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Global Wave-Front Properties for Fourier Integral Operators and Hyperbolic Problems

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Abstract We illustrate the composition properties for an extended family of SG Fourier integral operators. We prove continuity results on modulation spaces, and study mapping properties of global wave-front sets for such operators. These extend classical results to more general situations. For example, there are no requirements on homogeneity for the phase functions. Finally, we apply our results to the study of the propagation of singularities, in the context of modulation spaces, for the solutions to the Cauchy problems for the corresponding linear hyperbolic operators.

Keywords Wave-front · Fourier integral operator · Banach · Modulation · Micro-local

Mathematics Subject Classification 35A18 · 35S30 · 42B05 · 35H10

1 Introduction

In [26], global wave-front sets with respect to convenient Banach or Fréchet spaces were introduced, and global mapping properties of pseudo-differential operators of SG-type were established in terms of these wave-front sets (see, e.g. [14,16,18,

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19,25–27,38,40]). For any such Banach or Fréchet space $\mathcal{B}$ and tempered distribution f, the global wave-front set $\mathrm{WF}_{\mathcal{B}}(f)$ is the union of three components $\mathrm{WF}^m_{\mathcal{B}}(f)$, $m = 1, 2, 3$. The first component (for $m = 1$) describes the local wave-front set which informs where f locally fails to belong to $\mathcal{B}$, as well as the directions where the singularities (with respect to $\mathcal{B}$) propagates. The second and third components (for $m = 2$ or $m = 3$) inform where at infinity the growth and oscillations of f are strong enough such that f fails to belong to $\mathcal{B}$. We remark that $\mathrm{WF}^1_{\mathscr{S}}(f)$, $\mathrm{WF}^2_{\mathscr{S}}(f)$ and $\mathrm{WF}^3_{\mathscr{S}}(f)$ agree with $\mathrm{WF}^{\psi}_{\mathscr{S}}(f)$, $\mathrm{WF}^{e}_{\mathscr{S}}(f)$ and $\mathrm{WF}^{\psi e}_{\mathscr{S}}(f)$, respectively, in [19]. Note also that for admissible $\mathcal{B}$, these wave-front sets give suitable information for local and global behavior, since f belongs to $\mathcal{B}$ globally (locally), if and only if $\mathrm{WF}_{\mathcal{B}}(f) = \emptyset$ ($\mathrm{WF}^1_{\mathcal{B}}(f) = \emptyset$).

It is convenient to formulate mapping properties for pseudo-differential operators of SG-type in terms of SG-ordered pairs $(\mathcal{B}, \mathcal{C})$, where $\mathcal{B}$ and $\mathcal{C}$ should be appropriate target and image spaces of the involved pseudo-differential operators. (Cf. [26].) More precisely, the pair $(\mathcal{B}, \mathcal{C})$ of spaces $\mathcal{B}$ and $\mathcal{C}$ containing $\mathscr{S}$ and contained in $\mathscr{S}'$, is called SG-ordered with respect to the weight ω_0 if the mappings

$$
\begin{aligned}
\mathrm{Op}(a) &: \mathcal{B} \to \mathcal{C}, \quad \mathrm{Op}(b)^* : \mathcal{C} \to \mathcal{B},\\
\mathrm{Op}(c) &: \mathcal{B} \to \mathcal{B} \quad \text{and} \quad \mathrm{Op}(c) : \mathcal{C} \to \mathcal{C}
\end{aligned}
\tag{1.1}
$$

are continuous for every $a \in \mathrm{SG}^{(\omega_0)}$, $b \in \mathrm{SG}^{(1/\omega_0)}$ and $c \in \mathrm{SG}^{0,0}$. If it is only required that the first mapping property in (1.1) holds, then the pair $(\mathcal{B}, \mathcal{C})$ is called *weakly* SG-*ordered*. Here $\mathrm{SG}^{(\omega)}$, the set of all SG-symbols with respect to ω, belongs to an extended family of symbol classes of SG-type. We refer to [19] for the definition of (also classical) SG-symbols. We notice that (1.1) is true also after $\mathrm{Op}(b)$ is replaced by its adjoint $\mathrm{Op}(b)^*$, because $\mathrm{Op}(\mathrm{SG}^{(\omega)})^* = \mathrm{Op}(\mathrm{SG}^{(\omega)})$.

Important examples on function and distribution spaces which give rise to SG-ordered pair are the Schwartz space, or the set of tempered distributions. An other important example appears when these spaces are suitable modulation spaces, a family of function and distribution spaces, introduced by Feichtinger in [29] and further developed in [30,31] by Feichtinger and Gröchenig. More precisely, in [26] it is noticed that $(\mathscr{S}, \mathscr{S})$ and $(\mathscr{S}', \mathscr{S}')$ are SG-ordered pairs, and for any weight ω and any modulation space $\mathcal{B}$, there is a (unique) modulation space $\mathcal{C}$ such that $(\mathcal{B}, \mathcal{C})$ is an SG-ordered pair with respect to ω. In particular, the family of SG-ordered pairs is broad in the sense that $\mathcal{B}$ can be chosen as a Sobolev space, or, more general, as a Sobolev–Kato space, since such spaces are special cases of modulation spaces. Moreover, if $\mathrm{SG}^{(\omega)}$ is a classical symbol class of SG-type and $\mathcal{B}$ is a Sobolev–Kato space, then $\mathcal{C}$ is also a Sobolev–Kato space.

For any SG-ordered pairs $(\mathcal{B}, \mathcal{C})$ with respect to ω, it is proved in [25,26] that the wave-front sets with respect to $\mathcal{B}$ and $\mathcal{C}$ posses convenient mapping properties. For example, if $f \in \mathscr{S}'$ and $a \in \mathrm{SG}^{(\omega)}$, then (1.1) is refined as

$$
\begin{aligned}
&\mathrm{WF}_{\mathcal{C}}(\mathrm{Op}(a) f) \subseteq \mathrm{WF}_{\mathcal{B}}(f),\\
\text{i.e.,}\quad &\mathrm{WF}^m_{\mathcal{C}}(\mathrm{Op}(a) f) \subseteq \mathrm{WF}^m_{\mathcal{B}}(f), \quad m = 1, 2, 3,
\end{aligned}
\tag{1.2}
$$

and that reversed inclusions are obtained by adding the set of characteristic points to the left-hand sides in (1.2). In particular, since the set of characteristic points is empty for elliptic operators, it follows that equalities are attained in (1.2) for such operators.

In this paper we establish similar properties for Fourier integral operators. More precisely, for any symbol a in $\mathrm{SG}^{(\omega)}$ for some weight ω, the Fourier integral operator (or FIO) $\mathrm{Op}_\varphi(a)$ is given by

$$f \mapsto (\mathrm{Op}_\varphi(a) f)(x) \equiv (2\pi)^{-d} \int_{\mathbf{R}^d} e^{i\varphi(x,\xi)} a(x,\xi) \widehat{f}(\xi)\, d\xi,$$

and its formal L^2-adjoint by

$$f \mapsto (\mathrm{Op}_\varphi(a)^* f)(x) \equiv (2\pi)^{-d} \iint_{\mathbf{R}^{2d}} e^{i(\langle x,\xi\rangle - \varphi(y,\xi))} \overline{a(y,\xi)} f(y)\, dy d\xi.$$

The operator $\mathrm{Op}^*_\varphi(a) = \mathrm{Op}_\varphi(a)^*$ is here called Fourier integral operator of type II, while $\mathrm{Op}_\varphi(a)$ is called a Fourier integral operator of type I, with phase function φ and amplitude (or symbol) a. The phase function φ should be in $\mathrm{SG}^{1,1}_{1,1}$ and satisfy

$$\langle \varphi'_x(x,\xi)\rangle \asymp \langle \xi \rangle \quad \text{and} \quad \langle \varphi'_\xi(x,\xi)\rangle \asymp \langle x \rangle. \tag{1.3}$$

Here and in what follows, $A \asymp B$ means that $A \lesssim B$ and $B \lesssim A$, where $A \lesssim B$ means that $A \le c \cdot B$, for a suitable constant $c > 0$. Furthermore, φ should also fulfill the usual (global) non-degeneracy condition

$$|\det(\varphi''_{x\xi}(x,\xi))| \ge c, \quad x, \xi \in \mathbf{R}^d,$$

for some constant $c > 0$.

In Sect. 4, the notion on SG-ordered pair from [26] is reformulated to include such Fourier integral operators, where the operators $\mathrm{Op}(a)$ and $\mathrm{Op}(b)^*$ in (1.1) are replaced by $\mathrm{Op}_\varphi(a)$ and $\mathrm{Op}_\varphi(b)^*$, respectively, and takes into account the phase-function φ.

In order to establish wave-front results, similar to (1.2), it is also required that the phase functions fulfill some further natural conditions, namely, that they preserve shapes in certain ways near the points in the phase space $T^*\mathbf{R}^d \simeq \mathbf{R}^{2d}$ (see Sect. 5). In fact, the definitions of wave-front sets of appropriate distributions are based on the behavior in *cones* of corresponding Fourier transformations, after localizing the involved distributions near points or along certain directions.

In order to explain our main results, let ϕ be the canonical transformation of $T^*\mathbf{R}^d$ generated by φ, and consider an elliptic Fourier integral operator $\mathrm{Op}_\varphi(a)$ with amplitude $a \in \mathrm{SG}^{(\omega_0)}$. If $(\mathcal{B}, \mathcal{C})$ are (weakly) SG-ordered with respect to ω_0 and φ (see Sect. 4 for precise definitions), then, under some natural *invariance conditions* on the weight ω_0,

$$\mathrm{WF}_{\mathcal{C}}(\mathrm{Op}_\varphi(a)f) = \phi(\mathrm{WF}_{\mathcal{B}}(f)). \tag{1.4}$$

A similar result holds for $\mathrm{Op}^*_\varphi(a)f$, namely

$$\mathrm{WF}_{\widetilde{\mathcal{B}}}(\mathrm{Op}^*_\varphi(a)f) = \phi^{-1}(\mathrm{WF}_{\widetilde{\mathcal{C}}}(f)), \tag{1.5}$$

when $\mathrm{Op}^*_\varphi(a)\colon \widetilde{\mathcal{C}} \to \widetilde{\mathcal{B}}$, with a (in general, different) couple of admissible spaces $\widetilde{\mathcal{C}}, \widetilde{\mathcal{B}}$, and the inverse ϕ^{-1} of the canonical transformation in (1.4). More generally, by dropping the ellipticity of the amplitude functions, with $a \in \mathrm{SG}^{(\omega_1)}$, $b \in \mathrm{SG}^{(\omega_2)}$, $(\mathcal{B}_1, \mathcal{C}_1, \mathcal{B}_2, \mathcal{C}_2)$ being SG-ordered with respect to ω_1, ω_2 and φ, we show that

$$\mathrm{WF}_{\mathcal{C}_1}(\mathrm{Op}_\varphi(a)f) \subseteq \phi(\mathrm{WF}_{\mathcal{B}_1}(f))^{\mathrm{con}} \tag{1.6}$$

and

$$\mathrm{WF}_{\mathcal{B}_2}(\mathrm{Op}^*_\varphi(b)f) \subseteq \phi^{-1}(\mathrm{WF}_{\mathcal{C}_2}(f))^{\mathrm{con}}, \tag{1.7}$$

provided the phase function φ additionally fulfills conditions similar to those in Kumano-Go [36]. In (1.6) and (1.7), we denoted by W^{con} the union of the smallest m-conical subsets which include the three components W_m, $m \in \{1, 2, 3\}$, of W (see [36] and Sect. 5 below).

Notice that the required conditions on the phase function are automatically satisfied by all the phase functions arising from the short-time solutions to hyperbolic Cauchy problems in the SG-classical context, see [14,15,17,18]. We then apply our results to describe the propagation of singularities from the initial data to the solutions to such SG-hyperbolic Cauchy problems.

The results above are based on comprehensive investigations of algebraic and continuity properties of the involved Fourier integral operators. A significant part of these investigations concern compositions between Fourier integral operators of type I or II, with pseudo-differential operators. This is performed in [24], where it is proved that for any Fourier integral operators $\mathrm{Op}_\varphi(a)$ and $\mathrm{Op}^*_\varphi(b)$ with $a, b \in \mathrm{SG}^{(\omega_1)}$, and some $p \in \mathrm{SG}^{(\omega_2)}$, then, under suitable invariance conditions on the weights,

$$\begin{aligned}
\mathrm{Op}(p) \circ \mathrm{Op}_\varphi(a) &= \mathrm{Op}_\varphi(c_1) \quad \mathrm{mod}\ \ \mathrm{Op}(\mathcal{B}_0),\\
\mathrm{Op}(p) \circ \mathrm{Op}^*_\varphi(b) &= \mathrm{Op}^*_\varphi(c_2) \quad \mathrm{mod}\ \ \mathrm{Op}(\mathcal{B}_0),\\
\mathrm{Op}_\varphi(a) \circ \mathrm{Op}(p) &= \mathrm{Op}_\varphi(c_3) \quad \mathrm{mod}\ \ \mathrm{Op}(\mathcal{B}_0)\\
\mathrm{Op}^*_\varphi(b) \circ \mathrm{Op}(p) &= \mathrm{Op}^*_\varphi(c_4) \quad \mathrm{mod}\ \ \mathrm{Op}(\mathcal{B}_0),
\end{aligned}$$

for some $c_j \in \mathrm{SG}^{(\omega_{0,j})}$, $j = 1, \ldots, 4$, and suitable weights $\omega_{0,j}$. Here $\mathrm{Op}(\mathcal{B}_0)$ is a set of appropriate smoothing operators, depending on the symbols and the phase function. Furthermore, if $a \in \mathrm{SG}^{(\omega_1)}$ and $b \in \mathrm{SG}^{(\omega_2)}$, then it is also proved that

$\mathrm{Op}_\varphi^*(b) \circ \mathrm{Op}_\varphi(a)$ and $\mathrm{Op}_\varphi(a) \circ \mathrm{Op}_\varphi^*(b)$ are equal to pseudo-differential operators $\mathrm{Op}(c_5)$ and $\mathrm{Op}(c_6)$, respectively, for some $c_5, c_6 \in \mathrm{SG}^{(\omega_{0,j})}$, $j = 5, 6$. We also present asymptotic formulae for c_j, $j = 1, \dots, 6$, in terms of a and b, or of a, b and p, modulo smoothing terms. The extensions of the calculus of SG Fourier integral operators developed in [16] to the classes $\mathrm{SG}^{(\omega_0)}_{r,\rho}$, introduced and systematically used in [25–27], is recalled in Sect. 3.

The formulae (1.4)–(1.7), given by the calculus recalled in Sect. 3, also rely on certain asymptotic expansions in the framework of symbolic calculus of SG pseudo-differential operators, as well as on continuity properties for SG-ordered pairs.

