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Sharpness of Complex Interpolation on α-Modulation Spaces

Weichao Guo[1] · Dashan Fan[2] · Huoxiong Wu[1] · Guoping Zhao[3]

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Abstract In this paper, we solve a long standing problem on the modulation spaces, α-modulation spaces and Besov spaces. We establish sharp conditions for the complex interpolation between these function spaces. We show that no α-modulation space $M_{p,q}^{s,\alpha}$ can be regarded as the interpolation space between $M_{p_1,q_1}^{s_1,\alpha_1}$ and $M_{p_2,q_2}^{s_2,\alpha_2}$, unless α_1 is equal to α_2, essentially. Especially, our results show that the α-modulation spaces can not be obtained by complex interpolation between modulation spaces and Besov spaces.

Keywords Modulation space · α-Modulation space · Besov space · Sharpness · Complex interpolation

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✉ Guoping Zhao
guopingzhaomath@gmail.com

Weichao Guo
weichaoguomath@gmail.com

Dashan Fan
fan@uwm.edu

Huoxiong Wu
huoxwu@xmu.edu.cn

1 School of Mathematical Sciences, Xiamen University, Xiamen 361005, People's Republic of China

2 Department of Mathematics, University of Wisconsin-Milwaukee, Milwaukee, WI 53201, USA

3 Department of Mathematics, Zhejiang University, Hangzhou 310027, People's Republic of China

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1 Introduction

The modulation spaces $M^s_{p,q}$ were introduced by Feichtinger [10] in 1983 using the short-time Fourier transform. His initial motivation was to use a different space from the L^p to measure the smoothness of a function. Since their introduction, it became increasingly clear that the modulation spaces are quite natural and useful for the studying time-frequency behavior of functions and that they play a significant role in harmonic analysis and partial differential equations. Particularly, these spaces and their applications received extensive studies in the last 10 years. For instance, the reader may see [1,2,7,10,11,16,20–22] and the references therein.

The definition of classical Besov spaces $B^s_{p,q}$ is based on a dyadic decomposition of the frequency space, while the definition of modulation spaces is based on the unit square decomposition of the frequency space (uniform frequency decomposition). Thus, it is natural to build a bridge connecting modulation spaces and Besov spaces. To this end, under the guidance of Feichtinger, in his PhD thesis Gröbner introduced the α-modulation space $M^{s,\alpha}_{p,q}$, which form a family of intermediate spaces between these two types of spaces. The parameter α controls the 'mixture' between both kinds of spaces. Gröbner used the general framework of decomposition spaces considered by Feichtinger and Gröbner in [8] and [9] to build the α-modulation spaces. Borup and Nielsen [4] and Fornasier [12] constructed Banach frames for α-modulation spaces in the multivariate setting. Borup and Nielsen [5,6] also discussed, in the framework of α-modulation spaces, the boundedness of certain pseudo-differential operators with symbols in the Hörmander class.

The modulation spaces arise as special α-modulation spaces in the case $\alpha = 0$, and the (inhomogeneous) Besov space $B^s_{p,q}$ can be regarded as the limit case of $M^{s,\alpha}_{p,q}$ as $\alpha \to 1$ (see [13]). So, for the sake of convenience, we can view the Besov spaces as special α-modulation spaces and use $M^{s,1}_{p,q}$ to denote the inhomogeneous Besov space $B^s_{p,q}$. The interested reader should also consult the recent paper [15], which contains a more comprehensive study of α-modulation spaces.

As we mentioned above, the α-modulation space $M^{s,\alpha}_{p,q}$ plays the role of an intermediate space between the spaces $M^s_{p,q}$ and $B^s_{p,q}$. One may ask how it plays, or in what sense, as an intermediate space. For instance, in [23] it was shown that, from the view as the action of certain unimodular Fourier multipliers, the α-modulation space is an intermediate function space between modulation space and Besov space. But this intuition is false in some other cases. Thus, one motivation of this paper is to explore this fact in the sense of complex interpolation.

A natural long standing question on modulation, α-modulation and Besov spaces is: Can we obtain the α-modulation spaces by interpolation between certain modulation spaces and Besov spaces? More specifically, if $s = (1-\alpha)s_0 + \alpha s_1$, can we conclude

$$M^{s,\alpha}_{p,q} = [M^{s_1,0}_{p,q}, M^{s_2,1}_{p,q}]_\alpha \, ? \tag{1.1}$$

Here $[X, Y]_\theta$ denotes the complex interpolation space of exponent θ ($0 < \theta < 1$) between X and Y (see [3]).

In [15], the authors pointed out that the answer of the question is negative in some special cases. The main technique used in [15] is based on the fact that, for two α-modulation spaces $M^{s_1,\alpha_1}_{p_1,q_1}$ and $M^{s_2,\alpha_2}_{p_2,q_2}$ which are multiplication algebras, the complex interpolation spaces of $M^{s_1,\alpha_1}_{p_1,q_1}$ and $M^{s_2,\alpha_2}_{p_2,q_2}$ are also multiplication algebras. This method, which deeply depends on the multiplication algebra property of α-modulation spaces, leads to some unnatural constrains that seemingly can not be diminished. The algebra property seems not to be the most suitable tool for characterizing the complex interpolation between α-modulation spaces.

Instead, in our proof, the solution is obtained by taking full advantage of the properties of complex interpolation. Actually, in this paper we give a complete answer in a more general sense. We construct some specific functions and operators to test the operator interpolation inequalities, making the arguments more clear and efficient. As a consequence, we show that no α-modulation space can be regarded as the interpolation space between $M^{s_1,\alpha_1}_{p_1,q_1}$ and $M^{s_2,\alpha_2}_{p_2,q_2}$, unless α_1 is equal to α_2, essentially. Also, our conclusion gives the solution of the question mentioned above.

It is known that the theory of complex interpolation is a powerful tool in the study of linear and multi-linear operators on function spaces. In order to obtain the boundedness of a linear or multi-linear operator between certain function spaces, we only need to obtain its boundedness on endpoint spaces. Then boundedness on the full range of function spaces (interpolation spaces) can be easily obtained by complex interpolation. In view of this motivation, establishing a theory of complex interpolation for the α-modulation spaces seems worthwhile. We notice that the known results imply $[M^{s_1,\alpha}_{p_1,q_1}, M^{s_2,\alpha}_{p_2,q_2}]_\theta = M^{s_\theta,\alpha}_{p_\theta,q_\theta}$ for $\theta \in (0,1)$. This indicates that the complex interpolation theory indeed works for the α-modulation spaces if α is fixed. But the situation becomes complicated for the modulation spaces of different α. To clarify this matter, on the analogy of the known results, one might wonder whether $[M^{s_1,\alpha_1}_{p_1,q_1}, M^{s_2,\alpha_2}_{p_2,q_2}]_\theta = M^{s_\theta,\alpha_\theta}_{p_\theta,q_\theta}$ holds for $\theta \in (0,1)$, where $\alpha_\theta = (1-\theta)\alpha_1 + \theta\alpha_2$. First, we ask if the interpolation space between $M^{s_1,\alpha_1}_{p_1,q_1}$ and $M^{s_2,\alpha_2}_{p_2,q_2}$ exists if $\alpha_1 \neq \alpha_2$. Second, even if the interpolation space between $M^{s_1,\alpha_1}_{p_1,q_1}$ and $M^{s_2,\alpha_2}_{p_2,q_2}$ exists, for instance it is a certain α-modulation space, we do not know if $M^{s_\theta,\alpha_\theta}_{p_\theta,q_\theta}$ is just the right interpolation space. For these reasons, to approach our aim, we will not consider directly the discrimination of $[M^{s_1,\alpha_1}_{p_1,q_1}, M^{s_2,\alpha_2}_{p_2,q_2}]_\theta = M^{s_\theta,\alpha_\theta}_{p_\theta,q_\theta}$, but try to find conditions on the pairs (p_1,p_2), (q_1,q_2), (s_1,s_2) and (α_1,α_2), for which $[M^{s_1,\alpha_1}_{p_1,q_1}, M^{s_2,\alpha_2}_{p_2,q_2}]_\theta$ is an α-modulation space.

In the spirit of the abstract complex interpolation theory for Quasi-Banach spaces, we know that the complex interpolation space $[M^{s_1,\alpha_1}_{p_1,q_1}, M^{s_2,\alpha_2}_{p_2,q_2}]_\theta$ is well-defined for arbitrary values of the parameters, even for $\alpha_1 \neq \alpha_2$. However, under the condition $\alpha_1 \neq \alpha_2$, we will show that, for any $\theta \in (0,1)$, the interpolation space $[M^{s_1,\alpha_1}_{p_1,q_1}, M^{s_2,\alpha_2}_{p_2,q_2}]_\theta$ is not any α-modulation space. To achieve our target, for three α-modulation spaces $M^{s_1,\alpha_1}_{p_1,q_1}$, $M^{s_2,\alpha_2}_{p_2,q_2}$ and $M^{s,\alpha}_{p,q}$ to be checked, we will assume towards a contradiction that $[M^{s_1,\alpha_1}_{p_1,q_1}, M^{s_2,\alpha_2}_{p_2,q_2}]_\theta = M^{s,\alpha}_{p,q}$ holds and choose a known triplet of complex interpolation spaces Y_1, Y_2 and Y_θ satisfying $[Y_1, Y_2]_\theta = Y_\theta$.

By choosing a suitable operator T, we use the general property of interpolation spaces to obtain

$$\left\| T|M^{s,\alpha}_{p,q} \to [Y_1, Y_2]_\theta \right\| \lesssim \left\| T|M^{s_1,\alpha_1}_{p_1,q_1} \to Y_1 \right\|^{1-\theta} \left\| T|M^{s_2,\alpha_2}_{p_2,q_2} \to Y_2 \right\|^{\theta} \tag{1.2}$$

and

$$\left\| T|[Y_1, Y_2]_\theta \to M^{s,\alpha}_{p,q} \right\| \lesssim \left\| T|Y_1 \to M^{s_1,\alpha_1}_{p_1,q_1} \right\|^{1-\theta} \left\| T|Y_2 \to M^{s_2,\alpha_2}_{p_2,q_2} \right\|^{\theta}. \tag{1.3}$$

Theoretically, as long as we collect enough known complex interpolation triplets and operators, the information of α-modulation spaces to be checked can be characterized in full. This will yield stronger criteria for disproving the identity $[M^{s_1,\alpha_1}_{p_1,q_1}, M^{s_2,\alpha_2}_{p_2,q_2}]_\theta = M^{s,\alpha}_{p,q}$. So, the rest of the work is to establish suitable tools for achieving the goal mentioned above.

We use

$$\mathcal{M} = \{M^{s,\alpha}_{p,q} : \ p, q \in (0,\infty], s \in \mathbb{R}, \alpha \in [0,1]\} \tag{1.4}$$

to denote the set of all α-modulation spaces.

Now, we state our main theorems.

Theorem 1.1 (Banach case) *Let* $1 \le p_i, q_i \le \infty$, $s_i \in \mathbb{R}$, $\alpha_i \in [0,1]$ *for* $i = 1, 2$. *Then*

$$\left[M^{s_1,\alpha_1}_{p_1,q_1}, M^{s_2,\alpha_2}_{p_2,q_2}\right]_\theta \in \mathcal{M} \tag{1.5}$$

for some $\theta \in (0,1)$, *if and only if*

$$\alpha_1 = \alpha_2 \quad or \quad p_1 = q_1 = 2 \quad or \quad p_2 = q_2 = 2. \tag{1.6}$$

Moreover, we have

$$\left[M^{s_1,\alpha_1}_{p_1,q_1}, M^{s_2,\alpha_2}_{p_2,q_2}\right]_\theta = \begin{cases} M^{s_\theta,\alpha}_{p_\theta,q_\theta}, & \text{if } \alpha_1 = \alpha_2 = \alpha, \\ M^{s_\theta,\alpha_2}_{p_\theta,q_\theta}, & \text{if } p_1 = q_1 = 2, \\ M^{s_\theta,\alpha_1}_{p_\theta,q_\theta}, & \text{if } p_2 = q_2 = 2, \\ H^{s_\theta}, & \text{if } p_1 = q_1 = p_2 = q_2 = 2, \end{cases} \tag{1.7}$$

where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \ \frac{1}{q_\theta} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \ s_\theta = (1-\theta)s_1 + \theta s_2. \tag{1.8}$$

Theorem 1.2 (Quasi-Banach case) *Let* $0 < p, q \le \infty$, $s_i \in \mathbb{R}$, $\alpha_i \in [0,1]$ *for* $i = 1, 2$. *Then*

$$\left[M^{s_1,\alpha_1}_{p,q}, M^{s_2,\alpha_2}_{p,q}\right]_\theta \in \mathcal{M} \tag{1.9}$$

for some $\theta \in (0,1)$, *if and only if*

$$\alpha_1 = \alpha_2 \quad or \quad p = q = 2. \tag{1.10}$$

Moreover, we have

$$\left[M^{s_1,\alpha_1}_{p,q}, M^{s_2,\alpha_2}_{p,q}\right]_\theta = \begin{cases} M^{s_\theta,\alpha}_{p,q}, & \text{if } \alpha_1 = \alpha_2 = \alpha, \\ H^{s_\theta}, & \text{if } p = q = 2, \end{cases} \tag{1.11}$$

where $s_\theta = (1-\theta)s_1 + \theta s_2$.

Remark 1.3 In Theorem 1.1, the constrains $p_i, q_i \geq 1$ are convenient for us to use the dual method. If $p_i < 1$ or $q_i < 1$, then the dual method does not work in most cases. Heuristically, this indicates that we may not be able to catch enough information about p from it's dual p' in the case $p < 1$. So, we have to establish more delicate estimates to make up for the loss of duality. We handle this situation in Theorem 1.2 based on the restrictive conditions $p_1 = p_2$ and $q_1 = q_2$. However, we believe that the additional assumption in Theorem 1.1 or Theorem 1.2 can be eliminated. In other words, it is our conjecture that the results in Theorem 1.1 remain true when $0 < p_i, q_i \leq \infty$.

The organization of this paper is as follows. In Sect. 2, we introduce some notations and definitions that will be used throughout this paper. We recall the definitions of α-modulation spaces and Besov spaces and collect some of their properties that will be used later on. We also present some basic results about the technique of complex interpolation which will be our main tools in the proof. In Sect. 3, we establish some relations among p_i, q_i, s_i, under the assumption that the convexity inequality associated with certain α-modulation spaces holds for all Schwartz functions. These estimates are the key for the discrimination of the complex interpolation in Theorem 1.1. In Sect. 4, we establish some additional estimates for the proof of Theorem 1.2. These estimates allow us to obtain a new proof for the sharpness of embeddings between α-modulation spaces. We complete the proof of our main theorems in Sect. 5. Since the assumptions in our theorems are fairly weak, we must first obtain some priori estimates, then the estimates obtained in Sects. 3 and 4 can be used for further determination of the parameters. In combination with the positive results of complex interpolation for α-modulation spaces, we obtain the sufficient and necessary conditions and complete our proofs.

