{\epsilon|x|}\int_{\mathbb{R}^3}\frac{e^{i\sqrt{\lambda_j}|x-y|}}{4\pi|x-y|}|V_1(y)||\psi_j(y)|dy\\
&\le \int_{\mathbb{R}^3}\frac{e^{-(\sqrt{-\lambda_j}-\epsilon)|x-y|}}{4\pi|x-y|}e^{\epsilon|y|}|V_1(y)||\psi_j(y)|dy\\
&\le \|e^{\epsilon|\cdot|}V_1\psi_j\|_{L^{3/2,1}}\lesssim \|e^{\epsilon|\cdot|}V_1\|_{L^{6,2}}\|\psi_j\|_{L^2}.
\end{aligned}
\tag{3.24}
$$

Similarly, one can check that $e^{\epsilon|\cdot|}R_0(\lambda_j)V_2e^{-\epsilon|\cdot|}$ is bounded on L^∞ and its operator norm is strictly <1. Thus, we prove that

$$
\|e^{\epsilon|\cdot|}\psi_j\|_{L^\infty}\le\left(\sum_{n=0}^{\infty}\|e^{\epsilon|\cdot|}R_0(\lambda_j)V_2e^{-\epsilon|\cdot|}\|^n_{L^\infty\to L^\infty}\right)\|e^{\epsilon|\cdot|}R_0(\lambda_j)V_1\psi_j\|_{L^\infty}<\infty.
\tag{3.25}
$$

Therefore, $\psi_j\in L^p$ and $P_{\lambda_j}f=\langle\psi_j,f\rangle_{L^2}\psi_j$ is bounded on L^p for all $1\le p\le\infty$.

(ii) Let $\delta_1,\delta_2>0$ be arbitrarily small numbers. Then, since $\lambda_j<0$, by the inhomogeneous Sobolev inequality, we get

$$
\begin{aligned}
\|\nabla\psi_j\|_{L^{\frac{1}{1-\delta_1}}} &= \|\nabla R_0^+(\lambda_j)V\psi_j\|_{L^{\frac{1}{1-\delta_1}}}\lesssim\|V\psi_j\|_{L^{\frac{1}{1-\delta_1}}}\\
&\le\|V\|_{L^{3/2,\infty}}\|\psi_j\|_{L^{\frac{3}{1-3\delta_1},1}}<\infty,\\
\|\nabla\psi_j\|_{L^{\frac{3}{1+\delta_2}}} &= \|\nabla R_0^+(\lambda_j)V\psi_j\|_{L^{\frac{3}{1+\delta_2}}}\lesssim\|V\psi_j\|_{W^{-1,\frac{3}{1+\delta_2}}}\\
&\lesssim\|V\psi_j\|_{L^{\frac{3}{2+\delta_2}}}\le\|V\|_{L^{3/2,\infty}}\|\psi_j\|_{L^{\frac{3}{\delta_2},\frac{3}{2+\delta_2}}}<\infty.
\end{aligned}
\tag{3.26}
$$

Thus, interpolation gives (ii). □

4 High Frequency Estimate: Proof of Lemma 2.1 (i)

4.1 Construction of the Formal Series Expansion

For a large dyadic number N_1 to be chosen later, we construct a formal series for $\mathrm{Pb}_{\ge N_1}$ as follows. First, iterating the resolvent identity

$$
(I+VR_0^+(\lambda))^{-1}=I-(I+VR_0^+(\lambda))^{-1}VR_0^+(\lambda),
\tag{4.1}
$$

we generate a formal series expansion

$$
(I+VR_0^+(\lambda))^{-1}\text{“}=\text{”}\sum_{n=0}^{\infty}(-VR_0^+(\lambda))^n.
\tag{4.2}
$$

Plugging (4.2) into (2.7), we write

$$\mathrm{Pb}_{\geq N_1} \text{“}=\text{”} -\sum_{N\geq N_1}\sum_{n=0}^{\infty}\frac{1}{\pi}\int_0^{\infty} m(\lambda)\chi_N(\sqrt{\lambda})\,\mathrm{Im}[R_0^+(\lambda)(-VR_0^+(\lambda))^n VR_0^+(\lambda)]d\lambda. \tag{4.3}$$

Then, writing the first and the last free resolvents explicitly by the free resolvent formula $R_0^+(\lambda)(x,y) = \frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|}$ and collecting terms having λ by Fubini theorem, we write the kernel of $\mathrm{Pb}_{\geq N_1}$ as

$$\begin{aligned}
&\mathrm{Pb}_{\geq N_1}(x,y)\\
&\text{“}=\text{”}\sum_{N\geq N_1}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{\pi}\int_0^{\infty} m(\lambda)\chi_N(\sqrt{\lambda})\\
&\quad\times \mathrm{Im}\left[\int_{\mathbb{R}^6}\frac{e^{i\sqrt{\lambda}|x-\tilde{x}|}}{4\pi|x-\tilde{x}|}(VR_0^+(\lambda))^n(\tilde{x},\tilde{y})V(\tilde{y})\frac{e^{i\sqrt{\lambda}|\tilde{y}-y|}}{4\pi|\tilde{x}-y|}d\tilde{x}d\tilde{y}\right]d\lambda\\
&=\int_{\mathbb{R}^6}\frac{V(\tilde{y})}{16\pi^3|x-\tilde{x}||\tilde{y}-y|}\left\{\sum_{N\geq N_1}\sum_{n=0}^{\infty}(-1)^{n+1}\mathrm{Pb}_N^n(x,\tilde{x},\tilde{y},y)\right\}d\tilde{x}d\tilde{y},
\end{aligned} \tag{4.4}$$

where

$$\mathrm{Pb}_N^n(x,\tilde{x},\tilde{y},y) = \int_0^{\infty} m(\lambda)\chi_N(\sqrt{\lambda})\,\mathrm{Im}[e^{i\sqrt{\lambda}(|x-\tilde{x}|+|\tilde{y}-y|)}(VR_0^+(\lambda))^n(\tilde{x},\tilde{y})]d\lambda. \tag{4.5}$$

We note that the series (4.4) makes sense only formally at this moment, but it will be shown that the sum is absolutely convergent, and that it satisfies the bound we want to have.

4.2 Intermediate Kernel Estimates

We estimate the intermediate kernel $\mathrm{Pb}_N^n(x,\tilde{x},\tilde{y},y)$ in two ways. First, we show that the sum of $\mathrm{Pb}_N^n(x,\tilde{x},\tilde{y},y)$ in $N\geq N_1$ is absolutely convergent, and moreover each $\mathrm{Pb}_N^n(x,\tilde{x},\tilde{y},y)$ decays away from $x=\tilde{x}$ and $\tilde{y}=y$.

Lemma 4.1 (Summability in N) *For $s_1, s_2\geq 0$, we have*

$$|\mathrm{Pb}_N^n(x,\tilde{x},\tilde{y},y)| \lesssim \frac{N^2\|m\|_{\mathcal{H}(s_1+s_2)}}{\langle N(x-\tilde{x})\rangle^{s_1}\langle N(\tilde{y}-y)\rangle^{s_2}}k_1^n(\tilde{x},\tilde{y}) \tag{4.6}$$

and

$$\|k_1^n(\tilde{x},\tilde{y})\|_{L_{\tilde{y}}^{\infty}L_{\tilde{x}}^1} \leq \left(\frac{\|V\|_{\mathcal{K}}}{4\pi}\right)^n. \tag{4.7}$$

For the proof, we need the following lemma.

Lemma 4.2 (Oscillatory integral) *For $s \geq 0$,*

$$\left|\int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda})\,\mathrm{Im}(e^{i\sqrt{\lambda}\sigma})d\lambda\right| \lesssim \frac{N^2}{\langle N\sigma\rangle^s}\|m\|_{\mathcal{H}(s)}. \tag{4.8}$$

Proof By abuse of notation, we denote by χ the even extension of itself. Making change of variables $\lambda \mapsto N^2\lambda^2$, we write

$$\begin{aligned}\int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda})\,\mathrm{Im}(e^{i\sqrt{\lambda}\sigma})d\lambda &= N^2\int_0^\infty 2\lambda m(N^2\lambda^2)\chi(\lambda)\sin(N\lambda\sigma)d\lambda\\ &= N^2\int_{\mathbb{R}} \lambda m(N^2\lambda^2)\chi(\lambda)e^{i\lambda N\sigma}d\lambda\\ &= N^2\Big(m(N^2\lambda^2)\lambda\chi(\lambda)\Big)^\vee(N\sigma)\\ &= \frac{N^2}{\langle N\sigma\rangle^s}\Big(\langle\nabla\rangle^s(m(N^2\lambda^2)\lambda\chi(\lambda))\Big)^\vee(N\sigma). \end{aligned}\tag{4.9}$$

Thus, it follows from Hausdorff–Young inequality and the fractional Leibniz rule that

$$\begin{aligned}\left|\int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda})\,\mathrm{Im}(e^{i\sqrt{\lambda}\sigma})d\lambda\right| &\leq \frac{N^2}{\langle N\sigma\rangle^s}\|m(N^2\lambda^2)\lambda\chi(\lambda)\|_{W^{s,1}}\\ &\lesssim \frac{N^2}{\langle N\sigma\rangle^s}\|m(N^2\lambda^2)\chi(\lambda)\|_{W^{s,2}}\\ &\leq \frac{N^2}{\langle N\sigma\rangle^s}\|m\|_{\mathcal{H}(s)}. \end{aligned}\tag{4.10}$$