The first of the above two points concerns making sense of expansions of the form

$$a \sim \sum a_j,$$

in the framework of the generalised SG-classes $\mathrm{SG}^{(\omega_0)}_{r,\rho}$. The ideas are similar to the corresponding properties in the usual Hörmander calculus in Sect. 18.1 in [35]. For this reason, in [22] we have established properties of asymptotic expansions for symbols classes of the form $S(m, g)$, parameterized by the weight function m and Riemannian metric g on the phase space (cf. Sect. 18.4 in [35]). Note here that any SG-class is equal to $S(m, g)$ for some choice of m and g, and that similar facts hold for the Hörmander classes $S^r_{\rho,\delta}$. The results therefore cover several situations on asymptotic expansions for pseudo-differential operators.

With respect to the second point above, we study in Sect. 4 some specific spaces which are SG-ordered or weakly SG-ordered. For example, we present necessary and sufficient conditions for the involved weight functions and parameters, in order for Sobolev–Kato spaces, Sobolev spaces and modulation spaces should be SG-ordered or weakly SG-ordered. A direct proof of the continuity from $L^2(\mathbf{R}^d)$ to itself of SG Fourier integral operators with a uniformly bounded amplitude (that is, the amplitude is of order 0, 0, or, equivalently, the weight ω is bounded), similar to the one given in [16], can be found in [24]. Moreover, taking advantage of the calculus developed in [24], recalled in Sect. 3 for the convenience of the reader, and relying on results in [13,34], we prove that our classes of SG Fourier integral operators are continuous between suitable couples of weighted modulation spaces $(M^p_{(\omega_1)}(\mathbf{R}^d), M^p_{(\omega_2)}(\mathbf{R}^d))$.

Finally, in Sect. 5 we prove our main propagation results and illustrate their application to Cauchy problems, for SG-hyperbolic linear operators and first order systems with constant multiplicities. In view of the mapping properties proved in Sect. 4, we observe that such problems are "well-posed with a loss of regularity" when considered in the environment of Lebesgue and modulation spaces, differently from other known situations, see, e.g, Bényi et al. [2], Cordero and Nicola [12], Wang and Hudzik [45] and the references quoted therein.

2 Preliminaries

We begin by fixing the notation and recalling some basic concepts which will be needed below. In Sects. 2.1–2.4 we mainly summarize parts of the contents of Sect. 2 in [24,26,27]. Some of the results that we recall, compared with their original formulation in the SG context appeared in [16], are here given in a slightly more general form, adapted to the definitions given in Sect. 2.3.

2.1 Weight Functions

Let ω and v be positive measurable functions on $\mathbf{R}^d$. Then ω is called v-moderate if

$$\omega(x+y) \lesssim \omega(x)v(y) \tag{2.1}$$

If v in (2.1) can be chosen as a polynomial, then ω is called a function or weight of *polynomial type*. We let $\mathscr{P}(\mathbf{R}^d)$ be the set of all polynomial type functions on $\mathbf{R}^d$. If $\omega(x,\xi) \in \mathscr{P}(\mathbf{R}^{2d})$ is constant with respect to the x-variable or the ξ-variable, then we sometimes write $\omega(\xi)$, respectively $\omega(x)$, instead of $\omega(x,\xi)$, and consider ω as an element in $\mathscr{P}(\mathbf{R}^{2d})$ or in $\mathscr{P}(\mathbf{R}^d)$ depending on the situation. We say that v is submultiplicative if (2.1) holds for $\omega = v$. For convenience we assume that all submultiplicative weights are even, and v and v_j always stand for submultiplicative weights, if nothing else is stated.

Without loss of generality we may assume that every $\omega \in \mathscr{P}(\mathbf{R}^d)$ is smooth and satisfies the ellipticity condition $\partial^\alpha \omega/\omega \in L^\infty$. In fact, by Lemma 1.2 in [41] it follows that for each $\omega \in \mathscr{P}(\mathbf{R}^d)$, there is a smooth and elliptic $\omega_0 \in \mathscr{P}(\mathbf{R}^d)$ which is equivalent to ω in the sense

$$\omega \asymp \omega_0. \tag{2.2}$$

The weights involved in the sequel often have to satisfy additional conditions. More precisely let $r, \rho \geq 0$. Then $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ is the set of all $\omega(x,\xi)$ in $\mathscr{P}(\mathbf{R}^{2d}) \bigcap C^\infty(\mathbf{R}^{2d})$ such that

$$\langle x\rangle^{r|\alpha|}\langle \xi\rangle^{\rho|\beta|}\frac{\partial_x^\alpha \partial_\xi^\beta \omega(x,\xi)}{\omega(x,\xi)} \in L^\infty(\mathbf{R}^{2d}), \tag{2.3}$$

for every multi-indices α and β. Any weight $\omega \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ is called SG-moderate on $\mathbf{R}^{2d}$, of order r and ρ. Notice that $\mathscr{P}_{r,\rho}$ is different here compared to [25], and there are elements in $\mathscr{P}(\mathbf{R}^{2d})$ which have no equivalent elements in $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$. On the other hand, if $s, t \in \mathbf{R}$ and $r, \rho \in [0,1]$, then $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ contains all weights of the form

$$\vartheta_{m,\mu}(x,\xi) \equiv \langle x\rangle^m \langle \xi\rangle^\mu, \tag{2.4}$$

which are one of the most common type of weights.

It will also be useful to consider SG-moderate weights in one or three sets of variables. Let $\omega \in \mathscr{P}(\mathbf{R}^{3d}) \bigcap C^\infty(\mathbf{R}^{3d})$, and let $r_1, r_2, \rho \geq 0$. Then ω is called SG moderate on $\mathbf{R}^{3d}$, of order r_1, r_2 and ρ, if it fulfills

$$\langle x_1\rangle^{r_1|\alpha_1|}\langle x_2\rangle^{r_2|\alpha_2|}\langle \xi\rangle^{\rho|\beta|}\frac{\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_\xi^\beta \omega(x_1,x_2,\xi)}{\omega(x_1,x_2,\xi)} \in L^\infty(\mathbf{R}^{3d}).$$

The set of all SG-moderate weights on $\mathbf{R}^{3d}$ of order r_1, r_2 and ρ is denoted by $\mathscr{P}_{r_1,r_2,\rho}(\mathbf{R}^{3d})$. Finally, we denote by $\mathscr{P}_r(\mathbf{R}^d)$ the set of all SG-moderate weights of order $r \geq 0$ on $\mathbf{R}^d$, which are defined in a similar fashion.

2.2 Modulation Spaces

Let $\phi \in \mathscr{S}(\mathbf{R}^d)$. Then the *short-time Fourier transform* of $f \in \mathscr{S}(\mathbf{R}^d)$ with respect to (the window function) ϕ is defined by

$$V_\phi f(x,\xi) = (2\pi)^{-d/2}\int_{\mathbf{R}^d} f(y)\overline{\phi(y-x)}e^{-i\langle y,\xi\rangle}\,dy. \tag{2.5}$$

More generally, the short-time Fourier transform of $f \in \mathscr{S}'(\mathbf{R}^d)$ with respect to $\phi \in \mathscr{S}'(\mathbf{R}^d)$ is defined by

$$(V_\phi f) = \mathscr{F}_2 F, \quad \text{where} \quad F(x,y) = (f\otimes\overline{\phi})(y, y-x). \tag{2.5$'$}$$

Here $\mathscr{F}_2 F$ is the partial Fourier transform of $F(x,y) \in \mathscr{S}'(\mathbf{R}^{2d})$ with respect to the y-variable, and the Fourier transform $\mathscr{F}$ is the linear and continuous map on $\mathscr{S}'(\mathbf{R}^d)$ which takes the form

$$(\mathscr{F}f)(\xi) = \hat{f}(\xi) \equiv \int_{\mathbf{R}^d} f(x)e^{-i\langle x,\xi\rangle}dx$$

when $f \in L^1(\mathbf{R}^d)$. We refer to [32,33] for more facts about the short-time Fourier transform. To introduce the modulation spaces, we first recall that a Banach space $\mathscr{B}$, continuously embedded in $L^1_{\mathrm{loc}}(\mathbf{R}^d)$, is called a *(translation) invariant BF-space on* $\mathbf{R}^d$, with respect to a submultiplicative weight $v \in \mathscr{P}(\mathbf{R}^d)$, if there is a constant C such that the following conditions are fulfilled:

(1) $\mathscr{S}(\mathbf{R}^d) \subseteq \mathscr{B} \subseteq \mathscr{S}'(\mathbf{R}^d)$ (continuous embeddings);
(2) if $x \in \mathbf{R}^d$ and $f \in \mathscr{B}$, then $f(\cdot - x) \in \mathscr{B}$, and

$$\|f(\cdot - x)\|_{\mathscr{B}} \leq Cv(x)\|f\|_{\mathscr{B}}; \tag{2.6}$$

(3) if $f, g \in L^1_{\mathrm{loc}}(\mathbf{R}^d)$ satisfy $g \in \mathscr{B}$ and $|f| \leq |g|$ almost everywhere, then $f \in \mathscr{B}$ and

$$\|f\|_{\mathscr{B}} \leq C\|g\|_{\mathscr{B}};$$

(4) if $f \in \mathscr{B}$ and $\varphi \in C_0^\infty(\mathbf{R}^d)$, then $f * \varphi \in \mathscr{B}$, and

$$\|f * \varphi\|_{\mathscr{B}} \le \|\varphi\|_{L^1_{(v)}} \|f\|_{\mathscr{B}}. \tag{2.7}$$

The following definition of modulation spaces is due to Feichtinger [30]. Let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$ with respect to $v \in \mathscr{P}(\mathbf{R}^{2d})$, $\phi \in \mathscr{S}(\mathbf{R}^d)\backslash 0$ and let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ be such that ω is v-moderate. The *modulation space* $M(\omega, \mathscr{B})$ consists of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that $V_\phi f \cdot \omega \in \mathscr{B}$. We notice that $M(\omega, \mathscr{B})$ is a Banach space with the norm

$$\|f\|_{M(\omega,\mathscr{B})} \equiv \|V_\phi f \omega\|_{\mathscr{B}} \tag{2.8}$$

(cf. [31]).

Remark 2.1 Assume that $p, q \in [1, \infty]$, and let $L_1^{p,q}(\mathbf{R}^{2d})$ and $L_2^{p,q}(\mathbf{R}^{2d})$ be the sets of all $F \in L^1_{\rm loc}(\mathbf{R}^{2d})$ such that

$$\|F\|_{L_1^{p,q}} \equiv \left(\int \left(\int |F(x,\xi)|^p \, dx \right)^{q/p} d\xi \right)^{1/q} < \infty$$

and

$$\|F\|_{L_2^{p,q}} \equiv \left(\int \left(\int |F(x,\xi)|^q \, d\xi \right)^{p/q} dx \right)^{1/p} < \infty$$

(with obvious modifications when $p = \infty$ or $q = \infty$). Then $M(\omega, L_1^{p,q}(\mathbf{R}^{2d}))$ is equal to the *classical* modulation space $M^{p,q}_{(\omega)}(\mathbf{R}^d)$, and $M(\omega, L_2^{p,q}(\mathbf{R}^{2d}))$ is equal to the space $W^{p,q}_{(\omega)}(\mathbf{R}^d)$, related to Wiener-amalgam spaces (cf. [29–31,33]). We set $M^p_{(\omega)} = M^{p,p}_{(\omega)} = W^{p,p}_{(\omega)}$. Furthermore, if $\omega = 1$, then we write $M^{p,q}$, M^p and $W^{p,q}$ instead of $M^{p,q}_{(\omega)}$, $M^p_{(\omega)}$ and $W^{p,q}_{(\omega)}$ respectively.

Remark 2.2 Several important spaces agree with certain modulation spaces. In fact, let $s, \sigma \in \mathbf{R}$. If $\omega = \vartheta_{s,\sigma}$ (cf. (2.4)), then $M^2_{(\omega)}(\mathbf{R}^d)$ is equal to the weighted Sobolev space (or Sobolev–Kato space) $H^2_{\sigma,s}(\mathbf{R}^d)$ in [19,38], the set of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that $\langle x\rangle^s \langle D\rangle^\sigma f \in L^2(\mathbf{R}^d)$. In particular, if $s = 0$ ($\sigma = 0$), then $M^2_{(\omega)}(\mathbf{R}^d)$ equals to $H^2_\sigma(\mathbf{R}^d)$ ($L^2_s(\mathbf{R}^d)$). Furthermore, if instead $\omega(x,\xi) = \langle x, \xi\rangle^s \equiv \langle (x,\xi)\rangle^s$, then $M^2_{(\omega)}(\mathbf{R}^d)$ is equal to the Sobolev–Shubin space of order s. (Cf. e.g. [37]).

2.3 Pseudo-differential Operators and SG Symbol Classes

Let $a \in \mathscr{S}(\mathbf{R}^{2d})$, and $t \in \mathbf{R}$ be fixed. Then the pseudo-differential operator $\mathrm{Op}_t(a)$ is the linear and continuous operator on $\mathscr{S}(\mathbf{R}^d)$ defined by the formula

$$(\mathrm{Op}_t(a) f)(x) = (2\pi)^{-d} \iint e^{i\langle x-y,\xi\rangle} a((1-t)x + ty, \xi) f(y) \, dy d\xi \tag{2.9}$$

(cf. Chap. XVIII in [35]). For general $a \in \mathscr{S}'(\mathbf{R}^{2d})$, the pseudo-differential operator $\operatorname{Op}_t(a)$ is defined as the continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ with distribution kernel

$$K_{t,a}(x,y) = (2\pi)^{-d/2}(\mathscr{F}_2^{-1}a)((1-t)x+ty, x-y). \tag{2.10}$$

If $t=0$, then $\operatorname{Op}_t(a)$ is the Kohn–Nirenberg representation $\operatorname{Op}(a) = a(x,D)$, and if $t = 1/2$, then $\operatorname{Op}_t(a)$ is the Weyl quantization.

In most of our situations, a belongs to a generalized SG-symbol class, which we shall consider now. Let $m, \mu, r, \rho \in \mathbf{R}$ be fixed. Then the SG-class $\mathrm{SG}^{m,\mu}_{r,\rho}(\mathbf{R}^{2d})$ is the set of all $a \in C^\infty(\mathbf{R}^{2d})$ such that

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \lesssim \langle x\rangle^{m-r|\alpha|}\langle \xi\rangle^{\mu-\rho|\beta|},$$

for all multi-indices α and β. Usually we assume that $r, \rho \ge 0$ and $\rho + r > 0$.

