J Fourier Anal Appl (2016) 22:462–490
DOI 10.1007/s00041-015-9427-9

On the Boundedness of Singular Integrals in Morrey Spaces and its Preduals

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Received: 2 September 2014 / Published online: 16 September 2015

Abstract We reduce the boundedness of operators in Morrey spaces $L_p^r(\mathbb{R}^n)$, its preduals, $H^{\varrho}L_p(\mathbb{R}^n)$, and their preduals $\overset{\circ}{L}{}_p^r(\mathbb{R}^n)$ to the boundedness of the appropriate operators in Lebesgue spaces, $L_p(\mathbb{R}^n)$. Hereby, we need a weak condition with respect to the operators which is satisfied for a large set of classical operators of harmonic analysis including singular integral operators and the Hardy-Littlewood maximal function. The given vector-valued consideration of these issues is a key ingredient for various applications in harmonic analysis.

Keywords Singular integral operators · Calderón–Zygmund operators · Morrey spaces · Predual Morrey spaces

Mathematics Subject Classification Primary 42B35 · 46E30 · 42B15 · 42B20; Secondary 42B25

1 Introduction

Let $\overset{\circ}{L}{}_p^r(\mathbb{R}^n)$ be the completion of $D(\mathbb{R}^n)$ in $L_p^r(\mathbb{R}^n)$, where

Communicated by Loukas Grafakos.

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$$\left\| f|L_p^r\left(\mathbb{R}^n\right)\right\| = \sup_{\substack{x\in\mathbb{R}^n,\\ R>0}} R^{-\left(\frac{n}{p}+r\right)} \left\| f|L_p\left(B_R(x)\right)\right\|, \quad 1<p<\infty, -\frac{n}{p}\leq r<0.$$

Then we have

$$\left(\overset{\circ}{L}{}_p^r\left(\mathbb{R}^n\right)\right)'' \cong \left(H^{\varrho}L_{p'}\left(\mathbb{R}^n\right)\right)' \cong L_p^r\left(\mathbb{R}^n\right), \tag{1}$$

where the second duality assertion is due to [1,9,14,22,30] and the first assertion is observed by [2] and proved by [22]. Roughly speaking in this paper we prove that the $L_p(\mathbb{R}^n)$-boundedness of an operator T satisfying the condition

$$|(Tf)(y)| \leq c\int_{\mathbb{R}^n}\frac{|f(z)|}{|y-z|^n}\mathrm{d}z \quad \text{for all } f\in D\left(\mathbb{R}^n\right) \text{ and } y\notin \operatorname{supp}(f), \tag{2}$$

implies its boundedness in $\overset{\circ}{L}{}_p^r(\mathbb{R}^n)$. Therefrom, under some additional conditions with respect to T we get also the boundedness of T in $H^{\varrho}L_p(\mathbb{R}^n)$ and $L_p^r(\mathbb{R}^n)$ by (1) and duality arguments. Our paper can be considered as an extension of the new approach given in [21] and [22] to a wider class of operators and to the vector-valued situation. Let us mention that the extension of operators of this type and related norm estimates have to be treated with greater care than in many related papers investigating mapping properties of operators in $L_p^r(\mathbb{R}^n)$. We refer to Remark 4.5 for the relation of our paper to the existing literature. In particular, we cannot expect an unique extension to Morrey spaces $L_p^r(\mathbb{R}^n)$. On the contrary it turned out that there are infinitely many possible extension operators (cf. [22, Remark 5.3]). Let us also mention that the vector-valued situation under consideration is crucial having in mind applications as a Michlin–Hörmander type theorem (and hence applications to Navier–Stokes equations cf. [26] and [21, Remark 4.3]), Littlewood–Paley theory for Morrey spaces and its preduals as well as for Lizorkin representations of Triebel–Lizorkin–Morrey spaces. The given results are partially contained in [20]. Condition (2) is due to Soria and Weiss [23] who transferred the boundedness of singular operators on Lebesgue spaces to the boundedness of these operators in some weighted Lebesgue spaces.

The paper is organized as follows. Basic definitions and preliminaries which are needed later on are collected in Sect. 2. Duality theory for vector-valued Morrey-type spaces is treated in Sect. 3. The main results can be found in Theorem 3.1 (preduals of Morrey spaces) and Theorem 3.3 (Morrey spaces as bidual spaces). In final Sect. 4 we prove our main results concerning the transference of mapping properties of operators satisfying condition (2) to vector-valued Morrey type spaces. The general theorem is presented in Sect. 4.1 (Theorem 4.3) following the method developed in [21] and [22]. As a consequence of our main theorem we obtain mapping properties for various classes of operators in vector-valued Morrey-type spaces. Section 4.2 is concerned with Calderón–Zygmund operators. Here we present also an alternative approach via weighted spaces (Theorem 4.9). Maximal operators of Hardy-Littlewood and Calderón–Zygmund type as well as related vector-valued inequalities are considered in Sect. 4.3. The final Sect. 4.4 is devoted to some classes of Fourier multipliers such as characteristic functions, smooth multipliers and Bochner–Riesz mulipliers at the critical index.

2 Definitions and Preliminaries

2.1 Notation

We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be the Euclidean n-space, where $n \in \mathbb{N}$. Put $\mathbb{R} = \mathbb{R}^1$. Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ and let $S'(\mathbb{R}^n)$ be the space of all tempered distributions on $\mathbb{R}^n$. Let $D(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$ be the collection of all infinitely differentiable complex-valued functions with compact support in $\mathbb{R}^n$, where the support of a function f is abbreviated by $\operatorname{supp}(f)$. Moreover, denotes $C(\mathbb{R}^n)$ and $\mathrm{Lip}(\mathbb{R}^n)$ the collection of all continuous and Lipschitz continuous, respectively, and bounded complex-valued functions defined on $\mathbb{R}^n$.

Furthermore, $L_p(\mathbb{R}^n)$ with $1 \le p < \infty$, is the standard complex Banach space with respect to the Lebesgue measure, normed by

$$\|f\,|L_p\left(\mathbb{R}^n\right)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x\right)^{1/p}.$$

For a measurable subset M of $\mathbb{R}^n$ we similarly define $L_p(M)$. Moreover, $|M|$ stands for the Lebesgue measure of M and χ_M for the characteristic function on M. As usual $\mathbb{Z}$ is the collection of all integers; and $\mathbb{Z}^n$ where $n \in \mathbb{N}$ denotes the lattice of all points $m = (m_1, \dots, m_n) \in \mathbb{R}^n$ with $m_j \in \mathbb{Z}$. As usual, $L_p^{\mathrm{loc}}(\mathbb{R}^n)$ collects all equivalence classes of almost everywhere coinciding measurable complex locally p-integrable functions, hence $f \in L_p(M)$ for any bounded measurable set M in $\mathbb{R}^n$. For any $p \in (1, \infty)$ we denote by p' the conjugate index, namely, $1/p + 1/p' = 1$. For Banach spaces X and Y and an operator $T : X \to Y$

$$T : X \hookrightarrow Y$$

means, that the operator is bounded, that is,

$$\|Tx|Y\| \le c\,\|x|X\|$$

where the the constant c is independent of $x \in X$. Let $D(\mathbb{R}^n) \hookrightarrow X$. A bounded operator $\widetilde{T}$ acting in X, hence $\widetilde{T} : X \hookrightarrow X$, is called an extension of T to X if it coincides on $D(\mathbb{R}^n)$ with T. We denote the Fourier transform of f on $S(\mathbb{R}^n)$ or $S'(\mathbb{R}^n)$ by $\hat{f}$ and its inverse by $\check{f}$ where the normalisation of $\hat{f}$ does not matter for our estimates. The concrete value of constants may vary from one formula to the next, but remains the same within one chain of (in)equalities. Finally, $A \cong B$ is an abbreviation that there are two constants $c, C > 0$ such that $cA \le B \le CA$.

2.2 Morrey Spaces, Duals and Preduals

Definition 2.1 For $1 < p < \infty$ and $-\frac{n}{p} \le r < 0$ we define *Morrey spaces* as

$$L_p^r\left(\mathbb{R}^n\right) \equiv \{f \in L_p^{\mathrm{loc}}\left(\mathbb{R}^n\right) : \left\| f|L_p^r\left(\mathbb{R}^n\right)\right\| < \infty\}$$

with the norm

$$\begin{aligned}\left\| f|L_p^r\left(\mathbb{R}^n\right)\right\| &\equiv \sup_{M\in\mathbb{Z}^n}\sup_{J\in\mathbb{Z}} 2^{J\left(\frac{n}{p}+r\right)} \| f|L_p(Q_{JM})\| \\ &\cong \sup_{x\in\mathbb{R}^n}\sup_{R>0} R^{-\left(\frac{n}{p}+r\right)} \| f|L_p\left(B_R(x)\right)\|,\end{aligned}$$

where $Q_{JM} \equiv Q_{J,M} \equiv 2^{-J}\left(M + [-1,1]^n\right)$ and $B_R(x)$ denotes the ball with radius R centered at x.
Moreover, $\overset{\circ}{L}{}_p^r\left(\mathbb{R}^n\right)$ denotes the closure of $D(\mathbb{R}^n)$ with respect to $\left\|\cdot|L_p^r\left(\mathbb{R}^n\right)\right\|$.

Definition 2.2 Let $1 < p < \infty$ and $-n < \varrho < -n/p$. Then the *predual Morrey spaces* $H^{\varrho}L_p(\mathbb{R}^n)$ collects all $h \in S'(\mathbb{R}^n)$ which can be represented as

$$h = \sum_{J\in\mathbb{Z}, M\in\mathbb{Z}^n} \lambda_{J,M} a_{J,M} \quad \text{in} \quad S'(\mathbb{R}^n) \quad \text{with}$$

$$\operatorname{supp} a_{J,M} \subset Q_{J,M}, \qquad \|a_{J,M}\,|L_p(\mathbb{R}^n)\| \le 2^{-J\left(\frac{n}{p}+\varrho\right)}, \tag{3}$$

such that

$$\sum_{J\in\mathbb{Z}, M\in\mathbb{Z}^n} |\lambda_{J,M}| < \infty. \tag{4}$$

Furthermore,

$$\|h\,|H^{\varrho}L_p(\mathbb{R}^n)\| \equiv \inf \sum_{J\in\mathbb{Z}, M\in\mathbb{Z}^n} |\lambda_{J,M}|$$

where the infimum is taken over all representations (3), (4).

Remark 2.3 The notation of $H^{\varrho}L_p(\mathbb{R}^n)$ as a predual will be justified in the Theorem 3.1. By triangular and Hölder's inequality (3) and (4) ensure that the convergence in (3) is unconditionally in $L_u(\mathbb{R}^n)$, where $\varrho u = -n$. In particular it holds $H^{\varrho}L_p(\mathbb{R}^n) \hookrightarrow L_u(\mathbb{R}^n)$ and we have $1 < u < p$ (cf. [22, (3.10)]). Let $L_p(\mathbb{R}^n, w_\alpha)$ with $1 < p < \infty$ and $w_\gamma(x) = (1+|x|^2)^{\gamma/2}$, $\gamma \in \mathbb{R}$, be the weighted Lebesgue spaces, normed by

$$\|f\,|L_p(w_\alpha, \mathbb{R}^n)\| = \|w_\alpha f\,|L_p(\mathbb{R}^n)\|. \tag{5}$$

Then it holds

$$L_p(w_\alpha, \mathbb{R}^n) \hookrightarrow H^\varrho L_p(\mathbb{R}^n) \tag{6}$$

with $\alpha > n/p'$ (cf. [22, (3.5)]). Furthermore, $D(\mathbb{R}^n)$, $S(\mathbb{R}^n)$ are dense both in $\overset{\circ}{L}{}^r_p(\mathbb{R}^n)$ and $H^\varrho L_p(\mathbb{R}^n)$. $H^\varrho L_p(\mathbb{R}^n)$ and $L^r_p(\mathbb{R}^n)$ are Banach spaces and $L_u(\mathbb{R}^n) \hookrightarrow L^r_p(\mathbb{R}^n) \hookrightarrow L_p(w_\alpha, \mathbb{R}^n)$ for $u = -n/r$ and $\alpha < -n/p$ (cf. [22, Theorem 3.1]). The last embedding as well as (6) can be sharpened cf. (20) below.