□

Proof of Lemma 4.1 First, using the free resolvent formula, we write

$$\begin{aligned}&\mathrm{Pb}_N^n(x,\tilde{x},\tilde{y},y)\\ &= \int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda})\,\mathrm{Im}\left\{\int_{\mathbb{R}^{3(n-1)}}\prod_{k=1}^n V(x_k)\frac{\prod_{k=0}^{n+1}e^{i\sqrt{\lambda}|x_k-x_{k+1}|}}{\prod_{k=1}^n 4\pi|x_k-x_{k+1}|}d\mathbf{x}_{(2,n)}\right\}d\lambda\\ &= \int_{\mathbb{R}^{3(n-1)}}\frac{\prod_{k=1}^n V(x_k)}{\prod_{k=1}^n 4\pi|x_k-x_{k+1}|}\left\{\int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda})\,\mathrm{Im}(e^{i\sqrt{\lambda}\sigma_{n+1}})d\lambda\right\}d\mathbf{x}_{(2,n)}, \end{aligned}\tag{4.11}$$

where $x_0 := x$, $x_1 := \tilde{x}$, $x_{n+1} := \tilde{y}$, $x_{n+2} := y$, $d\mathbf{x}_{(2,n)} := dx_2\ldots dx_n$ and $\sigma_n := \sum_{j=0}^n |x_j - x_{j+1}|$. Then, by Lemma 4.2 with $s = s_1 + s_2$ and the trivial inequality

$$|x_0 - x_1| = |x-\tilde{x}|,\ |x_{n+1}-x_{n+2}| = |\tilde{y}-y| \leq \sigma_{n+1} = \sum_{j=0}^{n+1}|x_j - x_{j+1}|, \tag{4.12}$$

we obtain that

$$|\mathrm{Pb}_N^n(x,\tilde{x},\tilde{y},y)| \lesssim \frac{N^2\|m\|_{\mathcal{H}(s_1+s_2)}}{\langle N(x-\tilde{x})\rangle^{s_1}\langle N(\tilde{y}-y)\rangle^{s_2}} \int_{\mathbb{R}^{3(n-1)}} \frac{\prod_{k=1}^n |V(x_k)|}{\prod_{k=1}^n 4\pi|x_k-x_{k+1}|} d\mathbf{x}_{(2,n)}. \tag{4.13}$$

We define

$$k_1^n(\tilde{x},\tilde{y}) := \int_{\mathbb{R}^{3(n-1)}} \frac{\prod_{k=1}^n |V(x_k)|}{\prod_{k=1}^n 4\pi|x_k-x_{k+1}|} d\mathbf{x}_{(2,n)}. \tag{4.14}$$

Then, by the definition of the global Kato norm, we have

$$\begin{aligned}
\|k_1^n(\tilde{x},\tilde{y})\|_{L_{\tilde{y}}^\infty L_{\tilde{x}}^1} &\le \sup_{x_{n+1}\in\mathbb{R}^3} \int_{\mathbb{R}^{3n}} \frac{\prod_{k=1}^n |V(x_k)|}{\prod_{k=1}^n 4\pi|x_k-x_{k+1}|} d\mathbf{x}_{(1,n)} \\
&\le \Big(\sup_{x_n\in\mathbb{R}^3} \int_{\mathbb{R}^{3(n-1)}} \frac{\prod_{k=1}^{n-1} |V(x_k)|}{\prod_{k=1}^{n-1} 4\pi|x_k-x_{k+1}|} d\mathbf{x}_{(1,n-1)}\Big) \\
&\quad \times \Big(\sup_{x_{n+1}\in\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(x_n)|}{4\pi|x_n-x_{n+1}|} dx_n\Big) \\
&\le \Big(\sup_{x_n\in\mathbb{R}^3} \int_{\mathbb{R}^{3(n-1)}} \frac{\prod_{k=1}^{n-1} |V(x_k)|}{\prod_{k=1}^{n-1} 4\pi|x_k-x_{k+1}|} d\mathbf{x}_{(1,n-1)}\Big)\Big(\frac{\|V\|_{\mathcal{K}}}{4\pi}\Big) \\
&\le \cdots (\text{repeat}) \cdots \le \Big(\frac{\|V\|_{\mathcal{K}}}{4\pi}\Big)^n.
\end{aligned} \tag{4.15}$$

□

Next, we show summability of the intermediate kernel in n.

Lemma 4.3 (Summability in n) *For $\epsilon > 0$, there exist $N_1 = N_1(V,\epsilon) \gg 1$ and $k_2^n(\tilde{x},\tilde{y}) \in L_{\tilde{y}}^\infty L_{\tilde{x}}^1$ such that for $N \ge N_1$,*

$$|\mathrm{Pb}_N^n(x,\tilde{x},\tilde{y},y)| \lesssim \epsilon^n N^2\|m\|_{\mathcal{H}(0)} k_2^n(\tilde{x},\tilde{y}). \tag{4.16}$$

and

$$\|k_2^n(\tilde{x},\tilde{y})\|_{L_{\tilde{y}}^\infty L_{\tilde{x}}^1} \lesssim \epsilon^n. \tag{4.17}$$

Proof By Lemma 3.1 (iii), given $\epsilon > 0$, there exist $N_1 \gg 1$ and an operator $D : L^1 \to L^1$ such that $\|D(x,y)\|_{L_y^\infty L_x^1} \le \epsilon^4$ and $|(VR_0^+(\lambda))^4(x,y)| \le D(x,y)$. We also observe that

$$|(VR_0^+(\lambda))(x,y)| = \left|V(x)\frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|}\right| = \frac{|V(x)|}{4\pi|x-y|} = \Big(|V|(-\Delta)^{-1}\Big)(x,y). \tag{4.18}$$

We denote by $\lfloor a \rfloor$ the largest integer less than or equal to a. Then, we have

$$\begin{aligned}
|\mathrm{Pb}^n_N(x, \tilde{x}, \tilde{y}, y)| &\leq \int_0^\infty |m(\lambda)|\chi_N(\sqrt{\lambda})|(VR_0^+(\lambda))^n(\tilde{x}, \tilde{y})|d\lambda \\
&\leq \int_0^\infty |m(\lambda)|\chi_N(\sqrt{\lambda})\left|\left(D^{\lfloor \frac{n}{4} \rfloor}(|V|(-\Delta)^{-1})^{n-4\lfloor \frac{n}{4} \rfloor}\right)(\tilde{x}, \tilde{y})\right|d\lambda \\
&\lesssim N^2 \int_0^\infty |m(N^2\lambda)|\chi(\lambda)d\lambda \cdot \left|\left(D^{\lfloor \frac{n}{4} \rfloor}(|V|(-\Delta)^{-1})^{n-4\lfloor \frac{n}{4} \rfloor}\right)(\tilde{x}, \tilde{y})\right| \\
&\lesssim N^2 \|m\|_{\mathcal{H}(0)}\left|\left(D^{\lfloor \frac{n}{4} \rfloor}(|V|(-\Delta)^{-1})^{n-4\lfloor \frac{n}{4} \rfloor}\right)(\tilde{x}, \tilde{y})\right|. \qquad (4.19)
\end{aligned}$$

We define

$$k_2^n(\tilde{x}, \tilde{y}) := \left|\left(D^{\lfloor \frac{n}{4} \rfloor}(|V|(-\Delta)^{-1})^{n-4\lfloor \frac{n}{4} \rfloor}\right)(\tilde{x}, \tilde{y})\right|. \qquad (4.20)$$

Then, by Lemma 3.1, one can check (4.17). □

4.3 Proof of Lemma 2.1 (*i*)

Let $\delta > 0$ be a sufficiently small number to be chosen later. Let $\epsilon > 0$ be a small number depending on $\|V\|_{\mathcal{K}}$ and $\delta > 0$ [see (4.30)]. Then, we pick a large dyadic number N_1 from Lemma 4.3. We will show that $\mathrm{Pb}_{\geq N_1}$ is bounded from $L^{\frac{3}{3-\delta},1}$ to $L^{\frac{3}{3-\delta},\infty}$.

Let $s = \frac{2}{1-\delta} > 2$ and $\theta = \frac{2-\delta}{2}$ ($\Rightarrow 2\theta = 2 - \delta$, $(s-2)\theta > \delta$ and $s\theta > 2$). Then, by Lemma 4.1 with $s_1 = 2$ and $s_2 = s - 2$ and Lemma 4.3, we get

$$\begin{aligned}
|\mathrm{Pb}^n_N(x, \tilde{x}, \tilde{y}, y)| &= |\mathrm{Pb}^n_N(x, \tilde{x}, \tilde{y}, y)|^\theta |\mathrm{Pb}^n_N(x, \tilde{x}, \tilde{y}, y)|^{1-\theta} \\
&\lesssim \frac{N^2\|m\|_{\mathcal{H}(s)}}{\langle N(x-\tilde{x})\rangle^{2\theta}\langle N(\tilde{y}-y)\rangle^{(s-2)\theta}}\left(k_1^n(\tilde{x}, \tilde{y})\right)^\theta \left(k_2^n(\tilde{x}, \tilde{y})\right)^{1-\theta}. \qquad (4.21)
\end{aligned}$$

We claim that

$$\sum_{N \in 2^{\mathbb{Z}}} \frac{N^2}{\langle Nx\rangle^{2\theta}\langle Ny\rangle^{(s-2)\theta}} \lesssim \frac{1}{|x|^{2-\delta}|y|^\delta}. \qquad (4.22)$$

Fix $x, y \in \mathbb{R}^3$, and consider the following four cases.