More generally, assume that $\omega \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$. Then $\mathrm{SG}^{(\omega)}_{r,\rho}(\mathbf{R}^{2d})$ consists of all $a \in C^\infty(\mathbf{R}^{2d})$ such that

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \lesssim \omega(x,\xi)\langle x\rangle^{-r|\alpha|}\langle \xi\rangle^{-\rho|\beta|}, \quad x,\xi \in \mathbf{R}^d, \tag{2.11}$$

for all multi-indices α and β. We notice that

$$\mathrm{SG}^{(\omega)}_{r,\rho}(\mathbf{R}^{2d}) = S(\omega, g_{r,\rho}), \tag{2.12}$$

when $g = g_{r,\rho}$ is the Riemannian metric on $\mathbf{R}^{2d}$, defined by the formula

$$\big(g_{r,\rho}\big)_{(y,\eta)}(x,\xi) = \langle y\rangle^{-2r}|x|^2 + \langle \eta\rangle^{-2\rho}|\xi|^2 \tag{2.13}$$

(cf. Sect. 18.4–18.6 in [35]). Furthermore, $\mathrm{SG}^{(\omega)}_{r,\rho} = \mathrm{SG}^{m,\mu}_{r,\rho}$ when $\omega = \vartheta_{m,\mu}$ (see (2.4)).

For conveniency we set

$$\mathrm{SG}^{(\omega\vartheta_{-\infty,0})}_{\rho}(\mathbf{R}^{2d}) = \mathrm{SG}^{(\omega\vartheta_{-\infty,0})}_{r,\rho}(\mathbf{R}^{2d}) \equiv \bigcap_{N\ge 0} \mathrm{SG}^{(\omega\vartheta_{-N,0})}_{r,\rho}(\mathbf{R}^{2d}),$$
$$\mathrm{SG}^{(\omega\vartheta_{0,-\infty})}_{r}(\mathbf{R}^{2d}) = \mathrm{SG}^{(\omega\vartheta_{0,-\infty})}_{r,\rho}(\mathbf{R}^{2d}) \equiv \bigcap_{N\ge 0} \mathrm{SG}^{(\omega\vartheta_{0,-N})}_{r,\rho}(\mathbf{R}^{2d}),$$

and

$$\mathrm{SG}^{(\omega\vartheta_{-\infty,-\infty})}(\mathbf{R}^{2d}) = \mathrm{SG}^{(\omega\vartheta_{-\infty,-\infty})}_{r,\rho}(\mathbf{R}^{2d}) \equiv \bigcap_{N\ge 0} \mathrm{SG}^{(\omega\vartheta_{-N,-N})}_{r,\rho}(\mathbf{R}^{2d}).$$

We observe that $\mathrm{SG}^{(\omega\vartheta_{-\infty,0})}_{r,\rho}(\mathbf{R}^{2d})$ is independent of r, $\mathrm{SG}^{(\omega\vartheta_{0,-\infty})}_{r,\rho}(\mathbf{R}^{2d})$ is independent of ρ, and that $\mathrm{SG}^{(\omega\vartheta_{-\infty,-\infty})}_{r,\rho}(\mathbf{R}^{2d})$ is independent of both r and ρ. Furthermore, for any $x_0, \xi_0 \in \mathbf{R}^d$ we have

$$\begin{aligned}\mathrm{SG}_{\rho}^{(\omega\vartheta_{-\infty,0})}(\mathbf{R}^{2d}) &= \mathrm{SG}_{\rho}^{(\omega_0\vartheta_{-\infty,0})}(\mathbf{R}^{2d}), \quad \text{when} \quad \omega_0(\xi) = \omega(x_0,\xi),\\ \mathrm{SG}_{r}^{(\omega\vartheta_{0,-\infty})}(\mathbf{R}^{2d}) &= \mathrm{SG}_{r}^{(\omega_0\vartheta_{0,-\infty})}(\mathbf{R}^{2d}), \quad \text{when} \quad \omega_0(x) = \omega(x,\xi_0),\end{aligned}$$

and

$$\mathrm{SG}^{(\omega\vartheta_{-\infty,-\infty})}(\mathbf{R}^{2d}) = \mathscr{S}(\mathbf{R}^{2d}).$$

The following result shows that the concept of asymptotic expansion extends to the classes $\mathrm{SG}_{r,\rho}^{(\omega)}(\mathbf{R}^{2d})$. We refer to [22, Theorem 8] for the proof.

Proposition 2.3 *Let $r,\rho \ge 0$ satisfy $r+\rho > 0$, and let $\{s_j\}_{j\ge 0}$ and $\{\sigma_j\}_{j\ge 0}$ be sequences of non-positive numbers such that $\lim_{j\to\infty} s_j = -\infty$ when $r > 0$ and $s_j = 0$ otherwise, and $\lim_{j\to\infty}\sigma_j = -\infty$ when $\rho > 0$ and $\sigma_j = 0$ otherwise. Also let $a_j \in \mathrm{SG}_{r,\rho}^{(\omega_j)}(\mathbf{R}^{2d})$, $j = 0, 1, \ldots$, where $\omega_j = \omega\cdot\vartheta_{s_j,\sigma_j}$. Then there is a symbol $a \in \mathrm{SG}_{r,\rho}^{(\omega)}(\mathbf{R}^{2d})$ such that*

$$a - \sum_{j=0}^{N} a_j \in \mathrm{SG}_{r,\rho}^{(\omega_{N+1})}(\mathbf{R}^{2d}). \tag{2.14}$$

The symbol a is uniquely determined modulo a remainder h, where

$$\begin{aligned} h &\in \mathrm{SG}_{\rho}^{(\omega\vartheta_{-\infty,0})}(\mathbf{R}^{2d}) \quad \textit{when} \quad r > 0,\\ h &\in \mathrm{SG}_{r}^{(\omega\vartheta_{0,-\infty})}(\mathbf{R}^{2d}) \quad \textit{when} \quad \rho > 0,\\ h &\in \mathscr{S}(\mathbf{R}^{2d}) \quad \textit{when} \quad r > 0, \rho > 0. \end{aligned} \tag{2.15}$$

Definition 2.4 The notation $a \sim \sum a_j$ is used when a and a_j fulfill the hypothesis in Proposition 2.3. Furthermore, the formal sum

$$\sum_{j\ge 0} a_j$$

is called an *asymptotic expansion*.

It is a well-known fact that SG-operators give rise to linear continuous mappings from $\mathscr{S}(\mathbf{R}^d)$ to itself, extendable as linear continuous mappings from $\mathscr{S}'(\mathbf{R}^d)$ to itself. They also act continuously between modulation spaces, and in some situations between suitable Sobolev spaces $H_s^p(\mathbf{R}^d)$ and Lebesgue spaces $L_t^p(\mathbf{R}^d)$. Here $H_\sigma^p(\mathbf{R}^d)$ consists of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that $\langle D\rangle^\sigma f \in L^p(\mathbf{R}^d)$, and $L_s^p(\mathbf{R}^d)$ consists of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that $\langle\,\cdot\,\rangle^s f \in L^p(\mathbf{R}^d)$. We also define $H_{s,\sigma}^p(\mathbf{R}^d)$ as the set of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that $\langle\,\cdot\,\rangle^s\langle D\rangle^\sigma f \in L^p(\mathbf{R}^d)$. Indeed, in the first one of the following propositions, the first part is a special case of [44, Theorem 3.2], and the second part follows from [8, Corollary 6]. (See also [26] for the first part and the proof of [34, Theorem 3.1] for the second part.) The second proposition follows from [46, Theorem 10.7] or [28, Theorem 1.1]. The proofs are therefore omitted.

Proposition 2.5 *Let $r, \rho \geq 0$, $t \in \mathbf{R}$ and $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$. Then the following is true:*

(1) *if $a \in \mathrm{SG}_{r,\rho}^{(\omega_0)}(\mathbf{R}^{2d})$, then $\mathrm{Op}_t(a)$ is continuous from $M(\omega, \mathscr{B})$ to $M(\omega/\omega_0, \mathscr{B})$, for every choice of $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ and every translation invariant BF-space $\mathscr{B}$ on $\mathbf{R}^{2d}$;*
(2) *there exist $a \in \mathrm{SG}_{r,\rho}^{(\omega_0)}(\mathbf{R}^{2d})$ and $b \in \mathrm{SG}_{r,\rho}^{(1/\omega_0)}(\mathbf{R}^{2d})$ such that for every choice of $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ and every translation invariant BF-space $\mathscr{B}$ on $\mathbf{R}^{2d}$, the mappings*

$$\mathrm{Op}_t(a) : \mathscr{S}(\mathbf{R}^d) \to \mathscr{S}(\mathbf{R}^d), \quad \mathrm{Op}_t(a) : \mathscr{S}'(\mathbf{R}^d) \to \mathscr{S}'(\mathbf{R}^d)$$

and

$$\mathrm{Op}_t(a) : M(\omega, \mathscr{B}) \to M(\omega/\omega_0, \mathscr{B}).$$

are continuous bijections with inverses $\mathrm{Op}_t(b)$.

Proposition 2.6 *Let $r, \rho > 0$, $t \in \mathbf{R}$, $p \in (1, \infty)$ and $s, \sigma \in \mathbf{R}$. Then the following is true:*

(1) *if $\mu \in \mathbf{R}$ and $a \in \mathrm{SG}_{0,\rho}^{0,\mu}(\mathbf{R}^{2d})$, then $\mathrm{Op}_t(a)$ is continuous from $H_\sigma^p(\mathbf{R}^d)$ to $H_{\sigma-\mu}^p(\mathbf{R}^d)$;*
(2) *if $m \in \mathbf{R}$ and $a \in \mathrm{SG}_{r,0}^{m,0}(\mathbf{R}^{2d})$, then $\mathrm{Op}_t(a)$ is continuous from $L_s^p(\mathbf{R}^d)$ to $L_{s-m}^p(\mathbf{R}^d)$;*
(3) *if $m, \mu \in \mathbf{R}$ and $a \in \mathrm{SG}_{r,\rho}^{m,\mu}(\mathbf{R}^{2d})$, then $\mathrm{Op}_t(a)$ is continuous from $H_{s,\sigma}^p(\mathbf{R}^d)$ to $H_{s-m,\sigma-\mu}^p(\mathbf{R}^d)$.*

2.4 Composition and Further Properties of SG Classes of Symbols, Amplitudes, and Functions

We define families of *smooth functions with* SG *behaviour*, depending on one, two or three sets of real variables (cfr. also [21]). We then introduce pseudo-differential operators defined by means of SG amplitudes. Subsequently, we recall sufficient conditions for maps of $\mathbf{R}^d$ into itself to keep the invariance of the SG classes.

In analogy of SG amplitudes defined on $\mathbf{R}^{2d}$, we consider corresponding classes of amplitudes defined on $\mathbf{R}^{3d}$. More precisely, for any $m_1, m_2, \mu, r_1, r_2, \rho \in \mathbf{R}$, let $\mathrm{SG}_{r_1,r_2,\rho}^{m_1,m_2,\mu}(\mathbf{R}^{3n})$ be the set of all $a \in C^\infty\left(\mathbf{R}^{3d}\right)$ such that

$$|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_\xi^\beta a(x_1, x_2, \xi)| \lesssim \langle x_1\rangle^{m_1-r_1|\alpha_1|}\langle x_2\rangle^{m_2-r_2|\alpha_2|}\langle\xi\rangle^{\mu-\rho|\beta|}, \tag{2.16}$$

for every multi-indices $\alpha_1, \alpha_2, \beta$. We usually assume $r_1, r_2, \rho \geq 0$ and $r_1+r_2+\rho > 0$. More generally, let $\omega \in \mathscr{P}_{r_1,r_2,\rho}(\mathbf{R}^{3d})$. Then $\mathrm{SG}_{r_1,r_2,\rho}^{(\omega)}(\mathbf{R}^{3d})$ is the set of all $a \in C^\infty\left(\mathbf{R}^{3d}\right)$ which satisfy

$$|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_\xi^\beta a(x, y, \xi)| \lesssim \omega(x_1, x_2, \xi)\langle x_1\rangle^{-r_1|\alpha_1|}\langle x_2\rangle^{-r_2|\alpha_2|}\langle\xi\rangle^{-\rho|\beta|}, \tag{2.16′}$$

for every multi-indices $\alpha_1, \alpha_2, \beta$. The set $\mathrm{SG}^{(\omega)}_{r_1,r_2,\rho}(\mathbf{R}^{3n})$ is equipped with the usual Fréchet topology based upon the seminorms implicitly given in (2.16)$'$.

As above,

$$\mathrm{SG}^{(\omega)}_{r_1,r_2,\rho} = \mathrm{SG}^{m_1,m_2,\mu}_{r_1,r_2,\rho} \quad \text{when} \quad \omega(x_1, x_2, \xi) = \langle x_1\rangle^{m_1}\langle x_2\rangle^{m_2}\langle \xi\rangle^{\mu}.$$

Definition 2.7 Let $r_1, r_2, \rho \geq 0$, $r_1 + r_2 + \rho > 0$, and let $a \in \mathrm{SG}^{(\omega)}_{r_1,r_2,\rho}(\mathbf{R}^{3d})$, where $\omega \in \mathscr{P}_{r_1,r_2,\rho}(\mathbf{R}^{3d})$. Then the pseudo-differential operator $\mathrm{Op}(a)$ is the linear and continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ with distribution kernel

$$K_a(x, y) = (2\pi)^{-d/2}(\mathscr{F}_3^{-1}a)(x, y, x - y).$$

For $f \in \mathscr{S}(\mathbf{R}^d)$, we have

$$(\mathrm{Op}(a)f)(x) = (2\pi)^{-d} \iint e^{i\langle x-y,\xi\rangle} a(x, y, \xi) f(y)\, dyd\xi.$$

The operators introduced in Definition 2.7 have properties analogous to the usual SG operator families described in [14]. They coincide with the operators defined in the previous subsection, where corresponding symbols are obtained by means of asymptotic expansions, modulo remainders of the type given in (2.4). For the sake of brevity, we omit the details. Evidently, when neither the amplitude functions a, nor the corresponding weight ω, depend on x_2, we obtain the definition of SG symbols and pseudo-differential operators, given in the previous subsection.

Next we consider SG *functions*, also called *functions with* SG *behavior*. That is, amplitudes which depend only on one set of variables in $\mathbf{R}^d$. We denote them by $\mathrm{SG}^{(\omega)}_r(\mathbf{R}^d)$ and $\mathrm{SG}^m_r(\mathbf{R}^d)$, $r > 0$, respectively, for a general weight $\omega \in \mathscr{P}_r(\mathbf{R}^d)$ and for $\omega(x) = \langle x\rangle^m$. Furthermore, if $\phi\colon \mathbf{R}^{d_1} \to \mathbf{R}^{d_2}$, and each component ϕ_j, $j = 1, \dots, d_2$, of ϕ belongs to $\mathrm{SG}^{(\omega)}_r(\mathbf{R}^{d_1})$, we will occasionally write $\phi \in \mathrm{SG}^{(\omega)}_r(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$. We use similar notation also for other vector-valued SG symbols and amplitudes.