2 Preliminary

We recall some notations. Let C be a positive constant that may depend on $n, p_i, q_i, s_i, \alpha_i$, where $i = 1, 2$. The notation $X \lesssim Y$ denotes the statement $X \leq CY$. The notation $X \sim Y$ means the statement $X \lesssim Y \lesssim X$, and the notation $X \simeq Y$ denotes the statement $X = CY$. We write $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$. For a multi-index $k = (k_1, k_2, ..., k_n) \in \mathbb{Z}^n$, we denote $|k|_\infty := \max_{i=1,2...n} |k_i|$, and $\langle k \rangle := (1 + |k|^2)^{\frac{1}{2}}$. The translation operator is defined by $T_x f(t) = f(t - x)$, $t, x \in \mathbb{R}^n$. For any $p \in (0, \infty]$, we denote by p' the dual number of p, i.e.,

$$p' = \begin{cases} \frac{p}{p-1}, & 1 < p \leq \infty, \\ \infty, & 0 < p \leq 1. \end{cases} \tag{2.1}$$

Let $\mathscr{S} := \mathscr{S}(\mathbb{R}^n)$ be the set of all Schwartz functions and $\mathscr{S}' := \mathscr{S}'(\mathbb{R}^n)$ be the space of all tempered distributions. We define the Fourier transform $\mathscr{F}f$ and the inverse Fourier transform $\mathscr{F}^{-1}f$ of $f \in \mathscr{S}(\mathbb{R}^n)$ by

$$\mathscr{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx, \quad \mathscr{F}^{-1}f(x) = \widehat{f}(-x) = \int_{\mathbb{R}^n} f(\xi)e^{2\pi i x\cdot\xi}d\xi.$$

We use $L^p(\mathbb{R}^n)$, to denote the Banach space (or Quasi-Banach space when $0 < p \le 1$) of measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ whose norm (or Quasi-norm)

$$\|f\|_{L^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{\frac{1}{p}} \tag{2.2}$$

is finite, with the usual modification when $p = \infty$.

We recall some definitions and properties of the function spaces to be discussed in this paper. For the convenience of doing calculations pertaining to α-modulation spaces, we give the definition of α-modulation spaces based on decomposition methods, without introducing them in full generality. Now, we give the partition of unity on frequency space for $\alpha \in [0, 1)$. We suppose that $c > 0$ and $C > 0$ are two appropriate constants, and choose a Schwartz function sequence $\{\eta_k^\alpha\}_{k\in\mathbb{Z}^n}$ satisfying

$$\begin{cases} |\eta_k^\alpha(\xi)| \ge 1, \ \text{if } |\xi - \langle k\rangle^{\frac{\alpha}{1-\alpha}}k| < c\langle k\rangle^{\frac{\alpha}{1-\alpha}}; \\ \mathbf{supp}\eta_k^\alpha \subset \{\xi : |\xi - \langle k\rangle^{\frac{\alpha}{1-\alpha}}k| < C\langle k\rangle^{\frac{\alpha}{1-\alpha}}\}; \\ \sum_{k\in\mathbb{Z}^n} \eta_k^\alpha(\xi) \equiv 1, \forall\xi \in \mathbb{R}^n; \\ |\partial^\gamma \eta_k^\alpha(\xi)| \le C_\alpha \langle k\rangle^{-\frac{\alpha|\gamma|}{1-\alpha}}, \forall\xi \in \mathbb{R}^n, \gamma \in (\mathbb{Z}^+ \cup \{0\})^n. \end{cases} \tag{2.3}$$

Then $\{\eta_k^\alpha(\xi)\}_{k\in\mathbb{Z}^n}$ constitutes a smooth partition of unity on $\mathbb{R}^n$. The frequency decomposition operators associated with above function sequence can be defined by

$$\Box_k^\alpha := \mathscr{F}^{-1}\eta_k^\alpha\mathscr{F} \tag{2.4}$$

for $k \in \mathbb{Z}^n$. Let $0 < p, q \le \infty$, $s \in \mathbb{R}$, $\alpha \in [0, 1)$. The α-modulation space associated with the above decomposition is defined by

$$M_{p,q}^{s,\alpha}(\mathbb{R}^n) = \left\{ f \in \mathscr{S}'(\mathbb{R}^n) : \|f\|_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} = \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{sq}{1-\alpha}} \|\Box_k^\alpha f\|_{L^p}^q\right)^{\frac{1}{q}} < \infty \right\}$$

with the usual modification when $q = \infty$. For simplicity, we denote $M_{p,q}^s = M_{p,q}^{s,0}$ and $\eta_k(\xi) = \eta_k^0(\xi)$.

Remark 2.1 We recall that the above definition is independent of the choice of exact η_k^α (see [15]). Also, for sufficiently small $\delta > 0$, one can construct a function sequence $\{\eta_k^\alpha(\xi)\}_{k\in\mathbb{Z}^n}$ such that $\eta_k^\alpha(\xi) = 1$ and $\eta_k^\alpha(\xi)\eta_l^\alpha(\xi) = 0$ if $k \ne l$, when ξ lies in the ball $B(\langle k\rangle^{\frac{\alpha}{1-\alpha}}k, \langle k\rangle^{\frac{\alpha}{1-\alpha}}\delta)$ (see [4,12,14]).

Next we introduce the dyadic decomposition of $\mathbb{R}^n$. Let φ be a smooth bump function supported in the ball $\{\xi : |\xi| < \frac{3}{2}\}$ which is equal to 1 on the ball $\{\xi : |\xi| \leq \frac{4}{3}\}$. Denote

$$\psi(\xi) = \varphi(\xi) - \varphi(2\xi), \tag{2.5}$$

and a function sequence

$$\begin{cases} \psi_j(\xi) = \psi(2^{-j}\xi), \ j \in \mathbb{N} \\ \psi_0(\xi) = 1 - \sum_{j\in\mathbb{N}} \psi_j(\xi) = \varphi(\xi). \end{cases} \tag{2.6}$$

For integers $j \in \mathbb{N} \cup \{0\}$, we define the Littlewood-Paley operators

$$\widehat{\Delta_j f}(\xi) = \psi_j(\xi)\widehat{f}(\xi). \tag{2.7}$$

Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$. For $f \in \mathscr{S}'$ we set

$$\|f\|_{B^s_{p,q}} = \left(\sum_{j=0}^{\infty} 2^{jsq} \|\Delta_j f\|^q_{L^p} \right)^{1/q} \tag{2.8}$$

with the usual modification when $q = \infty$. The (inhomogeneous) Besov space is the space of all tempered distributions f for which the quantity $\|f\|_{B^s_{p,q}}$ is finite.

Remark 2.2 As for the α-modulation space, the definition of Besov space is independent of the choice of the bump functions φ. So one can choose an appropriate φ as one needs. Also, one can easily verify that the function sequence $\{\psi_j\}_{j\in\{0\}\bigcup\mathbb{N}}$ satisfies

$$\begin{cases} \mathbf{supp}\psi \subset \{\xi \in \mathbb{R}^n : \frac{2}{3} \leq |\xi| \leq \frac{3}{2}\}; \\ \psi_0(\xi) = 1 \text{ and, } \ \psi_0(\xi)\psi_l(\xi) = 0, \ \xi \in B(0,\delta); \\ \psi_j(\xi) = 1 \text{ and } \psi_j(\xi)\psi_l(\xi) = 0, \ 2^j - 2^j\delta \leq |\xi| \leq 2^j + 2^j\delta, \ l \neq j, \end{cases} \tag{2.9}$$

for $l, j \in \mathbb{N}$, where $\delta = 1/4$.

We list some basic properties about α-modulation spaces.

Lemma 2.3 (see [13,15]) *Let* $0 < p_i, q_i \leq \infty$, $s_i \in \mathbb{R}$ *for* $i = 1, 2$, $\alpha \in [0, 1]$. *Then we have*

$$[M^{s_1,\alpha}_{p_1,q_1}, M^{s_2,\alpha}_{p_2,q_2}]_\theta = M^{s_\theta,\alpha}_{p_\theta,q_\theta} \tag{2.10}$$

for $\theta \in (0, 1)$, *where*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \ \frac{1}{q_\theta} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \ s_\theta = (1-\theta)s_1 + \theta s_2.$$

Lemma 2.4 (see [15]) $M^{s,\alpha}_{2,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$ *with equivalent norms. Here* $H^s(\mathbb{R}^n)$ *denotes the Sobolev space of order* s.

We also need the following proposition which will be used in our proof.

Proposition 2.5 (Dual method for α modulation spaces) *Suppose* $0 < p, q \leq \infty$. *Let* $f \in \mathscr{S}$ *and define*

$$T_f(\varphi) = \langle \varphi, f \rangle \tag{2.11}$$

for $\varphi \in \mathscr{S}'$. *Then* T_f *is a bounded linear functional on* $M^{s,\alpha}_{p,q}$, *and*

$$\|T_f\|_{\left(M^{s,\alpha}_{p,q}\right)^*} \sim \|f\|_{M^{-s+n\alpha(\frac{1}{1\wedge p}-1),\alpha}_{p',q'}}. \tag{2.12}$$

The only thing we must point out is that this proposition works also in the endpoint case $p = \infty$ or $q = \infty$. One can verify this proposition by the same method used in determining the dual spaces of α-modulation spaces (see [15]). We omit the details here, but refer the reader to [15] for a further discussion.

We recall some basic results about complex interpolation. The following well-known results are the main reason why complex interpolation plays an important role for proving boundedness of linear operators. For a proof of the following result, see [15, Proposition 2.11] and the references therein.

Lemma 2.6 (Operator interpolation for complex interpolation) *Let* (X_1, X_2) *and* (Y_1, Y_2) *be two compatible couples of Quasi Banach spaces,* $\theta \in (0, 1)$. *If a linear operator* T *belongs to* $L(X_1, Y_1) \cap L(X_2, Y_2)$, *then we have*

$$\|T|\ [X_1, X_2]_\theta \to [Y_1, Y_2]_\theta\| \leq \|T|\ X_1 \to Y_1\|^{1-\theta}\ \|T|\ X_2 \to Y_2\|^\theta\ . \tag{2.13}$$

Taking $X_1 = X_2 = \mathbb{C}$ or $Y_1 = Y_2 = \mathbb{C}$, one can easily verify two direct corollaries from the above Lemma 2.6.

Lemma 2.7 (Convexity Inequality) *Let* (X_1, X_2) *be a compatible couple of Quasi Banach spaces. For every* $\theta \in (0, 1)$, *we have*

$$\|f\|_{[X_1,X_2]_\theta} \leq \|f\|^{1-\theta}_{X_1} \|f\|^\theta_{X_2} \tag{2.14}$$

for $f \in X_1 \cap X_2$.

Lemma 2.8 (Dual Convexity Inequality) *Let* (X_1, X_2) *be a compatible couple of Quasi Banach spaces. Let* T *be a linear functional defined in* X_1 *and* X_2. *Then for every* $\theta \in (0, 1)$, *we have*

$$\|T\|_{[X_1,X_2]^*_\theta} \leq \|T\|^{1-\theta}_{X_1^*} \|T\|^\theta_{X_2^*} \tag{2.15}$$

for $T \in X_1^* \cap X_2^*$.

3 The Convexity Inequality

In this section, we deduce some estimates about the indices p_i, q_i under the assumption that the convexity inequality $\|f\|_{M^{s,\alpha}_{p,q}} \lesssim \|f\|^{1-\theta}_{M^{s_1,\alpha_1}_{p_1,q_1}} \|f\|^{\theta}_{M^{s_2,\alpha_2}_{p_2,q_2}}$ holds for all Schwartz functions f. If $M^{s,\alpha}_{p,q}$ is the complex interpolation space between $M^{s_1,\alpha_1}_{p_1,q_1}$ and $M^{s_2,\alpha_2}_{p_2,q_2}$, the convexity inequality follows. We construct some specific functions to test the convexity inequality and obtain some relationship among the parameters.

For $\alpha \in (0,1)$, $j \in \{0\} \cup \mathbb{N}$, denote

$$\Gamma^{\alpha,1}_j = \{l \in \mathbb{Z}^n : \Delta_j \circ \Box^\alpha_l \neq 0\}. \tag{3.1}$$

For $\alpha_1, \alpha_2 \in (0,1)$, $k \in \mathbb{Z}^n$, we denote

$$\Gamma^{\alpha_1,\alpha_2}_k = \{l \in \mathbb{Z}^n : \Box^{\alpha_2}_k \circ \Box^{\alpha_1}_l \neq 0\}. \tag{3.2}$$

By the above definition, we deduce $\langle l\rangle^{\frac{1}{1-\alpha_1}} \sim \langle k\rangle^{\frac{1}{1-\alpha_2}}$ for $l \in \Gamma^{\alpha_1,\alpha_2}_k$. We also have $\Gamma^{\alpha_1,\alpha_2}_k \sim \langle k\rangle^{\frac{n(\alpha_2-\alpha_1)}{1-\alpha_2}}$ for $\alpha_1 \leq \alpha_2$, $\Gamma^{\alpha_1,\alpha_2}_k \sim 1$ for $\alpha_1 \geq \alpha_2$.

For $f \in \mathscr{S}$ with compact Fourier support and $\widehat{f}(\xi) = 1$ in a open subset of $\mathbb{R}^n$, we denote

$$\Gamma^\alpha_f = \{k \in \mathbb{Z}^n : \Box^\alpha_k f \neq 0\},\ \widetilde{\Gamma^\alpha_f} = \{k \in \mathbb{Z}^n : \Box^\alpha_k f = \mathscr{F}^{-1}\eta^\alpha_k\}. \tag{3.3}$$

For $g \in \mathscr{S}$ with compact support and $g(x) = 1$ in a open subset of $\mathbb{R}^n$, we denote

$$\mathbf{supp}g = \overline{\{x \in \mathbb{R}^n : g(x) \neq 0\}},\ \widetilde{\mathbf{supp}}g = \{x \in \mathbb{R}^n : g(x) = 1\}. \tag{3.4}$$

We recall a convolution lemma which will be used frequently in this paper.