(Case 1 $N < \min(|x|^{-1}, |y|^{-1})$**)**

$$\sum_{\text{Case 1}} \frac{N^2}{\langle Nx\rangle^{2\theta}\langle Ny\rangle^{(s-2)\theta}} \leq \sum_{\text{Case 1}} N^2 \leq \min\left(\frac{1}{|x|}, \frac{1}{|y|}\right)^2 \leq \frac{1}{|x|^{2-\delta}|y|^\delta}. \qquad (4.23)$$

(Case 2 $|x|^{-1} \leq N < |y|^{-1}$)

$$\sum_{\text{Case 2}} \frac{N^2}{\langle Nx\rangle^{2\theta} \langle Ny\rangle^{(s-2)\theta}} \leq \sum_{\text{Case 2}} \frac{N^2}{|Nx|^{2\theta}} = \sum_{\text{Case 2}} \frac{N^{2(1-\theta)}}{|x|^{2\theta}} = \sum_{\text{Case 2}} \frac{N^{\delta}}{|x|^{2-\delta}} \leq \frac{1}{|x|^{2-\delta}|y|^{\delta}}. \tag{4.24}$$

(Case 3 $|y|^{-1} \leq N < |x|^{-1}$)

$$\sum_{\text{Case 3}} \frac{N^2}{\langle Nx\rangle^{2\theta} \langle Ny\rangle^{(s-2)\theta}} \leq \sum_{\text{Case 3}} \frac{N^2}{|Ny|^{\delta}} = \sum_{\text{Case 3}} \frac{N^{2-\delta}}{|y|^{\delta}} \leq \frac{1}{|x|^{2-\delta}|y|^{\delta}}. \tag{4.25}$$

(Case 4 $N \geq \max(|x|^{-1}, |y|^{-1})$)

$$\sum_{\text{Case 4}} \frac{N^2}{\langle Nx\rangle^{2\theta} \langle Ny\rangle^{(s-2)\theta}} \lesssim \frac{1}{|x|^{2\theta}|y|^{(s-2)\theta}} \sum_{\text{Case 4}} \frac{1}{N^{s\theta-2}} \leq \frac{1}{|x|^{2-\delta}|y|^{(s-2)\theta}}|y|^{s\theta-2} \leq \frac{1}{|x|^{2-\delta}|y|^{2-2\theta}} = \frac{1}{|x|^{2-\delta}|y|^{\delta}}. \tag{4.26}$$

Collecting all, we prove the claim.

Applying (4.22) to (4.21) and summing in $N \geq N_1$, we obtain

$$\sum_{N\geq N_1} |\mathrm{Pb}_N^n(x, \tilde{x}, \tilde{y}, y)| \lesssim \frac{\|m\|_{\mathcal{H}(s)}}{|x-\tilde{x}|^{2-\delta}|\tilde{y}-y|^{\delta}} \Big(k_1^n(\tilde{x}, \tilde{y})\Big)^{\theta} \Big(k_2^n(\tilde{x}, \tilde{y})\Big)^{1-\theta}. \tag{4.27}$$

Let

$$K(x, y) = \sum_{n=0}^{\infty} \Big(k_1^n(\tilde{x}, \tilde{y})\Big)^{\theta} \Big(k_2^n(x, y)\Big)^{1-\theta}. \tag{4.28}$$

Then, $K \in L_y^{\infty} L_x^1$, since if N_1 is large enough,

$$\begin{aligned}\|K(x, y)\|_{L_y^\infty L_x^1} &\leq \sum_{n=0}^{\infty} \left\| \Big(k_1^n(\tilde{x}, \tilde{y})\Big)^{\theta} \Big(k_2^n(x, y)\Big)^{1-\theta} \right\|_{L_y^\infty L_x^1} \\ &\leq \sum_{n=0}^{\infty} \|k_1^n(\tilde{x}, \tilde{y})\|_{L_y^\infty L_x^1}^{\theta} \|k_2^n(x, y)\|_{L_y^\infty L_x^1}^{1-\theta} \\ &\leq \sum_{n=0}^{\infty} \left(\frac{\|V\|_{\mathcal{K}}}{4\pi}\right)^{n\theta} \epsilon^n < \infty,\end{aligned} \tag{4.29}$$

where $\epsilon > 0$ is chosen so that $(\frac{\|V\|_{\mathcal{K}}}{4\pi})^{\theta}\epsilon < 1$ with $\theta = \frac{2-\delta}{2}$. Therefore, we obtain the kernel estimates for $\mathrm{Pb}_{\geq N_1}(x, y)$,

$$
\begin{aligned}
|\mathrm{Pb}_{N_1}(x, y)| &\leq \int_{\mathbb{R}^6} \frac{V(\tilde{y})}{16\pi^3|x-\tilde{x}||\tilde{y}-y|} \sum_{N \geq N_1} \sum_{n=0}^{\infty} |\mathrm{Pb}_N^n(x, \tilde{x}, \tilde{y}, y)| d\tilde{x} d\tilde{y} \\
&\lesssim \int_{\mathbb{R}^6} \frac{|V(\tilde{y})|}{|x-\tilde{x}|^{3-\delta}|\tilde{y}-y|^{1+\delta}} K(\tilde{x}, \tilde{y}) d\tilde{x} d\tilde{y}, \qquad (4.30)
\end{aligned}
$$

with $K \in L_y^\infty L_x^1$.

Let T_K be the integral operator with kernel $K(x, y)$, which is bounded on L^1 by (4.29). By the fractional integration inequality and Hölder inequality in the Lorentz spaces (see "Appendix"), we conclude that

$$
\begin{aligned}
&\|\mathrm{Pb}_{\geq N_1} f\|_{L^{\frac{3}{3-\delta},\infty}} \\
&\lesssim \left\| \int_{\mathbb{R}^9} \frac{|V(\tilde{y})|}{|x-\tilde{x}|^{3-\delta}|\tilde{y}-y|^{1+\delta}} K(\tilde{x}, \tilde{y}) |f(y)| d\tilde{x} d\tilde{y} dy \right\|_{L_x^{\frac{3}{3-\delta},\infty}} \\
&\lesssim \||\nabla|^{-\delta} T_K(|V||\nabla|^{-(2-\delta)}(|f|))\|_{L^{\frac{3}{3-\delta},\infty}} \lesssim \|T_K(|V||\nabla|^{-(2-\delta)}(|f|))\|_{L_x^1} \qquad (4.31) \\
&\lesssim \||V||\nabla|^{-(2-\delta)}(|f|)\|_{L_x^1} \leq \|V\|_{L^{3/2,\infty}} \||\nabla|^{-(2-\delta)}|f|\|_{L^{3,1}} \lesssim \|f\|_{L^{\frac{3}{3-\delta},1}}.
\end{aligned}
$$

Remark 4.4 In (4.31), we only used the fractional integration inequality and the Hölder inequality. Note that after applying the fractional integration inequality, we always have the $L^{p,q}$-norm with smaller p on the right hand side, although we want to show the $L^{\frac{3}{3-\epsilon},1} - L^{\frac{3}{3-\epsilon},\infty}$ boundedness. Hence, one must have at least one chance to raise the number p to compensate the decrease of p caused by the fractional integration inequalities. In (4.31), the potential V plays such a role with the Hölder inequality. This is the main reason that we keep one extra potential term V in the spectral representation by considering the perturbation $m(H)P_c - m(-\Delta)$ instead of $m(H)P_c$, and introducing intermediated kernels $\mathrm{Pb}_N^n(x, \tilde{x}, \tilde{y}, y)$, even though they look rather artificial.

5 Low Frequency Estimate: Proof of Lemma 2.1 (ii)

5.1 Construction of the Formal Series Expansion

We prove Lemma 2.1 (ii) by modifying the argument in Sect. 4. Note that for small N, the formal series expansion (4.2) may not be convergent, since $(VR_0^+(\lambda))^4$ in (4.2) is not small anymore. Hence, we introduce a new series expansion for $(I + VR_0^+(\lambda))^{-1}$,

$$\begin{aligned}(I + VR_0^+(\lambda))^{-1} &= (I + VR_0^+(\lambda_0) + B_{\lambda,\lambda_0})^{-1}\\ &= [(I + B_{\lambda,\lambda_0}S_{\lambda_0})(I + VR_0^+(\lambda_0))]^{-1}\\ &= (I + VR_0^+(\lambda_0))^{-1}(I + B_{\lambda,0}S_{\lambda_0})^{-1}\\ \text{“}&=\text{”} S_{\lambda_0}\sum_{n=0}^{\infty}(-B_{\lambda,\lambda_0}S_{\lambda_0})^n, \end{aligned} \tag{5.1}$$

where $B_{\lambda,\lambda_0} = V(R_0^+(\lambda) - R_0^+(\lambda_0))$ and $S_{\lambda_0} = (I + VR_0^+(\lambda_0))^{-1}$. Plugging the formal series (5.1) with $\lambda_0 = 0$ into (2.7), we write

$$\mathrm{Pb}_N \text{“}=\text{”} \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{\pi}\int_0^{\infty} m(\lambda)\chi_N(\sqrt{\lambda})\,\mathrm{Im}[R_0^+(\lambda)S_0(B_{\lambda,0}S_0)^n VR_0^+(\lambda)]d\lambda. \tag{5.2}$$