In the sequel we need to consider compositions of SG amplitudes with functions with SG behavior. In particular, the latter will often be SG maps (or diffeomorphisms) with SG^0-parameter dependence, generated by phase functions (introduced in [16]), see Definitions 2.8 and 2.9, and Sect. 3.1 below. For the convenience of the reader, we first recall, in a form slightly more general than the one adopted in [16], the definition of SG diffeomorphisms with SG^0-parameter dependence.

Definition 2.8 Let $\Omega_j \subseteq \mathbf{R}^{d_j}$ be open, $\Omega = \Omega_1 \times \cdots \times \Omega_k$ and let $\phi \in C^\infty(\mathbf{R}^d \times \Omega; \mathbf{R}^d)$. Then ϕ is called an SG map (with SG^0-parameter dependence) when the following conditions hold:

(1) $\langle \phi(x, \eta)\rangle \asymp \langle x\rangle$, uniformly with respect to $\eta \in \Omega$;

(2) for all $\alpha \in \mathbf{Z}^d_+$, $\beta = (\beta_1, \dots, \beta_k)$, $\beta_j \in \mathbf{Z}^{d_j}_+$, $j = 1, \dots, k$, and any $(x, \eta) \in \mathbf{R}^d \times \Omega$,

$$|\partial^\alpha_x \partial^{\beta_1}_{\eta_1} \cdots \partial^{\beta_k}_{\eta_k} \phi(x, \eta)| \lesssim \langle x\rangle^{1-|\alpha|}\langle \eta_1\rangle^{-|\beta_1|} \cdots \langle \eta_k\rangle^{-|\beta_k|},$$

where $\eta = (\eta_1, \ldots, \eta_k)$ and $\eta_j \in \Omega_j$ for every j.

Definition 2.9 Let $\phi \in C^\infty(\mathbf{R}^d \times \Omega; \mathbf{R}^d)$ be an SG map. Then ϕ is called an SG diffeomorphism (with SG^0-parameter dependence) when there is a constant $\varepsilon > 0$ such that

$$|\det \phi'_x(x, \eta)| \geq \varepsilon, \tag{2.17}$$

uniformly with respect to $\eta \in \Omega$.

Remark 2.10 The condition (1) in Definition 2.8 and (2.17), together with abstract results (see, e.g., [3], page 221) and the inverse function theorem, imply that, for any $\eta \in \Omega$, an SG diffeomorphism $\phi(\,\cdot\,, \eta)$ is a smooth, global bijection from $\mathbf{R}^d$ to itself with smooth inverse $\psi(\,\cdot\,, \eta) = \phi^{-1}(\,\cdot\,, \eta)$. It can be proved that also the inverse mapping $\psi(y, \eta) = \phi^{-1}(y, \eta)$ fulfills Conditions (1) and (2) in Definition 2.8, as well as (2.17), see [16].

Definition 2.11 Let $r, \rho \geq 0$, $r + \rho > 0$, $\omega \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, and let $\phi, \phi_1, \phi_2 \in C^\infty(\mathbf{R}^d \times \mathbf{R}^{d_0}; \mathbf{R}^d)$ be SG mappings.

(1) ω is called $(\phi, 1)$-*invariant* when

$$\omega(\phi(x, \eta_1 + \eta_2), \xi) \lesssim \omega(\phi(x, \eta_1), \xi),$$

for any $x, \xi \in \mathbf{R}^d$, $\eta_1, \eta_2 \in \mathbf{R}^{d_0}$, uniformly with respect to $\eta_2 \in \mathbf{R}^{d_0}$. The set of all $(\phi, 1)$-invariant weights in $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ is denoted by $\mathscr{P}^{\phi,1}_{r,\rho}(\mathbf{R}^{2d})$;

(2) ω is called $(\phi, 2)$-*invariant* when

$$\omega(x, \phi(\xi, \eta_1 + \eta_2)) \lesssim \omega(x, \phi(\xi, \eta_1)),$$

for any $x, \xi \in \mathbf{R}^d$, $\eta_1, \eta_2 \in \mathbf{R}^{d_0}$, uniformly with respect to $\eta_2 \in \mathbf{R}^{d_0}$. The set of all $(\phi, 2)$-invariant weights in $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ is denoted by $\mathscr{P}^{\phi,2}_{r,\rho}(\mathbf{R}^{2d})$;

(3) ω is called (ϕ_1, ϕ_2)-*invariant* if ω is both $(\phi_1, 1)$-invariant and $(\phi_2, 2)$-invariant. The set of all (ϕ_1, ϕ_2)-invariant weights in $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ is denoted by $\mathscr{P}^{(\phi_1,\phi_2)}_{r,\rho}(\mathbf{R}^{2d})$

The next Lemma 2.12, proved in [24], shows that, under mild additional conditions, the families of weights introduced in Sect. 2.1 are indeed "invariant" under composition with SG maps with SG^0-parameter dependence. That is, the compositions introduced in Definition 2.11 are still weight functions in the sense of Sect. 2.1, belonging to suitable sets $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$.

Lemma 2.12 *Let* $r, \rho \in [0, 1]$, $r + \rho > 0$, $\omega \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, *and let* $\phi \colon \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^d$ *be an* SG *map as in Definition 2.8. The following statements hold true.*

(1) *Assume* $\omega \in \mathscr{P}^{\phi,1}_{1,\rho}(\mathbf{R}^{2d})$, *and set* $\omega_1(x, \xi) := \omega(\phi(x, \xi), \xi)$. *Then* $\omega_1 \in \mathscr{P}_{1,\rho}(\mathbf{R}^{2d})$.

(2) *Assume* $\omega \in \mathscr{P}^{\phi,2}_{r,1}(\mathbf{R}^{2d})$, *and set* $\omega_2(x, \xi) := \omega(x, \phi(\xi, x))$. *Then* $\omega_2 \in \mathscr{P}_{r,1}(\mathbf{R}^{2d})$.

Remark 2.13 It is obvious that, when dealing with Fourier integral operators, the requirements for ϕ and ω in Lemma 2.12 need to be satisfied only on the support of the involved amplitude. By Lemma 2.12, it also follows that if $a \in \mathrm{SG}^{(\omega)}_{1,1}(\mathbf{R}^{2d})$ and $\phi = (\phi_1, \phi_2)$, where $\phi_1 \in \mathrm{SG}^{1,0}_{1,1}(\mathbf{R}^{2d})$ and $\phi_2 \in \mathrm{SG}^{0,1}_{1,1}(\mathbf{R}^{2d})$ are SG maps with SG^0 parameter dependence, then $a \circ \phi \in \mathrm{SG}^{(\omega_0)}_{1,1}(\mathbf{R}^{2d})$ when $\omega_0 := \omega \circ \phi$, provided ω is (ϕ_1, ϕ_2)-invariant. Similar results hold for SG amplitudes and weights defined on $\mathbf{R}^{3d}$.

Remark 2.14 By the definitions it follows that any weight $\omega = \vartheta_{s,\sigma}$, $s, \sigma \in \mathbf{R}$, is $(\phi, 1)$-, $(\phi, 2)$-, and (ϕ_1, ϕ_2)-invariant with respect to any SG diffeomorphism with SG^0 parameter dependence ϕ, (ϕ_1, ϕ_2).

3 Symbolic Calculus for Generalised FIOs of SG Type

We here recall the class of Fourier integral operators we are interested in, generalizing those studied in [16]. The corresponding symbolic calculus has been obtained in [24], from which we recall the results listed below, and to which we refer the reader for the details. A key tool in the proofs of the composition theorems below are the results on asymptotic expansions in the Weyl–Hörmander calculus obtained in [22].

3.1 Phase Functions of SG Type

We recall the definition of the class of admissible phase functions in the SG context, as it was given in [16]. We then observe that the subclass of *regular phase functions* generates (parameter-dependent) mappings of $\mathbf{R}^d$ onto itself, which turn out to be SG maps with SG^0 parameter-dependence. Finally, we define some *regularizing operators*, which are used to prove the properties of the SG Fourier integral operators introduced in the next subsection.

Definition 3.1 A real-valued function $\varphi \in \mathrm{SG}^{1,1}_{1,1}(\mathbf{R}^{2d})$ is called a *simple phase function* (or *simple phase*), if

$$\langle \varphi'_\xi(x,\xi) \rangle \asymp \langle x \rangle \quad \text{and} \quad \langle \varphi'_x(x,\xi) \rangle \asymp \langle \xi \rangle, \tag{3.1}$$

are fulfilled, uniformly with respect to ξ and x, respectively. The set of all simple phase functions is denoted by $\mathfrak{F}$. Moreover, the simple phase function φ is called *regular*, if $\left|\det(\varphi''_{x\xi}(x,\xi))\right| \geq c$ for some $c > 0$ and all $x, \xi \in \mathbf{R}^d$. The set of all regular phases is denoted by $\mathfrak{F}^r$.

We observe that a regular phase function φ defines two globally invertible mappings, namely $\xi \mapsto \varphi'_x(x,\xi)$ and $x \mapsto \varphi'_\xi(x,\xi)$, see the analysis in [16]. Then the following result holds true for the mappings ϕ_1 and ϕ_2 generated by the first derivatives of the admissible regular phase functions.

Proposition 3.2 *Let* $\varphi \in \mathfrak{F}$. *Then* $\phi_1 \colon \mathbf{R}^d \to \mathbf{R}^d \colon x \mapsto \varphi'_\xi(x, \xi_0)$ *and* $\phi_2 \colon \mathbf{R}^d \to \mathbf{R}^d \colon \xi \mapsto \varphi'_x(x_0, \xi)$ *are* SG *maps (with* SG^0 *parameter dependence) from* $\mathbf{R}^d$ *to itself,* *for any* $x_0, \xi_0 \in \mathbf{R}^d$. *If* $\varphi \in \mathfrak{F}^r$, ϕ_1 *and* ϕ_2 *give rise to* SG *diffeomorphism with* SG^0 *parameter dependence.*

For any $\varphi \in \mathfrak{F}$, the operators $\Theta_{1,\varphi}$ and $\Theta_{2,\varphi}$ are defined by

$$(\Theta_{1,\varphi} f)(x,\xi) \equiv f(\varphi'_\xi(x,\xi),\xi) \quad \text{and} \quad (\Theta_{2,\varphi} f)(x,\xi) \equiv f(x,\varphi'_x(x,\xi)),$$

when $f \in C^1(\mathbf{R}^{2d})$, and remark that the modified weights

$$(\Theta_{1,\varphi}\omega)(x,\xi) = \omega(\varphi'_\xi(x,\xi),\xi) \quad \text{and} \quad (\Theta_{2,\varphi}\omega)(x,\xi) = \omega(x,\varphi'_x(x,\xi)), \tag{3.2}$$

will appear frequently in the sequel. In the following lemma we show that these weights belong to the same classes of weights as ω, provided they additionally fulfill

$$\Theta_{1,\varphi}\omega \asymp \Theta_{2,\varphi}\omega \tag{3.3}$$

when φ is the involved phase function. That is, (3.3) is a sufficient condition to obtain $(\phi_1, 1)$- and/or $(\phi_2, 2)$-invariance of ω in the sense of Definition 2.11, depending on the values of the parameters $r, \rho \geq 0$.

Lemma 3.3 *Let* φ *be a simple phase on* $\mathbf{R}^{2d}$, $r, \rho \in [0,1]$ *be such that* $r = 1$ *or* $\rho = 1$, *and let* $\Theta_{j,\varphi}\omega$, $j = 1, 2$, *be as in* (3.2), *where* $\omega \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ *satisfies* (3.3). *Then*

$$\Theta_{j,\varphi}\omega \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d}), \quad j = 1, 2.$$

In what follows we let

$${}^t a(x,\xi) = a(\xi,x) \quad \text{and} \quad (a^*)(x,\xi) = \overline{a(\xi,x)},$$

when $a(x,\xi)$ is a function.

3.2 Generalised Fourier Integral Operators of SG Type

In analogy with the definition of generalized SG pseudo-differential operators, recalled in Sect. 2.1, we define the class of Fourier integral operators we are interested in terms of their distributional kernels. These belong to a class of tempered oscillatory integrals, studied in [21]. Thereafter we prove that they posses convenient mapping properties.

Definition 3.4 Let $\omega \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ satisfy (3.3), $r, \rho \geq 0$, $r + \rho > 0$, $\varphi \in \mathfrak{F}$, $a, b \in \mathrm{SG}^{(\omega)}_{r,\rho}(\mathbf{R}^{2d})$.

(1) The generalized Fourier integral operator $A = \mathrm{Op}_\varphi(a)$ of SG *type I* (SG *FIOs of type I*) with phase φ and amplitude a is the linear continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ with distribution kernel $K_A \in \mathscr{S}'(\mathbf{R}^{2d})$ given by

$$K_A(x,y) = (2\pi)^{-d/2}(\mathscr{F}_2(e^{i\varphi}a))(x,y);$$

(2) The generalized Fourier integral operator $B = \mathrm{Op}^*_\varphi(b)$ of SG *type II* (SG *FIOs of type II*) with phase φ and amplitude b is the linear continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ with distribution kernel $K_B \in \mathscr{S}'(\mathbf{R}^{2d})$ given by

$$K_B(x,y) = (2\pi)^{-d/2}(\mathscr{F}_2^{-1}(e^{-i\varphi}\overline{b}))(y,x).$$

Evidently, if $f \in \mathscr{S}(\mathbf{R}^d)$, and A and B are the operators in Definition 3.4, then

$$Af(x) = \mathrm{Op}_\varphi(a)u(x) = (2\pi)^{-d/2}\int e^{i\varphi(x,\xi)}\, a(x,\xi)\,(\mathscr{F} f)(\xi)\,d\xi, \tag{3.4}$$

and

$$\begin{aligned} Bf(x) &= \mathrm{Op}^*_\varphi(b)u(x) \\ &= (2\pi)^{-d}\iint e^{i(\langle x,\xi\rangle - \varphi(y,\xi))}\,\overline{b(y,\xi)}\,f(y)\,dyd\xi. \end{aligned} \tag{3.5}$$

Remark 3.5 In the sequel the formal (L^2-)adjoint of an operator Q is denoted by Q^*. By straightforward computations it follows that the SG type I and SG type II operators are formal adjoints to each others, provided the amplitudes and phase functions are the same. That is, if b and φ are the same as in Definition 3.4, then $\mathrm{Op}^*_\varphi(b) = \mathrm{Op}_\varphi(b)^*$.