Definition 2.4 Let $1 < p < \infty$, $-n < \varrho < -n/p$. Let $H^\varrho L_p(\mathbb{R}^n)^\varepsilon_F$ be the following subspace of $H^\varrho L_p(\mathbb{R}^n)$ defined as

$$H^\varrho L_p(\mathbb{R}^n)^\varepsilon_F \equiv \Bigg\{\varphi \in H^\varrho L_p(\mathbb{R}^n) \Big| \text{ there exists an } L \in \mathbb{N} \text{ such that } \varphi = \sum_{\substack{J\in\mathbb{Z}, M\in\mathbb{Z}^n \\ |J|\le L, |M|\le L}} h_{J,M}, \quad \operatorname{supp} h_{J,M} \subset Q_{J,M} \text{ and } \sum_{\substack{|J|\le L \\ |M|\le L}} 2^{J\left(\frac{n}{p}+\varrho\right)} \left\| h_{J,M} \,|L_p(Q_{J,M})\right\| \le (1+\varepsilon)\|\varphi|H^\varrho L_p(\mathbb{R}^n)\| \Bigg\}.$$

Proposition 2.5 *Let $1 < p < \infty$, $-n < \varrho < -n/p$. Then $H^\varrho L_p(\mathbb{R}^n)^\varepsilon_F$ is dense in $H^\varrho L_p(\mathbb{R}^n)$.*

Proof Let $h \in H^\varrho L_p(\mathbb{R}^n)$ and $\varepsilon > 0$. Let $h = \sum_{J\in\mathbb{Z}, M\in\mathbb{Z}^n} \lambda_{J,M} a_{J,M}$ in $S'(\mathbb{R}^n)$ such that $\sum_{J\in\mathbb{Z}, M\in\mathbb{Z}^n} |\lambda_{J,M}| \le (1+\varepsilon/2)\|h\,|H^\varrho L_p(\mathbb{R}^n)\|$ with $\operatorname{supp} a_{J,M} \subset Q_{J,M}$ and $\|a_{J,M}\,|L_p(\mathbb{R}^n)\| \le 2^{-J\left(\frac{n}{p}+\varrho\right)}$. We define then $h_{J,M} \equiv \lambda_{J,M} a_{J,M}$ for $J \in \mathbb{Z}$, $M \in \mathbb{Z}^n$ and obtain

$$\sum_{J\in\mathbb{Z}, M\in\mathbb{Z}^n} 2^{J\left(\frac{n}{p}+\varrho\right)} \left\| h_{J,M}\,|L_p(Q_{J,M})\right\| \le \sum_{J\in\mathbb{Z}, M\in\mathbb{Z}^n} |\lambda_{J,M}| \le \left(1+\frac{\varepsilon}{2}\right) \left\| h\,|H^\varrho L_p(\mathbb{R}^n)\right\|.$$

Let

$$h^L = \sum_{|J|\le L, |M|\le L} h_{J,M}, \qquad L \in \mathbb{N}.$$

Then

$$\left\| h - h^L\,|H^\varrho L_p(\mathbb{R}^n)\right\| \to 0 \quad \text{if} \quad L \to \infty.$$

Hence,

$$\sum_{\substack{|J|\le L\\|M|\le L}} 2^{J\left(\frac{n}{p}+\varrho\right)} \left\|h_{J,M}\,|L_p(Q_{J,M})\right\| \le \left(1+\frac{\varepsilon}{2}\right)\left\|h\,|H^\varrho L_p(\mathbb{R}^n)\right\|$$

$$\le (1+\varepsilon)\left\|h^L\,|H^\varrho L_p(\mathbb{R}^n)\right\|.$$

□

2.3 Vector-Valued Morrey Spaces

Definition 2.6 Let $1 < p < \infty$, $-\frac{n}{p} \le r < 0$ and $1 < q < \infty$. Let $L^r_p(\ell_q, \mathbb{R}^n)$ be the collection of all sequences of functions f_j belonging to $L^r_p(\mathbb{R}^n)$ such that

$$\left\|f_j|L^r_p(\ell_q,\mathbb{R}^n)\right\| \equiv \left\|\{f_j\}\,|L^r_p(\ell_q,\mathbb{R}^n)\right\| \equiv \left\|\left(\sum_{j=0}^{\infty}|f_j(\cdot)|^q\right)^{\frac{1}{q}}\right|L^r_p\left(\mathbb{R}^n\right)\right\|$$

is finite. Moreover,

$$\overset{\circ}{L}{}^r_p(\ell_q,\mathbb{R}^n) \equiv \Bigg\{\{f_j\}_{j\in\mathbb{N}_0} \in L^r_p(\ell_q,\mathbb{R}^n)\Bigg|\ \text{there exist } f^k_j \in D(\mathbb{R}^n) \text{ for all } j\in\mathbb{N}_0, k\in\mathbb{N} \text{ and } f^k_j = 0 \text{ for } j>k \text{ with } \left\|\left\{f_j - f^k_j\right\}_j\right| L^r_p(\ell_q,\mathbb{R}^n)\right\| \to 0 \quad (k\to\infty)\Bigg\}.$$

Furthermore, for $\alpha \in \mathbb{R}$ we define the space $L_p(\ell_q, w_\alpha, \mathbb{R}^n)$ as $L^r_p(\ell_q, \mathbb{R}^n)$ using the norm of $L_p(w_\alpha, \mathbb{R}^n)$ instead the norm of $L^r_p(\mathbb{R}^n)$. If $\alpha = 0$, we simply write $L_p(\ell_q, \mathbb{R}^n)$.

Definition 2.7 Then $H^\varrho L_p(\ell_q, \mathbb{R}^n)$ denotes the collection of all sequences of functions g_j belonging to $H^\varrho L_p(\mathbb{R}^n)$ such that $\left\|g_j(\cdot)|\ell_q\right\|$ is in $H^\varrho L_p(\mathbb{R}^n)$. Moreover, $H^\varrho L_p(\ell_q, \mathbb{R}^n)^\varepsilon_F$ stands for the collection of all sequences of functions g_j belonging to $H^\varrho L_p(\ell_q, \mathbb{R}^n)$ such that $\left\|g_j(\cdot)|\ell_q\right\|$ is in $H^\varrho L_p(\mathbb{R}^n)^\varepsilon_F$.

3 Duals and Preduals: The Vector-Valued Case

3.1 Predual Spaces

The duality with respect to Morrey spaces is discussed in the scalar case in detail with complete proofs in [22]. Here we give complete proofs in the vector-valued case following their approach.

Theorem 3.1 *Let $1 < p < \infty$, $-\frac{n}{p} < r < 0$, $r + \varrho = -n$ and $1 < q < \infty$. Then the predual space of $L^r_p(\ell_q, \mathbb{R}^n)$ is $H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n)$. Moreover,*

$$g \in \left(H^{\varrho} L_{p'}(\ell_{q'}, \mathbb{R}^n)\right)'$$

if, and only if, it can be uniquely represented as

$$g(f) = \int_{\mathbb{R}^n} \sum_{j \in \mathbb{N}_0} g_j(x) f_j(x) \mathrm{d}x \tag{7}$$

for all $f \equiv \{f_j\} \in L_{p'}(\ell_{q'}, w_\alpha, \mathbb{R}^n) \hookrightarrow H^{\varrho} L_{p'}(\ell_{q'}, \mathbb{R}^n)$, $\alpha > n/p$, *where*

$$\{g_j\} \in L_p^r(\ell_q, \mathbb{R}^n) \text{ and } \left\| g \left| \left(H^{\varrho} L_{p'}(\ell_{q'}, \mathbb{R}^n)\right)' \right\| = \left\| g_j | L_p^r(\ell_q, \mathbb{R}^n) \right\| \right. .$$

Moreover, if $\{g_j\} \in L_p^r(\ell_q, \mathbb{R}^n)$, *then*

$$\left\| g_j | L_p^r(\ell_q, \mathbb{R}^n) \right\| = \sup_f \left| \int_{\mathbb{R}^n} \sum_{j \in \mathbb{N}_0} g_j(x) f_j(x) \mathrm{d}x \right| \tag{8}$$

where the supremum is taken over all $f \equiv \{f_j\} \in H^{\varrho} L_{p'}(\ell_{q'}, \mathbb{R}^n)$ *with* $\| f | H^{\varrho} L_{p'}(\ell_{q'}, \mathbb{R}^n) \| \leq 1$.

Proof Let $g \equiv \{g_j\} \in L_p^r(\ell_q, \mathbb{R}^n)$ and $\{\tilde{f}_j\} \in H^{\varrho} L_{p'}(\ell_{q'}, \mathbb{R}^n)$ such that $\left\| \tilde{f}_j(\cdot) \right| \ell_{q'} \|$ is in $H^{\varrho} L_{p'}(\mathbb{R}^n)_F^{\varepsilon}$. Hölder's inequality yields

$$\begin{aligned}
\int_{\mathbb{R}^n} \sum_{j \in \mathbb{N}_0} \left| g_j(y) \tilde{f}_j(y) \right| \mathrm{d}y &\leq \int_{\mathbb{R}^n} \left\| \left\{ g_j(y) \right\}_j | \ell_q \right\| \left\| \left\{ \tilde{f}_j(y) \right\}_j | \ell_{q'} \right\| \mathrm{d}y \\
&\leq \sum_{\substack{J \in \mathbb{Z}, M \in \mathbb{Z}^n \\ |J| \leq L, |M| \leq L}} \int_{\mathbb{R}^n} \left\| \left\{ g_j(y) \right\}_j | \ell_q \right\| h_{J,M}(y) \mathrm{d}y \\
&\leq \sum_{\substack{J \in \mathbb{Z}, M \in \mathbb{Z}^n \\ |J|, |M| \leq L}} 2^{J\left(\frac{n}{p} + r\right)} \left\| \left\| \left\{ g_j(\cdot) \right\}_j | \ell_q \right\| \bigg| L_p(Q_{J,M}) \right\| 2^{J\left(\frac{n}{p'} + \varrho\right)} \left\| h_{J,M} | L_{p'}\left(Q_{J,M}\right) \right\| \\
&\leq (1 + \varepsilon) \left\| g_j | L_p^r(\ell_q, \mathbb{R}^n) \right\| \left\| \tilde{f}_j | H^{\varrho} L_{p'}(\ell_{q'}, \mathbb{R}^n) \right\|
\end{aligned}$$

where $\left\| \tilde{f}_j(\cdot) \right| \ell_{q'} \|$ is represented as in Definition 2.4 and $r + \varrho + n = 0$. Therefore the operator T_g given by

$$T_g\left(\{\tilde{f}_j\}\right) \equiv \int_{\mathbb{R}^n} \sum_{j \in \mathbb{N}_0} \left| g_j(y) \tilde{f}_j(y) \right| \mathrm{d}y$$

is bounded on $H^{\varrho} L_{p'}(\ell_{q'}, \mathbb{R}^n)_F^{\varepsilon}$. We get the (unique) continuous extension $T_g : H^{\varrho} L_{p'}(\ell_{q'}, \mathbb{R}^n) \hookrightarrow \mathbb{R}$ by means of Proposition 2.5, where this extension is justified as in the linear case cf. (15) and (17) below. Let $\{f_j\} \in H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n)$. By Proposition 2.5 there is furthermore a sequence $\{f_j^k\}$ of $H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n)_F^\varepsilon$ such that $\{f_j^k\}$ tends to $\{f_j\}$ in $H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n)$ for $k \to \infty$. For u such that $\varrho u = -n$ by $H^\varrho L_{p'}(\mathbb{R}^n) \hookrightarrow L_u(\mathbb{R}^n)$ exists a subsequence such that $\left\| f_j^{k_l}(\cdot) - f_j(\cdot) \middle| \ell_{q'} \right\| \to 0$ almost everywhere with respect to the Lebesgue measure in $\mathbb{R}^n$ for $l \to \infty$. This implies $f_j^{k_l} \to f_j$ almost everywhere for all $j \in \mathbb{N}_0$ if $l \to \infty$. The Lemma of Fatou yields then

$$\begin{aligned}\int_{\mathbb{R}^n} \sum_{j\in\mathbb{N}_0} \left|g_j(y) f_j(y)\right| \mathrm{d}y &= \int_{\mathbb{R}^n} \sum_{j\in\mathbb{N}_0} \left|g_j(y) \lim_{l\to\infty} f_j^{k_l}(y)\right| \mathrm{d}y \\ &\le \lim_{l\to\infty} T_g(\{f_j^{k_l}\}) = T_g(\{f_j\})\end{aligned}$$

Thus, for $\varepsilon \searrow 0$ we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \sum_{j\in\mathbb{N}_0} g_j(y) f_j(y) \mathrm{d}y\right| &\le \int_{\mathbb{R}^n} \sum_{j\in\mathbb{N}_0} \left|g_j(y) f_j(y)\right| \mathrm{d}y \\ &\le \left\|g_j | L_p^r(\ell_q, \mathbb{R}^n)\right\| \left\| f_j | H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n)\right\| \end{aligned} \tag{9}$$

for $\{g_j\} \in L_p^r(\ell_q, \mathbb{R}^n)$ and $\{f_j\} \in H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n)$. Hence, in particular, any $\{g_j\} \in L_p^r(\ell_q, \mathbb{R}^n)$ induces a bounded linear functional on $H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n)$.