Lemma 3.1 (Convolution in L^p with $p < 1$, see Proposition 2.1 in [15]) *Let* $0 < p < 1$, $x_0 \in \mathbb{R}^n$, $r > 0$. *Suppose* $f, g \in L^p$ *with Fourier support in* $B(x_0, r)$. *Then*

$$\|f * g\|_{L^p} \lesssim r^{n(1/p-1)} \|f\|_{L^p} \|g\|_{L^p}. \tag{3.5}$$

Lemma 3.2 *Suppose* $0 < p_i, q_i \leq \infty$, $s_i \in \mathbb{R}$, $\alpha_i \in [0,1]$ *for* $i = 1, 2$. *We assume* $\alpha_1 < \alpha_2$. *For fixed* $\theta \in (0,1)$, *we denote*

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2},\ \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2},\ s = (1-\theta)s_1 + \theta s_2 \tag{3.6}$$

For any $\alpha \in [0,1]$, *if the convexity inequality*

$$\|f\|_{M^{s,\alpha}_{p,q}} \lesssim \|f\|^{1-\theta}_{M^{s_1,\alpha_1}_{p_1,q_1}} \|f\|^{\theta}_{M^{s_2,\alpha_2}_{p_2,q_2}} \tag{3.7}$$

holds for all $f \in \mathscr{S}$, *then we have*

$$\begin{cases} \frac{1}{p_1}+\frac{1}{q_1}\geq 1,\ \frac{1}{p_1}\leq\frac{1}{q_1}, & \textit{if}\ \alpha>\alpha_1,\\ \frac{1}{p_2}+\frac{1}{q_2}\leq 1,\ \frac{1}{p_2}\geq\frac{1}{q_2}, & \textit{if}\ \alpha=\alpha_1,\\ \frac{1}{p}+\frac{1}{q}\leq 1,\ \frac{1}{p}\geq\frac{1}{q}, & \textit{if}\ \alpha<\alpha_1. \end{cases} \tag{3.8}$$

Proof We only show the proof for the case $\alpha_1<\alpha_2<1$, since the proofs of the other cases are similar. For the sake of simplicity, we write $M=M^{s,\alpha}_{p,q}$ and $M_i=M^{s_i,\alpha_i}_{p_i,q_i}$ for $i=1,2$ in this proof. Let f be a smooth function with small Fourier support near the origin such that $\mathbf{supp}\widehat{f^\alpha_k}\subset\widetilde{\mathbf{supp}\eta^\alpha_k}$ for every $k\in\mathbb{Z}^n$, $\alpha\in[0,1]$, where we denote

$$\widehat{f^\alpha_k}=\widehat{f}\left(\frac{\xi-\langle k\rangle^{\frac{\alpha}{1-\alpha}}k}{\langle k\rangle^{\frac{\alpha}{1-\alpha}}}\right). \tag{3.9}$$

We divide the proof into several cases.

Case 1: $\alpha\in(\alpha_1,\alpha_2)$.

For each $k\in\mathbb{Z}^n$, we choose $j\in\{0\}\cup\mathbb{Z}^+$ such that $\langle k\rangle^{\frac{1}{1-\alpha}}\sim 2^j$.

Firstly, a direct calculation yields

$$\begin{cases} \|f^\alpha_k\|_M\sim 2^{js}\|f^\alpha_k\|_{L^p}\sim 2^{js}2^{jn\alpha(1-1/p)},\\ \|f^\alpha_k\|_{M_2}\sim 2^{js_2}\|f^\alpha_k\|_{L^{p_2}}\sim 2^{js_2}2^{jn\alpha(1-1/p_2)}. \end{cases} \tag{3.10}$$

To estimate $\|f^\alpha_k\|_{M_1}$, we use Young's inequality for $p_1\geq 1$ or Lemma 3.1 for $p_1<1$ to deduce

$$\begin{aligned} \|\Box^{\alpha_1}_l f^\alpha_k\|_{L^{p_1}} &= \|(\mathscr{F}^{-1}\eta^{\alpha_1}_l)*f^\alpha_k\|_{L^{p_1}}\\ &\lesssim 2^{jn\alpha(1/(p_1\wedge 1)-1)}\|f^\alpha_k\|_{L^{p_1\wedge 1}}\|\mathscr{F}^{-1}\eta^{\alpha_1}_l\|_{L^{p_1}}\\ &\lesssim \|\mathscr{F}^{-1}\eta^{\alpha_1}_l\|_{L^{p_1}}\\ &\lesssim 2^{jn\alpha_1(1-1/p_1)}. \end{aligned} \tag{3.11}$$

Noting that

$$|\Gamma^{\alpha_1}_{f^\alpha_k}|\sim 2^{jn(\alpha-\alpha_1)} \tag{3.12}$$

and

$$\langle l\rangle^{\frac{1}{1-\alpha_1}}\sim 2^j \tag{3.13}$$

for $l\in\Gamma^{\alpha_1}_{f^\alpha_k}$, we have

$$\begin{aligned} \|f^\alpha_k\|_{M_1} &= \left(\sum_{l\in\mathbb{Z}^n}\langle l\rangle^{\frac{s_1q_1}{1-\alpha_1}}\|\Box^{\alpha_1}_l f^\alpha_k\|^{q_1}_{L^{p_1}}\right)^{\frac{1}{q_1}}\\ &\lesssim 2^{js_1}2^{jn\alpha_1(1-1/p_1)}2^{jn(\alpha-\alpha_1)/q_1}. \end{aligned} \tag{3.14}$$

The convexity inequality

$$\|f_k\|_M \lesssim \|f_k\|_{M_1}^{1-\theta}\|f_k\|_{M_2}^{\theta} \tag{3.15}$$

then yields that

$$2^{js}2^{jn\alpha(1-1/p)} \lesssim \left(2^{js_1}2^{jn\alpha_1(1-1/p_1)}2^{jn(\alpha-\alpha_1)/q_1}\right)^{1-\theta}\left(2^{js_2}2^{jn\alpha(1-1/p_2)}\right)^{\theta} \tag{3.16}$$

as $j\to\infty$, which implies

$$\begin{aligned} s\ &+n\alpha(1-1/p) \le (1-\theta)\left(s_1+n\alpha_1(1-1/p_1)+n(\alpha-\alpha_1)/q_1\right)\\ &+\theta\left(s_2+n\alpha(1-1/p_2)\right). \end{aligned} \tag{3.17}$$

Recalling

$$\frac{1}{p}=\frac{1-\theta}{p_1}+\frac{\theta}{p_2},\ \ s=(1-\theta)s_1+\theta s_2,$$

we have

$$n(\alpha-\alpha_1)(1-\theta)(1-1/p_1)\le n(\alpha-\alpha_1)(1-\theta)/q_1. \tag{3.18}$$

Hence

$$1-1/p_1\le 1/q_1, \tag{3.19}$$

and we obtain

$$\frac{1}{p_1}+\frac{1}{q_1}\ge 1. \tag{3.20}$$

Secondly, we set

$$F_{k,N}=\sum_{l\in\Gamma_k^{\alpha_1,\alpha}} T_{Nl}f_l^{\alpha_1}. \tag{3.21}$$

Here T_{Nl} denotes the translation operator: $T_{Nl}f(x)=f(x-Nl)$.
Obviously for $l\in\Gamma_k^{\alpha_1,\alpha}$, we have

$$\Box_l^{\alpha_1}F_{k,N}=T_{Nl}f_l^{\alpha_1}, \tag{3.22}$$

and

$$\|\Box_l^{\alpha_1}F_{k,N}\|_{L_{p_1}}=\|T_{Nl}f_l^{\alpha_1}\|_{L^{p_1}}=\|f_l^{\alpha_1}\|_{L^{p_1}}\sim 2^{jn\alpha_1(1-1/p_1)}. \tag{3.23}$$

So

$$\begin{aligned} \|F_{k,N}\|_{M_1} &= \left(\sum_{l\in\Gamma_k^{\alpha_1,\alpha}}\langle l\rangle^{\frac{s_1q_1}{1-\alpha_1}}\|\Box_l^{\alpha_1}F_{k,N}\|_{L_{p_1}}^{q_1}\right)^{\frac{1}{q_1}}\\ &\sim 2^{js_1}2^{jn\alpha_1(1-1/p_1)}|\Gamma_k^{\alpha_1,\alpha}|^{1/q_1}\\ &\sim 2^{js_1}2^{jn\alpha_1(1-1/p_1)}2^{jn(\alpha-\alpha_1)/q_1}. \end{aligned} \tag{3.24}$$

On the other hand, we have

$$\begin{cases} \|F_{k,N}\|_M \sim 2^{js}\|F_{k,N}\|_{L^p}, \\ \|F_{k,N}\|_{M_2} \sim 2^{js_2}\|F_{k,N}\|_{L^{p_2}}. \end{cases} \tag{3.25}$$

By the almost orthogonality of $\{T_{Nl}f_l^{\alpha_1}\}_{l\in\Gamma_k^{\alpha_1,\alpha}}$ as $N\to\infty$, we deduce that

$$\begin{aligned} \lim_{N\to\infty}\int_{\mathbb{R}^n}|F_{k,N}|^p dx &= \lim_{N\to\infty}\int_{\mathbb{R}^n}\Big|\sum_{l\in\Gamma_k^{\alpha_1,\alpha}} T_{Nl}f_l^{\alpha_1}\Big|^p dx = \lim_{N\to\infty}\sum_{l\in\Gamma_k^{\alpha_1,\alpha}}\int_{\mathbb{R}^n}|T_{Nl}f_l^{\alpha_1}|^p dx \\ &= \sum_{l\in\Gamma_k^{\alpha_1,\alpha}}\int_{\mathbb{R}^n}|f_l^{\alpha_1}|^p dx \sim 2^{jn\alpha_1(p-1)}|\Gamma_k^{\alpha_1,\alpha}| \\ &\sim 2^{jn\alpha_1(p-1)}2^{jn(\alpha-\alpha_1)}. \end{aligned}$$

With a small modification when $p=\infty$, we obtain

$$\lim_{N\to\infty}\|F_{k,N}\|_{L^p} = 2^{jn\alpha_1(1-1/p)}2^{jn(\alpha-\alpha_1)/p} \tag{3.26}$$

for $p\in(0,\infty]$. So

$$\begin{cases} \|F_{k,N}\|_M \sim 2^{js}2^{jn\alpha_1(1-1/p)}2^{jn(\alpha-\alpha_1)/p}, \\ \|F_{k,N}\|_{M_2} \sim 2^{js_2}2^{jn\alpha_1(1-1/p_2)}2^{jn(\alpha-\alpha_1)/p_2} \end{cases} \tag{3.27}$$

as $N\to\infty$. Letting N tend to infinity in the convexity inequality

$$\|F_{k,N}\|_M \lesssim \|F_{k,N}\|_{M_1}^{1-\theta}\|F_{k,N}\|_{M_2}^{\theta}, \tag{3.28}$$

we deduce that

$$\begin{aligned} &2^{js}2^{jn\alpha_1(1-1/p)}2^{jn(\alpha-\alpha_1)/p} \\ &\lesssim \left(2^{js_1}2^{jn\alpha_1(1-1/p_1)}2^{jn(\alpha-\alpha_1)/q_1}\right)^{1-\theta}\left(2^{js_2}2^{jn\alpha_1(1-1/p_2)}2^{jn(\alpha-\alpha_1)/p_2}\right)^{\theta} \end{aligned} \tag{3.29}$$

as $j\to\infty$. Recalling

$$\frac{1}{p}=\frac{1-\theta}{p_1}+\frac{\theta}{p_2},\ \ s=(1-\theta)s_1+\theta s_2,$$

we obtain

$$(\alpha-\alpha_1)/p \le (1-\theta)(\alpha-\alpha_1)/q_1+\theta(\alpha-\alpha_1)/p_2, \tag{3.30}$$

which yields

$$(1-\theta)(\alpha-\alpha_1)/p_1 \le (1-\theta)(\alpha-\alpha_1)/q_1, \tag{3.31}$$

and

$$1/p_1 \le 1/q_1. \tag{3.32}$$

Case 2: $\alpha \geq \alpha_2$.

For each $k \in \mathbb{Z}^n$, we choose $j \in \{0\} \cup \mathbb{Z}^+$ such that $\langle k \rangle^{\frac{1}{1-\alpha_2}} \sim 2^j$.

Firstly, a direct calculation yields

$$\begin{cases} \|f_k^{\alpha_2}\|_M \sim 2^{js} \|f_k^{\alpha_2}\|_{L^p} \sim 2^{js} 2^{jn\alpha_2(1-1/p)}, \\ \|f_k^{\alpha_2}\|_{M_2} \sim 2^{js_2} \|f_k^{\alpha_2}\|_{L^{p_2}} \sim 2^{js_2} 2^{jn\alpha_2(1-1/p_2)}. \end{cases} \tag{3.33}$$

To estimate $\|f_k^{\alpha_2}\|_{M_1}$, we have

$$\|\Box_l^{\alpha_1} f_k^{\alpha_2}\|_{L^{p_1}} \lesssim \|\mathscr{F}^{-1} \eta_l^{\alpha_1}\|_{L^{p_1}} \lesssim 2^{jn\alpha_1(1-1/p_1)}. \tag{3.34}$$

Noting that

$$|\Gamma_{f_k^{\alpha_2}}^{\alpha_1}| \sim 2^{jn(\alpha_2-\alpha_1)} \tag{3.35}$$

and

$$\langle l \rangle^{\frac{1}{1-\alpha_1}} \sim 2^j \tag{3.36}$$

for $l \in \Gamma_{f_k^{\alpha_2}}^{\alpha_1}$, we have

$$\begin{aligned} \|f_k^{\alpha_2}\|_{M_1} &= \left(\sum_{l \in \mathbb{Z}^n} \langle l \rangle^{\frac{s_1 q_1}{1-\alpha_1}} \|\Box_l^{\alpha_1} f_k^{\alpha_2}\|_{L^{p_1}}^{q_1} \right)^{\frac{1}{q_1}} \\ &\lesssim 2^{js_1} 2^{jn\alpha_1(1-1/p_1)} 2^{jn(\alpha_2-\alpha_1)/q_1}. \end{aligned} \tag{3.37}$$