Obviously, for any $\omega \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, ${}^t\omega = \omega^*$ is also an admissible weight which belongs to $\mathscr{P}_{\rho,r}(\mathbf{R}^{2d})$. Similarly, for arbitrary $\varphi \in \mathfrak{F}$ and $a \in \mathrm{SG}^{(\omega)}_{r,\rho}(\mathbf{R}^{2d})$, we have ${}^t\varphi = \varphi^* \in \mathfrak{F}$ and ${}^ta, a^* \in \mathrm{SG}^{(\omega^*)}_{\rho,r}(\mathbf{R}^{2d})$. Furthermore, by Definition 3.4 we get

$$\begin{gathered} \mathrm{Op}^*_\varphi(b) = \mathscr{F}^{-1}\circ \mathrm{Op}_{-\varphi^*}(b^*)\circ \mathscr{F}^{-1} \\ \Longleftrightarrow \\ \mathrm{Op}_\varphi(a) = \mathscr{F}\circ \mathrm{Op}^*_{-\varphi^*}(a^*)\circ \mathscr{F}. \end{gathered} \tag{3.6}$$

The following result shows that type I and type II operators are linear and continuous from $\mathscr{S}(\mathbf{R}^d)$ to itself, and extendable to linear and continuous operators from $\mathscr{S}'(\mathbf{R}^d)$ to itself.

Theorem 3.6 *Let a, b and φ be the same as in Definition 3.4. Then $\mathrm{Op}_\varphi(a)$ and $\mathrm{Op}^*_\varphi(b)$ are linear and continuous operators on $\mathscr{S}(\mathbf{R}^d)$, and uniquely extendable to linear and continuous operators on $\mathscr{S}'(\mathbf{R}^d)$.*

3.3 Composition with Pseudo-differential Operators of SG Type

The composition theorems presented in this and the subsequent subsections are variants of those originally appeared in [16]. The notation used in the statements of the composition theorems are those introduced in Sects. 2.3, 3.1 and 3.2. The proofs and more details can be found in [24].

Theorem 3.7 *Let $r_j, \rho_j \in [0,1]$, $\varphi \in \mathfrak{F}$ and let $\omega_j \in \mathscr{P}_{r_j,\rho_j}(\mathbf{R}^{2d})$, $j = 0, 1, 2$, be such that*

$$\rho_2 = 1, \quad r_0 = \min\{r_1, r_2, 1\}, \quad \rho_0 = \min\{\rho_1, 1\}, \quad \omega_0 = \omega_1 \cdot (\Theta_{2,\varphi}\omega_2),$$

and $\omega_2 \in \mathscr{P}_{r,1}(\mathbf{R}^{2d})$ is $(\phi, 2)$-invariant with respect to $\phi\colon \xi \mapsto \varphi'_x(x,\xi)$. Also let $a \in \mathrm{SG}^{(\omega_1)}_{r_1,\rho_1}(\mathbf{R}^{2d})$, $p \in \mathrm{SG}^{(\omega_2)}_{r_2,1}(\mathbf{R}^{2d})$, and let

$$\psi(x,y,\xi) = \varphi(y,\xi) - \varphi(x,\xi) - \langle y - x, \varphi'_x(x,\xi)\rangle. \tag{3.7}$$

Then

$$\begin{aligned} &\mathrm{Op}(p) \circ \mathrm{Op}_\varphi(a) = \mathrm{Op}_\varphi(c) \,\mathrm{Mod}\, \mathrm{Op}_\varphi(\mathrm{SG}^{(\omega\vartheta_{0,-\infty})}_0), && r_1 = 0,\\ &\mathrm{Op}(p) \circ \mathrm{Op}_\varphi(a) = \mathrm{Op}_\varphi(c) \,\mathrm{Mod}\, \mathrm{Op}(\mathscr{S}), && r_1 > 0, \end{aligned}$$

where $c \in \mathrm{SG}^{(\omega_0)}_{r_0,\rho_0}(\mathbf{R}^{2d})$ admits the asymptotic expansion

$$c(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} (D^\alpha_\xi p)(x, \varphi'_x(x,\xi))\, D^\alpha_y \left[e^{i\psi(x,y,\xi)} a(y,\xi) \right]_{y=x}. \tag{3.8}$$

Theorem 3.8 *Let $r_j, \rho_j \in [0,1]$, $\varphi \in \mathfrak{F}$ and let $\omega_j \in \mathscr{P}_{r_j,\rho_j}(\mathbf{R}^{2d})$, $j = 0, 1, 2$, be such that*

$$r_2 = 1, \quad r_0 = \min\{r_1, 1\}, \quad \rho_0 = \min\{\rho_1, \rho_2, 1\}, \quad \omega_0 = \omega_1 \cdot (\Theta_{1,\varphi}\omega_2),$$

and $\omega_2 \in \mathscr{P}_{r,1}(\mathbf{R}^{2d})$ is $(\phi, 1)$-invariant with respect to $\phi\colon x \mapsto \varphi'_\xi(x,\xi)$. Also let $a \in \mathrm{SG}^{(\omega_1)}_{r_1,\rho_1}(\mathbf{R}^{2d})$ and $p \in \mathrm{SG}^{(\omega_2)}_{1,\rho_2}(\mathbf{R}^{2d})$. Then

$$\begin{aligned} &\mathrm{Op}_\varphi(a) \circ \mathrm{Op}(p) \ = \mathrm{Op}_\varphi(c) \,\mathrm{Mod}\, \mathrm{Op}_\varphi(\mathrm{SG}^{(\omega\vartheta_{-\infty,0})}_0), \quad \rho_1 = 0,\\ &\mathrm{Op}_\varphi(a) \circ \mathrm{Op}(p) \ = \mathrm{Op}_\varphi(c) \,\mathrm{Mod}\, \mathrm{Op}(\mathscr{S}), \quad \rho_1 > 0, \end{aligned}$$

where the transpose ${}^t c$ of $c \in \mathrm{SG}^{(\omega_0)}_{r_0,\rho_0}(\mathbf{R}^{2d})$ admits the asymptotic expansion (3.8), after p and a have been replaced by ${}^t p$ and ${}^t a$, respectively.

Theorem 3.9 *Let $r_j, \rho_j \in [0,1]$, $\varphi \in \mathfrak{F}$ and let $\omega_j \in \mathscr{P}_{r_j,\rho_j}(\mathbf{R}^{2d})$, $j = 0, 1, 2$, be such that*

$$\rho_2 = 1, \quad r_0 = \min\{r_1, r_2, 1\}, \quad \rho_0 = \min\{\rho_1, 1\}, \quad \omega_0 = \omega_1 \cdot (\Theta_{2,\varphi}\omega_2),$$

and $\omega_2 \in \mathscr{P}_{r,1}(\mathbf{R}^{2d})$ is $(\phi, 2)$-invariant with respect to $\phi\colon \xi \mapsto \varphi'_x(x,\xi)$. Also let $b \in \mathrm{SG}^{(\omega_1)}_{r_1,\rho_1}(\mathbf{R}^{2d})$, $p \in \mathrm{SG}^{(\omega_2)}_{r_2,1}(\mathbf{R}^{2d})$, ψ be the same as in (3.7), and let $q \in \mathrm{SG}^{(\omega_2)}_{r_2,1}(\mathbf{R}^{2d})$ be such that

$$q(x,\xi) \sim \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} D_x^\alpha D_\xi^\alpha \overline{p(x,\xi)}. \tag{3.9}$$

Then

$$\begin{aligned} \mathrm{Op}^*_\varphi(b) \circ \mathrm{Op}(p) &= \mathrm{Op}_\varphi(c) \,\mathrm{Mod}\, \mathrm{Op}^*_\varphi(\mathrm{SG}_0^{(\omega\vartheta_{0,-\infty})}), \quad r_1 = 0,\\ \mathrm{Op}^*_\varphi(b) \circ \mathrm{Op}(p) &= \mathrm{Op}_\varphi(c) \,\mathrm{Mod}\, \mathrm{Op}(\mathscr{S}), \quad r_1 > 0, \end{aligned}$$

where $c \in \mathrm{SG}^{(\omega_0)}_{r_0,\rho_0}(\mathbf{R}^{2d})$ *admits the asymptotic expansion*

$$c(x,\xi) \sim \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} (D_\xi^\alpha q)(x, \varphi'_x(x,\xi)) D_y^\alpha \Big[e^{i\psi(x,y,\xi)} b(y,\xi) \Big]_{y=x}. \tag{3.10}$$

Theorem 3.10 *Let* $r_j, \rho_j \in [0,1]$, $\varphi \in \mathfrak{F}$ *and let* $\omega_j \in \mathscr{P}_{r_j,\rho_j}(\mathbf{R}^{2d})$, $j = 0,1,2$, *be such that*

$$r_2 = 1, \quad r_0 = \min\{r_1, 1\}, \quad \rho_0 = \min\{\rho_1, \rho_2, 1\}, \quad \omega_0 = \omega_1 \cdot (\Theta_{1,\varphi}\omega_2),$$

and $\omega_2 \in \mathscr{P}_{r,1}(\mathbf{R}^{2d})$ *is* $(\phi, 1)$*-invariant with respect to* $\phi \colon x \mapsto \varphi'_\xi(x,\xi)$. *Also let* $a \in \mathrm{SG}^{(\omega_1)}_{r_1,\rho_1}(\mathbf{R}^{2d})$ *and* $p \in \mathrm{SG}^{(\omega_2)}_{1,\rho_2}(\mathbf{R}^{2d})$. *Then*

$$\begin{aligned} \mathrm{Op}(p) \circ \mathrm{Op}^*_\varphi(b) &= \mathrm{Op}_\varphi(c) \,\mathrm{Mod}\, \mathrm{Op}^*_\varphi(\mathrm{SG}_0^{(\omega\vartheta_{-\infty,0})}), \quad \rho_1 = 0,\\ \mathrm{Op}(p) \circ \mathrm{Op}^*_\varphi(b) &= \mathrm{Op}_\varphi(c) \,\mathrm{Mod}\, \mathrm{Op}(\mathscr{S}), \quad \rho_1 > 0, \end{aligned}$$

where the transpose ${}^t c$ *of* $c \in \mathrm{SG}^{(\omega_0)}_{r_0,\rho_0}(\mathbf{R}^{2d})$ *admits the asymptotic expansion* (3.10)*, after* q *and* b *have been replaced by* ${}^t q$ *and* ${}^t b$*, respectively.*

3.4 Composition Between SG FIOs of Type I and Type II

The subsequent Theorems 3.12 and 3.13 deal with the composition of a type I operator with a type II operator, and show that such compositions are pseudo-differential operators with symbols in natural classes.

The main difference, with respect to the arguments in [16] for the analogous composition results, is that we again make use, in both cases, of the generalized asymptotic expansions introduced in Definition 2.4. This allows to overcome the additional difficulty, not arising there, that the amplitudes appearing in the computations below involve weights which are still polynomially bounded, but which do not satisfy, in general, the moderateness condition (2.1). On the other hand, all the terms appearing in the associated asymptotic expansions belong to SG classes with weights of the form $\widetilde{\omega}_{2,\varphi} \cdot \vartheta_{-k,-k}$, where $\widetilde{\omega} = \omega_1 \cdot \omega_2$, which can be handled through the results in [22].

Let S_φ, $\varphi \in \mathfrak{F}$, be the operator defined by the formulae

$$(S_\varphi f)(x, y, \xi) = f(x, y, \Phi(x, y, \xi)) \cdot \left|\det \Phi'_\xi(x, y, \xi)\right|$$
$$\text{where} \quad \int_0^1 \varphi'_x(y + t(x-y), \Phi(x, y, \xi))\, dt = \xi. \tag{3.11}$$

That is, for every fixed $x, y \in \mathbf{R}^d$, $\xi \mapsto \Phi(x, y, \xi)$ is the inverse of the map

$$\xi \mapsto \int_0^1 \varphi'_x(y + t(x-y), \xi)\, dt. \tag{3.12}$$

Notice that, as proved in [16], the map (3.12) is indeed invertible for (x, y) belonging to a suitable neighborhood of the diagonal $y = x$ of $\mathbf{R}^d \times \mathbf{R}^d$, and it turns out to be an SG diffeomorphism with SG^0 parameter dependence. We also recall, from [16], the definition of the SG compatible cut-off functions localizing to such neighborhoods.

Definition 3.11 The sets $\Xi^\Delta(k)$, $k > 0$, of the SG compatible cut-off functions along the diagonal of $\mathbf{R}^d \times \mathbf{R}^d$, consist of all $\chi = \chi(x, y) \in \mathrm{SG}^{0,0}_{1,1}(\mathbf{R}^{2d})$ such that

$$\begin{aligned} |y - x| \le k\langle x\rangle/2 &\Longrightarrow \chi(x, y) = 1, \\ |y - x| > k\langle x\rangle &\Longrightarrow \chi(x, y) = 0. \end{aligned} \tag{3.13}$$

If not otherwise stated, we always assume $k \in (0, 1)$.

Theorem 3.12 *Let $r_j \in [0, 1]$, $\varphi \in \mathfrak{F}$ and let $\omega_j \in \mathscr{P}_{r_j,1}(\mathbf{R}^{2d})$, $j = 0, 1, 2$, be such that ω_1 and ω_2 are $(\phi, 2)$-invariant with respect to $\phi\colon \xi \mapsto (\varphi'_x)^{-1}(x, \xi)$,*

$$r_0 = \min\{r_1, r_2, 1\} \quad \textit{and} \quad \omega_0(x, \xi) = \omega_1(x, \phi(x, \xi))\omega_2(x, \phi(x, \xi)),$$

Also let $a \in \mathrm{SG}^{(\omega_1)}_{r_1,1}(\mathbf{R}^{2d})$ and $b \in \mathrm{SG}^{(\omega_2)}_{r_2,1}(\mathbf{R}^{2d})$. Then

$$\mathrm{Op}_\varphi(a) \circ \mathrm{Op}^*_\varphi(b) = \mathrm{Op}(c),$$

for some $c \in \mathrm{SG}^{(\omega_0)}_{r_0,1}(\mathbf{R}^{2d})$. Furthermore, if $\varepsilon \in (0, 1)$, $\chi \in \Xi^\Delta(\varepsilon)$, $c_0(x, y, \xi) = a(x, \xi)b(y, \xi)\chi(x, y)$ and S_φ is given by (3.11), then h admits the asymptotic expansion

$$c(x, \xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} (D^\alpha_y D^\alpha_\xi (S_\varphi c_0))(x, y, \xi)\Big|_{y=x}.$$

To formulate the next result we modify the operator S_φ in (3.11) such that it fulfills the formulae

$$\begin{aligned} &(S_\varphi f)(x, \xi, \eta) = f(\Phi(x, \xi, \eta), \xi, \eta) \cdot \left|\det \Phi'_x(x, \xi, \eta)\right| \\ &\text{where} \quad \textstyle\int_0^1 \varphi'_\xi(\Phi(x, \xi, \eta), \eta + t(\xi - \eta))\, dt = x. \end{aligned} \tag{3.14}$$

Theorem 3.13 *Let $\rho_j \in [0,1]$, $\varphi \in \mathfrak{F}^r$ and let $\omega_j \in \mathscr{P}_{1,\rho_j}(\mathbf{R}^{2d})$, $j = 0,1,2$, be such that ω_1 and ω_2 are $(\phi, 1)$-invariant with respect to $\phi\colon x \mapsto (\varphi'_\xi)^{-1}(x,\xi)$,*

$$\rho_0 = \min\{\rho_1, \rho_2, 1\} \quad \textit{and} \quad \omega_0(x,\xi) = \omega_1(\phi(x,\xi),\xi)\omega_2(\phi(x,\xi),\xi),$$

Also let $a \in \mathrm{SG}^{(\omega_1)}_{1,\rho_1}(\mathbf{R}^{2d})$ and $b \in \mathrm{SG}^{(\omega_2)}_{1,\rho_2}(\mathbf{R}^{2d})$. Then

$$\mathrm{Op}^*_\varphi(b) \circ \mathrm{Op}_\varphi(a) = \mathrm{Op}(c),$$

for some $c \in \mathrm{SG}^{(\omega_0)}_{1,\rho_0}(\mathbf{R}^{2d})$. Furthermore, if $\varepsilon \in (0,1)$, $\chi \in \Xi^\Delta(\varepsilon)$, $c_0(x,\xi,\eta) = a(x,\xi)b(x,\eta)\chi(\xi,\eta)$ and S_φ is given by (3.14), then h admits the asymptotic expansion

$$c(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}(D^\alpha_x D^\alpha_\eta(S_\varphi c_0))(x,\xi,\eta)\big|_{\eta=\xi}.$$

3.5 Elliptic FIOs of Generalized SG Type and Parametrices: Egorov's Theorem

The results about the parametrices of the subclass of generalized (SG) elliptic Fourier integral operators are achieved in the usual way, by means of the composition theorems in Sects. 3.3 and 3.4. The same holds for the versions of the Egorov's theorem adapted to the present situation. The additional conditions, compared with the statements in [16], concern the invariance of the weights, so that the hypotheses of the composition theorems above are fulfilled.