Conversely, suppose that g is a bounded linear functional on $H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n)$ with the norm $\|g\|$. Taking into account (6) the linear functional g induces a bounded linear functional on $L_{p'}(\ell_{q'}, w_\alpha, \mathbb{R}^n)$ for $\alpha > n/p$ and therefore we have the representation formula

$$g(\{f_j\}) = \int_{\mathbb{R}^n} \sum_j g_j(y) f_j(y) \mathrm{d}y \tag{10}$$

for some $\{g_j\} \in L_p(\ell_q, w_{-\alpha}, \mathbb{R}^n)$ and for all $\{f_j\} \in L_{p'}(\ell_{q'}, w_\alpha, \mathbb{R}^n)$. Let $\{\tilde{f}_j\} \in L_{p'}(\ell_{q'}, w_\alpha, \mathbb{R}^n)$ with supp $\tilde{f}_j \subset Q_{J,M}$ for all $j \in \mathbb{N}_0$. Then

$$\left\| \tilde{f}_j | H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n)\right\| \le 2^{J(\frac{n}{p'}+\varrho)} \left\| \|\{\tilde{f}_j(\cdot)\}_j | \ell_{q'}\| \middle| L_{p'}(Q_{J,M})\right\|.$$

With $\frac{n}{p'} + \varrho = -\frac{n}{p} - r$ one obtains

$$\begin{aligned}\left|g\left(\{f_j\}\right)\right| &\le \|g\| \left\| \tilde{f}_j \middle| H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n)\right\| \\ &\le \|g\| \, 2^{-J\left(\frac{n}{p}+r\right)} \left\| \|\{\tilde{f}_j(\cdot)\}_j | \ell_{q'}\| \middle| L_{p'}(Q_{J,M})\right\|.\end{aligned}$$

Then one has by duality in $L_{p'}(\ell_{q'}, Q_{J,M})$ and (10)

$$\|g_j \,| L_p(\ell_q, Q_{J,M})\| \le 2^{-J\left(\frac{n}{p'}+r\right)} \|g\|.$$

Hereby, $L_{p'}(\ell_{q'}, Q_{J,M})$ is defined similarly as $L^r_p(\ell_{q'}, \mathbb{R}^n)$ using $L_{p'}(Q_{J,M})$ instead of $L^r_p(\mathbb{R}^n)$. Note also that an element of $L_{p'}(\ell_{q'}, Q_{J,M})$, say $\{\tilde{f}_j\}$, is also in $L_{p'}(\ell_{q'}, w_\alpha, \mathbb{R}^n)$ if one extends $\tilde{f}_j$, $j \in \mathbb{N}_0$, outside of $Q_{J,M}$ by zero. The last inequality proves $\{g_j\} \in L^r_p(\ell_q, \mathbb{R}^n)$ and

$$\left\| g_j | L^r_p(\ell_q, \mathbb{R}^n) \right\| \leq \|g\|.$$

□

3.2 Dual Spaces

In the proof of the next theorem, which is a vector-valued extension of [22, Theorem 4.1, (4.5)], we benefit from the following general assertion.

Proposition 3.2 ([4, p. 73] and [24, Lemma in Sect. 1.11.1]) *Let $\{A_j\}_{j\in\mathbb{N}_0}$ be a sequence of complex Banach spaces and $\{A'_j\}_{j\in\mathbb{N}_0}$ their respective duals. Moreover, we put*

$$c_0(\{A_j\}) \equiv \Big\{ a \equiv \{a_j\}_{j\in\mathbb{N}_0} \Big| a_j \in A_j, \\ \|a|c_0(A_j)\| \equiv \|a|\ell_\infty(A_j)\| \equiv \sup_j \|a_j|A_j\| < \infty, \|a_j|A_j\| \to 0 \Big\},$$

$$\ell_1(\{A'_j\}) \equiv \left\{ a' \equiv \left\{a'_j\right\}_{j\in\mathbb{N}_0} \middle| a'_j \in A'_j, \left\|a'|\ell_1(A'_j)\right\| \equiv \sum_j \left\|a_j|A'_j\right\| < \infty \right\}.$$

Then

$$\big(c_0(\{A_j\})\big)' = \ell_1(\{A'_j\}) \text{ with } a'(a) = \sum_{j=0}^{\infty} a'_j(a_j) \text{ and}$$

$$\left\| \cdot \left| (c_0(A_j))' \right\| = \left\| \cdot \left| \ell_1(A'_j) \right\| \right. \right. .$$

Theorem 3.3 *Let $1 < p < \infty$, $-\frac{n}{p} < r < 0$, $r + \varrho = -n$ and $1 < q < \infty$. Then the dual space of $\mathring{L}^r_p(\ell_q, \mathbb{R}^n)$ is $H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n)$. Moreover, $g \in \left(\mathring{L}^r_p(\ell_q, \mathbb{R}^n)\right)'$ if, and only if, it can be uniquely represented as*

$$g(f) = \int_{\mathbb{R}^n} \sum_{j\in\mathbb{N}_0} g_j(x) f_j(x) \mathrm{d}x$$

for all $f \equiv \{f_j\} \in L_{-\frac{n}{r}}(\ell_q, \mathbb{R}^n) \hookrightarrow \mathring{L}^r_p(\ell_q, \mathbb{R}^n)$, where

$$\{g_j\} \in H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n) \text{ and } \left\| g \left| \left(\mathring{L}^r_p(\ell_q, \mathbb{R}^n)\right)' \right\| = \left\| g_j | H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n) \right\| \right. .$$

Moreover, if $\{g_j\} \in H^{\varrho} L_{p'}(\ell_{q'}, \mathbb{R}^n)$*, then*

$$\left\| g_j | H^{\varrho} L_{p'}(\ell_{q'}, \mathbb{R}^n) \right\| = \sup_f \left| \int_{\mathbb{R}^n} \sum_{j \in \mathbb{N}_0} g_j(x) f_j(x) \mathrm{d}x \right| \tag{11}$$

where the supremum is taken over all $f \equiv \{f_j\} \in \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n)$ *with* $\left\| f \left| L^r_p(\ell_q, \mathbb{R}^n) \right. \right\| \leq 1$.

Proof It follows from (9) that any $\{g_j\} \in H^{\varrho} L_{p'}(\ell_{q'}, \mathbb{R}^n)$ induces a bounded linear functional on $\overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n)$.

Conversely, suppose g is a bounded linear functional on $\overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n)$ with norm $\|g\|$. We observe that

$$\begin{aligned} \left\| \{f_j\} | L^r_p(\ell_q, \mathbb{R}^n) \right\| &= \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} \left(\int_{Q_{J,M}} \left(\sum_{j \in \mathbb{N}_0} |f_j(x)|^q \right)^{\frac{p}{q}} 2^{J(n+pr)} \mathrm{d}x \right)^{\frac{1}{p}} \\ &= \left\| f^j_{JM} | c_0 \left(L_p(\ell_q, \mu_J, Q_{JM}) \right) \right\|, \end{aligned}$$

where $f^j_{JM} \equiv f_j \chi_{Q_{JM}}$, $\mu_J(\mathrm{d}x) \equiv 2^{J(n+pr)}$ and

$$\begin{aligned} &\left\| f^j_{JM} | c_0 \left(L_p(\ell_q, \mu_J, Q_{JM}) \right) \right\| \\ &\equiv \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} \left(\int_{Q_{JM}} \left(\sum_{j \in \mathbb{N}_0} |f^j_{JM}(x)|^q \right)^{\frac{p}{q}} 2^{J(n+pr)} \mathrm{d}x \right)^{\frac{1}{p}}. \end{aligned}$$

This shows that $\overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n)$ is isomorphic to a closed subspace of $c_0 \left(L_p(\ell_q, \mu_J, Q_{JM}) \right)$ analogously to the scalar-valued case in [22, (4.18)–(4.20)]. More precisely, we have a linear, surjective and isometric map $I : \{f_j\} \mapsto \{f^j_{JM}\}$ from $\overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n)$ onto the closed subspace $\{\{f^j_{JM}\} | \{f_j\} \in \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n)\}$ of $c_0 \left(L_p(\ell_q, \mu_J, Q_{JM}) \right)$ and

$$I \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n) = \{\{f^j_{JM}\} | \{f_j\} \in \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n)\} \hookrightarrow c_0 \left(L_p(\ell_q, \mu_J, Q_{JM}) \right).$$

Hahn-Banach's theorem yields $g \in \left(\overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n) \right)'$ if, and only if, $g \in \left(c_0 \left(L_p(\ell_q, \mu_J, Q_{JM}) \right) \right)'$ and by Proposition 3.2 we have the representation

$$g(\{f_j\}) = \sum_{J\in\mathbb{Z},M\in\mathbb{Z}^n} \int_{Q_{JM}} \sum_{j\in\mathbb{N}_0} f_j(x) g^j_{JM}(x) 2^{J(n+pr)} \mathrm{d}x \tag{12}$$

for any $\{f_j\} \in \mathring{L}^r_p(\ell_q, \mathbb{R}^n)$ with $\{g^j_{JM}\} \in \ell_1\left(L_{p'}(\ell_{q'}, \mu_J, Q_{JM})\right)$, where

$$\left\|\{g^j_{JM}\}|\ell_1\left(L_{p'}(\ell_{q'}, \mu_J, Q_{JM})\right)\right\| = \sum_{J\in\mathbb{Z},M\in\mathbb{Z}^n} \left(\int_{Q_{JM}} \left(\sum_{j\in\mathbb{N}_0} |g^j_{JM}(x)|^{q'}\right)^{\frac{p'}{q'}} 2^{J(n+pr)}\,\mathrm{d}x\right)^{\frac{1}{p'}}.$$

Moreover, Hahn-Banach's theorem implies that

$$\begin{aligned}&\left\|g\left|\left(\mathring{L}^r_p(\mathbb{R}^n)\right)'\right.\right\|\\ &= \inf\Big\{\left\|g^j_{JM}\left|\ell_1\left(L_{p'}(\ell_{q'}, \mu_J, Q_{JM})\right)\right.\right\| \Big| g(\{f_j\}) = g^j_{JM}(\{f_j\})\\ &\quad \text{for all } \{f_j\} \in \mathring{L}^r_p(\ell_q, \mathbb{R}^n) \text{ and } g^j_{JM} \in \ell_1\left(L_{p'}(\ell_{q'}, \mu_J, Q_{JM})\right)\Big\}.\end{aligned}$$

Using Lebesgue's dominated convergence theorem we deduce from (12) (cf. (13) for an integrable majorant) the representation

$$g(\{f_j\}) = \int_{\mathbb{R}^n} \sum_{j\in\mathbb{N}_0} f_j(x) \sum_{J\in\mathbb{Z},M\in\mathbb{Z}^n} g^j_{JM}(x)\chi_{Q_{JM}}(x) 2^{J(n+pr)} \mathrm{d}x$$

for $\{f_j\} \in L_{-\frac{n}{r}}(\ell_q, \mathbb{R}^n)$. Let $\varepsilon > 0$. For $h^j_{JM} \equiv g^j_{JM}\chi_{Q_{JM}} 2^{J(n+pr)}$ we obtain

$$\left\|g^j_{JM}|L_{p'}(\ell_{q'}, \mu_J, Q_{JM})\right\| = 2^{-J\left(\frac{n}{p}+r\right)} \left\|h^j_{JM}|L_{p'}(\ell_{q'}, \mathbb{R}^n)\right\| \equiv \lambda_{JM}.$$

Therefore $\{\lambda_{JM}\}_{J,M} \in \ell_1$ and for an appropriate choice of g^j_{JM} we obtain also $\|\lambda|\ell_1\| \le (1+\varepsilon)\|g\|$. For a^j_{JM} given by $h^j_{JM} = \lambda_{JM} a^j_{JM}$ we have then $\left\|a^j_{JM}\right| L_{p'}(\ell_{q'}, \mathbb{R}^n)\Big\| \le 2^{J\left(\frac{n}{p}+r\right)}$ with supp $\left(a^j_{JM}\right) \subset Q_{JM}$. Finally, it holds $\left\{\sum_{J\in\mathbb{Z},M\in\mathbb{Z}^n} \lambda_{JM} a^j_{JM}\right\}_j \in H^{\varrho} L_{p'}(\ell_{q'}, \mathbb{R}^n)$ and

$$\left\{\sum_{J\in\mathbb{Z},M\in\mathbb{Z}^n} g^j_{JM}\chi_{Q_{JM}} 2^{J(n+pr)}\right\}_j \in H^{\varrho} L_{p'}(\ell_{q'}, \mathbb{R}^n).$$