The convexity inequality

$$\|f_k^{\alpha_2}\|_M \lesssim \|f_k^{\alpha_2}\|_{M_1}^{1-\theta} \|f_k^{\alpha_2}\|_{M_2}^{\theta} \tag{3.38}$$

then yields that

$$2^{js} 2^{jn\alpha_2(1-1/p)} \lesssim \left(2^{js_1} 2^{jn\alpha_1(1-1/p_1)} 2^{jn(\alpha_2-\alpha_1)/q_1}\right)^{1-\theta} \left(2^{js_2} 2^{jn\alpha_2(1-1/p_2)}\right)^{\theta}$$

as $j \to \infty$, which implies that

$$\begin{aligned} s + n\alpha_2(1-1/p) \ &\leq (1-\theta)\,(s_1 + n\alpha_1(1-1/p_1) \\ &\qquad + n(\alpha_2-\alpha_1)/q_1) + \theta\,(s_2 + n\alpha_2(1-1/p_2)). \end{aligned}$$

Recalling

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \ \ s = (1-\theta)s_1 + \theta s_2,$$

we have

$$n(\alpha_2-\alpha_1)(1-\theta)(1-1/p_1) \leq n(\alpha_2-\alpha_1)(1-\theta)/q_1. \tag{3.39}$$

Hence, we obtain

$$1 - 1/p_1 \leq 1/q_1, \tag{3.40}$$

that is,

$$\frac{1}{p_1} + \frac{1}{q_1} \geq 1. \tag{3.41}$$

Secondly, we set

$$F_{k,N} = \sum_{l \in \Gamma_k^{\alpha_1,\alpha_2}} T_{Nl} f_l^{\alpha_1}. \tag{3.42}$$

Obviously for $l \in \Gamma_k^{\alpha_1,\alpha_2}$, we have

$$\Box_l^{\alpha_1} F_{k,N} = T_{Nl} f_l^{\alpha_1}, \tag{3.43}$$

and

$$\|\Box_l^{\alpha_1} F_{k,N}\|_{L_{p_1}} = \|T_{Nl} f_l^{\alpha_1}\|_{L^{p_1}} = \|f_l^{\alpha_1}\|_{L^{p_1}} \sim 2^{jn\alpha_1(1-1/p_1)}. \tag{3.44}$$

So

$$\begin{aligned} \|F_{k,N}\|_{M_1} &= \left(\sum_{l \in \Gamma_k^{\alpha_1,\alpha_2}} \langle l \rangle^{\frac{s_1 q_1}{1-\alpha_1}} \|\Box_l^{\alpha_1} F_{k,N}\|_{L_{p_1}}^{q_1} \right)^{\frac{1}{q_1}} \\ &\sim 2^{js_1} 2^{jn\alpha_1(1-1/p_1)} |\Gamma_k^{\alpha_1,\alpha_2}|^{1/q_1} \\ &\sim 2^{js_1} 2^{jn\alpha_1(1-1/p_1)} 2^{jn(\alpha_2-\alpha_1)/q_1}. \end{aligned} \tag{3.45}$$

On the other hand, we have

$$\begin{cases} \|F_{k,N}\|_M \sim 2^{js} \|F_{k,N}\|_{L^p} \\ \|F_{k,N}\|_{M_2} \sim 2^{js_2} \|F_{k,N}\|_{L^{p_2}}. \end{cases} \tag{3.46}$$

By an orthogonality argument as above, we have that

$$\begin{cases} \|F_{k,N}\|_M \sim 2^{js} 2^{jn\alpha_1(1-1/p)} 2^{jn(\alpha_2-\alpha_1)/p} \\ \|F_{k,N}\|_{M_2} \sim 2^{js_2} 2^{jn\alpha_1(1-1/p_2)} 2^{jn(\alpha_2-\alpha_1)/p_2} \end{cases} \tag{3.47}$$

as $N \to \infty$. Letting N tend to infinity in the convexity inequality

$$\|F_{k,N}\|_M \lesssim \|F_{k,N}\|_{M_1}^{1-\theta} \|F_{k,N}\|_{M_2}^{\theta}, \tag{3.48}$$

we deduce that

$$\begin{aligned} &2^{js} 2^{jn\alpha_1(1-1/p)} 2^{jn(\alpha_2-\alpha_1)/p} \\ &\quad \lesssim \left(2^{js_1} 2^{jn\alpha_1(1-1/p_1)} 2^{jn(\alpha_2-\alpha_1)/q_1}\right)^{1-\theta} \left(2^{js_2} 2^{jn\alpha_1(1-1/p_2)} 2^{jn(\alpha_2-\alpha_1)/p_2}\right)^{\theta} \end{aligned} \tag{3.49}$$

as $j \to \infty$. Recalling

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \ s = (1-\theta)s_1 + \theta s_2,$$

we obtain

$$(\alpha_2 - \alpha_1)/p \le (1-\theta)(\alpha_2-\alpha_1)/q_1 + \theta(\alpha_2-\alpha_1)/p_2, \tag{3.50}$$

and

$$(1-\theta)(\alpha_2-\alpha_1)/p_1 \le (1-\theta)(\alpha_2-\alpha_1)/q_1. \tag{3.51}$$

So it follows

$$1/p_1 \le 1/q_1. \tag{3.52}$$

Case 3: $\alpha = \alpha_1$.

For each $k \in \mathbb{Z}^n$, we choose $j \in \{0\} \cup \mathbb{Z}^+$ such that $\langle k \rangle^{\frac{1}{1-\alpha_2}} \sim 2^j$.

Firstly, direct calculations give that

$$\|f_k^{\alpha_2}\|_M = \left(\sum_{l\in\mathbb{Z}^n} \langle l \rangle^{\frac{sq}{1-\alpha}} \|\Box_l^\alpha f_k^{\alpha_2}\|_{L^p}^q\right)^{\frac{1}{q}} \gtrsim \left(\sum_{l\in\widetilde{\Gamma^\alpha_{f_k^{\alpha_2}}}} \langle l \rangle^{\frac{sq}{1-\alpha}} \|\mathscr{F}^{-1}\eta_l^\alpha\|_{L^p}^q\right)^{\frac{1}{q}} \gtrsim 2^{js}2^{j\alpha n(1-1/p)}2^{j(\alpha_2-\alpha)n/q}, \tag{3.53}$$

$$\|f_k^{\alpha_2}\|_{M_1} = \left(\sum_{l\in\mathbb{Z}^n} \langle l \rangle^{\frac{s_1q_1}{1-\alpha_1}} \|\Box_l^{\alpha_1} f_k^{\alpha_2}\|_{L^{p_1}}^{q_1}\right)^{\frac{1}{q_1}} \lesssim \left(\sum_{l\in\Gamma^\alpha_{f_k^{\alpha_2}}} \langle l \rangle^{\frac{s_1q_1}{1-\alpha}} \|\mathscr{F}^{-1}\eta_l^\alpha\|_{L^{p_1}}^{q_1}\right)^{\frac{1}{q_1}} \lesssim 2^{js_1}2^{j\alpha n(1-1/p_1)}2^{j(\alpha_2-\alpha)n/q_1}, \tag{3.54}$$

and

$$\|f_k^{\alpha_2}\|_{M_2} \sim 2^{js_2}\|f_k^{\alpha_2}\|_{L^{p_2}} \sim 2^{js_2}2^{j\alpha_2 n(1-1/p_2)}. \tag{3.55}$$

The convexity inequality

$$\|f_k^{\alpha_2}\|_M \lesssim \|f_k^{\alpha_2}\|_{M_1}^{1-\theta}\|f_k^{\alpha_2}\|_{M_2}^\theta \tag{3.56}$$

then yields that

$$2^{js}2^{j\alpha n(1-1/p)}2^{j(\alpha_2-\alpha)n/q} \lesssim \left(2^{js_1}2^{j\alpha n(1-1/p_1)}2^{j(\alpha_2-\alpha)n/q_1}\right)^{1-\theta}\left(2^{js_2}2^{j\alpha_2 n(1-1/p_2)}\right)^\theta$$

as $j \to \infty$, which implies

$$\theta(\alpha-\alpha_2)(1-1/p_2) + \theta(\alpha_2-\alpha)/q_2 \le 0. \tag{3.57}$$

So

$$\frac{1}{p_2} + \frac{1}{q_2} \le 1. \tag{3.58}$$

Secondly, we set

$$F_{k,N} = \sum_{l \in \Gamma_k^{\alpha,\alpha_2}} T_{Nl} f_l^{\alpha}. \tag{3.59}$$

Obviously for $l \in \Gamma_k^{\alpha,\alpha_2}$, we have

$$\Box_l^{\alpha_1} F_{k,N} = T_{Nl} f_l^{\alpha}, \tag{3.60}$$

and

$$\|\Box_l^{\alpha_1} F_{k,N}\|_{L_{p_1}} = \|T_{Nl} f_l^{\alpha}\|_{L^{p_1}} = \|f_l^{\alpha}\|_{L^{p_1}} \sim 2^{jn\alpha(1-1/p_1)}. \tag{3.61}$$

So

$$\begin{aligned} \|F_{k,N}\|_{M_1} &= \left(\sum_{l \in \Gamma_k^{\alpha,\alpha_2}} \langle l \rangle^{\frac{s_1 q_1}{1-\alpha_1}} \|\Box_l^{\alpha_1} F_{k,N}\|_{L_{p_1}}^{q_1} \right)^{\frac{1}{q_1}} \\ &\sim 2^{js_1} 2^{jn\alpha(1-1/p_1)} |\Gamma_k^{\alpha,\alpha_2}|^{1/q_1} \\ &\sim 2^{js_1} 2^{jn\alpha(1-1/p_1)} 2^{jn(\alpha_2-\alpha)/q_1}. \end{aligned} \tag{3.62}$$

Similarly, we have

$$\|F_{k,N}\|_{M} \sim 2^{js} 2^{jn\alpha(1-1/p)} 2^{jn(\alpha_2-\alpha)/q}. \tag{3.63}$$

On the other hand, we have

$$\|F_{k,N}\|_{M_2} \sim 2^{js_2} \|F_{k,N}\|_{L^{p_2}}. \tag{3.64}$$

By an orthogonality argument as above, we have that

$$\|F_{k,N}\|_{M_2} \sim 2^{js_2} 2^{jn\alpha(1-1/p_2)} 2^{jn(\alpha_2-\alpha)/p_2} \tag{3.65}$$

as $N \to \infty$. Letting N tend to infinity in the convexity inequality

$$\|F_{k,N}\|_{M} \lesssim \|F_{k,N}\|_{M_1}^{1-\theta} \|F_{k,N}\|_{M_2}^{\theta}, \tag{3.66}$$

we deduce that

$$\begin{aligned} &2^{js} 2^{jn\alpha(1-1/p)} 2^{jn(\alpha_2-\alpha)/q} \\ &\quad \lesssim \left(2^{js_1} 2^{jn\alpha(1-1/p_1)} 2^{jn(\alpha_2-\alpha)/q_1}\right)^{1-\theta} \left(2^{js_2} 2^{jn\alpha(1-1/p_2)} 2^{jn(\alpha_2-\alpha)/p_2}\right)^{\theta} \end{aligned} \tag{3.67}$$

as $j \to \infty$. We obtain

$$1/q_2 \le 1/p_2. \tag{3.68}$$

Case 4: $\alpha < \alpha_1$.

For $k \in \mathbb{Z}^n$, we choose $j \in \{0\} \cup \mathbb{Z}^+$ such that $\langle k\rangle^{\frac{1}{1-\alpha_1}} \sim 2^j$.

Firstly, a direct calculation yields

$$\begin{cases} \|f_k^{\alpha_1}\|_{M_1} \sim 2^{js_1}\|f_k^{\alpha_1}\|_{L^{p_1}} \sim 2^{js_1}2^{jn\alpha_1(1-1/p_1)} \\ \|f_k^{\alpha_1}\|_{M_2} \sim 2^{js_2}\|f_k^{\alpha_1}\|_{L^{p_2}} \sim 2^{js_2}2^{jn\alpha_1(1-1/p_2)}. \end{cases} \tag{3.69}$$

To estimate $\|f_k^{\alpha_1}\|_M$, we have

$$\begin{aligned} \|f_k^{\alpha_1}\|_M &= \left(\sum_{l\in\mathbb{Z}^n} \langle l\rangle^{\frac{sq}{1-\alpha}} \|\Box_l^\alpha f_k^{\alpha_1}\|_{L^p}^q\right)^{\frac{1}{q}} \gtrsim \left(\sum_{l\in\widetilde{\Gamma^\alpha_{f_k^{\alpha_1}}}} \langle l\rangle^{\frac{sq}{1-\alpha}} \|\mathscr{F}^{-1}\eta_l^\alpha\|_{L^p}^q\right)^{\frac{1}{q}} \\ &\gtrsim 2^{js}2^{jn\alpha(1-1/p)}2^{jn(\alpha_1-\alpha)/q}. \end{aligned} \tag{3.70}$$

The convexity inequality

$$\|f_k^{\alpha_1}\|_M \lesssim \|f_k^{\alpha_1}\|_{M_1}^{1-\theta}\|f_k^{\alpha_1}\|_{M_2}^{\theta} \tag{3.71}$$

then yields that

$$2^{js}2^{jn\alpha(1-1/p)}2^{jn(\alpha_1-\alpha)/q} \lesssim \left(2^{js_1}2^{jn\alpha_1(1-1/p_1)}\right)^{1-\theta}\left(2^{js_2}2^{jn\alpha_1(1-1/p_2)}\right)^{\theta} \tag{3.72}$$

as $j \to \infty$, which implies

$$(\alpha_1-\alpha)/q \le (\alpha_1-\alpha)(1-1/p). \tag{3.73}$$

So we obtain

$$\frac{1}{p}+\frac{1}{q} \le 1. \tag{3.74}$$

Secondly, we set

$$F_{k,N} = \sum_{l\in\Gamma_k^{\alpha,\alpha_1}} T_{Nl} f_l^\alpha. \tag{3.75}$$

Obviously for $l \in \Gamma_k^{\alpha,\alpha_1}$, we have

$$\Box_l^\alpha F_{k,N} = T_{Nl} f_l^\alpha, \tag{3.76}$$

and

$$\|\Box_l^\alpha F_{k,N}\|_{L_p} = \|T_{Nl} f_l^\alpha\|_{L^p} = \|f_l^\alpha\|_{L^p} \sim 2^{jn\alpha(1-1/p)}. \tag{3.77}$$

So

$$\begin{aligned}\|F_{k,N}\|_M &= \Bigg(\sum_{l\in\Gamma_k^{\alpha,\alpha_1}} \langle l\rangle^{\frac{sq}{1-\alpha}} \|\Box_l^\alpha F_{k,N}\|_{L_p}^q\Bigg)^{\frac{1}{q}}\\ &\sim 2^{js}2^{jn\alpha(1-1/p)}|\Gamma_k^{\alpha,\alpha_1}|^{1/q}\\ &\sim 2^{js}2^{jn\alpha(1-1/p)}2^{jn(\alpha_1-\alpha)/q}.\end{aligned} \tag{3.78}$$