Definition 3.14 Let $r, \rho > 0$, $\omega \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, $\varphi \in \mathfrak{F}^r$ and let $a, b \in \mathrm{SG}^{(\omega)}_{r,\rho}(\mathbf{R}^{2d})$. The operators $\mathrm{Op}_\varphi(a)$ and $\mathrm{Op}^*_\varphi(b)$ are called *elliptic*, if a and b are SG-elliptic (cf. Sect. 1 in [26].)

Lemma 3.15 *Let $\phi = (\phi_1, \phi_2)$, where ϕ_2 and ϕ_1 are the* SG *diffeomorphisms in Theorems 3.12 and 3.13, respectively, and let $\omega \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ be ϕ-invariant. Also let $a \in \mathrm{SG}^{(\omega)}_{1,1}(\mathbf{R}^{2d})$ be such that $\mathrm{Op}_\varphi(a)$ is elliptic.*

Then the pseudo-differential operators $\mathrm{Op}_\varphi(a) \circ \mathrm{Op}^*_\varphi(a)$ and $\mathrm{Op}^*_\varphi(a) \circ \mathrm{Op}_\varphi(a)$ are SG *elliptic.*

Theorem 3.16 *Let ω be ϕ-invariant, $\phi = (\phi_1, \phi_2)$, where ϕ_2 and ϕ_1 are the* SG *diffeomorphisms in Theorems 3.12 and 3.13, respectively. Also let $\varphi \in \mathfrak{F}^r$, and let $a \in \mathrm{SG}^{(\omega)}_{1,1}(\mathbf{R}^{2d})$ be* SG *elliptic. Then $\mathrm{Op}_\varphi(a)$ and $\mathrm{Op}^*_\varphi(a)$ admit parametrices which are elliptic* SG *FIOs of type II and type I, respectively.*

In the next two results we need the canonical transformation $\phi\colon (x,\xi) \mapsto (y,\eta)$ generated by the phase function φ, given by

$$\begin{cases} \xi = \varphi'_x(x,\eta) \\ y = \varphi'_\xi(x,\eta) = \varphi'_\eta(x,\eta). \end{cases} \tag{3.15}$$

Theorem 3.17 *Let ϕ be the canonical transformation (3.15), $\phi_0 : \xi \mapsto (\varphi'_x)^{-1}(x,\xi)$, $\omega_1, \omega_2 \in \mathscr{P}_{1,1}(\mathbf{R}^{2d})$ be such that ω_1 is $(\phi_0, 2)$-invariant and ω_2 is ϕ-invariant, and let*

$$\omega_0(x,\xi) = \omega_1(x, (\varphi'_x)^{-1}(x,\xi))^2 \cdot \omega_2(\phi(x,\xi)).$$

Also let $a \in \mathrm{SG}^{(\omega_1)}_{1,1}(\mathbf{R}^{2d})$ and $p \in \mathrm{SG}^{(\omega_2)}_{1,1}(\mathbf{R}^{2d})$. Then

$$\mathrm{Op}_\varphi(a) \circ \mathrm{Op}(p) \circ \mathrm{Op}^*_\varphi(a) = \mathrm{Op}(p_0),$$

where $p_0 \in \mathrm{SG}^{(\omega_0)}_{1,1}(\mathbf{R}^{2d})$ satisfies

$$\begin{aligned} p_0(x,\xi) &= p(\varphi'_\xi(x,\eta),\eta)\,|a(x,\eta)|^2\,|\det \varphi''_{x\xi}(x,\eta)|^{-1} \bmod \mathrm{SG}^{(\omega_0 \cdot \vartheta_{-1,-1})}_{1,1}(\mathbf{R}^{2d}), \\ &\quad \textit{with} \quad \eta = (\varphi'_x)^{-1}(x,\xi). \end{aligned}$$

Theorem 3.18 *Let ϕ be the canonical transformation (3.15), $\omega_1, \omega_2 \in \mathscr{P}_{1,1}(\mathbf{R}^{2d})$ be such that ω_2 is ϕ-invariant, and let*

$$\omega_0(x,\xi) = \omega_2(\phi(x,\xi)).$$

Also let $a \in \mathrm{SG}^{(\omega_1)}_{1,1}(\mathbf{R}^{2d})$ be elliptic, $p \in \mathrm{SG}^{(\omega_2)}_{1,1}(\mathbf{R}^{2d})$, and let b be chosen such that $\mathrm{Op}^*_\varphi(b)$ is a parametrix to $\mathrm{Op}_\varphi(a)$. Then

$$\mathrm{Op}_\varphi(a) \circ \mathrm{Op}(p) \circ \mathrm{Op}^*_\varphi(b) = \mathrm{Op}(p_0),$$

where $p_0 \in \mathrm{SG}^{(\omega_0)}_{1,1}(\mathbf{R}^{2d})$ satisfies

$$p_0(x,\xi) = p(\phi(x,\xi)) \bmod \mathrm{SG}^{(\omega_0 \cdot \vartheta_{-1,-1})}_{1,1}(\mathbf{R}^{2d}).$$

4 Continuity on Lebesgue and Modulation Spaces

In this section we recall some basic facts about continuity properties for Fourier integral operators when acting on Lebesgue and modulation spaces. We also use the analysis in previous sections in combination with certain lifting properties for modulation spaces in order to establish weighted versions of continuity results for Fourier integral operators on modulation spaces.

4.1 Continuity on Lebesgue Spaces

We start by considering the following result, which, for trivial Sobolev parameters, is related to Theorem 2.6 in [23]. A direct proof of the $L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d)$ boundedness of $\mathrm{Op}_\varphi(a)$ for $a \in \mathrm{SG}^{0,0}_{1,1}(\mathbf{R}^d)$ and a regular phase function $\varphi \in \mathfrak{F}^r$ was given in [16].

A similar argument actually holds for $a \in \mathrm{SG}^{0,0}_{r,\rho}(\mathbf{R}^d)$, $r, \rho \geq 0$, and is given in [24] (see also [39]). Here $B_r(x_0)$ is the open ball with center at $x_0 \in \mathbf{R}^d$ and radius r.

Theorem 4.1 *Let* $\sigma_1, \sigma_2 \in \mathbf{R}$, $p \in (1, \infty)$ *and* $m, \mu \in \mathbf{R}$ *be such that*

$$m \leq -(d-1)\left|\frac{1}{p} - \frac{1}{2}\right|, \quad \mu \leq -(d-1)\left|\frac{1}{p} - \frac{1}{2}\right| + \sigma_1 - \sigma_2.$$

Also let $\varphi \in \mathrm{SG}^{1,1}_{1,1}(\mathbf{R}^{2d})$ *be such that for some constants* $c > 0$ *and* $R > 0$ *and every multi-index* α *it holds*

$$|\det \varphi''_{x,\xi}(x,\xi)| \geq c, \quad |\partial_x^\alpha \varphi(x,\xi)| \lesssim \langle x\rangle^{1-|\alpha|}\langle \xi\rangle$$
$$\langle \varphi'_x(x,\xi)\rangle \asymp \langle \xi\rangle, \quad \langle \varphi'_\xi(x,\xi)\rangle \asymp \langle x\rangle,$$

and

$$\varphi(x, t\xi) = t\varphi(x,\xi), \quad x, \xi \in \mathbf{R}^d, \ |\xi| \geq R, \ t \geq 1.$$

If $a \in \mathrm{SG}^{m,\mu}_{1,1}(\mathbf{R}^{2d})$ *is supported outside* $\mathbf{R}^d \times B_r(0)$ *for some* $r > 0$, *then* $\mathrm{Op}_\varphi(a)$ *extends to a continuous operator from* $H^p_{\sigma_1}(\mathbf{R}^d)$ *to* $H^p_{\sigma_2}(\mathbf{R}^d)$.

Proof Let $T = \langle D\rangle^{\sigma_2} \circ \mathrm{Op}_\varphi(a) \circ \langle D\rangle^{-\sigma_1}$. Since

$$\langle D\rangle^{\sigma_2} : H^p_{\sigma_2} \to L^p \quad \text{and} \quad \langle D\rangle^{-\sigma_1} : L^p \to H^p_{\sigma_1}$$

are continuous bijections, the result follows if we prove that T is continuous on L^p.

By Theorems 3.8 and 3.9 it follows that

$$T = \mathrm{Op}_\varphi(a_1) \mod \mathrm{Op}(\mathscr{S}),$$

where $a_1 \in \mathrm{SG}^{m,\mu_0}_{1,1}(\mathbf{R}^{2d})$ with

$$\mu_0 \leq -(d-1)\left|\frac{1}{p} - \frac{1}{2}\right|.$$

Furthermore, by the symbolic calculus and the fact that a is supported outside $\mathbf{R}^d \times B_r(0)$ we get

$$\mathrm{Op}_\varphi(a_1) = \mathrm{Op}_\varphi(a_2) \mod \mathrm{Op}(\mathscr{S}),$$

where $a_2 \in \mathrm{SG}^{m,\mu_0}_{1,1}(\mathbf{R}^{2d})$ is supported outside $\mathbf{R}^d \times B_r(0)$. Hence

$$T = \mathrm{Op}_\varphi(a_2) + \mathrm{Op}(c),$$

where $c \in \mathscr{S}$, giving that $\mathrm{Op}(c)$ is continuous on L^p.

Since $\mathrm{Op}_\varphi(a_2)$ is continuous on L^p, by [23, Theorem 2.6] and its proof, the result follows. □

Remark 4.2 Let φ be a phase function satisfying the hypotheses of Theorem 4.1, $s_1, s_2, \sigma_1, \sigma_2 \in \mathbf{R}$, $p \in (1,\infty)$, and assume that $a \in \mathrm{SG}^{m,\mu}_{1,1}(\mathbf{R}^{2d})$ is supported outside $\mathbf{R}^d \times B_r(0)$ for some $r > 0$, with $m, \mu \in \mathbf{R}$ satisfying

$$m \le -(d-1)\left|\frac{1}{p} - \frac{1}{2}\right| + s_1 - s_2, \quad \mu \le -(d-1)\left|\frac{1}{p} - \frac{1}{2}\right| + \sigma_1 - \sigma_2.$$

Then $\mathrm{Op}_\varphi(a)$ extends to a continuous operator from $H^p_{s_1,\sigma_1}(\mathbf{R}^d)$ to $H^p_{s_2,\sigma_2}(\mathbf{R}^d)$, which follows by similar arguments as in the proof of Theorem 4.1 (see [23]).

4.2 Continuity on Modulation Spaces

Next we consider continuity properties on modulation spaces. The following result extends Theorem 1.2 in [13]. Here we let $M^\infty_{0,(\omega)}(\mathbf{R}^d)$ be the completion of $\mathscr{S}(\mathbf{R}^d)$ under the norm $\|\cdot\|_{M^\infty_{(\omega)}}$. We also say that a (complex-valued) Gauss function Ψ is non-degenerate, if $|\Psi|$ tends to zero at infinity.

Theorem 4.3 *Let $m, \mu \in \mathbf{R}$ and $1 \le p < \infty$ be such that*

$$m \le -d\left|\frac{1}{2} - \frac{1}{p}\right|, \quad \mu \le -d\left|\frac{1}{2} - \frac{1}{p}\right|,$$

and let $\omega_j \in \mathscr{P}_{1,1}(\mathbf{R}^{2d})$, $j = 0, 1, 2$, be such that

$$\omega_0(x,\xi) \lesssim \frac{\omega_1(\varphi'_\xi(x,\xi),\xi)}{\omega_2(x,\varphi'_x(x,\xi))}\langle x\rangle^m\langle\xi\rangle^\mu.$$

Also let $a \in \mathrm{SG}^{(\omega_0)}_{1,1}(\mathbf{R}^{2d})$ and $\varphi \in \mathfrak{F}^r$, and assume that ω_j is (ϕ_j, j)-invariant, $j = 1, 2$, with $\phi_1 \colon x \mapsto \varphi'_\xi(x,\xi)$ and $\phi_2 \colon \xi \mapsto \varphi'_x(x,\xi)$. Then $\mathrm{Op}_\varphi(a)$ is uniquely extendable to a continuous map from $M^p_{(\omega_1)}(\mathbf{R}^d)$ to $M^p_{(\omega_2)}(\mathbf{R}^d)$ and from $M^\infty_{0,(\omega_1)}(\mathbf{R}^d)$ to $M^\infty_{0,(\omega_2)}(\mathbf{R}^d)$.