As in the previous section, writing the first and the last free resolvents explicitly by the free resolvent formula $R_0^+(\lambda)(x,y) = \frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|}$ and collecting terms having λ by Fubini theorem, we write the kernel of Pb_N as

$$\begin{aligned}&\mathrm{Pb}_N(x,y)\\ &\text{“}=\text{”} \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{\pi}\int_0^{\infty} m(\lambda)\chi_N(\sqrt{\lambda})\\ &\qquad\times \mathrm{Im}\Big[\iint_{\mathbb{R}^6}\frac{e^{i\sqrt{\lambda}|x-\tilde{x}|}}{4\pi|x-\tilde{x}|}[S_0(B_{\lambda,0}S_0)^n](\tilde{x},\tilde{y})V(\tilde{y})\frac{e^{i\sqrt{\lambda}|\tilde{y}-y|}}{4\pi|\tilde{x}-y|}d\tilde{x}d\tilde{y}\Big]d\lambda\\ &= \iint_{\mathbb{R}^6}\frac{V(\tilde{y})}{16\pi^3|x-\tilde{x}||\tilde{y}-y|}\Big[\sum_{n=0}^{\infty}(-1)^{n+1}\mathrm{Pb}_N^n(x,\tilde{x},\tilde{y},y)\Big]d\tilde{x}d\tilde{y},\end{aligned} \tag{5.3}$$

where

$$\mathrm{Pb}_N^n(x,\tilde{x},\tilde{y},y) = \int_0^{\infty} m(\lambda)\chi_N(\sqrt{\lambda})\,\mathrm{Im}[e^{i\sqrt{\lambda}(|x-\tilde{x}|+|\tilde{y}-y|)}[S_0(B_{\lambda,0}S_0)^n](\tilde{x},\tilde{y})]d\lambda. \tag{5.4}$$

By Lemma 3.1 (ii), $B_{\lambda,0}$ in (5.3) is small for sufficiently small N. This fact will guarantee the convergence of the formal series.

5.2 Intermediate Kernel Estimates

We will show the kernel estimates analogous to Lemmas 4.1 and 4.3. Then, Lemma 2.1 (ii) will follow from exactly the same argument in Sect. 4.3, thus we omit the proof.

Lemma 5.1 (Summability in N) *There exists $k_1^n(\tilde{x},\tilde{y})$ such that for $s_1, s_2 \geq 0$,*

$$|\mathrm{Pb}_N^n(x,\tilde{x},\tilde{y},y)| \lesssim \frac{N^2\|m\|_{\mathcal{H}(s_1+s_2)}}{\langle N(x-\tilde{x})\rangle^{s_1}\langle N(\tilde{y}-y)\rangle^{s_2}}k_1^n(\tilde{x},\tilde{y}) \tag{5.5}$$

and

$$\|k_1^n(\tilde{x},\tilde{y})\|_{L^\infty_{\tilde{y}}L^1_{\tilde{x}}} \le (\tilde{S}+1)^{n+1}\left(\frac{\|V\|_{\mathcal{K}}}{2\pi}\right)^n, \tag{5.6}$$

where $\tilde{S}$ is the positive number given by (3.20).

Proof First, splitting $B_{\lambda,0}$ into $VR_0^+(\lambda) - VR_0^+(0)$ in

$$\mathrm{Pb}_N^n(x,\tilde{x},\tilde{y},y) = \int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda})\,\mathrm{Im}[e^{i\sqrt{\lambda}(|x-\tilde{x}|+|\tilde{y}-y|)}[S_0(B_{\lambda,0}S_0)^n](\tilde{x},\tilde{y})]d\lambda, \tag{5.7}$$

we write $\mathrm{Pb}_N^n(x,\tilde{x},\tilde{y},y)$ as the sum of 2^n copies of

$$\int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda})\,\mathrm{Im}[e^{i\sqrt{\lambda}(|x-\tilde{x}|+|\tilde{y}-y|)}[S_0VR_0^+(\alpha_1\lambda)S_0\cdots VR_0^+(\alpha_n\lambda)S_0](\tilde{x},\tilde{y})]d\lambda \tag{5.8}$$

up to $\pm$, where $\alpha_k = 0$ or 1 for each $k = 1,\ldots,n$. Next, splitting all S_0 into I and $\tilde{S}_0$ in (5.8), we further decompose (5.8) into the sum of 2^{n+1} kernels.

Among them, let us consider the two representative terms,

$$\mathrm{Im}\int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda})e^{i\sqrt{\lambda}(|x-\tilde{x}|+|\tilde{y}-y|)}[\tilde{S}_0VR_0^+(\alpha_1\lambda)\tilde{S}_0\ldots VR_0^+(\alpha_n\lambda)\tilde{S}_0](\tilde{x},\tilde{y})d\lambda, \tag{5.9}$$

$$\mathrm{Im}\int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda})e^{i\sqrt{\lambda}(|x-\tilde{x}|+|\tilde{y}-y|)}[VR_0^+(\alpha_1\lambda)\ldots VR_0^+(\alpha_n\lambda)](\tilde{x},\tilde{y})d\lambda. \tag{5.10}$$

For the first term, by the free resolvent formula (1.10), we write (5.9) in the integral form,

$$\begin{aligned}
&\mathrm{Im}\int_0^\infty\int_{\mathbb{R}^{6n}} m(\lambda)\chi_N(\sqrt{\lambda})\prod_{k=1}^{n+1}\tilde{S}_0(x_{2k-1},x_{2k})\\
&\quad\times\prod_{k=1}^{n}V(x_{2k})\frac{\prod_{k=0}^{n+1}e^{i\alpha_k\sqrt{\lambda}|x_{2k}-x_{2k+1}|}}{\prod_{k=1}^{n}4\pi|x_{2k}-x_{2k+1}|}d\mathbf{x}_{(2,2n+1)}d\lambda\\
&=\int_{\mathbb{R}^{6n}}\frac{\prod_{k=1}^{n+1}\tilde{S}_0(x_{2k-1},x_{2k})\prod_{k=1}^{n}V(x_{2k})}{\prod_{k=1}^{n}4\pi|x_{2k}-x_{2k+1}|}\\
&\quad\times\left\{\int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda})\,\mathrm{Im}(e^{i\sqrt{\lambda}\tilde{\sigma}_{n+1}})d\lambda\right\}d\mathbf{x}_{(2,2n+1)}
\end{aligned} \tag{5.11}$$

where $x_0 := x$, $x_1 := \tilde{x}$, $x_{2n+2} := \tilde{y}$, $x_{2n+3} := y$, $d\mathbf{x}_{(2,n)} := dx_2\cdots dx_n$, $\tilde{\sigma}_n := \sum_{k=0}^{n+1}\alpha_k|x_{2k}-x_{2k+1}|$ and $\alpha_0 = \alpha_{n+1} = 1$. Then, by Lemma 4.2 with $s = s_1 + s_2$ and $|x_0 - x_1|, |x_{2n+2} - x_{2n+3}| \le \tilde{\sigma}_{n+1}$, we obtain that

$$\Big| \int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda})\,\mathrm{Im}(e^{i\sqrt{\lambda}\tilde{\sigma}_{n+1}})d\lambda \Big| \lesssim \frac{N^2\|m\|_{\mathcal{H}(s_1+s_2)}}{\langle N(x_0-x_1)\rangle^{s_1}\langle N(x_{2n+2}-x_{2n+3})\rangle^{s_2}}. \tag{5.12}$$

Applying (5.12) to (5.9), we get the arbitrary polynomial decay away from $x_0 = x_1$,

$$|(5.9)| \lesssim \frac{N^2\|m\|_{\mathcal{H}(s_1+s_2)}k^n_{(5.9)}(\tilde{x},\tilde{y})}{\langle N(x_0-x_1)\rangle^{s_1}\langle N(x_{2n+2}-x_{2n+3})\rangle^{s_2}} = \frac{N^2\|m\|_{\mathcal{H}(s_1+s_2)}k^n_{(5.9)}(\tilde{x},\tilde{y})}{\langle N(x-\tilde{x})\rangle^{s_1}\langle N(\tilde{y}-y)\rangle^{s_2}}, \tag{5.13}$$

where

$$\begin{aligned} k^n_{(5.9)}(\tilde{x},\tilde{y}) :&= \int_{\mathbb{R}^{6n}} \frac{\prod_{k=1}^{n+1}|\tilde{S}_0(x_{2k-1},x_{2k})|\prod_{k=1}^n|V(x_{2k})|}{\prod_{k=1}^n 4\pi|x_{2k}-x_{2k+1}|}d\mathbf{x}_{(2,2n+1)} \\ &= [|\tilde{S}_0|(|V|(-\Delta)^{-1}|\tilde{S}_0|)^n](\tilde{x},\tilde{y}) \end{aligned} \tag{5.14}$$

and $|\tilde{S}_0|$ is the integral operator with kernel $|\tilde{S}_0(x,y)|$. We claim that

$$\|k^n_{(5.9)}(\tilde{x},\tilde{y})\|_{L^\infty_y L^1_{x_1}} \lesssim \tilde{S}^{n+1}(\|V\|_{\mathcal{K}}/4\pi)^n. \tag{5.15}$$

Indeed, since $\||\tilde{S}_0|(|V|(-\Delta)^{-1}|\tilde{S}_0|)^n f\|_{L^1} \le \tilde{S}^{n+1}(\frac{\|V\|_{\mathcal{K}}}{4\pi})^n\|f\|_{L^1}$ and $|\tilde{S}_0|(|V|(-\Delta)^{-1}|\tilde{S}_0|)^n$ is an integral operator, sending $f \to \delta(\cdot - y)$, we prove the claim.