On the other hand, we have

$$\begin{cases}\|F_{k,N}\|_{M_1} \sim 2^{js_1}\|F_{k,N}\|_{L^{p_1}}\\ \|F_{k,N}\|_{M_2} \sim 2^{js_2}\|F_{k,N}\|_{L^{p_2}}.\end{cases} \tag{3.79}$$

By an orthogonality argument as above, we have

$$\begin{cases}\|F_{k,N}\|_{M_1} \sim 2^{js_1}2^{jn\alpha(1-1/p_1)}2^{jn(\alpha_1-\alpha)/p_1}\\ \|F_{k,N}\|_{M_2} \sim 2^{js_2}2^{jn\alpha(1-1/p_2)}2^{jn(\alpha_1-\alpha)/p_2}\end{cases} \tag{3.80}$$

as $N\to\infty$. Letting N tend to infinity in the convexity inequality

$$\|F_{k,N}\|_M \lesssim \|F_{k,N}\|_{M_1}^{1-\theta}\|F_{k,N}\|_{M_2}^{\theta}, \tag{3.81}$$

we deduce that

$$\begin{aligned}&2^{js}2^{jn\alpha(1-1/p)}2^{jn(\alpha_1-\alpha)/q}\\ &\quad\lesssim \left(2^{js_1}2^{jn\alpha(1-1/p_1)}2^{jn(\alpha_1-\alpha)/p_1}\right)^{1-\theta}\left(2^{js_2}2^{jn\alpha(1-1/p_2)}2^{jn(\alpha_1-\alpha)/p_2}\right)^{\theta}\end{aligned} \tag{3.82}$$

as $j\to\infty$. Thus we obtain

$$(\alpha_1-\alpha)/q \le (\alpha_1-\alpha)/p. \tag{3.83}$$

The desired inequality

$$1/q \le 1/p \tag{3.84}$$

follows. □

4 Additional Operator Norm Estimates

As we mentioned before, duality arguments are often not applicable in the context of Quasi-Banach spaces. So the key point for the discrimination of complex interpolation is how to regain the information without the full duality. To this end, in this section we bring some additional known complex interpolation spaces into the operator interpolation inequalities (2.13). Our purpose is to establish some asymptotic estimates for certain operators between the additional spaces and our target spaces. As a corollary, we give a direct proof for the sharpness of embedding between α-modulation spaces (see [15] for an alternative proof).

Lemma 4.1 (Additional operator norm estimates) *Let* $0 < p, q \le \infty$, $s_i \in \mathbb{R}$, $\alpha_i \in [0, 1]$ *for* $i = 1, 2$. *We have*

$$\left\| \Box_k^{\alpha_1 \vee \alpha_2} |\ M_{p,q}^{s_1,\alpha_1} \to M_{p,q}^{s_2,\alpha_2} \right\| \sim \langle k \rangle^{\frac{(s_2-s_1)+\left(0\vee[n(\alpha_2-\alpha_1)(1/p-1/q)]\vee[n(\alpha_2-\alpha_1)(1-1/p-1/q)]\right)}{1-(\alpha_1\vee\alpha_2)}}, \tag{4.1}$$

for $\alpha_1 \vee \alpha_2 < 1$, $k \in \mathbb{Z}^n$, *and*

$$\left\| \Delta_j |\ M_{p,q}^{s_1,\alpha_1} \to M_{p,q}^{s_2,\alpha_2} \right\| \sim 2^{j\left[(s_2-s_1)+\left(0\vee[n(\alpha_2-\alpha_1)(1/p-1/q)]\vee[n(\alpha_2-\alpha_1)(1-1/p-1/q)]\right)\right]}, \tag{4.2}$$

for $\alpha_1 \vee \alpha_2 = 1$, $j \in \{0\} \cup \mathbb{Z}^+$.

Proof We only give the proof for the case $\alpha_1 < \alpha_2 = 1$, which will be used in the proof of Theorem 1.2. The other cases can be handled similarly. For instance, in the case $\alpha_1 < \alpha_2 < 1$, one can repeat the following process by replacing Δ_j with $\Box_k^{\alpha_2}$.

In the case $\alpha_1 < \alpha_2 = 1$, we need to show

$$\left\| \Delta_j |\ M_{p,q}^{s_1,\alpha_1} \to B_{p,q}^{s_2} \right\| \sim 2^{j\left[(s_2-s_1)+\left(0\vee[n(1-\alpha_1)(1/p-1/q)]\vee[n(1-\alpha_1)(1-1/p-1/q)]\right)\right]}. \tag{4.3}$$

For Lower Bound Estimates. We only need to construct some special functions to test the operator inequalities. Take a smooth function f whose Fourier transform $\widehat{f}$ has small support near the origin such that $\mathbf{supp}\widehat{f_k^\alpha} \subset \widetilde{\mathbf{supp}\eta_k^\alpha}$ for every $k \in \mathbb{Z}^n$, $\alpha \in [0, 1]$, where we denote

$$\widehat{f_k^\alpha} = \widehat{f}\left(\frac{\xi - \langle k \rangle^{\frac{\alpha}{1-\alpha}} k}{\langle k \rangle^{\frac{\alpha}{1-\alpha}}}\right). \tag{4.4}$$

Firstly, we have

$$\left\| \Delta_j |\ M_{p,q}^{s_1,\alpha_1} \to B_{p,q}^{s_2} \right\| \gtrsim \frac{\|\Delta_j f_k^0\|_{B_{p,q}^{s_2}}}{\|f_k^0\|_{M_{p,q}^{s_1,\alpha_1}}} \sim \frac{2^{js_2}\|f_k^0\|_{L^p}}{2^{js_1}\|f_k^0\|_{L^p}} \gtrsim 2^{j(s_2-s_1)} \tag{4.5}$$

for some suitable $k \in \mathbb{Z}^n$ such that $\langle k \rangle \sim 2^j$.

Next, we choose a smooth function h whose Fourier transform has sufficiently small support near the origin, such that $\widehat{h_j}(\xi) = \widehat{h}(\frac{\xi}{2^j})$ satisfies

$$\mathbf{supp}\widehat{h_j} \subset \widetilde{\mathbf{supp}}\psi_j. \tag{4.6}$$

A direct calculation yields that

$$\|\Delta_j h_j\|_{B_{p,q}^{s_2}} = \|h_j\|_{B_{p,q}^{s_2}} \sim 2^{js_2}\|h_j\|_{L^p} \sim 2^{j(s_2+n(1-1/p))} \tag{4.7}$$

and

$$\|h_j\|_{M^{s_1,\alpha_1}_{p,q}} = \left(\sum_{l\in\mathbb{Z}^n} \langle l\rangle^{\frac{s_1 q}{1-\alpha_1}} \|\Box^{\alpha_1}_l h_j\|^q_{L^p}\right)^{\frac{1}{q}} \lesssim \left(\sum_{l\in\Gamma^{\alpha_1}_{h_j}} \langle l\rangle^{\frac{s_1 q}{1-\alpha_1}} \|\mathscr{F}^{-1}\eta^{\alpha_1}_l\|^q_{L^p}\right)^{\frac{1}{q}} \\ \lesssim 2^{js_1} 2^{j\alpha_1 n(1-1/p)} 2^{j(1-\alpha_1)n/q}. \tag{4.8}$$

So we have

$$\left\|\Delta_j |\ M^{s_1,\alpha_1}_{p,q} \to B^{s_2}_{p,q}\right\| \gtrsim \frac{\|\Delta_j h_j\|_{B^{s_2}_{p,q}}}{\|h_j\|_{M^{s_1,\alpha_1}_{p,q}}} \gtrsim \frac{2^{j(s_2+n(1-1/p))}}{2^{js_1}2^{j\alpha_1 n(1-1/p)}2^{j(1-\alpha_1)n/q}} \\ = 2^{j(s_2-s_1+n(1-\alpha_1)(1-1/p-1/q))}. \tag{4.9}$$

Finally, let

$$F_{j,N} = \sum_{l\in\Gamma^{\alpha_1,1}_j} T_{Nl} f^{\alpha_1}_l, \tag{4.10}$$

where T_{Nl} denotes the translation operator: $T_{Nl} f(x) = f(x - Nl)$. Using the same method as in the proof of Lemma 3.2, we have

$$\|F_{j,N}\|_{M^{s_1,\alpha_1}_{p,q}} = \left(\sum_{l\in\Gamma^{\alpha_1,1}_j} \langle l\rangle^{\frac{s_1 q}{1-\alpha_1}} \|\Box^{\alpha_1}_l F_{j,N}\|^q_{L^p}\right)^{\frac{1}{q}} \\ \sim 2^{js_1} 2^{jn\alpha_1(1-1/p)} |\Gamma^{\alpha_1,1}_j|^{1/q} \\ \sim 2^{js_1} 2^{jn\alpha_1(1-1/p)} 2^{jn(1-\alpha_1)/q}. \tag{4.11}$$

Also, we have that

$$\|\Delta_j F_{j,N}\|_{B^{s_2}_{p,q}} \lesssim \|F_{j,N}\|_{B^{s_2}_{p,q}} \sim 2^{js_2}\|F_{j,N}\|_{L^p} \sim 2^{js_2} 2^{jn\alpha_1(1-1/p)} 2^{jn(1-\alpha_1)/p} \tag{4.12}$$

as $N \to \infty$. So by definition of the operator norm,

$$\left\|\Delta_j |\ M^{s_1,\alpha_1}_{p,q} \to B^{s_2}_{p,q}\right\| \gtrsim \lim_{N\to\infty} \frac{\|\Delta_j F_{j,N}\|_{B^{s_2}_{p,q}}}{\|F_{j,N}\|_{M^{s_1,\alpha_1}_{p,q}}} \sim \frac{2^{js_2}2^{jn\alpha_1(1-1/p)}2^{jn(1-\alpha_1)/p}}{2^{js_1}2^{jn\alpha_1(1-1/p)}2^{jn(1-\alpha_1)/q}} \\ \gtrsim 2^{j(s_2-s_1)} 2^{jn(1-\alpha_1)(1/p-1/q)}. \tag{4.13}$$

Now, we have the lower bound

$$\left\|\Delta_j |\ M^{s_1,\alpha_1}_{p,q} \to B^{s_2}_{p,q}\right\| \gtrsim 2^{j\left[(s_2-s_1)+\left(0\vee[n(1-\alpha_1)(1/p-1/q)]\vee[n(1-\alpha_1)(1-1/p-1/q)]\right)\right]}. \tag{4.14}$$

For Upper Bound Estimates. We only handle the following cases, then the other cases can be deduced by an easy interpolation argument.

Case 1. $p = q = 2$.

$$\|\Delta_j f\|_{B^{s_2}_{2,2}} \sim 2^{js_2}\|\Delta_j f\|_{L^2} \sim 2^{j(s_2-s_1)}\|\Delta_j f\|_{M^{s_1,\alpha_1}_{2,2}} \lesssim 2^{j(s_2-s_1)}\|f\|_{M^{s_1,\alpha_1}_{2,2}}. \quad (4.15)$$

Moreover, in this case we may write

$$2^{j(s_2-s_1)} = 2^{j(s_2-s_1+n(1-\alpha_1)(1/p-1/q))} = 2^{j(s_2-s_1+n(1-\alpha_1)(1-1/p-1/q))}. \quad (4.16)$$

Case 2. $p = \infty, q \le 1$.

$$\|\Delta_j f\|_{B^{s_2}_{\infty,q}} \sim 2^{js_2}\|\Delta_j f\|_{L^\infty} = 2^{js_2}\Big\|\sum_{l\in\Gamma_j^{\alpha_1,1}} \Box_l^{\alpha_1}\Delta_j f\Big\|_{L^\infty} \lesssim 2^{js_2}\sum_{l\in\Gamma_j^{\alpha_1,1}} \|\Box_l^{\alpha_1} f\|_{L^\infty}$$
$$\lesssim 2^{js_2}\left(\sum_{l\in\Gamma_j^{\alpha_1,1}} \|\Box_l^{\alpha_1} f\|_{L^\infty}^q\right)^{1/q} \lesssim 2^{j(s_2-s_1)}\|f\|_{M^{\alpha_1,s_1}_{\infty,q}}.$$

Moreover, for $p = \infty$ and $q = 1$, we can write,

$$2^{j(s_2-s_1)} = 2^{j(s_2-s_1+n(1-\alpha_1)(1-1/p-1/q))}. \quad (4.17)$$

Case 3. $p = q \le 1$.

$$\|\Delta_j f\|_{B^{s_2}_{p,q}} \sim 2^{js_2}\|\Delta_j f\|_{L^p} = 2^{js_2}\|\sum_{l\in\Gamma_j^{\alpha_1,1}} \Box_l^{\alpha_1}\Delta_j f\|_{L^p} \lesssim 2^{js_2}\left(\sum_{l\in\Gamma_j^{\alpha_1,1}} \|\Box_l^{\alpha_1} f\|_{L^p}^p\right)^{1/p}$$
$$= 2^{js_2}\left(\sum_{l\in\Gamma_j^{\alpha_1,1}} \|\Box_l^{\alpha_1} f\|_{L^p}^q\right)^{1/q} \lesssim 2^{j(s_2-s_1)}\|f\|_{M^{\alpha_1,s_1}_{p,q}}.$$

Moreover, in this case we have

$$2^{j(s_2-s_1)} = 2^{j(s_2-s_1+n(1-\alpha_1)(1/p-1/q))}. \quad (4.18)$$

Case 4. $p = q = \infty$.

$$\|\Delta_j f\|_{B^{s_2}_{\infty,\infty}} \sim 2^{js_2}\|\Delta_j f\|_{L^\infty} = 2^{js_2}\Big\|\sum_{l\in\Gamma_j^{\alpha_1,1}} \Box_l^{\alpha_1}\Delta_j f\Big\|_{L^\infty}$$
$$\lesssim 2^{js_2}\sum_{l\in\Gamma_j^{\alpha_1,1}} \|\Box_l^{\alpha_1} f\|_{L^\infty}$$

$$\lesssim 2^{js_2}|\Gamma_j^{\alpha_1,1}| \sup_{l\in\Gamma_j^{\alpha_1,1}} \|\Box_l^{\alpha_1} f\|_{L^\infty}$$
$$\lesssim 2^{j(s_2-s_1)}2^{jn(1-\alpha_1)}\|f\|_{M^{\alpha_1,s_1}_{\infty,\infty}}. \tag{4.19}$$

Moreover, we have

$$2^{j(s_2-s_1)}2^{jn(1-\alpha_1)} = 2^{j(s_2-s_1)}2^{jn(1-\alpha_1)(1-1/p-1/q)} \tag{4.20}$$

in this case.