Proof Let Ψ be a Gaussian, and let T_1 and T_2 be the (Toeplitz) operators, defined by the formulas

$$(T_1 f, g) = (\omega_1^{-1} V_\Psi f, V_\Psi g) \quad \text{and} \quad (T_2 f, g) = (\omega_2 V_\Psi f, V_\Psi g).$$

Then it follows from Theorem 1.1 in [34] that T_1 and T_2 on $\mathscr{S}$ are uniquely extendable to continuous bijections between M^p to $M^p_{(\omega_1)}$, and from $M^p_{(\omega_2)}$ to M^p. Since $\mathscr{S}$ is dense in $M^p_{(\omega_j)}$ and in $M^\infty_{(\omega_j)}$, the result follows if we prove

$$\|(T_2 \circ \mathrm{Op}_\varphi(a) \circ T_1) f\|_{M^p} \lesssim \|f\|_{M^p}, \quad f \in \mathscr{S}.$$

For some non-degenerate Gauss function Φ which depends on Ψ we have

$$T_j = \mathrm{Op}(a_j), \ \ j = 1, 2, \quad \text{where} \quad a_1 = ((\omega_1)^{-1}) * \Phi \quad \text{and} \quad a_2 = \omega_2 * \Phi.$$

Furthermore, using the fact that $\omega_j \in \mathscr{P}_{1,1}$, it follows by straight-forward computations that $a_1 \in \mathrm{SG}_{1,1}^{(1/\omega_1)}$ and $a_2 \in \mathrm{SG}_{1,1}^{(\omega_2)}$.

By using these facts in combination with Theorems 3.8 and 3.9, we get

$$T_2 \, \mathrm{Op}_\varphi(a) \circ T_1 = T_2 \circ (\mathrm{Op}_\varphi(h_1) + S_1) = \mathrm{Op}_\varphi(h_2) + S_2 + T_2 \circ S_1,$$

for some operators $S_j \in \mathrm{Op}(\mathscr{S})$, $j = 1, 2$, where

$$h_1 \in \mathrm{SG}_{1,1}^{(\omega_0/\widetilde{\omega}_1)} \quad \text{and} \quad h_2 \in \mathrm{SG}_{1,1}^{(\omega_0\widetilde{\omega}_2/\widetilde{\omega}_1)} \subseteq \mathrm{SG}_{1,1}^{m,\mu},$$

$\widetilde{\omega}_1(x,\xi) = \omega_1(\varphi'_\xi(x,\xi),\xi)$, $\widetilde{\omega}_2(x,\xi) = \omega_2(x,\varphi'_x(x,\xi))$. Since

$$T_2 \circ S_1 \in \mathrm{Op}(\mathrm{SG}_{1,1}^{(\omega_2)}) \circ \mathrm{Op}(\mathscr{S}) \subseteq \mathrm{Op}(\mathscr{S}),$$

it follows that

$$T_2 \circ \mathrm{Op}_\varphi(a) \circ T_1 = \mathrm{Op}_\varphi(h_2) + S_0,$$

where $S_0 \in \mathrm{Op}(\mathscr{S})$, giving that S_0 is continuous on M^p. Furthermore, the fact that $h_2 \in \mathrm{SG}_{1,1}^{m,\mu}$ and Theorem 1.2 in [13] imply that

$$\| \mathrm{Op}_\varphi(h_2) f \|_{M^p} \lesssim \| f \|_{M^p}, \quad f \in \mathscr{S}.$$

This gives the result. □

Remark 4.4 Let $\rho \in [1,2]$, $p, q \in [1,\infty]$ and $t \in \mathbf{R}$. A Fourier integral operator which frequently appears in the literature is the continuous map from $\mathscr{S}(\mathbf{R}^d)$ to $L^\infty(\mathbf{R}^d)$, given by $\mathrm{Op}_{\varphi(t)}(a)$, with symbol $a(x,\xi) = 1$ and a family of phase function $\varphi(t)$, parameterized by t, given by $\varphi(t,x,\xi) = it|\xi|^\rho$. That is, $\mathrm{Op}_{\varphi(t)}(a)$ is the operator

$$f \mapsto (2\pi)^{-d/2} \int_{\mathbf{R}^d} e^{it|\xi|^\rho} \widehat{f}(\xi) e^{i\langle x,\xi\rangle} \, d\xi,$$

for admissible f.

We remark that in [2] it is proved that $\mathrm{Op}_{\varphi(t)}(a)$ is uniquely extendable to a continuous map on $M^{p,q}(\mathbf{R}^d)$. In particular, Theorem 4.3 holds for $\omega_0 = \omega_1 = \omega_2$, without the loss of regularity, imposed by the conditions on m and μ. We also remark that the latter result was proved in the case $\rho = 2$ already in [42]. (See also [12,45] and the references therein for other related results and approaches.)

The continuity properties of SG pseudo-differential operators on modulation spaces, as well as the propagation of the global wave-front sets under their action, shortly recalled in the next Sect. 5, motivate the next definition, originally given in [26].

Definition 4.5 Let $r, \rho \in [0, 1]$, $t \in \mathbf{R}$, and let $\mathcal{B}$ be a topological vector space of distributions on $\mathbf{R}^d$ such that

$$\mathscr{S}(\mathbf{R}^d) \subseteq \mathcal{B} \subseteq \mathscr{S}'(\mathbf{R}^d)$$

with continuous embeddings. Then $\mathcal{B}$ is called SG-*admissible (with respect to* (r, ρ)) when $\mathrm{Op}_t(a)$ maps $\mathcal{B}$ continuously into itself, for every $a \in \mathrm{SG}^{0,0}_{r,\rho}(\mathbf{R}^d)$. If $\mathcal{B}$ and $\mathcal{C}$ are SG-admissible with respect to (r, ρ), and $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, then the pair $(\mathcal{B}, \mathcal{C})$ is called SG-*ordered (with respect to* (r, ρ, ω_0)), when the mappings

$$\mathrm{Op}_t(a) : \mathcal{B} \to \mathcal{C} \quad \text{and} \quad \mathrm{Op}_t(b) : \mathcal{C} \to \mathcal{B}$$

are continuous for every $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ and $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$.

The following definition, which extends Definition 4.5 to the case of generalized Fourier integral operators, is justified by Theorems 4.1 and 4.3.

Definition 4.6 Let $\varphi \in \mathrm{SG}^{1,1}_{1,1}(\mathbf{R}^{2d})$ be a regular phase function, and $\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{C}, \mathcal{C}_1, \mathcal{C}_2$, be SG-admissible with respect to r, ρ and d. Also let $\omega_0, \omega_1, \omega_2 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, and $\Omega \subseteq \mathbf{R}^d$ be open. Then the pair $(\mathcal{B}, \mathcal{C})$ is called *weakly-I* SG-*ordered (with respect to* $(r, \rho, \omega_0, \varphi, \Omega)$), when the mapping

$$\mathrm{Op}_\varphi(a) : \mathcal{B} \to \mathcal{C}$$

is continuous for every $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ which is supported outside $\mathbf{R}^d \times \Omega$. Similarly, the pair $(\mathcal{B}, \mathcal{C})$ is called *weakly-II* SG-*ordered (with respect to* $(r, \rho, \omega_0, \varphi, \Omega)$), when the mapping

$$\mathrm{Op}^*_\varphi(b) : \mathcal{C} \to \mathcal{B}$$

is continuous for every $b \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ which is supported outside $\Omega \times \mathbf{R}^d$. Furthermore, $(\mathcal{B}_1, \mathcal{C}_1, \mathcal{B}_2, \mathcal{C}_2)$ are called SG-*ordered (with respect to* $r, \rho, \omega_1, \omega_2, \varphi$, *and* Ω), when $(\mathcal{B}_1, \mathcal{C}_1)$ is a weakly-I SG-ordered pair with respect to $(r, \rho, \omega_1, \varphi, \Omega)$, and $(\mathcal{B}_2, \mathcal{C}_2)$ is a weakly-II SG-ordered pair with respect to $(r, \rho, \omega_2, \varphi, \Omega)$.

Remark 4.7 Let σ_1, σ_2, p, m and μ be the same as in Theorem 4.1. Then it follows from [26, Remark 1.9], and Theorem 4.1, Remark 4.2, and Theorem 4.3, that the following is true.

(1) $(H^p_{\sigma_1,\sigma_2}, H^p_{\sigma_1-\mu,\sigma_2-m})$ are weakly-I SG-ordered with respect to $(r, \rho, \omega_0, \varphi, \Omega)$, when

$$\omega_0(x, \xi) = \langle x\rangle^m \langle \xi\rangle^\mu \quad \text{and} \quad \Omega = B_\varepsilon(0), \varepsilon > 0.$$

(2) If p, m, μ and ω_j, $j = 0, 1, 2$ are the same as in Theorem 4.3, then it follows that $(M^p_{(\omega_1)}, M^p_{(\omega_2)})$ are weakly-I SG-ordered with respect to $(r, \rho, \omega_0, \varphi, \emptyset)$. If, in addition, $\varphi(x, \xi) = \langle x, \xi \rangle$ and $\mathscr{B}$ is an invariant BF-space, then $(M(\omega_1, \mathscr{B}), M(\omega_2, \mathscr{B}))$ are SG-ordered with respect to ω_0.

5 Propagation Results for Global Wave-Front Sets and Generalised FIOs of SG Type

We first recall the definition of the global wave-front sets, given in [26]. The content of Sect. 5.1 again comes from [27]. In Sect. 5.2 we prove our main results about the propagation of singularities in the SG context, under the action of the Fourier integral operators described above.

5.1 Global Wave-Front Sets

Here we recall the definition given in [26] of global wave-front sets for temperate distributions with respect to appropriate Banach or Fréchet spaces and state some of their properties (see also [27]). First we recall the definitions of the sets of characteristic points. Notice that if $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$, then

$$|a(x, \xi)| \lesssim \omega_0(x, \xi).$$

On the other hand, a is invertible, in the sense that $1/a$ is a symbol in $\mathrm{SG}^{(1/\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$, if and only if

$$\omega_0(x, \xi) \lesssim |a(x, \xi)|. \tag{5.1}$$

We need to deal with the situations where (5.1) holds only in certain (conic-shaped) subset of $\mathbf{R}^d \times \mathbf{R}^d$. Here we let Ω_m, $m = 1, 2, 3$, be the sets

$$\begin{aligned} \Omega_1 &= \mathbf{R}^d \times (\mathbf{R}^d \setminus 0), \quad \Omega_2 = (\mathbf{R}^d \setminus 0) \times \mathbf{R}^d, \\ \Omega_3 &= (\mathbf{R}^d \setminus 0) \times (\mathbf{R}^d \setminus 0), \end{aligned} \tag{5.2}$$

Definition 5.1 Let $r, \rho \geq 0$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, Ω_m, $m = 1, 2, 3$ be as in (5.2), and let $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$.

(1) a is called *locally* or *type*-1 *invertible* with respect to ω_0 at the point $(x_0, \xi_0) \in \Omega_1$, if there exist a neighbourhood X of x_0, an open conical neighbourhood Γ of ξ_0 and a positive constant R such that (5.1) holds for $x \in X$, $\xi \in \Gamma$ and $|\xi| \geq R$.
(2) a is called *Fourier-locally* or *type*-2 *invertible* with respect to ω_0 at the point $(x_0, \xi_0) \in \Omega_2$, if there exist an open conical neighbourhood Γ of x_0, a neighbourhood X of ξ_0 and a positive constant R such that (5.1) holds for $x \in \Gamma$, $|x| \geq R$ and $\xi \in X$.

(3) a is called *oscillating* or *type*-3 *invertible* with respect to ω_0 at the point $(x_0, \xi_0) \in \Omega_3$, if there exist open conical neighbourhoods Γ_1 of x_0 and Γ_2 of ξ_0, and a positive constant R such that (5.1) holds for $x \in \Gamma_1$, $|x| \geq R$, $\xi \in \Gamma_2$ and $|\xi| \geq R$.

If $m \in \{1, 2, 3\}$ and a is *not* type-m invertible with respect to ω_0 at $(x_0, \xi_0) \in \Omega_m$, then (x_0, ξ_0) is called *type-m characteristic* for a with respect to ω_0. The set of type-m characteristic points for a with respect to ω_0 is denoted by $\mathrm{Char}^m_{(\omega_0)}(a)$.

The *(global) set of characteristic points* (the characteristic set), for a symbol $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ with respect to ω_0 is defined as

$$\mathrm{Char}(a) = \mathrm{Char}_{(\omega_0)}(a) = \mathrm{Char}^1_{(\omega_0)}(a) \bigcup \mathrm{Char}^2_{(\omega_0)}(a) \bigcup \mathrm{Char}^3_{(\omega_0)}(a).$$

Remark 5.2 In the case $\omega_0 = 1$ we exclude the phrase "with respect to ω_0" in Definition 5.1. For example, $a \in \mathrm{SG}^{0,0}_{r,\rho}(\mathbf{R}^{2d})$ is *type*-1 *invertible* at $(x_0, \xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \backslash 0)$ if $(x_0, \xi_0) \notin \mathrm{Char}^1_{(\omega_0)}(a)$ with $\omega_0 = 1$. This means that there exist a neighbourhood X of x_0, an open conical neighbourhood Γ of ξ_0 and $R > 0$ such that (5.1) holds for $\omega_0 = 1$, $x \in X$ and $\xi \in \Gamma$ satisfies $|\xi| \geq R$.

In the next definition we introduce different classes of cutoff functions (see also Definition 1.9 in [25]).

Definition 5.3 Let $X \subseteq \mathbf{R}^d$ be open, $\Gamma \subseteq \mathbf{R}^d \setminus 0$ be an open cone, $x_0 \in X$ and let $\xi_0 \in \Gamma$.

(1) A smooth function φ on $\mathbf{R}^d$ is called a *cutoff (function)* with respect to x_0 and X, if $0 \leq \varphi \leq 1$, $\varphi \in C_0^\infty(X)$ and $\varphi = 1$ in an open neighbourhood of x_0. The set of cutoffs with respect to x_0 and X is denoted by $\mathscr{C}_{x_0}(X)$ or $\mathscr{C}_{x_0}$.
(2) A smooth function ψ on $\mathbf{R}^d$ is called a *directional cutoff (function)* with respect to ξ_0 and Γ, if there is a constant $R > 0$ and open conical neighbourhood $\Gamma_1 \subseteq \Gamma$ of ξ_0 such that the following is true:
   - $0 \leq \psi \leq 1$ and $\operatorname{supp} \psi \subseteq \Gamma$;
   - $\psi(t\xi) = \psi(\xi)$ when $t \geq 1$ and $|\xi| \geq R$;
   - $\psi(\xi) = 1$ when $\xi \in \Gamma_1$ and $|\xi| \geq R$.

   The set of directional cutoffs with respect to ξ_0 and Γ is denoted by $\mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma)$ or $\mathscr{C}^{\mathrm{dir}}_{\xi_0}$.

Remark 5.4 Let $X \subseteq \mathbf{R}^d$ be open and $\Gamma, \Gamma_1, \Gamma_2 \subseteq \mathbf{R}^d \backslash 0$ be open cones. Then the following is true.