Indeed, we have

$$\left(\sum_{j\in\mathbb{N}_0}\left|\sum_{\substack{J\in\mathbb{Z},\\ M\in\mathbb{Z}^n}}\lambda_{JM}a^j_{JM}\right|^{q'}\right)^{\frac{1}{q'}}\leq\sum_{\substack{J\in\mathbb{Z},\\ M\in\mathbb{Z}^n}}\lambda_{JM}\left(\sum_{j\in\mathbb{N}_0}\left|a^j_{JM}\right|^{q'}\right)^{\frac{1}{q'}}\in H^{\varrho}L_{p'}(\mathbb{R}^n)$$

using $b_{JM}\equiv\left(\sum_{j\in\mathbb{N}_0}\left|a^j_{JM}\right|^{q'}\right)^{\frac{1}{q'}}$ with supp $(b_{JM})\subset Q_{JM}$ and $\left\|b_{JM}|L_{p'}(\mathbb{R}^n)\right\|\leq 2^{J\left(\frac{n}{p}+r\right)}=2^{-J\left(\frac{n}{p'}+\varrho\right)}$. Finally,

$$\left\|\left\{\sum_{J\in\mathbb{Z},M\in\mathbb{Z}^n}g^j_{JM}(x)\chi_{Q_{JM}}(x)2^{J(n+pr)}\right\}_j\left|H^{\varrho}L_{p'}(\ell_{q'},\mathbb{R}^n)\right.\right\|\leq\|\lambda|\ell_1\|$$
$$\leq(1+\varepsilon)\,\|g\|\,.$$

By the same argumentation we obtain also

$$\left\{\sum_{J\in\mathbb{Z},M\in\mathbb{Z}^n}\left|g^j_{JM}\chi_{Q_{JM}}\right|2^{J(n+pr)}\right\}_j\in H^{\varrho}L_{p'}(\ell_{q'},\mathbb{R}^n)\hookrightarrow L_{-\frac{n}{\varrho}}(\ell_q,\mathbb{R}^n).$$

Together with $\{f_j\}\in L_{-\frac{n}{r}}(\ell_q,\mathbb{R}^n)$ and Hölder's inequality

$$\sum_{j\in\mathbb{N}_0}|f_j|\sum_{J\in\mathbb{Z},M\in\mathbb{Z}^n}\left|g^j_{JM}\chi_{Q_{JM}}\right|2^{J(n+pr)} \tag{13}$$

is an integrable majorant. Moreover, we observe $L_{-\frac{n}{r}}(\ell_q,\mathbb{R}^n)\hookrightarrow\overset{\circ}{L}{}^r_p(\ell_q,\mathbb{R}^n)$. Indeed, for $\{f_j\}\in L_{-\frac{n}{r}}(\ell_q,\mathbb{R}^n)$ there is a sequence $\{f^k_j\}_j$ tending to $\{f_j\}$ in $L_{-\frac{n}{r}}(\ell_q,\mathbb{R}^n)$ as $k\to\infty$ with $f^k_j\in D(\mathbb{R}^n)$ and $f^k_j=0$ for $j>k$ (and $f^k_j\nearrow f_j$ as $k\to\infty$) which also implies $\{f^k_j\}_j\to\{f_j\}$ in $\overset{\circ}{L}{}^r_p(\ell_q,\mathbb{R}^n)$ as $k\to\infty$ by $L_{-\frac{n}{r}}(\mathbb{R}^n)\hookrightarrow L^r_p(\mathbb{R}^n)$. □

4 Mapping Properties of Operators

4.1 The Main Theorem

Next we extend the approach developed in [21] and [22] to a wider class of operators and to vector-valued spaces.

Proposition 4.1 *Let* $1 < p < \infty$, $-\frac{n}{p} < r < 0$ *and* $1 < q < \infty$. *Then* $\overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n)$ *coincides with the completion of finite sequences of continuous compactly supported functions. More precisely, it holds*

$$\overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n) = \Big\{ \{f_j\}_{j\in\mathbb{N}_0} \in L^r_p(\ell_q, \mathbb{R}^n) \Big| \text{ there exist } f_j^k \in C(\mathbb{R}^n) \text{ compactly supported for all } j \in \mathbb{N}_0, k \in \mathbb{N} \text{ and } f_j^k = 0 \text{ for } j > k \text{ with } \Big\| \big\{ f_j - f_j^k \big\}_j \Big| L^r_p(\ell_q, \mathbb{R}^n) \Big\| \to 0 \Big\} \equiv \overline{C_0(\ell_q, \mathbb{R}^n)}^{\|\cdot|L^r_p(\mathbb{R}^n)\|}.$$

Proof Let $\{f_j\} \in \overline{C_0(\ell_q, \mathbb{R}^n)}^{\|\cdot|L^r_p(\mathbb{R}^n)\|}$. Let $\varepsilon > 0$. Then there exists a sequence $\{g_j\}$ with $g_j \in C(\mathbb{R}^n)$ compactly supported with $g_j = 0$ for $|j| > k$ and some $k \in \mathbb{N}$ such that $\left\| f_j - g_j | L^r_p(\ell_q, \mathbb{R}^n) \right\| < \varepsilon$. Let $x \in \mathbb{R}^n$, $R > 0$. Let $\bar{R} > 1$ such that $\operatorname{supp} \sum_{j=0}^k |g_j|^q \subset B_{\bar{R}-1}(0)$. Let $y \in \mathbb{R}^n$ with $|y| < 1$. Moreover, for $R \geq \bar{R}$

$$\left\| g_j(\cdot - y) - g_j(\cdot) | L_p(\ell_q, B_R(x)) \right\| \leq \frac{\varepsilon \bar{R}^r}{c} \left| B_{\bar{R}}(0) \right|^{\frac{1}{p}} \leq \varepsilon R^{\frac{n}{p}+r}$$

whenever

$$\sum_{j=0}^k |g_j(z - y) - g_j(z)| < \frac{\varepsilon \bar{R}^r}{c} \qquad \text{for all } z \in \mathbb{R}^n \tag{14}$$

which holds by the uniform continuity of g_j, $j = 0, \ldots, k$, for $|y| < \delta = \delta(\varepsilon, \bar{R}, r, g_0, \ldots, g_k)$, where c is a constant depending on n. Furthermore, for $R < \bar{R}$ again by (14)

$$\left\| g_j(\cdot - y) - g_j(\cdot) | L_p(\ell_q, B_R(x)) \right\| \leq \frac{\varepsilon \bar{R}^r}{c} |B_R(x)|^{\frac{1}{p}} \leq \varepsilon R^{\frac{n}{p}+r}$$

Let $\psi \in D(\mathbb{R}^n)$ with $\operatorname{supp} \psi \subset B_1(0)$, $\int_{\mathbb{R}^n} \psi(y) \mathrm{d}y = 1$, $0 \leq \psi \leq 1$ and $\psi_l(\cdot) \equiv l^n \psi(l \cdot)$, $l \in \mathbb{N}$. Then it holds $\left\| \{g_j * \psi_l - g_j\}_j | L^r_p(\ell_q, \mathbb{R}^n) \right\| < \varepsilon$ for l sufficient large where $g_j * \psi_l \in D(\mathbb{R}^n)$, $j \in \mathbb{N}_0$. Indeed, by means of Minkowski's inequality and the properties of ψ_l we find

$$\begin{aligned} &\left\| g_j * \psi_l - g_j | L_p(\ell_q, B_R(x)) \right\| \\ &\leq R^{\frac{n}{p}+r} \int_{|y| \leq \frac{1}{l}} |\psi_l(y)| R^{-\left(\frac{n}{p}+r\right)} \left\| g_j(\cdot - y) - g_j(\cdot) | L_p(\ell_q, B_R(x)) \right\| \mathrm{d}y \\ &\leq \varepsilon R^{\frac{n}{p}+r} \end{aligned}$$

where l is sufficiently large (depending on ε). □

Remark 4.2 In the last Proposition we adapted the proof in scalar-valued case $L^r_p(\mathbb{R}^n)$ given in [30, Proposition 3] to the vector-valued situation.

Theorem 4.3 *Let* $1 < p < \infty$, $-\frac{n}{p} < r < 0$, $r + \varrho = -n$, $1 < q < \infty$ *and let* $\{T_j\}_{j\in\mathbb{N}_0}$ *be a sequence of operators with the following properties:*

(i) $T_j : D(\mathbb{R}^n) \to C(\mathbb{R}^n)$, $j \in \mathbb{N}_0$, *and* T_j, $j \in \mathbb{N}_0$, *are*
  (a) *either linear or*
  (b)

$$\begin{aligned} \left(T_j(f_1 + f_2)\right)(y) &\le (T_j f_1)(y) + (T_j f_2)(y), \\ (T_j f)(y) &= \left(T_j(-f)\right)(y), \ T_j 0 = 0 \end{aligned} \tag{15}$$

  for $f, f_1, f_2 \in D(\mathbb{R}^n)$ *and* $y \in \mathbb{R}^n$*;*

(ii) *we have*

$$|(T_j f)(y)| \le c_1 \int_{\mathbb{R}^n} \frac{|f(z)|}{|y-z|^n} dz \tag{16}$$

  for all $f \in D(\mathbb{R}^n)$ *and all* $y \notin \operatorname{supp}(f)$*, where* c_1 *does not depend on* $j \in \mathbb{N}_0$, f *and* y*;*

(iii) *there is a constant* c_2 *such that*

$$\left\| T_j f_j | L_p(\ell_q, \mathbb{R}^n)\right\| \le c_2 \left\| f_j | L_p(\ell_q, \mathbb{R}^n)\right\|$$

  for all $\{f_j\}_{j\in\mathbb{N}_0} \subset D(\mathbb{R}^n)$.

Then, the following statements hold true.

(1) *There are unique continuous and bounded extensions* $\widetilde{T}_j$ *of* T_j *to* $\overset{\circ}{L}{}^r_p(\mathbb{R}^n)$ *for* $j \in \mathbb{N}_0$ *such that*

$$\left\{\widetilde{T}_j\right\}_{j\in\mathbb{N}_0} : \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n) \hookrightarrow \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n).$$

(2) *If* T_j *are linear for* $j \in \mathbb{N}_0$*, then the dual operators of the unique linear and bounded extensions* $\widetilde{T}_j$ *of* T_j *to* $\overset{\circ}{L}{}^r_p(\mathbb{R}^n)$, $\widetilde{T}_j{}' : H^\varrho L_{p'}(\mathbb{R}^n) \hookrightarrow H^\varrho L_{p'}(\mathbb{R}^n)$, *satisfy*

$$\left\{\widetilde{T}_j{}'\right\}_{j\in\mathbb{N}_0} : H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n) \hookrightarrow H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n).$$

  If the extensions of T_j *to* $L_p(\mathbb{R}^n)$ *due to assumption (iii) are formally self-adjoint for all* $j \in \mathbb{N}_0$*, then* $\widetilde{T}_j{}'$ *are the unique linear and bounded extensions of* T_j *acting in* $H^\varrho L_{p'}(\mathbb{R}^n)$.

(3) *If* T_j *are linear for* $j \in \mathbb{N}_0$*, then there are linear and bounded extensions* $\widetilde{T}_j$ *of* T_j *to* $L^r_p(\mathbb{R}^n)$ *such that*

$$\left\{\widetilde{T}_j\right\}_{j\in\mathbb{N}_0} : L^r_p(\ell_q, \mathbb{R}^n) \hookrightarrow L^r_p(\ell_q, \mathbb{R}^n).$$

Proof Step 1 We start showing Assertion (1).

At first we will show that $\{T_j\}_{j\in\mathbb{N}_0} : \overset{\circ}{L}{}^r_p(\ell_q,\mathbb{R}^n) \hookrightarrow L^r_p(\ell_q,\mathbb{R}^n)$. Let $\{f_j\}_{j=0}^\infty \in \overset{\circ}{L}{}^r_p(\ell_q,\mathbb{R}^n)$ with $f_j \in D(\mathbb{R}^n)$ for all j. Let $x \in \mathbb{R}^n$ and $R > 0$. We decompose

$$f_j = f_j^0 + \sum_{i=1}^\infty f_j^i,$$

where $f_j^0 \equiv \varphi_0 f_j$ and $f_j^i \equiv \varphi_i f_j$ for $i, j \in \mathbb{N}$ with $\{\varphi_i\}_{i\in\mathbb{N}_0} \subset D(\mathbb{R}^n)$ such that

$$\varphi_0 = 1 \text{ on } B_{2R}(x), \qquad \operatorname{supp}\varphi_0 \subset B_{4R}(x)$$

and

$$\operatorname{supp}\varphi_i \subset B_{2^{i+2}R}(x) \setminus B_{2^iR}(x), \qquad \sum_{i\in\mathbb{N}_0}\varphi_i = 1.$$

By means of (iii) we obtain

$$\left(\int_{B_R(x)}\left(\sum_{j=0}^\infty \left|T_j f_j^0(y)\right|^q\right)^{\frac{p}{q}} \mathrm{d}y\right)^{\frac{1}{p}} \le cR^{n\left(\frac{1}{p}+\frac{r}{n}\right)} \left\| f_j | L^r_p(\ell_q,\mathbb{R}^n)\right\|.$$

Let $i \in \mathbb{N}$ and $y \in B_R(x)$. It follows from (16) that

$$\left(\sum_{j=0}^\infty \left|T_j f_j^i(y)\right|^q\right)^{\frac{1}{q}} \le c(2^{i-1}R)^{-n}\int_{\mathbb{R}^n}\left(\sum_{j=0}^\infty \left|f_j^i(z)\right|^q\right)^{\frac{1}{q}} \mathrm{d}z.$$