Case 5. $p=2, q=\infty$.

$$\begin{aligned}\|\Delta_j f\|_{B^{s_2}_{2,\infty}} \sim 2^{js_2}\|\Delta_j f\|_{L^2} \sim 2^{js_2}\|\Delta_j f\|_{M^{0,\alpha_1}_{2,2}} \\ \lesssim 2^{js_2}\left(\sum_{l\in\Gamma_j^{\alpha_1,1}} \|\Box_l^{\alpha_1} f\|_{L^2}^2\right)^{1/2} \\ \lesssim 2^{js_2}|\Gamma_j^{\alpha_1,1}|^{1/2} \sup_{l\in\Gamma_j^{\alpha_1,1}} \|\Box_l^{\alpha_1} f\|_{L^2} \\ \lesssim 2^{j(s_2-s_1)}2^{jn(1-\alpha_1)(1/2)}\|f\|_{M^{\alpha_1,s_1}_{2,\infty}}.\end{aligned} \tag{4.21}$$

Moreover, we have

$$2^{j(s_2-s_1)}2^{jn(1-\alpha_1)(1/2)} = 2^{j(s_2-s_1+n(1-\alpha_1)(1/p-1/q))} = 2^{j(s_2-s_1+n(1-\alpha_1)(1-1/p-1/q))}$$

in this case.

Case 6. $p\le 1, q=\infty$.

$$\begin{aligned}\|\Delta_j f\|_{B^{s_2}_{p,\infty}} \sim 2^{js_2}\|\Delta_j f\|_{L^p} = 2^{js_2}\Big\|\sum_{l\in\Gamma_j^{\alpha_1,1}} \Box_l^{\alpha_1}\Delta_j f\Big\|_{L^p} \\ \lesssim 2^{js_2}\left(\sum_{l\in\Gamma_j^{\alpha_1,1}} \|\Box_l^{\alpha_1} f\|_{L^p}^p\right)^{1/p} \\ \lesssim 2^{js_2}|\Gamma_j^{\alpha_1,1}|^{1/p} \sup_{l\in\Gamma_j^{\alpha_1,1}} \|\Box_l^{\alpha_1} f\|_{L^p} \\ \lesssim 2^{j(s_2-s_1)}2^{jn(1-\alpha_1)(1/p)}\|f\|_{M^{\alpha_1,s_1}_{p,\infty}}.\end{aligned} \tag{4.22}$$

Moreover, we have

$$2^{j(s_2-s_1)}2^{jn(1-\alpha_1)(1/p)} = 2^{j(s_2-s_1+n(1-\alpha_1)(1/p-1/q))} \tag{4.23}$$

in this case. □

In order to characterize the existence of embeddings between α-modulation spaces, we only need to establish the following proposition, which can be viewed as a mild characterization of embedding.

Proposition 4.2 (Mild characterization for the embedding between α-modulation spaces) *Let* $0 < p, q \le \infty$, $s_i \in \mathbb{R}$, $\alpha_i \in [0,1]$ *for* $i = 1, 2$. *Then the embedding relationship*

$$M_{p,q}^{s_1,\alpha_1} \subset M_{p,q}^{s_2,\alpha_2} \tag{4.24}$$

holds if and only if

$$\sup_{k\in\mathbb{Z}^n} \left\| \Box_k^{\alpha_1\vee\alpha_2} \big| \, M_{p,q}^{s_1,\alpha_1} \to M_{p,q}^{s_2,\alpha_2} \right\| \lesssim 1 \tag{4.25}$$

for $\alpha_1 \vee \alpha_2 < 1$, *and*

$$\sup_{j\in\{0\}\cup\mathbb{Z}^+} \left\| \Delta_j \big| \, M_{p,q}^{s_1,\alpha_1} \to M_{p,q}^{s_2,\alpha_2} \right\| \lesssim 1 \tag{4.26}$$

for $\alpha_1 \vee \alpha_2 = 1$.

Proof We only give the proof for the case $\alpha_1, \alpha_2 < 1$, since the proofs of other cases are similar.

Case 1: $\alpha_1 \le \alpha_2$.

If the embedding $M_{p,q}^{s_1,\alpha_1} \subset M_{p,q}^{s_2,\alpha_2}$ holds, we have

$$\|\Box_k^{\alpha_2} f\|_{M_{p,q}^{s_2,\alpha_2}} \lesssim \|\Box_k^{\alpha_2} f\|_{M_{p,q}^{s_1,\alpha_1}} \lesssim \|f\|_{M_{p,q}^{s_1,\alpha_1}}. \tag{4.27}$$

On the other hand, if $\sup_{k\in\mathbb{Z}^n} \left\| \Box_k^{\alpha_2} \big| \, M_{p,q}^{s_1,\alpha_1} \to M_{p,q}^{s_2,\alpha_2} \right\| \lesssim 1$ holds, we deduce

$$\begin{aligned}
\langle k\rangle^{\frac{s_2}{1-\alpha_2}} \|\Box_k^{\alpha_2} f\|_{L^p} &\sim \|\Box_k^{\alpha_2} f\|_{M_{p,q}^{s_2,\alpha_2}} \\
&= \|\Box_k^{\alpha_2} \sum_{l\in\Gamma_k^{\alpha_2,\alpha_2}} \Box_l^{\alpha_2} f\|_{M_{p,q}^{s_2,\alpha_2}} \\
&\lesssim \|\sum_{l\in\Gamma_k^{\alpha_2,\alpha_2}} \Box_l^{\alpha_2} f\|_{M_{p,q}^{s_1,\alpha_1}} \\
&\lesssim \left(\sum_{l\in\Gamma_k^{\alpha_2,\alpha_2}} \sum_{m\in\Gamma_l^{\alpha_1,\alpha_2}} \langle m\rangle^{\frac{s_1 q}{1-\alpha_1}} \|\Box_m^{\alpha_1} f\|_{L^p}^q \right)^{1/q}.
\end{aligned} \tag{4.28}$$

Observing that $|\Gamma_m^{\alpha_2,\alpha_1}| \lesssim 1$ and $|\Gamma_l^{\alpha_2,\alpha_2}| \lesssim 1$, we obtain

$$\|f\|_{M_{p,q}^{s_2,\alpha_2}} = \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{s_2 q}{1-\alpha_2}} \|\Box_k^{\alpha_2} f\|_{L^p}^q \right)^{1/q}$$

$$\lesssim \left(\sum_{k\in\mathbb{Z}^n} \sum_{l\in\Gamma_k^{\alpha_2,\alpha_2}} \sum_{m\in\Gamma_l^{\alpha_1,\alpha_2}} \langle m\rangle^{\frac{s_1 q}{1-\alpha_1}} \|\Box_m^{\alpha_1} f\|_{L^p}^q \right)^{1/q}$$
$$= \left(\sum_{m\in\mathbb{Z}^n} \sum_{l\in\Gamma_m^{\alpha_2,\alpha_1}} \sum_{k\in\Gamma_l^{\alpha_2,\alpha_2}} \langle m\rangle^{\frac{s_1 q}{1-\alpha_1}} \|\Box_m^{\alpha_1} f\|_{L^p}^q \right)^{1/q}$$
$$\lesssim \left(\sum_{m\in\mathbb{Z}^n} \langle m\rangle^{\frac{s_1 q}{1-\alpha_1}} \|\Box_m^{\alpha_1} f\|_{L^p}^q \right)^{1/q} = \|f\|_{M_{p,q}^{s_1,\alpha_1}}. \tag{4.29}$$

Case 2: $\alpha_2 \le \alpha_1$.
If the embedding $M_{p,q}^{s_1,\alpha_1} \subset M_{p,q}^{s_2,\alpha_2}$ holds, we have

$$\|\Box_k^{\alpha_1} f\|_{M_{p,q}^{s_2,\alpha_2}} \lesssim \|\Box_k^{\alpha_1} f\|_{M_{p,q}^{s_1,\alpha_1}} \lesssim \|f\|_{M_{p,q}^{s_1,\alpha_1}}. \tag{4.30}$$

On the other hand, if $\sup_{k\in\mathbb{Z}^n} \left\|\Box_k^{\alpha_1} |\, M_{p,q}^{s_1,\alpha_1} \to M_{p,q}^{s_2,\alpha_2}\right\| \lesssim 1$ holds, we use $|\Gamma_k^{\alpha_1,\alpha_2}| \lesssim 1$ to deduce deduce

$$\begin{aligned} \|f\|_{M_{p,q}^{s_2,\alpha_2}} &= \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{s_2 q}{1-\alpha_2}} \|\Box_k^{\alpha_2} f\|_{L^p}^q \right)^{1/q} \\ &\lesssim \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{s_2 q}{1-\alpha_2}} \sum_{l\in\Gamma_k^{\alpha_1,\alpha_2}} \|\Box_l^{\alpha_1}\Box_k^{\alpha_2} f\|_{L^p}^q \right)^{1/q} \\ &\sim \left(\sum_{l\in\mathbb{Z}^n} \sum_{k\in\Gamma_l^{\alpha_2,\alpha_1}} \langle k\rangle^{\frac{s_2 q}{1-\alpha_2}} \|\Box_k^{\alpha_2}\Box_l^{\alpha_1} f\|_{L^p}^q \right)^{1/q} \\ &\sim \left(\sum_{l\in\mathbb{Z}^n} \|\Box_l^{\alpha_1} f\|_{M_{p,q}^{s_2,\alpha_2}}^q \right)^{1/q}. \end{aligned} \tag{4.31}$$

Then

$$\|f\|_{M_{p,q}^{s_2,\alpha_2}} \lesssim \left(\sum_{k\in\mathbb{Z}^n} \|\Box_k^{\alpha_1} f\|_{M_{p,q}^{s_2,\alpha_2}}^q \right)^{1/q}$$
$$= \left(\sum_{k\in\mathbb{Z}^n} \|\Box_k^{\alpha_1} \sum_{l\in\Gamma_k^{\alpha_1,\alpha_1}} \Box_l^{\alpha_1} f\|_{M_{p,q}^{s_2,\alpha_2}}^q \right)^{1/q}$$

$$\lesssim \left(\sum_{k\in\mathbb{Z}^n} \| \sum_{l\in\Gamma_k^{\alpha_1,\alpha_1}} \Box_l^{\alpha_1} f \|_{M_{p,q}^{s_1,\alpha_1}}^q \right)^{1/q}$$

$$\lesssim \left(\sum_{k\in\mathbb{Z}^n} \sum_{l\in\Gamma_k^{\alpha_1,\alpha_1}} \langle l \rangle^{\frac{s_1 q}{1-\alpha_1}} \|\Box_l^{\alpha_1} f\|_{L^p}^q \right)^{1/q}$$

$$= \left(\sum_{l\in\mathbb{Z}^n} \sum_{k\in\Gamma_l^{\alpha_1,\alpha_1}} \langle l \rangle^{\frac{s_1 q}{1-\alpha_1}} \|\Box_l^{\alpha_1} f\|_{L^p}^q \right)^{1/q} \lesssim \|f\|_{M_{p,q}^{s_1,\alpha_1}}, \tag{4.32}$$

where the last inequality holds for the reason that $|\Gamma_k^{\alpha_1,\alpha_1}| \lesssim 1$. □

Combining Lemma 4.1 and Proposition 4.2, we obtain the following corollary.

Corollary 4.3 (see Theorems 4.1, 4.2 in [15]) *Let* $0 < p, q \leq \infty$, $s_i \in \mathbb{R}$, $\alpha_i \in [0, 1]$ *for* $i = 1, 2$. *Then*

$$M_{p,q}^{s_1,\alpha_1} \subset M_{p,q}^{s_2,\alpha_2} \tag{4.33}$$

holds if and only if

$$s_2 + \big(0 \vee [n(\alpha_2 - \alpha_1)(1/p - 1/q)] \vee [n(\alpha_2 - \alpha_1)(1 - 1/p - 1/q)]\big) \leq s_1.$$

We remark that the embedding results between α_1-modulation and α_2-modulation spaces go back to Gröbner's thesis [13] in which he considered the case $1 \leq p, q \leq \infty$. In [18], Toft and Wahlberg then obtained some partial sufficient conditions, as well as some partial necessary conditions for such embedding. Finally, Wang and Han gave a complete characterization [15]. Embeddings between modulation and Besov spaces are considered in [17] and [19].

5 Proof of Theorems 1.1 and 1.2

5.1 Proof of Theorem 1.1

In this subsection, we suppose $1 \leq p_i, q_i \leq \infty$, $s_i \in \mathbb{R}$, $\alpha_i \in [0, 1]$ for $i = 1, 2$. For a fixed $\theta \in (0, 1)$, if

$$\left[M_{p_1,q_1}^{s_1,\alpha_1}, M_{p_2,q_2}^{s_2,\alpha_2}\right]_\theta \in \mathcal{M}, \tag{5.1}$$

then there exists a α-modulation space $M_{p,q}^{s,\alpha}$ such that

$$\left[M_{p_1,q_1}^{s_1,\alpha_1}, M_{p_2,q_2}^{s_2,\alpha_2}\right]_\theta = M_{p,q}^{s,\alpha}, \tag{5.2}$$

where $p, q \in (0, \infty]$, $s \in \mathbb{R}$, $\alpha \in [0, 1]$. We first make some priori estimates to determine the values of p, q and s.

Step 1: Priori estimates for p, q, s.

For fixed p_i, q_i, s_i under the assumption of Theorem 1.1, we denote

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \ \frac{1}{q_\theta} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \ s_\theta = (1-\theta)s_1 + \theta s_2.$$

We want to check that

$$p = p_\theta, q = q_\theta, s = s_\theta \tag{5.3}$$

under the assumption

$$\left[M^{s_1,\alpha_1}_{p_1,q_1}, M^{s_2,\alpha_2}_{p_2,q_2}\right]_\theta = M^{s,\alpha}_{p,q} \tag{5.4}$$

for some $\theta \in (0, 1)$.