Similarly, we write (5.10) as

$$\begin{aligned} &\mathrm{Im}\int_0^\infty\int_{\mathbb{R}^{3n-3}} m(\lambda)\chi_N(\sqrt{\lambda})\prod_{k=1}^n V(x_k)\frac{\prod_{k=0}^{n+1}e^{i\alpha_k\sqrt{\lambda}|x_k-x_{k+1}|}}{\prod_{k=1}^n 4\pi|x_k-x_{k+1}|}d\mathbf{x}_{(2,n)}d\lambda \\ &= \int_{\mathbb{R}^{3n-3}} \frac{\prod_{k=1}^n V(x_k)}{\prod_{k=1}^n 4\pi|x_k-x_{k+1}|}\left\{\int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda})\,\mathrm{Im}(e^{i\sqrt{\lambda}\tilde{\tilde{\sigma}}_{n+1}})d\lambda\right\}d\mathbf{x}_{(2,n)} \end{aligned} \tag{5.16}$$

where $x_0 := x$, $x_1 := \tilde{x}$, $x_{n+1} := \tilde{y}$, $x_{n+2} := y$, $\alpha_0 = \alpha_{n+2} = 1$ and $\tilde{\tilde{\sigma}}_n := \sum_{k=0}^n \alpha_k|x_k - x_{k+1}|$. Then, by Lemma 4.2 with $s = s_1 + s_2$ and $|x_0 - x_1|, |x_{n+1} - x_{n+2}| \le \tilde{\tilde{\sigma}}_{n+1}$, we obtain that

$$|(5.10)| \lesssim \frac{N^2\|m\|_{\mathcal{H}(s_1+s_2)}k^n_{(5.10)}(\tilde{x},\tilde{y})}{\langle N(x_0-x_1)\rangle^{s_1}\langle N(x_{n+1}-x_{n+2})\rangle^{s_2}} = \frac{N^2\|m\|_{\mathcal{H}(s_1+s_2)}k^n_{(5.10)}(\tilde{x},\tilde{y})}{\langle N(x-\tilde{x})\rangle^{s_1}\langle N(\tilde{y}-y)\rangle^{s_2}} \tag{5.17}$$

where

$$k^n_{(5.10)}(\tilde{x},\tilde{y}) := \int_{\mathbb{R}^{3n-3}}\frac{\prod_{k=1}^n|V(x_k)|}{\prod_{k=1}^n 4\pi|x_k-x_{k+1}|}d\mathbf{x}_{(2,n)} = (4\pi)^{-n}(|V|(-\Delta)^{-1})^n(\tilde{x},\tilde{y}). \tag{5.18}$$

Then by the definition of the global Kato norm, we prove that

$$\|k^n_{(5.10)}(\tilde{x},\tilde{y})\|_{L^\infty_y L^1_{x_1}} \le (\|V\|_{\mathcal{K}}/4\pi)^n. \tag{5.19}$$

Similarly, we estimate other kernels, and define $k_1^n(\tilde{x}, \tilde{y})$ as the sum of all 2^{2n+1} many upper bounds including $K_{(5.9)}(\tilde{x}, \tilde{y})$ and $K_{(5.10)}(\tilde{x}, \tilde{y})$. Then, $k_1^n(\tilde{x}, \tilde{y})$ satisfies (5.5) and (5.6). □

Lemma 5.2 (Summability in n) *For any $\epsilon > 0$, there exist a small number $N_0 = N_0(V, \epsilon) \ll 1$ and $k_2^n(\tilde{x}, \tilde{y}) \in L^\infty_{\tilde{y}} L^1_{\tilde{x}}$ such that for $N \le N_0$,*

$$|\mathrm{Pb}_N^n(x, \tilde{x}, \tilde{y}, y)| \lesssim N^2 \|m\|_{\mathcal{H}(s)} k_2^n(\tilde{x}, \tilde{y}) \tag{5.20}$$

and

$$\|k_2^n(\tilde{x}, \tilde{y})\|_{L^\infty_{\tilde{y}} L^1_{\tilde{x}}} \le \epsilon^n. \tag{5.21}$$

Proof Fix small $\epsilon > 0$. Then, by Lemma 3.1 (ii), we choose small $N_0 := \delta = \delta(\epsilon) > 0$ and an integral operator B such that $|B_{\lambda,0}(x, y)| \le B(x, y)$ for $0 \le \lambda \le N_0$, and

$$\|B\|_{L^1 \to L^1} \le \epsilon(\tilde{S} + 1)^{-1}, \tag{5.22}$$

where $\tilde{S}$ is a positive number given from (3.20). We define

$$k_2^n(\tilde{x}, \tilde{y}) := [(I + |\tilde{S}_0|)(B(I + |\tilde{S}_0|))^n](\tilde{x}, \tilde{y}),$$

where $|\tilde{S}_0|$ is the integral operator with $|\tilde{S}_0(x, y)|$ as kernel. Then, by definitions [see (5.4)], one can check that $k_2^n(\tilde{x}, \tilde{y})$ satisfies (5.20). For (5.21), splitting $(I + |\tilde{S}_0|)$ into I and $|\tilde{S}_0|$ in $k_2^n(\tilde{x}, \tilde{y})$, we get 2^{n+1} terms,

$$k_2^n(\tilde{x}, \tilde{y}) = [|\tilde{S}_0|(B|\tilde{S}_0|)^n](\tilde{x}, \tilde{y}) + \cdots + B^n(\tilde{x}, \tilde{y}). \tag{5.23}$$

For example, we consider $|\tilde{S}_0|(B|\tilde{S}_0|)^n$ and B^n. Since both $|\tilde{S}_0|$ and B are integral operators, by Lemma 3.4 and (5.12), we obtain

$$\begin{aligned} \|[|\tilde{S}_0|(B|\tilde{S}_0|)^n](\tilde{x}, \tilde{y})\|_{L^\infty_{\tilde{y}} L^1_{\tilde{x}}} &= \||\tilde{S}_0|(B|\tilde{S}_0|)^n\|_{L^1 \to L^1} \le \tilde{S}^{n+1}\Big(\epsilon(\tilde{S} + 1)^{-1}\Big)^n, \\ \|B^n(\tilde{x}, \tilde{y})\|_{L^\infty_{\tilde{y}} L^1_{\tilde{x}}} &= \|B^n\|_{L^1 \to L^1} \le \Big(\epsilon(\tilde{S} + 1)^{-1}\Big)^n. \end{aligned} \tag{5.24}$$

Similarly, we estimate other $2^{n+1} - 2$ terms. Summing them up, we prove (5.21). □

6 Medium Frequency Estimate: Proof of Lemma 2.1 (iii)

The proof closely follows from that of Lemma 2.1 (ii), so we only sketch the proof. For $\epsilon > 0$, we take $\delta = \delta(\epsilon) > 0$ from Lemma 3.1 (ii). We choose a partition of unity function $\psi \in C_c^\infty$ such that $\operatorname{supp}\psi \subset [-\delta, \delta]$, $\psi(\lambda) = 1$ if $|\lambda| \le \frac{\delta}{3}$ and $\sum_{j=1}^\infty \psi(\cdot - \lambda_j) \equiv 1$ on $(0, +\infty)$, where $\lambda_j = j\delta$.

Let N_0 and N_1 be dyadic numbers chosen in the previous sections. For $N_0 \le N \le N_1$, we first decompose $\chi_N(\sqrt{\lambda})$ in Pb_N [see (2.7)] into $\chi_N(\sqrt{\lambda}) = \sum_{j=N/2\delta}^{2N/\delta} \chi_N^j(\lambda)$

where $\chi_N^j(\lambda) = \chi_N(\sqrt{\lambda})\psi(\lambda - \lambda_j)$. Plugging the formal series (5.1) with $\lambda_0 = \lambda_j$ into each integral, we write the kernel of Pb_N as

$$\mathrm{Pb}_N(x, y)\text{“}=\text{”}\iint_{\mathbb{R}^6} \frac{V(\tilde{y})}{16\pi^3|x-\tilde{x}||\tilde{y}-y|}\Big[\sum_{n=0}^{\infty}(-1)^{n+1}\mathrm{Pb}_N^n(x, \tilde{x}, \tilde{y}, y)\Big]d\tilde{x}d\tilde{y}, \tag{6.1}$$

where

$$\mathrm{Pb}_N^n(x, \tilde{x}, \tilde{y}, y) = \sum_{j=N/2\delta}^{2N/\delta}\int_0^\infty m(\lambda)\chi_N^j(\sqrt{\lambda})\,\mathrm{Im}[e^{i\sqrt{\lambda}(|x-\tilde{x}|+|\tilde{y}-y|)}[S_{\lambda_j}(B_{\lambda,\lambda_j}S_{\lambda_j})^n](\tilde{x}, \tilde{y})]d\lambda. \tag{6.2}$$

By the arguments in the previous sections, for Lemma 2.1 (iii), it suffices to show the following two lemmas:

Lemma 6.1 (Summability in N) *For $N_0 < N < N_1$, there exists $k_{N,1}^n(\tilde{x}, \tilde{y})$ such that for $s_1, s_2 \geq 0$,*

$$|\mathrm{Pb}_N^n(x, \tilde{x}, \tilde{y}, y)| \lesssim \frac{N^2\|m\|_{\mathcal{H}(s_1+s_2)}k_{N,1}^n(\tilde{x}, \tilde{y})}{\langle N(x-\tilde{x})\rangle^{s_1}\langle N(\tilde{y}-y)\rangle^{s_2}}, \tag{6.3}$$

and

$$\|k_{N,1}^n(\tilde{x}, \tilde{y})\|_{L_{\tilde{y}}^\infty L_{\tilde{x}}^1} \leq (\tilde{S}+1)^{n+1}\Big(\frac{\|V\|_{\mathcal{K}}}{2\pi}\Big)^n. \tag{6.4}$$

Proof For instance, consider

$$\int_0^\infty m(\lambda)\chi_N^j(\lambda)\,\mathrm{Im}[e^{i\sqrt{\lambda}(|x-\tilde{x}|+|\tilde{y}-y|)}\{S_{\lambda_j}(B_{\lambda,\lambda_j}S_{\lambda_j})^n\}(\tilde{x}, \tilde{y})]d\lambda \tag{6.5}$$

among $O(N)$-many similar integrals in (6.2). As we did in Lemma 4.2, we show that

$$\Big|\int_0^\infty m(\lambda)\chi_N^j(\lambda)\,\mathrm{Im}(e^{i\sqrt{\lambda}\sigma})d\lambda\Big| \lesssim_{N_0,N_1} \frac{N\|m\|_{\mathcal{H}(s)}}{\langle N\sigma\rangle^s}. \tag{6.6}$$

Repeating the proof of Lemma 5.1 [but replacing S_0 and $B_{\lambda,0}$ by S_{λ_j} and B_{λ,λ_j} and applying (6.6) instead of Lemma 4.2], one can find $k_{N,j,1}^n(\tilde{x}, \tilde{y})$ such that for $s_1, s_2 \geq 0$,

$$|(6.5)| \lesssim \frac{N\|m\|_{\mathcal{H}(s_1+s_2)}k_{N,j,1}^n(\tilde{x}, \tilde{y})}{\langle N(x-\tilde{x})\rangle^{s_1}\langle N(\tilde{y}-y)\rangle^{s_2}}, \tag{6.7}$$

$$\|k_{N,j,1}^n(\tilde{x}, \tilde{y})\|_{L_{\tilde{y}}^\infty L_{\tilde{x}}^1} \leq (\tilde{S}+1)^{n+1}\Big(\frac{\|V\|_{\mathcal{K}}}{2\pi}\Big)^n. \tag{6.8}$$

Define

$$k^n_{N,1}(\tilde{x}, \tilde{y}) := \frac{\delta}{N} \sum_{j=N/2\delta}^{2N/\delta} k^n_{N,j,1}(\tilde{x}, \tilde{y}),$$

then it satisfies (6.3) and (6.4). □

Lemma 6.2 (Summability in n) *Let $\epsilon > 0$ be a small number chosen at the beginning of this section. For $N_0 < N < N_1$, there exists $k^n_{N,2}(\tilde{x}, \tilde{y})$ such that*

$$|\mathrm{Pb}^n_N(x, \tilde{x}, \tilde{y}, y)| \lesssim N^2 \|m\|_{\mathcal{H}(s)} k^n_{N,2}(\tilde{x}, \tilde{y}), \tag{6.9}$$

and

$$\|k^n_{N,2}(\tilde{x}, \tilde{y})\|_{L^\infty_{\tilde{y}} L^1_{\tilde{x}}} \le (1 + \tilde{S})^{n+1} \epsilon^n. \tag{6.10}$$

Proof Again, we consider (6.5). By the choice of ϵ and δ and Lemma 3.1 (ii), there exists an integral operator B such that $|B_{\lambda,\lambda_j}(x, y)| \le B(x, y)$ for $|\lambda - \lambda_j| < \delta$, $\lambda, \lambda_j \ge 0$, and $\|B\|_{L^1 \to L^1} \le \epsilon$. Let $|\tilde{S}_{\lambda_j}|$ be the integral operator with integral kernel $|\tilde{S}_{\lambda_j}(x, y)|$. Then, we have

$$|(6.5)| \lesssim N \|m\|_{\mathcal{H}(s)} [(I + |\tilde{S}_{\lambda_j}|)(B(I + |\tilde{S}_{\lambda_j}|))^n](\tilde{x}, \tilde{y}) \tag{6.11}$$

and

$$\|[(I + |\tilde{S}_{\lambda_j}|)(B(I + |\tilde{S}_{\lambda_j}|))^n](\tilde{x}, \tilde{y})\|_{L^\infty_{\tilde{y}} L^1_{\tilde{x}}} \le (1 + \tilde{S})^{n+1} \epsilon^n. \tag{6.12}$$

Therefore, we define

$$k^n_2(\tilde{x}, \tilde{y}) := \frac{\delta}{N} \sum_{j=N/2\delta}^{2N/\delta} [(I + |\tilde{S}_{\lambda_j}|)(B(I + |\tilde{S}_{\lambda_j}|))^n](\tilde{x}, \tilde{y}), \tag{6.13}$$

then it satisfies (6.9) and (6.10). □

7 Application to the Nonlinear Schrödinger Equation

7.1 Two Norm Estimates

Following the argument in [6], we begin with proving the boundedness of the imaginary power operators. For $\alpha \in \mathbb{R}$, the imaginary power operator $H^{i\alpha} P_c$ is defined as a spectral multiplier of symbol $\lambda^{i\alpha} 1_{[0,+\infty)}$. We consider $H^{i\alpha} P_c$ instead of $H^{i\alpha}$ just for convenience's sake. Indeed, by the assumptions, H has only finitely many negative eigenvalues, and the projection P_{λ_j} is bounded on L^r for any $1 < r < \infty$ (see Lemma 3.6). Therefore, the boundedness of $H^{i\alpha} P_c$ implies that of $H^{i\alpha} = H^{i\alpha} P_c + \sum \lambda_j^{i\alpha} P_{\lambda_j}$, where λ_j's are negative eigenvalues of H.

Lemma 7.1 (Imaginary power operator) *If* $V \in \mathcal{K}_0 \cap L^{3/2,\infty}$ *and* H *has no eigenvalue or resonance on* $[0, +\infty)$*, then for* $\alpha \in \mathbb{R}$*,*

$$\|H^{i\alpha} P_c\|_{L^r \to L^r} \lesssim \langle\alpha\rangle^3, \ 1 < r < \infty. \tag{7.1}$$

Proof Since $\|\lambda^{i\alpha} 1_{[0,+\infty)}\|_{\mathcal{H}(3)} \lesssim \langle\alpha\rangle^3$, the lemma follows from Theorem 1.1. □

Proposition 7.2 (Two norm estimates) *If* $V \in \mathcal{K}_0 \cap L^{3/2,\infty}$ *and* H *has no eigenvalue or resonance on* $[0, +\infty)$*, then for* $0 \le s \le 2$ *and* $1 < r < \frac{3}{s}$*,*

$$\|H^{\frac{s}{2}} P_c (-\Delta)^{-\frac{s}{2}} f\|_{L^r} \lesssim \|f\|_{L^r}, \tag{7.2}$$

$$\|(-\Delta)^{\frac{s}{2}} H^{-\frac{s}{2}} P_c f\|_{L^r} \lesssim \|f\|_{L^r}. \tag{7.3}$$

Proof (7.2) Pick $f, g \in L^1 \cap L^\infty$ such that supp $\hat{f} \subset B(0, R) \setminus B(0, r)$, $P_{n \le \cdot \le N} g = P_c g$ for some $R, r, N, n > 0$. Note that by Lemma 3.5, the collection of such f (g, resp) is dense in L^r ($L^{r'}$, resp). We define

$$F(z) := \langle H^z P_c (-\Delta)^{-z} f, g\rangle_{L^2} = \langle (-\Delta)^{-\operatorname{Re} z - i \operatorname{Im} z} f, H^{-i \operatorname{Im} z} H^{\operatorname{Re} z} g\rangle_{L^2}. \tag{7.4}$$

Indeed, $F(z)$ is well-defined, since $(-\Delta)^{-\operatorname{Re} z - i \operatorname{Im} z} f, H^{-i \operatorname{Im} z} H^{\operatorname{Re} z} g \in L^2$. Moreover, $F(z)$ is continuous on $S = \{z : 0 \le \operatorname{Re} z \le 1\} \subset \mathbb{C}$, and it is analytic in the interior of S. We claim that $H P_c (-\Delta)^{-1}$ is bounded on L^r for $1 < r < \frac{3}{2}$. Indeed, by Lemma 3.6 (i),

$$\|H P_c (-\Delta)^{-1} f\|_{L^r} \lesssim \|(-\Delta + V)(-\Delta)^{-1} f\|_{L^r} \le \|f\|_{L^r} + \|V(-\Delta)^{-1} f\|_{L^r}. \tag{7.5}$$

By the Hölder inequality (Lemma 7.5) and the Sobolev inequality in the Lorentz norms (Corollary 7.9), we have

$$\|V(-\Delta)^{-1} f\|_{L^r} \le \|V\|_{L^{3/2,\infty}} \|(-\Delta)^{-1} f\|_{L^{\frac{3r}{3-2r},r}} \lesssim \|f\|_{L^r}. \tag{7.6}$$