Hölder's inequality yields

$$\begin{aligned}
&\left(\int_{B_R(x)}\left(\sum_{j=0}^\infty \left|T_j\left(\sum_{i=1}^\infty f_j^i\right)(y)\right|^q\right)^{\frac{p}{q}} \mathrm{d}y\right)^{\frac{1}{p}} \\
&\le c\sum_{i=1}^\infty (2^{i-1}R)^{-n}\int_{\mathbb{R}^n}\left(\sum_{j=0}^\infty \left|f_j^i(z)\right|^q\right)^{\frac{1}{q}} \mathrm{d}z\, |B_R(x)|^{\frac{1}{p}} \\
&\le c'\sum_{i=1}^\infty (2^{i-1}R)^{-n} R^{\frac{n}{p}} (2^{i+2}R)^{n(1-\frac{1}{p})}(2^{i+2}R)^{\left(\frac{n}{p}+r\right)} \left\| f_j | L^r_p(\ell_q,\mathbb{R}^n)\right\| \\
&\le c'' R^{\left(\frac{n}{p}+r\right)} \left\| f_j | L^r_p(\ell_q,\mathbb{R}^n)\right\|.
\end{aligned}$$

By subadditivity of the operators we obtain

$$\left\| T_j f_j | L^r_p(\ell_q, \mathbb{R}^n) \right\| \le c \left\| f_j | L^r_p(\ell_q, \mathbb{R}^n) \right\|$$

where c does not depend on $\{f_j\}_j$. We get the unique continuous extension T : $\overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n) \hookrightarrow L^r_p(\ell_q, \mathbb{R}^n)$ of $\{T_j\}_j$ whenever T_j are linear. If T_j fulfills (15), then we have for $f_1, f_2 \in D(\mathbb{R}^n)$, $y \in \mathbb{R}^n$, $j \in \mathbb{N}_0$

$$|(T_j f_1)(y) - (T_j f_2)(y)| \le (T_j(f_1 - f_2))(y)$$

and hence for $\{f_j\}, \{\widetilde{f}_j\} \in \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n)$ with $f_j, \widetilde{f}_j \in D(\mathbb{R}^n)$ for all $j \in \mathbb{N}_0$

$$\begin{aligned} \left\| T_j f_j - T_j \widetilde{f}_j | L^r_p(\ell_q, \mathbb{R}^n) \right\| &\le \left\| T_j(f_j - \widetilde{f}_j) | L^r_p(\ell_q, \mathbb{R}^n) \right\| \\ &\le c \left\| f_j - \widetilde{f}_j | L^r_p(\ell_q, \mathbb{R}^n) \right\|. \end{aligned} \tag{17}$$

Therefore, $\{T_j\}_j$ is (Lipschitz-)continuous and moreover we get the unique continuous extension $T : \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n) \hookrightarrow L^r_p(\ell_q, \mathbb{R}^n)$ of $\{T_j\}_j$ using (17) in the same way as in the linear case.

Step 2 It remains to justify that also $T : \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n) \hookrightarrow \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n)$. By means of a density argument we may assume that $\{f_j\}_j \in \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n)$ with $f_j \in D(\mathbb{R}^n)$ for all $j \in \mathbb{N}_0$ and a k such that $f_j = 0$ for $|j| > k$. There is an $\bar{R}$ such that $\operatorname{supp} f_j \subset B_{\bar{R}}(0)$ for all $j \in \mathbb{N}_0$. Then

$$\left(\sum_{j=0}^{k} |(T_j f_j)(x)|^q \right)^{\frac{1}{q}} \le c\, |x|^{-n} \qquad \text{if} \quad |x| \ge 2\bar{R} \tag{18}$$

using (16) and triangle inequality. Here the constant c depends on $\{f_j\}_j$. Let $R \ge 2\bar{R}$. Then one has for cubes Q_{JM} with $Q_{JM} \subset \{x \in \mathbb{R}^n : |x| > R\}$,

$$2^{J(\frac{n}{p}+r)} \left\| \left(\sum_{j=0}^{k} |(T_j f_j)(x)|^q \right)^{\frac{1}{q}} \middle| L_p(Q_{JM}) \right\| \le c\, 2^{Jr}\, R^{-n} \qquad \text{if} \quad J \in \mathbb{N}_0.$$

Using in addition $1 < p < \infty$ we obtain

$$2^{J(\frac{n}{p}+r)} \left\| \left(\sum_{j=0}^{k} |(T_j f_j)(x)|^q \right)^{\frac{1}{q}} \middle| L_p(Q_{JM}) \right\| \le c\, 2^{J(\frac{n}{p}+r)}\, R^{-n(1-\frac{1}{p})}$$

if $-J \in \mathbb{N}$. Let $\psi_R \in D(\mathbb{R}^n)$ be a smooth cut-off function with $\psi_R(x) = 1$ if $|x| \leq R$. Then $\psi_R T_j f_j \in C(\mathbb{R}^n)$ compactly supported for $0 \leq j \leq k$ (by triangle inequality using $T_j : D(\mathbb{R}^n) \to L_\infty(\mathbb{R}^n)$) and it follows

$$\lim_{R\to\infty} \|\{T_j f_j - \psi_R T_j f_j\}_j | L^r_p(\ell_q, \mathbb{R}^n)\| = 0$$

with $1 < p < \infty$ and $0 < \frac{n}{p} + r < \frac{n}{p}$. Here one should mention that cubes which are not completely inside of $\{x \in \mathbb{R}^n : |x| > R\}$ are treated analogously and that $T_j f_j = 0$ for $j > k$ by $T_j 0 = 0$. Hence $\{T_j f_j\} \in \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n)$ by Proposition 4.1 and therefore $T : \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n) \hookrightarrow \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n)$ by the unique extension of $\{T_j\}_j$ to $\overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n)$. Futhermore, we observe that the projection of T to its $k-th$ component ($k \in \mathbb{N}_0$) coincides with $\widetilde{T}_k$ where $\widetilde{T}_k : \overset{\circ}{L}{}^r_p(\mathbb{R}^n) \hookrightarrow \overset{\circ}{L}{}^r_p(\mathbb{R}^n)$ is the unique continuous and bounded extension of T_k. This yields Assertion (1).

Step 3 Finally, Assertions (2) and (3) follow by duality (Theorems 3.1 and 3.3). As for the abstract background of duality one may consult [28, pp. 112/113] and [18, pp. 35/36]. We get firstly $\widetilde{T}_j{}' : H^\varrho L_{p'}(\mathbb{R}^n) \hookrightarrow H^\varrho L_{p'}(\mathbb{R}^n)$ and $\left(\widetilde{T}_j{}'\right)' : L^r_p(\mathbb{R}^n) \hookrightarrow L^r_p(\mathbb{R}^n)$ for all $j \in \mathbb{N}_0$. Moreover, by the linearity of $T = \{\widetilde{T}_j\}_{j\in\mathbb{N}_0} : \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n) \hookrightarrow \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n)$ duality also implies $T' : H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n) \hookrightarrow H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n)$ as well as $(T')' : L^r_p(\ell_q, \mathbb{R}^n) \hookrightarrow L^r_p(\ell_q, \mathbb{R}^n)$. The projection of T' to its $k-th$ component ($k \in \mathbb{N}_0$) coincides with $\widetilde{T}_k{}'$. Indeed, let $f_j \equiv g_j \equiv 0$ for all $j \neq k$ and let $f_k, g_k \in D(\mathbb{R}^n)$. By means of the definition of T' and $\widetilde{T}_k{}'$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} f_k(x) \left(T'(\{g_j\})\right)_k (x)\mathrm{d}x = \left\langle \{f_j\}, T'(\{g_j\})\right\rangle_{\left(\overset{\circ}{L}{}^r_p(\ell_q,\mathbb{R}^n), H^\varrho L_{p'}(\ell_{q'},\mathbb{R}^n)\right)} \\
&= \left\langle T(\{f_j\}), \{g_j\}\right\rangle_{\left(\overset{\circ}{L}{}^r_p(\ell_q,\mathbb{R}^n), H^\varrho L_{p'}(\ell_{q'},\mathbb{R}^n)\right)} \\
&= \left\langle \{\widetilde{T}_j f_j\}, \{g_j\}\right\rangle_{\left(L_p(\ell_q,w_\alpha,\mathbb{R}^n), L_{p'}(\ell_{q'},w_{-\alpha},\mathbb{R}^n)\right)} = \int_{\mathbb{R}^n} (\widetilde{T}_k f_k)(x) g_k(x)\mathrm{d}x \\
&= \left\langle \widetilde{T}_k f_k, g_k\right\rangle_{\left(L_p(w_\alpha,\mathbb{R}^n), L_{p'}(w_{-\alpha},\mathbb{R}^n)\right)} = \left\langle \widetilde{T}_k f_k, g_k\right\rangle_{\left(\overset{\circ}{L}{}^r_p(\mathbb{R}^n), H^\varrho L_{p'}(\mathbb{R}^n)\right)} \\
&= \left\langle f_k, \widetilde{T}_k{}' g_k\right\rangle_{\left(\overset{\circ}{L}{}^r_p(\mathbb{R}^n), H^\varrho L_{p'}(\mathbb{R}^n)\right)} = \int_{\mathbb{R}^n} f_k(x)(\widetilde{T}_k{}' g_k)(x)\mathrm{d}x
\end{aligned}$$

for $\alpha < -n/p$. Analogously, using $H^\varrho L_{p'}(\mathbb{R}^n) \hookrightarrow L_{-n/\varrho}(\mathbb{R}^n)$ we deduce that the projection of $(T')'$ to its $k-th$ component ($k \in \mathbb{N}_0$) coincides with $\left(\widetilde{T}_k{}'\right)'$ on $D(\mathbb{R}^n)$.

Moreover, we assume that the extensions of T_j to $L_p(\mathbb{R}^n)$ due to assumption (iii) are formally self-adjoint for all $j \in \mathbb{N}_0$. Then we obtain

$$
\begin{aligned}
&\left\langle f, \widetilde{T}_j{}' g\right\rangle_{\left(\overset{\circ}{L}{}^r_p(\mathbb{R}^n), H^\varrho L_{p'}(\mathbb{R}^n)\right)} = \left\langle \widetilde{T}_j f, g\right\rangle_{\left(\overset{\circ}{L}{}^r_p(\mathbb{R}^n), H^\varrho L_{p'}(\mathbb{R}^n)\right)} \\
&= \left\langle T_j f, g\right\rangle_{\left(\overset{\circ}{L}{}^r_p(\mathbb{R}^n), H^\varrho L_{p'}(\mathbb{R}^n)\right)} = \left\langle T_j f, g\right\rangle_{\left(L_p(w_\alpha,\mathbb{R}^n), L_{p'}(w_{-\alpha},\mathbb{R}^n)\right)} \\
&= \int_{\mathbb{R}^n} (T_j f)(x) g(x) \mathrm{d}x = \left\langle T_j f, g\right\rangle_{\left(L_p, L_{p'}\right)} = \left\langle f, T_j g\right\rangle_{\left(L_p, L_{p'}\right)} \qquad (19)
\end{aligned}
$$

for all $f, g \in D(\mathbb{R}^n)$ and $\alpha < -n/p$. Therefore, $\widetilde{T}_j{}' g = T_j g$ almost everywhere for all $g \in D(\mathbb{R}^n)$ and $j \in \mathbb{N}_0$ which means that $\widetilde{T}_j{}'$ are extensions of T_j to $H^\varrho L_{p'}(\mathbb{R}^n)$. Moreover, the biduals $\widetilde{T}_j{}'' = \left(\widetilde{T}_j{}'\right)'$, $j \in \mathbb{N}_0$, are extensions of T_j to $L^r_p(\mathbb{R}^n)$ by

$$
\begin{aligned}
&\left\langle g, \widetilde{T}_j{}'' f\right\rangle_{\left(H^\varrho L_{p'}(\mathbb{R}^n), L^r_p(\mathbb{R}^n)\right)} = \left\langle \widetilde{T}_j{}' g, f\right\rangle_{\left(H^\varrho L_{p'}(\mathbb{R}^n), L^r_p(\mathbb{R}^n)\right)} \\
&= \left\langle \widetilde{T}_j{}' g, f\right\rangle_{(L_u, L_{u'})} = \int_{\mathbb{R}^n} (\widetilde{T}_j{}' g)(x) f(x) \mathrm{d}x = \left\langle f, \widetilde{T}_j{}' g\right\rangle_{(\overset{\circ}{L}{}^r_p(\mathbb{R}^n), H^\varrho L_{p'}(\mathbb{R}^n))} \\
&= \left\langle \widetilde{T}_j f, g\right\rangle_{\left(\overset{\circ}{L}{}^r_p(\mathbb{R}^n), H^\varrho L_{p'}(\mathbb{R}^n)\right)} = \left\langle T_j f, g\right\rangle_{\left(\overset{\circ}{L}{}^r_p(\mathbb{R}^n), H^\varrho L_{p'}(\mathbb{R}^n)\right)} \\
&= \left\langle T_j f, g\right\rangle_{\left(L_p, L_{p'}\right)}
\end{aligned}
$$

for all $f, g \in D(\mathbb{R}^n)$, $u = -n/\varrho$ and $j \in \mathbb{N}_0$. Therefore, $\widetilde{T}_j{}'' = T_j$ on $D(\mathbb{R}^n)$ for $j \in \mathbb{N}_0$. □

Remark 4.4 The extension in Part 3 of the theorem is not unique. There exist infinitely many extensions of T_j acting in $L^r_p(\mathbb{R}^n)$. This can be seen following the same arguments as in [22, Remark 5.3]. Assumption (iii) can be replaced by

$$T : L_p(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n)$$

if $T_j = T$ for all j, T is linear and if q is between 2 and p (including 2 and p) which holds by the fact that the L_p-boundedness of a linear operator implies the $L_p(\ell_q, \mathbb{R}^n)$-boundedness for $q \in [p, 2]$ for $p \leq 2$ and $q \in [2, p]$ for $p > 2$ cf. [10, Corollary 4.5.4].