By the convexity inequality, we have

$$\|f\|_{M^{s,\alpha}_{p,q}} \lesssim \|f\|^{1-\theta}_{M^{s_1,\alpha_1}_{p_1,q_1}} \|f\|^{\theta}_{M^{s_2,\alpha_2}_{p_2,q_2}} \tag{5.5}$$

for all $f \in \mathscr{S}(\mathbb{R}^n) \subset M^{s_1,\alpha_1}_{p_1,q_1} \cap M^{s_2,\alpha_2}_{p_2,q_2}$. On the other hand, take $f \in \mathscr{S}(\mathbb{R}^n)$, and define

$$T_f(\varphi) = \langle \varphi, f \rangle \tag{5.6}$$

for $\varphi \in \mathscr{S}'$. Then T_f is a bounded linear functional on $M^{s_1,\alpha_1}_{p_1,q_1}$, $M^{s_2,\alpha_2}_{p_2,q_2}$ and $M^{s,\alpha}_{p,q}$. We use Proposition 2.5 and Lemma 2.8 to deduce that

$$\|f\|_{M^{-s,\alpha}_{p',q'}} \lesssim \|f\|^{1-\theta}_{M^{-s_1,\alpha_1}_{p'_1,q'_1}} \|f\|^{\theta}_{M^{-s_2,\alpha_2}_{p'_2,q'_2}} \tag{5.7}$$

for $p \geq 1$.

Proving $p = p_\theta$. Take a smooth function h whose Fourier transform $\widehat{h}$ has small compact support with $\widehat{h}(\xi) = 1$ near the origin, such that

$$\mathbf{supp}\widehat{h} \subset \widetilde{\mathbf{supp}}\eta_0^\alpha \tag{5.8}$$

for any $\alpha \in [0, 1]$. Let

$$\widehat{h_\lambda}(\xi) = \widehat{h}(\frac{\xi}{\lambda}) \tag{5.9}$$

for $\lambda \in (0, 1)$. We use (5.5) to deduce

$$\|h_\lambda\|_{L^p} \lesssim \|h_\lambda\|^{1-\theta}_{L^{p_1}} \|h_\lambda\|^{\theta}_{L^{p_2}}. \tag{5.10}$$

We then have

$$\lambda^{n(1-\frac{1}{p})} \lesssim \lambda^{n(1-\frac{1}{p_1})(1-\theta)} \lambda^{n(1-\frac{1}{p_2})(\theta)} \tag{5.11}$$

as $\lambda \downarrow 0$. This yields

$$\frac{1}{p} \leq \frac{1-\theta}{p_1} + \frac{\theta}{p_2} = \frac{1}{p_\theta} \leq 1, \tag{5.12}$$

and hence $p \geq p_\theta \geq 1$. Using (5.7), a dual argument then yields that

$$\lambda^{n(1-\frac{1}{p'})} \lesssim \lambda^{n(1-\frac{1}{p_1'})(1-\theta)} \lambda^{n(1-\frac{1}{p_2'})(\theta)} \tag{5.13}$$

as $\lambda \to 0$. So

$$\frac{1}{p} \geq \frac{1}{p_\theta}, \tag{5.14}$$

and then

$$p = p_\theta. \tag{5.15}$$

Proving $s = s_\theta$. Take h to be the same function as above. For $j \in \mathbb{N}$, define $h_j(x) = e^{2\pi i \langle \rho_j, x\rangle} h(x)$ for an arbitrary $\rho_j \in \mathbb{R}^n$, such that

$$|\rho_j| \sim 2^j \tag{5.16}$$

and

$$\mathbf{supp}\widehat{h_j} \subset \widetilde{\mathbf{supp}}\eta^{\alpha}_{k_j}, \widetilde{\mathbf{supp}}\eta^{\alpha_1}_{k_{1,j}}, \widetilde{\mathbf{supp}}\eta^{\alpha_2}_{k_{2,j}} \tag{5.17}$$

for some suitable $k_j, k_{1,j}, k_{2,j} \in \mathbb{Z}^n$. Clearly, we have

$$\langle k_j \rangle^{\frac{1}{1-\alpha}} \sim \langle k_{1,j} \rangle^{\frac{1}{1-\alpha_1}} \sim \langle k_{2,j} \rangle^{\frac{1}{1-\alpha_2}} \sim 2^j. \tag{5.18}$$

We use inequality (5.5) to deduce

$$\|h_j\|_{M^{s,\alpha}_{p,q}} \lesssim \|h_j\|^{1-\theta}_{M^{s_1,\alpha_1}_{p_1,q_1}} \|h_j\|^{\theta}_{M^{s_2,\alpha_2}_{p_2,q_2}}. \tag{5.19}$$

A direct calculation (using $p = p_\theta$) now yields

$$2^{js} \lesssim 2^{(1-\theta)js_1} 2^{\theta j s_2} = 2^{js_\theta} \tag{5.20}$$

as $j \to \infty$. Then $s \leq s_\theta$ follows. Using a dual argument as in Step 1, we obtain $s = s_\theta$.

Proving $q = q_\theta$. Let h_j be the functions as above. We denote

$$T_j(f) = h_j * f \tag{5.21}$$

and

$$\mathscr{T}_N = \sum_{j=1}^{N} T_j \tag{5.22}$$

for $N \in \mathbb{N}$. Recall the complex interpolation of modulation spaces

$$\left[M^{s_1}_{p_1,q_1}, M^{s_2}_{p_2,q_2}\right]_\theta = M^{s_\theta}_{p_\theta,q_\theta}. \tag{5.23}$$

Using Lemma 2.6, we have

$$\|\mathscr{T}_N|\ M^{s_\theta}_{p_\theta,q_\theta}\to M^{s_\theta,\alpha}_{p_\theta,q}\|\lesssim\|\mathscr{T}_N|\ M^{s_1}_{p_1,q_1}\to M^{s_1,\alpha_1}_{p_1,q_1}\|^{1-\theta}\|\mathscr{T}_N|\ M^{s_2}_{p_2,q_2}\to M^{s_2,\alpha_2}_{p_2,q_2}\|^{\theta}$$

and

$$\|\mathscr{T}_N|\ M^{s_\theta,\alpha}_{p_\theta,q}\to M^{s_\theta}_{p_\theta,q_\theta}\|\lesssim\|\mathscr{T}_N|\ M^{s_1,\alpha_1}_{p_1,q_1}\to M^{s_1}_{p_1,q_1}\|^{1-\theta}\|\mathscr{T}_N|\ M^{s_2,\alpha_2}_{p_2,q_2}\to M^{s_2}_{p_2,q_2}\|^{\theta}.$$

Because of $p_1, p_2\geq 1$, one can verify that

$$\|\mathscr{T}_N|\ M^{s_1}_{p_1,q_1}\to M^{s_1,\alpha_1}_{p_1,q_1}\|\sim\|\mathscr{T}_N|\ M^{s_2}_{p_2,q_2}\to M^{s_2,\alpha_2}_{p_2,q_2}\|\sim 1 \tag{5.24}$$

and

$$\|\mathscr{T}_N|\ M^{s_1,\alpha_1}_{p_1,q_1}\to M^{s_1}_{p_1,q_1}\|\sim\|\mathscr{T}_N|\ M^{s_2,\alpha_2}_{p_2,q_2}\to M^{s_2}_{p_2,q_2}\|\sim 1 \tag{5.25}$$

for all $N\in\mathbb{N}$. So

$$\|\mathscr{T}_N|\ M^{s_\theta}_{p_\theta,q_\theta}\to M^{s_\theta,\alpha}_{p_\theta,q}\|\lesssim 1 \tag{5.26}$$

and

$$\|\mathscr{T}_N|\ M^{s_\theta,\alpha}_{p_\theta,q}\to M^{s_\theta}_{p_\theta,q_\theta}\|\lesssim 1 \tag{5.27}$$

for all N. These inequalities imply

$$l^{q_\theta}\subset l^q \tag{5.28}$$

and

$$l^q\subset l^{q_\theta}. \tag{5.29}$$

However, the above two embedding relationships are true if and only if $q=q_\theta$.

Step 2: Dual argument. From the previous discussion, we know

$$p=p_\theta, q=q_\theta, s=s_\theta. \tag{5.30}$$

So we obtain

$$\|f\|_{M^{s_\theta,\alpha}_{p_\theta,q_\theta}}\lesssim\|f\|^{1-\theta}_{M^{s_1,\alpha_1}_{p_1,q_1}}\|f\|^{\theta}_{M^{s_2,\alpha_2}_{p_2,q_2}} \tag{5.31}$$

and

$$\|f\|_{M^{-s_\theta,\alpha}_{p'_\theta,q'_\theta}}\lesssim\|f\|^{1-\theta}_{M^{-s_1,\alpha_1}_{p'_1,q'_1}}\|f\|^{\theta}_{M^{-s_2,\alpha_2}_{p'_2,q'_2}} \tag{5.32}$$

for all $f\in\mathscr{S}$. We use Lemma 3.2 to deduce that

$$\begin{cases} p_1=q_1=2, & \text{if } \alpha>\alpha_1,\\ p_2=q_2=2, & \text{if } \alpha=\alpha_1,\\ p_\theta=q_\theta=2, & \text{if } \alpha<\alpha_1. \end{cases} \tag{5.33}$$

Step 3: Completion of proof for Theorem 1.1. The proof for $\alpha_1=\alpha_2$ is trivial. By symmetry of α_1 and α_2, it suffices to consider the case $\alpha_1<\alpha_2$. We divide this proof into several cases.

Case 1: $\alpha < \alpha_1$. We have $p_\theta = q_\theta = 2$, so

$$\left[M^{s_1,\alpha_1}_{p_1,q_1}, M^{s_2,\alpha_2}_{p_2,q_2}\right]_\theta = M^{s,\alpha}_{p_\theta,q_\theta} = M^{s,\alpha_1}_{2,2} = M^{s,\alpha_2}_{2,2}. \tag{5.34}$$

By the argument in the previous subsection, we have

$$p_1 = q_1 = p_2 = q_2 = 2. \tag{5.35}$$

Hence we have

$$\left[M^{s_1,\alpha_1}_{p_1,q_1}, M^{s_2,\alpha_2}_{p_2,q_2}\right]_\theta = \left[M^{s_1,\alpha_1}_{2,2}, M^{s_2,\alpha_2}_{2,2}\right]_\theta = \left[H^{s_1}, H^{s_2}\right]_\theta = H^{s_\theta}. \tag{5.36}$$

Case 2: $\alpha = \alpha_1$. We have $p_2 = q_2 = 2$. Thus

$$\left[M^{s_1,\alpha_1}_{p_1,q_1}, M^{s_2,\alpha_2}_{p_2,q_2}\right]_\theta = \left[M^{s_1,\alpha_1}_{p_1,q_1}, M^{s_2,\alpha_1}_{2,2}\right]_\theta = M^{s_\theta,\alpha_1}_{p_\theta,q_\theta}. \tag{5.37}$$

Case 3: $\alpha > \alpha_1$, $\alpha \neq \alpha_2$. We have $p_1 = q_1 = 2$. Thus

$$\left[M^{s_1,\alpha_1}_{p_1,q_1}, M^{s_2,\alpha_2}_{p_2,q_2}\right]_\theta = \left[M^{s_1,\alpha}_{2,2}, M^{s_2,\alpha_2}_{p_2,q_2}\right]_\theta. \tag{5.38}$$

Since

$$\left[M^{s_1,\alpha_1}_{p_1,q_1}, M^{s_2,\alpha_2}_{p_2,q_2}\right]_\theta = M^{s_\theta,\alpha}_{p_\theta,q_\theta},$$

we have

$$\left[M^{s_1,\alpha}_{2,2}, M^{s_2,\alpha_2}_{p_2,q_2}\right]_\theta = M^{s_\theta,\alpha}_{p_\theta,q_\theta}.$$

which is in Case 2, so we have

$$p_2 = q_2 = 2, \tag{5.39}$$

and

$$\left[M^{s_1,\alpha_1}_{p_1,q_1}, M^{s_2,\alpha_2}_{p_2,q_2}\right]_\theta = \left[M^{s_1,\alpha_1}_{2,2}, M^{s_2,\alpha_2}_{2,2}\right]_\theta = \left[H^{s_1}, H^{s_2}\right]_\theta = H^{s_\theta}. \tag{5.40}$$

Case 4: $\alpha = \alpha_2$. We have $p_1 = q_1 = 2$. Thus we obtain

$$\left[M^{s_1,\alpha_1}_{p_1,q_1}, M^{s_2,\alpha_2}_{p_2,q_2}\right]_\theta = \left[M^{s_1,\alpha_2}_{2,2}, M^{s_2,\alpha_2}_{p_2,q_2}\right]_\theta = M^{s_\theta,\alpha_2}_{p_\theta,q_\theta}. \tag{5.41}$$

5.2 Proof of Theorem 1.2

In this subsection, we suppose $0 < p, q \leq \infty$, $s_i \in \mathbb{R}$, $\alpha_i \in [0, 1]$ for $i = 1, 2$. For $\alpha_1 = \alpha_2$, the claim is trivial, so that we can assume $\alpha_1 \neq \alpha_2$. By symmetry, we can furthermore assume $\alpha_1 < \alpha_2$, which implies $\alpha_\theta < 1$.

For a fixed $\theta \in (0, 1)$, if

$$\left[M_{p,q}^{s_1,\alpha_1}, M_{p,q}^{s_2,\alpha_2}\right]_\theta \in \mathcal{M}, \tag{5.42}$$

then there exists a modulation space $M_{\tilde{p},\tilde{q}}^{s,\alpha}$ such that

$$\left[M_{p,q}^{s_1,\alpha_1}, M_{p,q}^{s_2,\alpha_2}\right]_\theta = M_{\tilde{p},\tilde{q}}^{s,\alpha}, \tag{5.43}$$

where $\tilde{p}, \tilde{q} \in (0, \infty]$, $s \in \mathbb{R}$, $\alpha \in [0, 1]$.