Hence, by the claim and Lemma 7.1, we get

$$|F(1+i\alpha)| \le \|H^{1+i\alpha} P_c (-\Delta)^{-1-i\alpha} f\|_{L^r} \|g\|_{L^{r'}} \lesssim \langle\alpha\rangle^6 \|f\|_{L^r} \|g\|_{L^{r'}}, \quad (1 < r < \tfrac{3}{2}), \tag{7.7}$$

$$|F(i\alpha)| \le \|H^{i\alpha} P_c (-\Delta)^{-i\alpha} f\|_{L^r} \|g\|_{L^{r'}} \lesssim \langle\alpha\rangle^6 \|f\|_{L^r} \|g\|_{L^{r'}}, \quad (1 < r < \infty). \tag{7.8}$$

Therefore (7.2) follows from the Stein's complex interpolation theorem.
(7.3) Pick f and g as above, and consider

$$G(z) := \langle (-\Delta)^z H^{-z} P_c g, f\rangle_{L^2}. \tag{7.9}$$

We claim that $(-\Delta) H^{-1} P_c g$ is bounded on L^r for $1 < r < \frac{3}{2}$. By the triangle inequality,

$$\|(-\Delta)H^{-1}P_c g\|_{L^r} = \|(H-V)H^{-1}P_c g\|_{L^r} \leq \|P_c g\|_{L^r} + \|VH^{-1}P_c g\|_{L^r}. \quad (7.10)$$

By Lemma 3.6 (i), $\|P_c g\|_{L^r} \lesssim \|g\|_{L^r}$. By the Hölder inequality in the Lorentz norms (Lemma 7.5) and the Sobolev inequality associated with H [12, Theorem 1.9], we get

$$\|VH^{-1}P_c g\|_{L^r} \leq \|V\|_{L^{3/2,\infty}}\|H^{-1}P_c g\|_{L^{\frac{3r}{3-2r},r}} \lesssim \|V\|_{L^{3/2,\infty}}\|g\|_{L^r}. \quad (7.11)$$

Repeating the above argument with the complex interpolation, we complete the proof. □

7.2 Local Well-Posedness

Now we are ready to show the local well-posedness (LWP) of a 3d quintic nonlinear Schrödinger equation

$$iu_t + \Delta u - Vu \pm |u|^4 u = 0;\ u(0) = u_0. \quad (\mathrm{NLS}_V)$$

Theorem 7.3 (LWP) *If $V \in \mathcal{K}_0 \cap L^{3/2,\infty}$ and H has no eigenvalue or resonance on $[0, +\infty)$, then* (NLS_V) *is locally well-posed in $\dot{H}^1$. Precisely, for $A > 0$, there exists $\delta = \delta(A) > 0$ such that for an initial data $u_0 \in \dot{H}^1$ obeying*

$$\|\nabla u_0\|_{L^2} \leq A \quad \text{and} \quad \|e^{-itH}u_0\|_{L^{10}_{t\in[0,T_0]}L^{10}_x} < \delta, \quad (7.12)$$

(NLS_V) *has a unique solution $u \in C_t(I; \dot{H}^1_x)$, with $I = [0, T) \subset [0, T_0]$, such that*

$$\|\nabla u\|_{L^{10}_{t\in I}L^{30/13}_x} < \infty \quad \text{and} \quad \|u\|_{L^{10}_{t\in I}L^{10}_x} < 2\delta. \quad (7.13)$$

Proof (Step 1 Contraction mapping argument) Let ψ_j be the eigenfunction corresponding to the negative eigenvalue λ_j normalized so that $\|\psi_j\|_{L^2} = 1$. Choose small $T \in (0, T_0)$ such that $\|\psi_j\|_{L^{10}_{t\in I}L^{10}_x}, \|\psi_j\|_{L^2_{t\in I}L^2_x} \leq 1$ for all j, where $I = [0, T]$ and $\psi_j(t, x) = \psi_j(x)$ for all $t \in I$. For notational convenience, we omit the time interval I in the norm $\|\cdot\|_{L^p_{t\in I}}$ if there is no confusion. Following a standard contraction mapping argument [3,21], we aim to show that

$$\Phi_{u_0}(v)(t) := e^{-itH}u_0 \pm i\int_0^t e^{-i(t-s)H}(|v|^4 v(s))ds \quad (7.14)$$

is a contraction map on the set

$$B_{a,b} := \{v : \|v\|_{L^{10}_{t,x}} \leq a,\ \|\nabla v\|_{L^{10}_t L^{30/13}_x} \leq b\}, \quad (7.15)$$

equipped with the metric $d(u, v) = \|u - v\|_{L^{10}_{t,x}} + \|\nabla(u - v)\|_{L^{10}_t L^{30/13}_x}$, where a, b and δ will be chosen later.

We claim that Φ_{u_0} maps from $B_{a,b}$ to itself. We write

$$\begin{aligned} \|\Phi_{u_0}(v)\|_{L^{10}_{t,x}} &\leq \|e^{-itH}u_0\|_{L^{10}_{t,x}} + \Big\| \int_0^t e^{-i(t-s)H} P_c(|v|^4 v(s))ds \Big\|_{L^{10}_{t,x}} \\ &\quad + \sum_{j=1}^J \Big\| \int_0^t e^{-i(t-s)H}(\langle |v|^4 v(s), \psi_j\rangle_{L^2}\psi_j)ds \Big\|_{L^{10}_{t,x}} \\ &= I + II + \sum_{j=1}^J III_j. \end{aligned} \tag{7.16}$$

By assumption, $I \leq \delta$. For II, by the Sobolev inequality associated with H [12, Theorem 1.6], Strichartz estimates (Proposition 1.2) and the two norm estimates, we get

$$\begin{aligned} II &\lesssim \Big\| \int_0^t e^{-i(t-s)H} P_c H^{1/2}(|v|^4 v(s))ds \Big\|_{L^{10}_t L^{30/13}_x} \lesssim \|H^{1/2}P_c(|v|^4 v)\|_{L^2_t L^{6/5}_x} \\ &\lesssim \|\nabla(|v|^4 v)\|_{L^2_t L^{6/5}_x} \leq 3\|(v^2\nabla v)(\bar{v})^2\|_{L^2_t L^{6/5}_x} + 2\|(v^2\nabla v)(\bar{v})^2\|_{L^2_t L^{6/5}_x} \\ &\lesssim \|v\|^4_{L^{10}_{t,x}} \|\nabla v\|_{L^{10}_t L^{30/13}_x} \leq a^4 b. \end{aligned} \tag{7.17}$$

For the last term, by the Hölder inequality, the choice of T and (7.17), we obtain

$$\begin{aligned} III_j &= \Big\| \int_0^t e^{-i(t-s)\lambda_j}(\langle |v|^4 v(s), \psi_j\rangle_{L^2}\psi_j)ds \Big\|_{L^{10}_{t,x}} \\ &\leq \Big(\int_0^T |\langle |v|^4 v(s), \psi_j\rangle_{L^2}|ds \Big) \|\psi_j\|_{L^{10}_{t,x}} \\ &\leq \|\nabla(|v|^4 v)\|_{L^2_t L^{6/5}_x} \||\nabla|^{-1}\psi_j\|_{L^2_t L^6_x} \\ &\lesssim \|\nabla(|v|^4 v)\|_{L^2_t L^{6/5}_x} \|\psi_j\|_{L^2_t L^2_x} \leq a^4 b. \end{aligned} \tag{7.18}$$

Therefore, we prove that

$$\|\Phi_{u_0}(v)\|_{L^{10}_{t,x}} \leq \delta + Ca^4 b. \tag{7.19}$$

Next, we write

$$\begin{aligned} \|\nabla \Phi_{u_0}(v)\|_{L^{10}_t L^{30/13}_x} &\leq \|\nabla P_c \Phi_{u_0}(v)\|_{L^{10}_t L^{30/13}_x} + \sum_{j=1}^J \|\nabla P_{\lambda_j}\Phi_{u_0}(v)\|_{L^{10}_t L^{30/13}_x} \\ &= \tilde{I} + \sum_{j=1}^J \tilde{I}I_j. \end{aligned} \tag{7.20}$$

For $\tilde{I}$, by the two norm estimates, Strichartz estimates and (7.17), we obtain

$$\begin{aligned}\tilde{I} &\lesssim \|H^{1/2}P_c\Phi_{u_0}(v)\|_{L_t^{10}L_x^{30/13}} \\ &\lesssim \|H^{1/2}P_c u_0\|_{L^2} + \|H^{1/2}P_c(|v|^4 v)\|_{L_t^2 L_x^{6/5}} \\ &\lesssim \|\nabla u_0\|_{L^2} + \|H^{1/2}P_c(|v|^4 v)\|_{L_t^2 L_x^{6/5}} \lesssim A + a^4 b.\end{aligned} \tag{7.21}$$

For $\tilde{II}$, by the Hölder inequality, (7.19) and Lemma 3.6, we get

$$\begin{aligned}\tilde{II}_j &\le \|\langle\Phi_{u_0}(v), \psi_j\rangle_{L^2}\|_{L_t^{10}}\|\psi_j\|_{L^{30/13}} \\ &\le \|\Phi_{u_0}(v)\|_{L_{t,x}^{10}}\|\psi_j\|_{L_x^{10/9}} \lesssim \delta + a^4 b.\end{aligned} \tag{7.22}$$