Remark 4.5 There are a lot of papers dealing with singular integrals in Morrey spaces. However, its well-definedness on the Morrey-type spaces under consideration as well as the norm estimates in these spaces have to be treated with greater care than usually done. On the one-hand one has to investigate how to extend singular integrals to Morrey spaces and on the other hand the estimates (16) are not available in general for functions belonging to Morrey spaces. Let us emphasize that we used (16) just for functions of $D(\mathbb{R}^n)$. The question if the estimate (16) holds for some singular integrals also for all $f \in L^r_p(\mathbb{R}^n)$ leads to an investiation of its maximal truncated versions (cf. [27, Proposition 2.25, Remark 2.26] as well as Sects. 4.2.2 and 4.3). Indeed, for these reasons in many papers one can only find the weaker mapping property $T : \overset{\circ}{L}{}^r_p(\mathbb{R}^n) \hookrightarrow L^r_p(\mathbb{R}^n)$

(see, for example [5,17], and [6,12,16] for operators satisfying (16)). In this sense our results on $\overset{\circ}{L}{}^r_p(\mathbb{R}^n)$ and $\overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n)$ are new (including even the boundedness of the Hardy-Littlewood maximal operator), in particular with respect to their generality. Note that results for Calderón–Zygmund operators in $\overset{\circ}{L}{}^r_p(\mathbb{R}^n)$ (scalar case) have been proved already in [21] and [22]. Moreover, some results in $H^\varrho L_{p'}(\mathbb{R}^n)$ and in $H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n)$ and $L^r_p(\ell_q, \mathbb{R}^n)$ seem to be new. The paper [2] made the important observation that the bidual of the completion of $D(\mathbb{R}^n)$ with respect to the Morrey norm coincides with the Morrey space itself (cf. (1)) and provided the basis of our investigations. To overcome the above mentioned problems investigating Calderón–Zygmund operators in $L^r_p(\mathbb{R}^n)$ they considered Muckenhoupt weighted characterizations of Morrey spaces and their preduals. However, this approach has also some weak points with respect to norm estimates since it does not take into account that the operator norm of classical operators of harmonic analysis (as the Hilbert transform) in Muckenhoupt weighted spaces usually depends on the Muckenhoupt weight.

We want to refer also to a less known forerunner result which can be found in [3]. There a solution for the above mentioned difficulties has been given for some Calderón–Zygmund operators in $H^\varrho L_p(\mathbb{R}^n)$.

4.2 Calderón–Zygmund Operators

4.2.1 Duality Approach

Definition 4.6 We define *Calderón–Zygmund operators* with homogeneous kernels with degree $-n$ setting,

$$(T^\Omega f)(y) \equiv \text{ p.v.} \int_{\mathbb{R}^n} \frac{\Omega(z/|z|)}{|z|^n} f(y-z)\mathrm{d}z,$$

where $f \in S(\mathbb{R}^n)$ and $\Omega \in L_\infty(\mathcal{S}^{n-1})$ with zero integral and $\mathcal{S}^{n-1}$ denotes the unit sphere.

Corollary 4.7 *Let* $1 < p < \infty$, $-\frac{n}{p} \leq r < 0$, $-n < \varrho < -\frac{n}{p'}$, $1 < q < \infty$. *Then the following statements hold true.*

(1) *There are unique linear and bounded extensions of* T^Ω *to* $\overset{\circ}{L}{}^r_p(\mathbb{R}^n)$ *and to* $H^\varrho L_{p'}(\mathbb{R}^n)$ *denoted again by* T^Ω *such that*

$$\left\{T^\Omega\right\}_{j\in\mathbb{N}_0} : \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n) \hookrightarrow \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n) \quad \textit{and}$$
$$\left\{T^\Omega\right\}_{j\in\mathbb{N}_0} : H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n) \hookrightarrow H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n).$$

(2) *There are infinitely many linear and bounded extensions of* T^Ω *to* $L^r_p(\mathbb{R}^n)$ *denoted again by* T^Ω *such that*

$$\left\{T^\Omega\right\}_{j\in\mathbb{N}_0} : L^r_p(\ell_q, \mathbb{R}^n) \hookrightarrow L^r_p(\ell_q, \mathbb{R}^n).$$

Proof We observe that $T^{\Omega} : D(\mathbb{R}^n) \rightarrow \mathrm{Lip}(\mathbb{R}^n)$. Indeed, by the same arguments as in [21, proof of Step 2 of Theorem 1.1, p. 8] we get the mapping properties $T^{\Omega} : W_p^k(\mathbb{R}^n) \hookrightarrow W_p^k(\mathbb{R}^n)$ for Sobolev spaces which lead to the above assertion by means of Sobolev type embeddings. Moreover, for $\Omega \in L_\infty(\mathcal{S}^{n-1})$ it holds $T^{\Omega} : L_p(\ell_q, \mathbb{R}^n) \hookrightarrow L_p(\ell_q, \mathbb{R}^n)$ by [7]. Now we obtain $T^{\Omega} : L_p^r(\ell_q, \mathbb{R}^n) \hookrightarrow L_p^r(\ell_q, \mathbb{R}^n)$ for $\Omega \in L_\infty(\mathcal{S}^{n-1})$ applying Theorem 4.3. Note that the dual of the extension of $T^{\Omega(-\cdot)}$ to $\overset{\circ}{L}{}_p^r(\mathbb{R}^n)$ coincides with T^{Ω} on $D(\mathbb{R}^n)$ by the same arguments as in (19). □

4.2.2 Alternative Approach Using Some Muckenhoupt Weights

The following alternative method due to Triebel [27, Sect. 2.5.3, Proposition 2.25, Remark 2.26] yields extensions of Calderón–Zygmund operators which are bounded in $L_p^r(\mathbb{R}^n)$. He studied the boundedness of T^{Ω} with $\Omega \in C^1(\mathcal{S}^{n-1})$. Here we generalize his approach to some non-convolution type Calderón–Zygmund operators. At first we observe that Morrey spaces $L_p^r(\mathbb{R}^n)$ are continuously embedded into some Muckenhoupt weighted L_p-spaces. Recall that $w_\alpha(\cdot) = (1 + |\cdot|^2)^{\frac{\alpha}{2}}$, and that $L_p(\mathbb{R}^n, w_\alpha)$ be the corresponding weighted L_p-space, normed as in (5).

Proposition 4.8 ([27, Proposition 2.10]) *Let* $1 < p < \infty$, $-\frac{n}{p} \leq r < 0$, $-n < \alpha p < -n - rp$. *Then it holds*

$$L_p^r(\mathbb{R}^n) \hookrightarrow L_p(w_\alpha, \mathbb{R}^n). \tag{20}$$

Proof Let $f \in L_p^r(\mathbb{R}^n)$. Then (20) follows from

$$\begin{aligned}
&\int_{\mathbb{R}^n} |f(x) w_\alpha(x)|^p \mathrm{d}x \\
&\leq c \left(\int_{|x|\leq 1} |f(x)|^p \mathrm{d}x + \sum_{j\in\mathbb{N}_0} 2^{j\alpha p} \int_{2^j \leq |x| \leq 2^{j+1}} |f(x)|^p \mathrm{d}x \right) \\
&\leq \hat{c} \left(\int_{|x|\leq 1} |f(x)|^p \mathrm{d}x + \sum_{j\in\mathbb{N}_0} 2^{j(\alpha p + n + rp)} \left\| f | L_p^r\left(\mathbb{R}^n\right) \right\|^p \right) \\
&\leq \bar{c} \left\| f | L_p^r\left(\mathbb{R}^n\right) \right\|^p .
\end{aligned}$$

□

Theorem 4.9 *Let* $1 < p < \infty$, $-\frac{n}{p} < r < 0$, $-n < \varrho < -\frac{n}{p'}$. *Let* T *be an operator with domain* $D(\mathbb{R}^n)$ *satisfying*

$$\left\| Tf | L_2(\mathbb{R}^n) \right\| \leq c_1 \left\| f | L_2(\mathbb{R}^n) \right\|$$

where the constant c_1 *is independent of* $f \in D(\mathbb{R}^n)$ *and*

$$(Tf)(y) = \lim_{\varepsilon \searrow 0} \int_{z \in \mathbb{R}^n, |y-z| \geq \varepsilon} K(y,z) f(z) \mathrm{d}z \tag{21}$$

almost everywhere for all $f \in D(\mathbb{R}^n)$*, where the function* $K(\cdot,\cdot)$ *defined* $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\}$ *satisfies the conditions* $|K(x,y)| \leq c_2 |x-y|^{-n}$ *and*

$$\begin{aligned} |K(x,y) - K(x',y)| &\leq c_2 \frac{|x-x'|^\delta}{(|x-y| + |x'-y|)^{n+\delta}}, \\ &\textit{whenever } 2|x-x'| \leq \max(|x-y|, |x'-y|), \\ |K(x,y) - K(x,y')| &\leq c_2 \frac{|y-y'|^\delta}{(|x-y| + |x-y'|)^{n+\delta}}, \\ &\textit{whenever } 2|y-y'| \leq \max(|x-y|, |x-y'|). \end{aligned}$$

Then the following statements hold true.

(1) *There are linear and bounded extensions of* T *to* $L^r_p(\mathbb{R}^n)$.

(2) *There is an unique linear and bounded extension of* T *to* $\overset{\circ}{L}{}^r_p(\mathbb{R}^n)$ *and to* $H^\varrho L_{p'}(\mathbb{R}^n)$.

Proof By [11, Corollary 9.4.7] there is an unique linear and bounded extension $\widetilde{T}$ of T to $L_p(w_\alpha, \mathbb{R}^n)$ with $-n < \alpha p < n(p-1)$. Therefore, Proposition 4.8 yields $\widetilde{T} : L^r_p(\mathbb{R}^n) \hookrightarrow L_p(w_\alpha, \mathbb{R}^n)$. We have even

$$\sup_{\varepsilon > 0} \left| \int_{z \in \mathbb{R}^n, |y-z| \geq \varepsilon} K(y,z) f(z) \mathrm{d}z \right| : L_p(w_\alpha, \mathbb{R}^n) \hookrightarrow L_p(w_\alpha, \mathbb{R}^n)$$

by [11, Theorem 9.4.6]. Together with (21) we see that

$$(\widetilde{T} f)(y) = \lim_{\varepsilon \searrow 0} \int_{z \in \mathbb{R}^n, |y-z| \geq \varepsilon} K(y,z) f(z) \mathrm{d}z$$

almost everywhere for all $f \in L_p(w_\alpha, \mathbb{R}^n)$ by [11, Theorem 2.1.14]. Now (16) holds for all $f \in L^r_p(\mathbb{R}^n)$ with $y \notin \operatorname{supp} f$. As in Step 1 of the proof of Theorem 4.3 we obtain $\widetilde{T} : L^r_p(\mathbb{R}^n) \hookrightarrow L^r_p(\mathbb{R}^n)$ that is Assertion (1). In particular, $\widetilde{T} : \overset{\circ}{L}{}^r_p(\mathbb{R}^n) \hookrightarrow L^r_p(\mathbb{R}^n)$. For $\alpha = 0$ by [11, Corollary 9.4.7] we have especially

$$\left\| Tf \left| L_{-\frac{n}{r}}(\mathbb{R}^n) \right\| \leq c \left\| f \right| L_{-\frac{n}{r}}(\mathbb{R}^n) \right\|$$

for all $f \in D(\mathbb{R}^n)$ where the constant c does not depend on f. Hence, $T : D(\mathbb{R}^n) \to L_{-n/r}(\mathbb{R}^n)$. Because of the embedding $L_{-n/r}(\mathbb{R}^n) \hookrightarrow L^r_p(\mathbb{R}^n)$ and the density of $D(\mathbb{R}^n)$ in $L_{-n/r}(\mathbb{R}^n)$ we even have $T : D(\mathbb{R}^n) \to \overset{\circ}{L}{}^r_p(\mathbb{R}^n)$. Indeed, let $f \in D(\mathbb{R}^n)$.