Take a smooth function h whose Fourier transform $\widehat{h}$ has small compact support with $\widehat{h}(\xi) = 1$ near the origin. Denote $T_h(f) = h * f$. One can easily verify that

$$T_h : M_{p,q}^{s_1,\alpha_1} \to M_{p,q}^{s_1,0}, \qquad T_h : M_{p,q}^{s_2,\alpha_2} \to M_{p,q}^{s_2,0}, \tag{5.44}$$

and

$$T_h : M_{p,q}^{s_1,0} \to M_{p,q}^{s_1,\alpha_1}, \qquad T_h : M_{p,q}^{s_2,0} \to M_{p,q}^{s_2,\alpha_2}. \tag{5.45}$$

By the operator interpolation inequality (Lemma 2.6), we deduce that

$$T_h : M_{\tilde{p},\tilde{q}}^{s,\alpha} \to M_{p,q}^{s_\theta,0} \quad \text{and} \quad T_h : M_{p,q}^{s_\theta,0} \to M_{\tilde{p},\tilde{q}}^{s,\alpha}. \tag{5.46}$$

Let g be a smooth function with compact support near the origin, $g_\lambda(\xi) = g(\frac{\xi}{\lambda})$. For sufficiently small λ, we have

$$T_h(g_\lambda) = g_\lambda, \ \|g_\lambda\|_{M_{\tilde{p},\tilde{q}}^{s,\alpha}} = \|g_\lambda\|_{L^{\tilde{p}}}, \ \|g_\lambda\|_{M_{p,q}^{s_\theta,0}} = \|g_\lambda\|_{L^p}. \tag{5.47}$$

Hence (5.46) implies

$$\|g_\lambda\|_{L^p} \lesssim \|g_\lambda\|_{L^{\tilde{p}}}, \ \|g_\lambda\|_{L^{\tilde{p}}} \lesssim \|g_\lambda\|_{L^p}. \tag{5.48}$$

Letting $\lambda \downarrow 0$, we conclude $1/p \le 1/\tilde{p}$ and $1/\tilde{p} \le 1/p$. So we have $\tilde{p} = p$.

In this subsection, since p might be smaller than 1, the dual convexity inequality (5.7) is replaced by

$$\|f\|_{M_{p',\tilde{q}'}^{-s+n\alpha(\frac{1}{1\wedge p}-1),\alpha}} \lesssim \|f\|^{1-\theta}_{M_{p',q'}^{-s_1+n\alpha_1(\frac{1}{1\wedge p}-1),\alpha_1}} \|f\|^{\theta}_{M_{p',q'}^{-s_2+n\alpha_2(\frac{1}{1\wedge p}-1),\alpha_2}}. \tag{5.49}$$

In the case that $p < 1$, the dual form of Lemma 3.2 is not applicable without an a priori estimate on α, even in the process of determining s. Additionally, by checking the proof, one can find that the method for obtaining priori estimates in the last subsection does not work in the case $p < 1$. The main difficulty is that we are not able to determine the values of $\tilde{q}, s, \alpha$ individually as we did in the last subsection. It seems that we need to handle all the indices simultaneously.

We denote

$$s_\theta = (1-\theta)s_1 + \theta s_2, \alpha_\theta = (1-\theta)\alpha_1 + \theta\alpha_2.$$

By Theorem 1.1, we only need to handle the case for $p < 1$ or $q < 1$. We divide the proof into three cases.

Case 1: $p < 1$, $\frac{1}{p} > \frac{1}{q}$.

By Lemma 4.1, we have

$$\begin{aligned}\|\Delta_j|\ B^0_{p,q} \to M^{s_1,\alpha_1}_{p,q}\| &\sim 2^{js_1}2^{jn(1-\alpha_1)(1/p+1/q-1)},\\ \|\Delta_j|\ B^0_{p,q} \to M^{s_2,\alpha_2}_{p,q}\| &\sim 2^{js_2}2^{jn(1-\alpha_2)(1/p+1/q-1)},\\ \|\Delta_j|\ B^0_{p,q} \to M^{s,\alpha}_{p,\tilde{q}}\| &\sim 2^{js}2^{jn(1-\alpha)(1/p+1/\tilde{q}-1)}.\end{aligned} \tag{5.50}$$

We now use Lemma 2.6 to deduce that

$$\|\Delta_j|\ B^0_{p,q} \to M^{s,\alpha}_{p,\tilde{q}}\| \lesssim \|\Delta_j|\ B^0_{p,q} \to M^{s_1,\alpha_1}_{p,q}\|^{1-\theta}\|\Delta_j|\ B^0_{p,q} \to M^{s_2,\alpha_2}_{p,q}\|^{\theta}, \tag{5.51}$$

which implies

$$2^{js}2^{jn(1-\alpha)(1/p+1/\tilde{q}-1)} \lesssim 2^{js_\theta}2^{jn(1-\alpha_\theta)(1/p+1/q-1)}. \tag{5.52}$$

Letting $j \to \infty$, we obtain

$$s + n(1-\alpha)(1/p + 1/\tilde{q} - 1) \le s_\theta + n(1-\alpha_\theta)(1/p + 1/q - 1). \tag{5.53}$$

On the other hand, we use Lemma 4.1 to deduce

$$\begin{aligned}\|\Delta_j|\ M^{s_1,\alpha_1}_{p,q} \to B^0_{p,q}\| &\sim 2^{-js_1}2^{jn(1-\alpha_1)(1/p-1/q)},\\ \|\Delta_j|\ M^{s_2,\alpha_2}_{p,q} \to B^0_{p,q}\| &\sim 2^{-js_2}2^{jn(1-\alpha_2)(1/p-1/q)},\end{aligned} \tag{5.54}$$

and

$$\|\Delta_j|\ M^{s,\alpha}_{p,\tilde{q}} \to B^0_{p,q}\| \sim \begin{cases} 2^{-js}2^{jn(1-\alpha)(1/p-1/\tilde{q})}, & \frac{1}{p} > \frac{1}{\tilde{q}},\\ 2^{-js}, & \frac{1}{p} \le \frac{1}{\tilde{q}}.\end{cases} \tag{5.55}$$

Using the operator interpolation inequality, one can deduce that

$$\|\Delta_j|\ M^{s,\alpha}_{p,\tilde{q}} \to B^0_{p,q}\| \lesssim \|\Delta_j|\ M^{s_1,\alpha_1}_{p,q} \to B^0_{p,q}\|^{1-\theta}\|\Delta_j|\ M^{s_2,\alpha_2}_{p,q} \to B^0_{p,q}\|^{\theta}, \tag{5.56}$$

which implies

$$-s + n(1-\alpha)(1/p - 1/\tilde{q}) \le -s_\theta + n(1-\alpha_\theta)(1/p - 1/q). \tag{5.57}$$

Addition of (5.53) and (5.57) yields

$$n(1-\alpha)(2/p - 1) \le n(1-\alpha_\theta)(2/p - 1). \tag{5.58}$$

So we have

$$\alpha \ge \alpha_\theta. \tag{5.59}$$

On the other hand, by the same method as in the last subsection (see the section "Proving $s = s_\theta$"), we can use

$$\|f\|_{M^{s,\alpha}_{p,\tilde{q}}} \lesssim \|f\|^{1-\theta}_{M^{s_1,\alpha_1}_{p,q}} \|f\|^{\theta}_{M^{s_2,\alpha_2}_{p,q}} \tag{5.60}$$

and

$$\|f\|_{M^{-s+n\alpha(\frac{1}{1\wedge p}-1),\alpha}_{p',\tilde{q}'}} \lesssim \|f\|^{1-\theta}_{M^{-s_1+n\alpha_1(\frac{1}{1\wedge p}-1),\alpha_1}_{p',q'}} \|f\|^{\theta}_{M^{-s_2+n\alpha_2(\frac{1}{1\wedge p}-1),\alpha_2}_{p',q'}} \tag{5.61}$$

to deduce

$$s \le s_\theta \tag{5.62}$$

and

$$-s + n\alpha(1/p - 1) \le -s_\theta + n\alpha_\theta(1/p - 1). \tag{5.63}$$

Adding the above two inequalities (5.62) and (5.63), we conclude

$$\alpha(1/p - 1) \le \alpha_\theta(1/p - 1) \tag{5.64}$$

which implies $\alpha \le \alpha_\theta$. So we have $\alpha = \alpha_\theta$. Putting $\alpha = \alpha_\theta$ into (5.63), we deduce $s \ge s_\theta$. So we have $s = s_\theta$.

Finally, we put $\alpha = \alpha_\theta$, $s = s_\theta$ into (5.53) and (5.57) and deduce $\tilde{q} = q$. Now, we have verified

$$\left[M^{s_1,\alpha_1}_{p,q}, M^{s_2,\alpha_2}_{p,q}\right]_\theta = M^{s_\theta,\alpha_\theta}_{p,q}. \tag{5.65}$$

Lemma 3.2 (together with $\alpha = \alpha_\theta \in (\alpha_1, \alpha_2)$) immediately yields $1/p \le 1/q$, which contradicts the assumption $1/p > 1/q$. We complete the proof for this case.

Case 2: $p < 1$, $\frac{1}{p} \le \frac{1}{q}$.

As in Case 1, one can deduce $s \le s_\theta$ and $\alpha \le \alpha_\theta$. Using Lemma 4.1, we have

$$\|\Delta_j | M^{s_1,\alpha_1}_{p,q} \to B^0_{p,q}\| \sim 2^{-js_1} \tag{5.66}$$

and

$$\|\Delta_j | M^{s_2,\alpha_2}_{p,q} \to B^0_{p,q}\| \sim 2^{-js_2}. \tag{5.67}$$

Then we use the operator interpolation inequality to deduce

$$\|\Delta_j | M^{s,\alpha}_{p,\tilde{q}} \to B^0_{p,q}\| \lesssim 2^{-js_\theta}. \tag{5.68}$$

However, Lemma 4.1 implies

$$\|\Delta_j | M^{s,\alpha}_{p,\tilde{q}} \to B^0_{p,q}\| \sim \begin{cases} 2^{-js}2^{jn(1-\alpha)(1/p-1/\tilde{q})}, & \frac{1}{p} > \frac{1}{\tilde{q}}, \\ 2^{-js}, & \frac{1}{p} \le \frac{1}{\tilde{q}}. \end{cases} \tag{5.69}$$

So, we have

$$-s \le -s + n(1-\alpha)\max\{0, 1/p - 1/\tilde{q}\} \le -s_\theta, \tag{5.70}$$

which implies $s \ge s_\theta$. Recalling the fact $s \le s_\theta$, we conclude $s = s_\theta$.

Additionally, by Corollary 4.3, one can deduce the embedding

$$M_{p,q}^{s_1,\alpha_1} \subset B_{p,q}^{s_1},\ M_{p,q}^{s_2,\alpha_2} \subset B_{p,q}^{s_2}. \tag{5.71}$$

Then, we use Lemma 2.6 to deduce

$$M_{p,\tilde{q}}^{s_\theta,\alpha} \subset B_{p,q}^{s_\theta}. \tag{5.72}$$

This implies

$$l^{\tilde{q}} \subset l^q, \tag{5.73}$$

and hence $1/q \le 1/\tilde{q}$.

On the other hand, we use Lemma 4.1 to deduce that

$$\begin{aligned} \|\Delta_j|\ B_{p,q}^0 \to M_{p,q}^{s_1,\alpha_1}\| &\sim 2^{js_1}2^{jn(1-\alpha_1)(1/p+1/q-1)}, \\ \|\Delta_j|\ B_{p,q}^0 \to M_{p,q}^{s_2,\alpha_2}\| &\sim 2^{js_2}2^{jn(1-\alpha_2)(1/p+1/q-1)}, \\ \|\Delta_j|\ B_{p,q}^0 \to M_{p,\tilde{q}}^{s_\theta,\alpha}\| &\sim 2^{js}2^{jn(1-\alpha)(1/p+1/\tilde{q}-1)}. \end{aligned} \tag{5.74}$$

As in Case 1, we use Lemma 2.6 (and the fact $s = s_\theta$ shown above) to deduce

$$s_\theta + n(1-\alpha)(1/p + 1/\tilde{q} - 1) \le s_\theta + n(1-\alpha_\theta)(1/p + 1/q - 1), \tag{5.75}$$

which implies

$$n(1-\alpha)(1/p + 1/\tilde{q} - 1) \le n(1-\alpha_\theta)(1/p + 1/q - 1). \tag{5.76}$$

Recalling $\alpha \le \alpha_\theta < 1$, one can deduce

$$n(1-\alpha)(1/p + 1/\tilde{q} - 1) \le n(1-\alpha)(1/p + 1/q - 1) \tag{5.77}$$

which implies $1/\tilde{q} \le 1/q$. So we have $\tilde{q} = q$. Putting $\tilde{q} = q$ into (5.76), we get

$$n(1-\alpha)(1/p + 1/q - 1) \le n(1-\alpha_\theta)(1/p + 1/q - 1) \tag{5.78}$$

which implies $\alpha \ge \alpha_\theta$. Recalling $\alpha \le \alpha_\theta$ again, we conclude $\alpha = \alpha_\theta$.

Now, we have verified that

$$\left[M_{p,q}^{s_1,\alpha_1}, M_{p,q}^{s_2,\alpha_2}\right]_\theta = M_{p,q}^{s_\theta,\alpha_\theta}. \tag{5.79}$$

We use Lemma 3.2 in the dual convexity inequality

$$\|f\|_{M_{p',q'}^{-s_\theta+n\alpha_\theta(\frac{1}{1\wedge p}-1),\alpha_\theta}} \lesssim \|f\|^{1-\theta}_{M_{p',q'}^{-s_1+n\alpha_1(\frac{1}{1\wedge p}-1),\alpha_1}} \|f\|^{\theta}_{M_{p',q'}^{-s_2+n\alpha_2(\frac{1}{1\wedge p}-1),\alpha_2}} \tag{5.80}$$

to deduce

$$\frac{1}{p'}+\frac{1}{q'}\geq 1, \tag{5.81}$$

which contradicts the identity $p'=q'=\infty$. This completes the proof for the present case.

Case 3: $p\geq 1, q<1$.

By the same method used in the proof of Theorem 1.1, one can verify the relationship

$$\tilde{p}=p,\ \tilde{q}=q,\ s_\theta=s. \tag{5.82}$$

Then we have

$$\left[M_{p,q}^{s_1,\alpha_1}, M_{p,q}^{s_2,\alpha_2}\right]_\theta = M_{p,q}^{s_\theta,\alpha}. \tag{5.83}$$

If $\alpha\leq\alpha_1$, we use Lemma 3.2 to deduce $p\leq q$, which contradicts to $q<1\leq p$.

If $\alpha>\alpha_1$, we use Lemma 3.2 in the dual convexity inequality

$$\|f\|_{M_{p',q'}^{-s_\theta+n\alpha(\frac{1}{1\wedge p}-1),\alpha}} \lesssim \|f\|^{1-\theta}_{M_{p',q'}^{-s_1+n\alpha_1(\frac{1}{1\wedge p}-1),\alpha_1}} \|f\|^{\theta}_{M_{p',q'}^{-s_2+n\alpha_2(\frac{1}{1\wedge p}-1),\alpha_2}} \tag{5.84}$$

to deduce

$$\frac{1}{p'}+\frac{1}{q'}\geq 1,\ \ \frac{1}{p'}\leq\frac{1}{q'}, \tag{5.85}$$

which also contradicts to the assumption in this case. This completes the proof. □

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