Collecting all, we prove that

$$\|\nabla\Phi_{u_0}(v)\|_{L_t^{10}L_x^{30/13}} \le CA + Ca^4 b. \tag{7.23}$$

Let $b = 2AC$, $a = \min((2C)^{-\frac{1}{4}}, (2Cb)^{-\frac{1}{3}})$ and $\delta = \frac{a}{2}$ ($\Rightarrow$ $Ca^4 b \le AC$ and $Ca^3 b \le \frac{1}{2}$). Then, by (7.19) and (7.23), Φ_{u_0} maps from $B_{a,b}$ to itself. Similarly, one can show that Φ_{u_0} is contractive in $B_{a,b}$. Thus, we conclude that there exists unique $u \in B_{a,b}$ such that

$$u(t) = \Phi_{u_0}(u) = e^{-itH}u_0 + i\int_0^t e^{-i(t-s)H}(|u|^4 u)(s)ds. \tag{7.24}$$

(Step 2 Continuity) In order to show that $u(t) \in C_t(I; \dot{H}_x^1)$, we write

$$\begin{aligned}u(t) &= e^{-itH}\left(P_c u_0 + \sum_{j=1}^J P_{\lambda_j}u_0\right) \\ &\quad \pm i\int_0^t e^{-i(t-s)H}\Big(P_c(|u|^4 u)(s) + \sum_{j=1}^J P_{\lambda_j}(|u|^4 u)(s)\Big)ds \\ &= e^{-itH}P_c u_0 + \sum_{j=1}^J e^{-it\lambda_j}P_{\lambda_j}u_0 \pm i\int_0^t e^{-i(t-s)H}P_c(|u|^4 u)(s)ds \\ &\quad \pm i\sum_{j=1}^J\int_0^t e^{-i(t-s)\lambda_j}P_{\lambda_j}(|u|^4 u)(s)ds \\ &=: I(t) + \sum_{j=1}^J II_j(t) + III(t) + \sum_{j=1}^J IV_j(t).\end{aligned} \tag{7.25}$$

For $I(t)$, by the two norm estimates and L^2-continuity of e^{-itH}, we have

$$\|I(t) - I(t_0)\|_{\dot{H}^1} \lesssim \|(e^{-itH} - e^{-it_0H})H^{1/2}P_c u_0\|_{L^2} \to 0 \quad \text{as } t \to t_0, \tag{7.26}$$

since $\|H^{1/2}P_c u_0\|_{L^2} \lesssim \|u\|_{\dot{H}^1} < \infty$. $II_j(t)$ is continuous in $\dot{H}^1$, since

$$\|P_{\lambda_j}u_0\|_{\dot{H}^1} = |\langle u_0, \psi_j\rangle_{L^2}|\|\psi_j\|_{\dot{H}^1} \lesssim \|u_0\|_{\dot{H}^1}\|\psi_j\|_{\dot{H}^{-1}} \lesssim \|u_0\|_{\dot{H}^1}\|\psi_j\|_{L^{6/5}} < \infty. \tag{7.27}$$

For $III(t)$, by the two norm estimates, Strichartz estimates and (7.17), we have

$$\begin{aligned}\|III(t) - III(t_0)\|_{\dot{H}^1} &\lesssim \|H^{1/2}(III(t) - III(t_0))\|_{L^2} \\ &\lesssim \|H^{1/2}P_c(|u|^4u)\|_{L^2_{s\in[t_0,t]}L^{6/5}_x} \to 0 \quad \text{as } t \to t_0.\end{aligned} \tag{7.28}$$

For $IV_j(t)$, by the Hölder inequality and (7.17), we write

$$\begin{aligned}\|IV_j(t) - IV_j(t_0)\|_{\dot{H}^1} &\leq \|\psi_j\|_{\dot{H}^1}\|\nabla(|u|^4u)(s)\|_{L^2_{t\in[t_0,t]}L^{6/5}_x}\||\nabla|^{-1}\psi_j\|_{L^2_{s\in[t_0,t]}L^6_x} \\ &\lesssim \|\nabla(|u|^4u)(s)\|_{L^2_{s\in[t_0,t]}L^{6/5}_x}\|\psi_j\|_{L^2_tL^2_x} \to 0 \quad \text{as } t \to t_0.\end{aligned} \tag{7.29}$$

Collecting all, we conclude that $u(t)$ is continuous in $\dot{H}^1$. □

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Appendix: Lorentz Spaces and Interpolation Theorem

Following [21], we summarize useful properties of the Lorentz spaces. Let (X, μ) be a measure space. The Lorentz (quasi) norm is defined by

$$\|f\|_{\tilde{L}^{p,q}} := \begin{cases} p^{1/q}\|\lambda\mu(\{|f| \geq \lambda\})^{1/p}\|_{L^q\left((0,+\infty), \frac{d\lambda}{\lambda}\right)} & \text{when } 1 \leq p < \infty \text{ and } 1 \leq q \leq \infty; \\ \|f\|_{L^\infty} & \text{when } p = q = \infty. \end{cases} \tag{7.30}$$

Lemma 7.4 (Properties of the Lorentz spaces) *Let* $1 \leq p \leq \infty$ *and* $1 \leq q, q_1, q_2 \leq \infty$.

(i) $L^{p,p} = L^p$, *and* $L^{p,\infty}$ *is the weak* L^p*-space.*
(ii) If $q_1 \leq q_2$, $L^{p,q_1} \subset L^{p,q_2}$.

Lemma 7.5 (Hölder inequality) *If* $1 \leq p, p_1, p_2, q, q_1, q_2 \leq \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, *then*

$$\|fg\|_{L^{p,q}} \lesssim \|f\|_{L^{p_1,q_1}}\|fg\|_{L^{p_2,q_2}}. \tag{7.31}$$

Lemma 7.6 (Dual characterization of $L^{p,q}$) *If* $1 < p < \infty$ *and* $1 \le q \le \infty$, *then*

$$\|f\|_{L^{p,q}} \sim \sup_{\|g\|_{L^{p',q'}} \le 1} \Big| \int_X f \bar{g} d\mu \Big|. \tag{7.32}$$

A measurable function f is called a *sub-step function* of height H and width W if f is supported on a set E with measure $\mu(E) = W$ and $|f(x)| \le H$ almost everywhere. Let T be a linear operator that maps the functions on a measure space (X, μ_X) to functions on another measure space (Y, μ_Y). We say that T is *restricted weak-type* $(p, \tilde{p})$ if

$$\|Tf\|_{L^{\tilde{p},\infty}} \lesssim H W^{1/p} \tag{7.33}$$

for all sub-step functions f of height H and width W.

Theorem 7.7 (Marcinkiewicz interpolation theorem) *Let T be a linear operator such that*

$$\langle Tf, g\rangle_{L^2} = \int_Y Tf \bar{g} d\mu_Y \tag{7.34}$$

is well-defined for all simple functions f and g. Let $1 \le p_0, p_1, \tilde{p}_0, \tilde{p}_1 \le \infty$. Suppose that T is restricted weak-type $(p_i, \tilde{p}_i)$ with constant $A_i > 0$ for $i = 0, 1$. Then,

$$\|Tf\|_{L^{\tilde{p}_\theta,q}} \lesssim A_0^{1-\theta} A_1^{\theta} \|f\|_{L^{p_\theta,q}}, \tag{7.35}$$

where $0 < \theta < 1$, $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{\tilde{p}_\theta} = \frac{1-\theta}{\tilde{p}_0} + \frac{\theta}{\tilde{p}_1}$, $\tilde{p}_\theta > 1$ *and* $1 \le q \le \infty$.

In this paper, we use the interpolation theorem of the following form.

Corollary 7.8 (Marcinkiewicz interpolation theorem) *Let T be a linear operator. Let $1 \le p_1 < p_2 \le \infty$. Suppose that for $i = 0, 1$, T is bounded from $L^{p_i,1}$ to $L^{p_i,\infty}$. Then T is bounded on L^p for $p_1 < p < p_2$.*

Proof The corollary follows from Theorem 7.7, since T is restricted weak-type (p_i, p_i):

$$\|f\|_{L^{p_i,1}} = p_i \int_0^\infty \mu(|f| \ge \lambda)^{1/p_i} d\lambda \le p_i \int_0^H W^{1/p_i} d\lambda = p_i H W^{1/p_i}, \tag{7.36}$$

for a sub-step function f of height H and width W. □

Corollary 7.9 (Fractional integration inequality in the Lorentz spaces)

$$\Big\| \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-s}} dy \Big\|_{L^{q,r}(\mathbb{R}^d)} \lesssim \|f\|_{L^{p,r}}, \tag{7.37}$$

where $1 < p < q < \infty$, $1 \le r \le \infty$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$. *At the endpoints, we have*

$$\Big\| \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-s}} dy \Big\|_{L^{\frac{d}{d-s},\infty}(\mathbb{R}^d)} \lesssim \|f\|_{L^1}, \quad \Big\| \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-s}} dy \Big\|_{L^\infty(\mathbb{R}^d)} \lesssim \|f\|_{L^{d/s,1}}. \tag{7.38}$$

Proof (7.38) follows from [18, Theorem 1, p. 119] and duality. Then, (7.37) follows from Corollary 7.8. □

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