Then $Tf \in L_{-n/r}(\mathbb{R}^n)$ and thus there is a sequence of functions of $D(\mathbb{R}^n)$ which tends to Tf in $L_{-n/r}(\mathbb{R}^n)$ and hence in $L^r_p(\mathbb{R}^n)$ which shows $Tf \in \overset{\circ}{L}{}^r_p(\mathbb{R}^n)$. Thus, $\widetilde{T} : \overset{\circ}{L}{}^r_p(\mathbb{R}^n) \hookrightarrow \overset{\circ}{L}{}^r_p(\mathbb{R}^n)$. The adjoint kernel of $K(x, y)$ given by $\overline{K(y, x)}$ also satisfies the required assumptions on the kernel. Hence, its corresponding operator is also bounded in $L_p(\mathbb{R}^n)$ (cf. [11, Definition 8.1.2]) but its dual coincides by the same arguments as in Step 3 of the proof of Theorem 4.3 with the operator T (with the kernel $K(x, y)$) on $D(\mathbb{R}^n)$ which implies Assertion (2).

Remark 4.10 Let us point out that for this method the embedding of $L^r_p(\mathbb{R}^n)$ in some Muckenhoupt weighted space is crucial for extending the domain of the considered Calderón–Zygmund operators to $L^r_p(\mathbb{R}^n)$. Recall the fact that the Hilbert transform is acting in $L_p(w_\alpha, \mathbb{R}^n)$ if, and only if, w_α is a Muckenhoupt weight. Moreover, we needed

$$(\widetilde{T} f)(y) = \lim_{\varepsilon \searrow 0} \int_{z \in \mathbb{R}^n, |y-z| \geq \varepsilon} K(y, z) f(z) \mathrm{d}z$$

almost everywhere for all $f \in L^r_p(\mathbb{R}^n)$. This is a rather deep result in comparison to the L_p-boundedness which we require in Theorem 4.3. Finally, let us emphasize again that the extension in Part 1 is by no means unique.

4.3 Vector-Valued Maximal Inequalities and Maximal Calderón–Zygmund Operators

Definition 4.11 We define *maximal Calderón–Zygmund operators* with homogeneous kernels with degree $-n$ by setting

$$\left(T^\Omega_* f\right)(y) \equiv \sup_{\varepsilon > 0} \left| \int_{|z| \geq \varepsilon} \frac{\Omega(z/|z|)}{|z|^n} f(y - z) \mathrm{d}z \right|$$

where $f \in \bigcup_{1 \leq p < \infty} L_p(\mathbb{R}^n)$ and $\Omega \in L_\infty(\mathcal{S}^{n-1})$ with zero integral and $\mathcal{S}^{n-1}$ denotes the unit sphere. As usual, the Hardy-Littlewood maximal operator M is given by

$$(Mf)(y) \equiv \sup_{R>0} \frac{1}{|B_R(y)|} \int_{B_R(y)} |f(z)| \, \mathrm{d}z, \qquad f \in L^{\mathrm{loc}}_1\left(\mathbb{R}^n\right).$$

Remark 4.12 If $f \in \bigcup_{1 \leq p < \infty} L_p(\mathbb{R}^n)$ then

$$\left| \int_{|z| \geq \varepsilon} \frac{\Omega(z/|z|)}{|z|^n} f(y - z) \mathrm{d}z \right|$$

is bounded for each $\varepsilon > 0$ and $y \in \mathbb{R}^n$ by Hölder's inequality. Hence $(T^\Omega_* f)(y)$ is well-defined for all $y \in \mathbb{R}^n$, but might be infinite.

Corollary 4.13 *Let* $1 < p < \infty$, $-\frac{n}{p} \leq r < 0$, $-n < \varrho < -\frac{n}{p'}$, $1 < q < \infty$. *Then*

$$\{M\}_{j\in\mathbb{N}_0} : \overset{\circ}{L}{}^r_p(\ell_q,\mathbb{R}^n) \hookrightarrow \overset{\circ}{L}{}^r_p(\ell_q,\mathbb{R}^n) \quad \textit{and}$$
$$\{M\}_{j\in\mathbb{N}_0} : L^r_p(\ell_q,\mathbb{R}^n) \hookrightarrow L^r_p(\ell_q,\mathbb{R}^n). \tag{22}$$

Moreover, if $\Omega \in C^1(\mathcal{S}^{n-1})$, *then*

$$T^\Omega_* : \overset{\circ}{L}{}^r_p\left(\mathbb{R}^n\right) \hookrightarrow \overset{\circ}{L}{}^r_p\left(\mathbb{R}^n\right) \quad \textit{and} \quad T^\Omega_* : L^r_p\left(\mathbb{R}^n\right) \hookrightarrow L^r_p\left(\mathbb{R}^n\right). \tag{23}$$

Proof At first we show that $M : D(\mathbb{R}^n) \to \mathrm{Lip}(\mathbb{R}^n)$. Let $f \in D(\mathbb{R}^n)$ and $f_h(\cdot) \equiv f(\cdot + h)$ for $h \in \mathbb{R}^n$. By sublinearity of M we obtain

$$Mf_h = M(f_h - f + f) \leq M(f_h - f) + Mf \text{ and } |Mf_h - Mf| \leq M(f_h - f).$$

It follows that

$$\begin{aligned}|(Mf)(x+h) - (Mf)(x)| = |(Mf_h)(x) - (Mf)(x)| &\leq [M(f_h - f)](x)\\ &\leq Lh,\end{aligned}$$

where L is the Lipschitz constant of f and $x \in \mathbb{R}^n$ (we even showed $M : \mathrm{Lip}(\mathbb{R}^n) \to \mathrm{Lip}(\mathbb{R}^n)$ with the arguments due to [15, Remark 2.2]). A version of Cotlar's inequality leads to the estimate

$$(T^\Omega_* f)(x) \leq c([M(|T^\Omega f|)](x) + (Mf)(x))$$

for $x \in \mathbb{R}^n$ (cf. [8, Lemma 5.15]). As above we obtain

$$|(T^\Omega_* f)(x+h) - (T^\Omega_* f)(x)| = |(T^\Omega_* f_h)(x) - (T^\Omega_* f)(x)| \leq [T^\Omega_*(f_h - f)](x).$$

Together with $T^\Omega : D(\mathbb{R}^n) \to \mathrm{Lip}(\mathbb{R}^n)$ (cf. proof of Corollary 4.7) it follows from $M : \mathrm{Lip}(\mathbb{R}^n) \to \mathrm{Lip}(\mathbb{R}^n)$ also that $T^\Omega_* : D(\mathbb{R}^n) \to \mathrm{Lip}(\mathbb{R}^n)$. Moreover, we claim that (16) holds also for M. Indeed, let $f \in D(\mathbb{R}^n)$ with $y \notin \mathrm{supp}\,(f)$. Then there exists an $i \in \mathbb{Z}$ such that $B_{2^i}(y) \cap \mathrm{supp}\, f = \emptyset$. Let $f^j \equiv \chi_{B_{2^{j+1}}(y)\setminus B_{2^j}(y)} f$ for $j \geq i$. Hence,

$$\begin{aligned}|(Mf)(y)| &\leq \sup_{R>0} \frac{1}{|B_R(y)|}\int_{B_R(y)} |f(z)|\mathrm{d}z\\ &\leq \sum_{j=i}^{\infty} \sup_{R>0} \frac{1}{|B_R(y)|}\int_{B_R(y)} |f^j(z)|\mathrm{d}z \leq \sum_{j=i}^{\infty} \frac{1}{|B_{2^j}(y)|}\int_{\mathbb{R}^n} |f^j(z)|\mathrm{d}z\\ &\leq c\sum_{j=i}^{\infty}\int_{B_{2^{j+1}}(y)\setminus B_{2^j}(y)} \frac{|f(z)|}{2^{jn}}\mathrm{d}z \leq c'\sum_{j=i}^{\infty}\int_{B_{2^{j+1}}(y)\setminus B_{2^j}(y)} \frac{|f(z)|}{|y-z|^n}\mathrm{d}z\\ &= c'\int_{\mathbb{R}^n} \frac{|f(z)|}{|y-z|^n}\mathrm{d}z.\end{aligned}$$

Now Theorem 4.3 implies the existence of unique continuous and bounded extensions of M to $\overset{\circ}{L}{}^r_p(\ell_q,\mathbb{R}^n)$ and of T_*^Ω to $\overset{\circ}{L}{}^r_p(\mathbb{R}^n)$. Since (16) also holds for M and all $f \in L^r_p(\mathbb{R}^n)$ in place of all $f \in D(\mathbb{R}^n)$ we achieve at (22) as in Step 1 of the proof of Theorem 4.3. Moreover, T_*^Ω is also well-defined on $L_p(w_\alpha,\mathbb{R}^n)$ if $-n < \alpha p < n(p-1)$ by [11, Theorem 9.4.6] (and not only on $\bigcup_{1\le p<\infty} L_p(\mathbb{R}^n)$) and hence on $L^r_p(\mathbb{R}^n)$ by Proposition 4.8. Thus, (16) also holds for T_*^Ω and all $f \in L^r_p(\mathbb{R}^n)$ in place of all $f \in D(\mathbb{R}^n)$. This yields (23). □

4.4 Fourier Multipliers

4.4.1 Multipliers Generated by Characteristic and Smooth Functions

Corollary 4.14 *Let* $1 < p < \infty$, $-\frac{n}{p} \le r < 0$, $-n < \varrho < -\frac{n}{p'}$, $1 < q < \infty$. *Let* $\{I_j\}_{j\in\mathbb{N}_0}$ *be a sequence of intervals on the real line, finite or infinite, and let* $\{S_j\}_j$ *be the sequence of operators defined by*

$$(S_j f)\hat{}(\xi) = \chi_{I_j}(\xi)\hat{f}(\xi), \quad f \in D(\mathbb{R}),\ \xi \in \mathbb{R}.$$

Moreover, let $\psi \in S(\mathbb{R}^n)$ *with* $\psi(0) = 0$. *We define*

$$\psi_j(\xi) = \psi(2^{-j}\xi) \text{ and } (\tilde{S}_j f)\hat{} = \psi_j \hat{f} \quad \text{for } j \in \mathbb{Z},\ \xi \in \mathbb{R}^n,\ f \in S'(\mathbb{R}^n).$$

Then the following statements hold true.

(1) *There are unique linear and bounded extensions of* S_j *to* $\overset{\circ}{L}{}^r_p(\mathbb{R}^n)$ *and to* $H^\varrho L_{p'}(\mathbb{R}^n)$ *denoted again by* S_j *and satisfying the mapping properties*

$$\begin{aligned} \{S_j\}_{j\in\mathbb{N}_0} &: \overset{\circ}{L}{}^r_p(\ell_q,\mathbb{R}) \hookrightarrow \overset{\circ}{L}{}^r_p(\ell_q,\mathbb{R}) \quad \text{and} \\ \{S_j\}_{j\in\mathbb{N}_0} &: H^\varrho L_{p'}(\ell_{q'},\mathbb{R}) \hookrightarrow H^\varrho L_{p'}(\ell_{q'},\mathbb{R}) \end{aligned}$$

(2) *There are infinitely many linear and bounded extensions of* S_j *to* $L^r_p(\mathbb{R}^n)$ *denoted again by* S_j *such that*

$$\{S_j\}_{j\in\mathbb{N}_0} : L^r_p(\ell_q,\mathbb{R}) \hookrightarrow L^r_p(\ell_q,\mathbb{R}).$$

(3) *We have the mapping properties*

$$\begin{aligned} \left\{\tilde{S}_j\right\}_{j\in\mathbb{Z}} &: \overset{\circ}{L}{}^r_p(\ell_q,\mathbb{R}^n) \hookrightarrow \overset{\circ}{L}{}^r_p(\ell_q,\mathbb{R}^n), \\ \left\{\tilde{S}_j\right\}_{j\in\mathbb{Z}} &: H^\varrho L_{p'}(\ell_{q'},\mathbb{R}^n) \hookrightarrow H^\varrho L_{p'}(\ell_{q'},\mathbb{R}^n), \end{aligned}$$

and

$$\left\{\tilde{S}_j\right\}_{j\in\mathbb{Z}} : L^r_p(\ell_q,\mathbb{R}^n) \hookrightarrow L^r_p(\ell_q,\mathbb{R}^n)\ .$$

Here we used the notation $\ell_q = \ell_q(\mathbb{Z})$.

Proof Part (1) and part (2) are consequences of Theorem 4.3 (see also Corollary 4.7). The required $L_p(\ell_q, \mathbb{R}^n)$-boundedness follows from [8, Corollary 8.2]. Alternatively, it suffices the $L_p(\ell_q, \mathbb{R}^n)$-boundedness of the Hilbert transform $(Hf)(y) \equiv \frac{1}{\pi} \lim_{\varepsilon \searrow 0} \int_{|z-y|\geq\varepsilon} \frac{f(z)}{y-z} \mathrm{d}z$, $f \in S(\mathbb{R}^n)$ (see e.g., [10, Corollary 4.6.3]) This can be seen using the formula

$$S_j f_j = \frac{i}{2}\left(M_{a_j} H M_{-a_j} f_j - M_{b_j} H M_{-b_j} f_j\right),$$

where $I_j = (a_j, b_j)$ (with the obvious modifications if the interval is unbounded) and where $M_a f(\cdot) \equiv e^{2\pi i a \cdot} f(\cdot)$ [8, (3.9)]. Now the desired result follows from Theorem 4.3 (for $n = 1$) taking into account also that the dual of the extension of the multiplier generated by $-I_j$ coincides with S_j on $D(\mathbb{R}^n)$ by the same arguments as for (19).

Moreover, we observe that

$$\left\{\tilde{S}_j\right\}_{j\in\mathbb{Z}} : L_p(\ell_q, \mathbb{R}^n) \hookrightarrow L_p(\ell_q, \mathbb{R}^n)\,. \tag{24}$$

This follows, for example, from [8, (8.1), p. 158]. The needed Hörmander condition is fulfilled by (26). Indeed, we have

$$\left\| \left\{|\nabla\Psi_j(x)|\right\}_j \middle| \ell_2 \right\| \leq \frac{c}{|x|^{n+1}}, \qquad x \in \mathbb{R}^n \tag{25}$$

(cf. [8, p. 161]). Hölder's inequality yields

$$|\Psi_j(x-y) - \Psi_j(x)| \leq |y| \left(\int_0^1 |(\nabla\Psi_j)(x-ty)|^2 \mathrm{d}t\right)^{\frac{1}{2}}$$

and furthermore using (25)

$$\begin{aligned} \left\|\left\{\Psi_j(x-y) - \Psi_j(x)\right\} |\ell_2\right\| &\leq |y| \left(\int_0^1 \left\|\left|(\nabla\Psi_j)(x-ty)\right| |\ell_2\right\|^2 \mathrm{d}t\right)^{\frac{1}{2}} \\ &\leq c\, \frac{|y|}{|x|^{n+1}} \end{aligned} \tag{26}$$

for $|x| \geq 2|y|$. Using (24) we find $\left\{\tilde{S}_j\right\}_{j\in\mathbb{Z}} : \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n) \hookrightarrow \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n)$ and $\left\{\tilde{S}'_j\right\}_{j\in\mathbb{Z}} : H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n) \hookrightarrow H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n)$ by means of Theorem 4.3. Here we have to show that in particular assumption (16) is fulfilled. If $\hat{\Psi} \equiv \psi$ and $\Psi_j(\cdot) \equiv 2^{jn}\Psi(2^j \cdot)$, then $\hat{\Psi}_j = \psi_j$ and $\tilde{S}_j f = \Psi_j * f \in C^\infty(\mathbb{R}^n) \cap S'(\mathbb{R}^n)$ by $L^r_p(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n)$. In particular, $\tilde{S}_j f = \Psi_j * f$ makes sense pointwise for all $f \in L^r_p(\mathbb{R}^n)$. Furthermore,

$$\left(\sum_{j\in\mathbb{Z}}\left|\Psi_j * f(x)\right|^2\right)^{\frac{1}{2}} \le \int_{\mathbb{R}} |f(y)| \left(\sum_{j\in\mathbb{Z}}\left|\Psi_j(x-y)\right|^2\right)^{\frac{1}{2}} \mathrm{d}y$$
$$\le c\int_{\mathbb{R}} \frac{|f(y)|}{|x-y|^n}\mathrm{d}y$$

for all $x \in \mathbb{R}^n$ and all $f \in L^r_p(\mathbb{R}^n)$ with $x \notin \operatorname{supp}(f)$ where $\left\|\left\{\Psi_j(\cdot)\right\}|\ell_2\right\| \le c|\cdot|^{-n}$ (cf. [8, p. 161]). This implies (16). We note that dual of the extension of the multiplier of $\psi_j(-\cdot)$ coincides with $\tilde{S}_j$ on $D(\mathbb{R}^n)$ by the same arguments as for (19). Hence,

$$\left\{\tilde{S}_j\right\}_{j\in\mathbb{Z}} : H^{\varrho}L_{p'}(\ell_{q'}, \mathbb{R}^n) \hookrightarrow H^{\varrho}L_{p'}(\ell_{q'}, \mathbb{R}^n).$$

We obtain

$$\left\{\tilde{S}_j\right\}_{j\in\mathbb{Z}} : L^r_p(\ell_q, \mathbb{R}^n) \hookrightarrow L^r_p(\ell_q, \mathbb{R}^n)$$

with the same norm estimates as in Theorem 4.3. Hereby we emphasize that the operator $\tilde{S}_j$ is well-defined on $S'(\mathbb{R}^n)$, in particular on $L^r_p(\mathbb{R}^n)$. Moreover, (16) holds for $f \in L^r_p(\mathbb{R}^n)$ in place of $f \in D(\mathbb{R}^n)$. □

Remark 4.15 The vector-valued Fourier multiplier assertion proved Corollary 4.14 paves the way to introduce predual Morrey versions $H^{\varrho}A^s_{p,q}(\mathbb{R}^n)$ of the Besov–Triebel–Lizorkin spaces $A^s_{p,q}(\mathbb{R}^n)$. In particular it implies the independence of admitted resolutions of unity. One replaces the $L_p(\mathbb{R}^n)$-norm in the definition of $A^s_{p,q}(\mathbb{R}^n)$ by the $H^{\varrho}L_{p'}(\mathbb{R}^n)$-norm in order to define $H^{\varrho}A^s_{p,q}(\mathbb{R}^n)$. The vector-valued Fourier multiplier assertion in Corollary 4.14 is also the key ingredient to obtain as in [25, Sect. 2.3.3] the density of $S(\mathbb{R}^n)$ in $H^{\varrho}A^s_{p,q}(\mathbb{R}^n)$. Moreover as in [25, Sect. 2.11.2] one can show using our vector-valued duality assertions (Theorems 3.1 and 3.3) also that the dual of $H^{\varrho}A^{-s}_{p',q'}(\mathbb{R}^n)$ is $L^rA^s_{p,q}(\mathbb{R}^n)$ and furthermore that the dual of the completion of $S(\mathbb{R}^n)$ with respect to $L^rA^s_{p,q}(\mathbb{R}^n)$ is $H^{\varrho}A^{-s}_{p',q'}(\mathbb{R}^n)$. Here $L^rA^s_{p,q}(\mathbb{R}^n)$ stands for the morreyfied versions of $A^s_{p,q}(\mathbb{R}^n)$ which are defined by replacing the $L_p(\mathbb{R}^n)$-norm in the definition of $A^s_{p,q}(\mathbb{R}^n)$ by the $L^r_p(\mathbb{R}^n)$-norm.

Moreover, in the one-dimensional case ($n = 1$) Corollary 4.14 implies also Lizorkin representations of the Triebel–Lizorkin–Morrey spaces. $L^rA^s_{p,q}(\mathbb{R})$ (cf. [25, Sect. 2.5.4]) using in addition Nikol'skij inequalities for Morrey spaces published in [20, Theorem 2.2.9, Theorem 2.2.20]. We want to mention that the spaces $L^rA^s_{p,q}(\mathbb{R}^n)$ are studied, in particular, in [13,19,20,26,29].

4.4.2 Strongly Singular Integrals

Definition 4.16 Let $0 < b < 1$ and let φ be a smooth cut-off function with $\varphi = 1$ on $\{|\xi| \ge 1\}$ and $\varphi = 0$ on $\{|\xi| \le 1/2\}$. If $f \in S(\mathbb{R}^n)$, then we define *strongly singular integrals* as

$$(T_b f)(x) \equiv \int_{\xi \in \mathbb{R}^n} \frac{e^{i|\xi|^b}}{|\xi|^{nb/2}} \varphi(|\xi|) \hat{f}(\xi) e^{2\pi i x \xi} \mathrm{d}\xi$$

(cf. [23, p. 192]).

Corollary 4.17 *Let $1 < p < \infty$, $-\frac{n}{p} \leq r < 0$, $-n < \varrho < -\frac{n}{p'}$ and let $q \in [p, 2]$ for $p \leq 2$ and $q \in [2, p]$ for $p > 2$. Then the following statements hold true.*

(1) *There are unique linear and bounded extensions of T_b to $\overset{\circ}{L}{}^r_p(\mathbb{R}^n)$ and to $H^\varrho L_{p'}(\mathbb{R}^n)$ denoted again by T_b and satisfying*

$$\{T_b\}_{j \in \mathbb{N}_0} : \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n) \hookrightarrow \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n) \quad \text{and}$$
$$\{T_b\}_{j \in \mathbb{N}_0} : H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n) \hookrightarrow H^\varrho L_{p'}(\ell_{q'}, \mathbb{R}^n)$$

(2) *There are infinitely many linear and bounded extensions of T_b to $L^r_p(\mathbb{R}^n)$ denoted again by T_b such that*

$$\{T_b\}_{j \in \mathbb{N}_0} : L^r_p(\ell_q, \mathbb{R}^n) \hookrightarrow L^r_p(\ell_q, \mathbb{R}^n).$$

Proof T_b satisfies (16) by [23, p. 192], see also [8, Chapter 5, Sect. 6.8]. The strongly singular integrals T_b are bounded on $L_p(\mathbb{R}^n)$, $1 < p < \infty$, by [8, Sect. 6.8] and the references given there. Moreover, $T_b : W^k_p(\mathbb{R}^n) \hookrightarrow W^k_p(\mathbb{R}^n)$ for all $k \in \mathbb{N}$ using the lift operator (which is the Fourier multiplier corresponding to $(1+|\cdot|^2)^{\sigma/2}$ for $\sigma \in \mathbb{R}$) and, in particular, it follows that $T_b f \in C^\infty$ for $f \in D(\mathbb{R}^n)$ by well-known Sobolev embeddings. Having in mind Remark 4.4 we obtain the assertion by Theorem 4.3. □

4.4.3 Bochner–Riesz Multipliers

Definition 4.18 Let $\lambda > 0$ and let $f \in S(\mathbb{R}^n)$. We define *Bochner–Riesz multipliers* as

$$(B^\lambda f)(x) \equiv \int_{|\xi| \leq 1} \left(1 - |\xi|^2\right)^\lambda \hat{f}(\xi) e^{2\pi i x \xi} \mathrm{d}\xi \, .$$

It is well-known that $B^\lambda f$ can be reformulated as

$$\left(B^\lambda f\right)(x) = c \lim_{\varepsilon \searrow 0} \int_{|x-y| \geq \varepsilon} \frac{J_{n/2+\lambda}(2\pi|x-y|)}{|x-y|^{n/2+\lambda}} f(y) \mathrm{d}y \tag{27}$$

where J_α stands for the Bessel function (cf. [8, Lemma 8.18] or [10, (10.2.1)]).

Corollary 4.19 *If $\lambda \geq (n-1)/2$ then the statements of Corollary 4.17 hold with B^λ in place of T_b.*

Proof Let $\lambda = (n-1)/2$ be the critical index. Using (27) as well as the estimate $J_{n/2+\lambda}(|x|) \le c|x|^{-1/2}$ (see e.g., [10, Appendix B.6]) we see that B^λ satisfies (16). The L_p-boundedness of B^λ at the critical index for $1 < p < \infty$ is known (cf. [8, Theorem 8.15]). We also have $B^\lambda : D(\mathbb{R}^n) \to \mathrm{Lip}(\mathbb{R}^n)$ by the same arguments as in [21, proof of Step 2 of Theorem 1.1, p. 8] taking into account the convolution structure of B^λ. Thus the assertion for $\lambda = (n-1)/2$ is a consequence of Theorem 4.3. Let $\lambda > (n-1)/2$. Then $|(B^\lambda f)|$ can be dominated pointwise by the Hardy-Littlewood maximal function Mf for $f \in D(\mathbb{R}^n)$ (cf. [10, Exercise 10.2.8]). Hence,

$$\left\| B^\lambda f_j | L^r_p(\ell_q, \mathbb{R}^n) \right\| \le c \left\| f_j | L^r_p(\ell_q, \mathbb{R}^n) \right\|$$

where c does not depend on $\{f_j\}_{j=0}^\infty \in \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n)$ with $f_j \in D(\mathbb{R}^n)$ for all j. As above we have $B^\lambda : D(\mathbb{R}^n) \to \mathrm{Lip}(\mathbb{R}^n)$. As in the proof of Theorem 4.3 we find an unique extension of B^λ denoted again as B^λ such that

$$B^\lambda : \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n) \hookrightarrow \overset{\circ}{L}{}^r_p(\ell_q, \mathbb{R}^n).$$

Hereby, we mention that the constant in (18) is allowed to depend on the fixed sequence of functions. The proof of the reaming parts of the corollary for $\lambda > (n-1)/2$ follows the same lines as in the proof of Theorem 4.3. □

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