(1) if $x_0 \in X$, $\xi_0 \in \Gamma$, $\varphi \in \mathscr{C}_{x_0}(X)$ and $\psi \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma)$, then $c_1 = \varphi \otimes \psi$ belongs to $\mathrm{SG}^{0,0}_{1,1}(\mathbf{R}^{2d})$, and is type-1 invertible at (x_0, ξ_0);
(2) if $x_0 \in \Gamma$, $\xi_0 \in X$, $\psi \in \mathscr{C}^{\mathrm{dir}}_{x_0}(\Gamma)$ and $\varphi \in \mathscr{C}_{\xi_0}(X)$, then $c_2 = \varphi \otimes \psi$ belongs to $\mathrm{SG}^{0,0}_{1,1}(\mathbf{R}^{2d})$, and is type-2 invertible at (x_0, ξ_0);
(3) if $x_0 \in \Gamma_1$, $\xi_0 \in \Gamma_2$, $\psi_1 \in \mathscr{C}^{\mathrm{dir}}_{x_0}(\Gamma_1)$ and $\psi_2 \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma_2)$, then $c_3 = \psi_1 \otimes \psi_2$ belongs to $\mathrm{SG}^{0,0}_{1,1}(\mathbf{R}^{2d})$, and is type-3 invertible at (x_0, ξ_0).

The next proposition shows that $\mathrm{Op}_t(a)$ for $t \in \mathbf{R}$ satisfies convenient invertibility properties of the form

$$\mathrm{Op}_t(a) \circ \mathrm{Op}_t(b) = \mathrm{Op}_t(c) + \mathrm{Op}_t(h), \tag{5.3}$$

outside the set of characteristic points for a symbol a. Here $\mathrm{Op}_t(b)$, $\mathrm{Op}_t(c)$ and $\mathrm{Op}_t(h)$ have the roles of "local inverse", "local identity" and smoothing operators respectively. From these statements it also follows that our set of characteristic points in Definition 5.1 are related to those in [19,35]. We let $\mathbb{I}_m$, $m = 1, 2, 3$, be the sets

$$\mathbb{I}_1 \equiv [0,1] \times (0,1],\ \mathbb{I}_2 \equiv (0,1] \times [0,1],\ \mathbb{I}_3 \equiv (0,1] \times (0,1] = \mathbb{I}_1 \cap \mathbb{I}_2. \tag{5.4}$$

which will be useful in the sequel.

Proposition 5.5 *Let $m \in \{1,2,3\}$, $(r,\rho) \in \mathbb{I}_m$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ and let $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$. Also let Ω_m be as in (5.2), $(x_0,\xi_0) \in \Omega_m$, and let (r_0,ρ_0) be equal to $(r,0)$, $(0,\rho)$ and (r,ρ) when m is equal to 1, 2 and 3, respectively. Then the following conditions are equivalent:*

(1) $(x_0,\xi_0) \notin \mathrm{Char}^m_{(\omega_0)}(a)$;
(2) *there is an element $c \in \mathrm{SG}^{0,0}_{r,\rho}$ which is type-m invertible at (x_0,ξ_0), and an element $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$ such that $ab = c$;*
(3) *(5.3) holds for some $c \in \mathrm{SG}^{0,0}_{r,\rho}$ which is type-m invertible at (x_0,ξ_0), and some elements $h \in \mathrm{SG}^{-r_0,-\rho_0}_{r,\rho}$ and $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$;*
(4) *(5.3) holds for some $c_m \in \mathrm{SG}^{0,0}_{r,\rho}$ in Remark 5.4 which is type-m invertible at (x_0,ξ_0), and some elements h and $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$, where $h \in \mathscr{S}$ when $m \in \{1,3\}$ and $h \in \mathrm{SG}^{-\infty,0}$ when $m = 2$.*
Furthermore, if $t = 0$, then the supports of b and h can be chosen to be contained in $X \times \mathbf{R}^d$ when $m = 1$, in $\Gamma \times \mathbf{R}^d$ when $m = 2$, and in $\Gamma_1 \times \mathbf{R}^d$ when $m = 3$.

We now introduce the complements of the wave-front sets. More precisely, let Ω_m, $m \in \{1,2,3\}$, be given by (5.2), $\mathcal{B}$ be an SG-admissible (Banach or Fréchet) space, and let $f \in \mathscr{S}'(\mathbf{R}^d)$. We recall that $\mathscr{S}(\mathbf{R}^d) \subseteq \mathcal{B} \subseteq \mathscr{S}'(\mathbf{R}^d)$ by the definitions, and that $\mathscr{S}(\mathbf{R}^d)$, Sobolev–Kato spaces and, more generally, modulation spaces, are SG-admissible, see [26,27]. Then the point $(x_0,\xi_0) \in \Omega_m$ is called *type-m* regular for f with respect to $\mathcal{B}$, if

$$\mathrm{Op}(c_m) f \in \mathcal{B}, \tag{5.5}$$

for some c_m in Remark 5.4. The set of all type-m regular points for f with respect to $\mathcal{B}$, is denoted by $\Theta^m_{\mathcal{B}}(f)$.

Definition 5.6 Let $m \in \{1,2,3\}$, Ω_m be as in (5.2), and let $\mathcal{B}$ be an SG-admissible (Banach or Fréchet) space.

(1) The *type-m wave-front set* of $f \in \mathscr{S}'(\mathbf{R}^d)$ with respect to $\mathcal{B}$ is the complement of $\Theta^m_{\mathcal{B}}(f)$ in Ω_m, and is denoted by $\mathrm{WF}^m_{\mathcal{B}}(f)$;

(2) The *global wave-front set* $\mathrm{WF}_{\mathcal{B}}(f) \subseteq (\mathbf{R}^d \times \mathbf{R}^d)\backslash 0$ is the set

$$\mathrm{WF}_{\mathcal{B}}(f) \equiv \mathrm{WF}^1_{\mathcal{B}}(f) \bigcup \mathrm{WF}^2_{\mathcal{B}}(f) \bigcup \mathrm{WF}^3_{\mathcal{B}}(f).$$

The sets $\mathrm{WF}^1_{\mathcal{B}}(f)$, $\mathrm{WF}^2_{\mathcal{B}}(f)$ and $\mathrm{WF}^3_{\mathcal{B}}(f)$ in Definition 5.6, are also called the *local*, *Fourier-local* and *oscillating* wave-front set of f with respect to $\mathcal{B}$.

Remark 5.7 Let Ω_m, $m = 1, 2, 3$ be the same as in (5.2).

(1) If $\Omega \subseteq \Omega_1$, and $(x_0, \xi_0) \in \Omega \iff (x_0, \sigma\xi_0) \in \Omega$ for $\sigma \geq 1$, then Ω is called *1-conical*;
(2) If $\Omega \subseteq \Omega_2$, and $(x_0, \xi_0) \in \Omega \iff (sx_0, \xi_0) \in \Theta^2_{\mathcal{B}}(f)$ for $s \geq 1$, then Ω is called *2-conical*;
(3) If $\Omega \subseteq \Omega_3$, and $(x_0, \xi_0) \in \Omega \iff (sx_0, \sigma\xi_0) \in \Omega$ for $s, \sigma \geq 1$, then Ω is called *3-conical*.

By (5.5) and the paragraph before Definition 5.6, it follows that if $m = 1, 2, 3$, then $\Theta^m_{\mathcal{B}}(f)$ is m-conical. The same holds for $\mathrm{WF}^m_{\mathcal{B}}(f)$, $m = 1, 2, 3$, by Definition 5.6, noticing that, for any $x_0 \in \mathbf{R}^r \setminus \{0\}$, any open cone $\Gamma \ni x_0$, and any $s > 0$, $\mathscr{C}^{\mathrm{dir}}_{x_0}(\Gamma) = \mathscr{C}^{\mathrm{dir}}_{sx_0}(\Gamma)$. For any $R > 0$ and $m \in \{1, 2, 3\}$, we set

$$\begin{aligned}
\Omega_{1,R} &\equiv \{\, (x, \xi) \in \Omega_1 \,;\, |\xi| \geq R \,\}, \quad \Omega_{2,R} \equiv \{\, (x, \xi) \in \Omega_2 \,;\, |x| \geq R \,\}, \\
\Omega_{3,R} &\equiv \{\, (x, \xi) \in \Omega_3 \,;\, |x|, |\xi| \geq R \,\}
\end{aligned}$$

Evidently, $\Omega_{m,R}$ is m-conical for every $m \in \{1, 2, 3\}$.

The next result describes the relation between "regularity with respect to $\mathcal{B}$" of temperate distributions and global wave-front sets.

Proposition 5.8 *Let $\mathcal{B}$ be* SG*-admissible, and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Then*

$$f \in \mathcal{B} \iff \mathrm{WF}_{\mathcal{B}}(f) = \emptyset.$$

For the sake of completeness, we recall that microlocality and microellipticity hold for these global wave-front sets and pseudo-differential operators in $\mathrm{Op}(\mathrm{SG}^{(\omega_0)}_{r,\rho})$, see [26]. This implies that operators which are elliptic with respect to $\omega_0 \in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$ when $0 < r, \rho \leq 1$ preserve the global wave-front set of temperate distributions. The next result is an immediate corollary of microlocality and microellipticity for operators in $\mathrm{Op}(\mathrm{SG}^{(\omega_0)}_{r,\rho})$:

Proposition 5.9 *Let $m \in \{1, 2, 3\}$, $(r, \rho) \in \mathbb{I}_m$, $t \in \mathbf{R}$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ be* SG*-elliptic with respect to ω_0 and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Moreover, let $(\mathcal{B}, \mathcal{C})$ be an* SG*-ordered pair with respect to ω_0. Then*

$$\mathrm{WF}^m_{\mathcal{C}}(\mathrm{Op}_t(a) f) = \mathrm{WF}^m_{\mathcal{B}}(f).$$

5.2 Action of Generalised FIOs of SG Type on Global Wave-Front Sets

We let ϕ be the canonical transformation of $T^*\mathbf{R}^d$ into itself generated by the phase function $\varphi \in \mathfrak{F}^r$. This means that $\phi = (\phi_1, \phi_2)$ is the smooth function on $T^*\mathbf{R}^d$ into itself, defined by the relations

$$(x, \xi) = \phi(y, \eta) \Longleftrightarrow \begin{cases} y = \varphi'_\xi(x, \eta) = \varphi'_\eta(x, \eta), \\ \xi = \varphi'_x(x, \eta), \end{cases} \tag{5.6}$$

As we have seen in Sect. 3.5, such transformations appear in the Egorov's theorem, through which we prove Theorems 5.14 and Corollaries 5.15 and 5.16 below. This justifies the following definition of admissibility of phase functions. Namely, the latter are required to generate transformations of the type (5.6) which "preserve the shape" of the different kinds of neighborhoods appearing in the Definition 5.1 of the set of characteristic points.

Definition 5.10 Let $\varphi \in \mathfrak{F}^r$ and let ϕ be the canonical transformation (5.6), generated by φ. Let $m \in \{1, 2, 3\}$ and Ω_m be as in (5.2).

(1) φ is called *1-admissible* at $(y_0, \eta_0) \in \Omega_1$ if, for every 1-cone $X \times \Gamma$ containing $\phi(y_0, \eta_0)$ and $r > 0$, there is a 1-cone $Y \times \Gamma_0$ containing (y_0, η_0) and $R > 0$ such that

$$\phi(y, \eta) \in (X \times \Gamma) \bigcap \Omega_{1,r} \quad \text{when} \quad (y, \eta) \in (Y \times \Gamma_0) \bigcap \Omega_{1,R};$$

(2) φ is called *2-admissible* at $(y_0, \eta_0) \in \Omega_2$ if, for every 2-cone $\Gamma \times X$ containing $\phi(y_0, \eta_0)$ and $r > 0$, there is a 2-cone $\Gamma_0 \times Y$ containing (y_0, η_0) and $R > 0$ such that

$$\phi(y, \eta) \in (\Gamma \times X) \bigcap \Omega_{2,r} \quad \text{when} \quad (y, \eta) \in (\Gamma_0 \times Y) \bigcap \Omega_{2,R};$$

(3) φ is called *3-admissible* at $(y_0, \eta_0) \in \Omega_3$ if, for every 3-cone $\Gamma_1 \times \Gamma_2$ containing $\phi(y_0, \eta_0)$ and $r > 0$, there is a 3-cone $\Gamma_{0,1} \times \Gamma_{0,2}$ containing (y_0, η_0) and $R > 0$ such that

$$\phi(y, \eta) \in (\Gamma_1 \times \Gamma_2) \bigcap \Omega_{3,r} \quad \text{when} \quad (y, \eta) \in (\Gamma_{0,1} \times \Gamma_{0,2}) \bigcap \Omega_{3,R}.$$

Furthermore, φ is called *m-admissible* if it is m-admissible at all points $(y, \eta) \in \Omega_m$, and φ is called *admissible* if it is m-admissible for all $m = 1, 2, 3$.

Remark 5.11 Notice that the inverse transformation ϕ^{-1} is defined as in (5.6), by exchanging the role of (x, ξ) and (y, η). If φ is m-admissible, $m = 1, 2, 3$, then both ϕ and ϕ^{-1} satisfy the corresponding property in Definition 5.10.

Remark 5.12 Let φ be m-admissible, $m = 1, 2, 3$, and let $\omega_0 \in \mathscr{P}_{1,1}(\mathbf{R}^{2d})$ be invariant with respect to the canonical transformation (5.6). For any $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$, setting $\widetilde{\omega}_0 = \omega_0 \circ \phi$, we have

$$(y_0, \eta_0) \in \mathrm{Char}^m_{(\widetilde{\omega}_0)}(a \circ \phi) \quad \Longleftrightarrow \quad (x_0, \xi_0) = \phi(y_0, \eta_0) \in \mathrm{Char}^m_{(\omega_0)}(a).$$

By Remark 5.11, similar properties hold with ϕ^{-1} in place of ϕ.

Remark 5.13 Let φ_m, $m = 1, 2, 3$, be phase functions such that

- $\xi \mapsto \varphi_1(x, \xi)$ is homogeneous of order 1 for large $|\xi|$;
- $x \mapsto \varphi_2(x, \xi)$ is homogeneous of order 1 for large $|x|$;
- $x \mapsto \varphi_3(x, \xi)$ and $\xi \mapsto \varphi_3(x, \xi)$ are homogeneous of order 1 for large $|x|$ and $|\xi|$.

Such phase functions are common in the literature. An example of admissible phase functions, which is not necessarily homogeneous, is given by the so called SG-*classical phase functions*. Families of such objects, smoothly depending on a parameter $t \in [-T, T]$, $T > 0$, are obtained by solving Cauchy problems associated with classical SG-hyperbolic systems with diagonal principal part.

In fact, omitting the dependence on the *time variable* t, an SG-classical phase functions φ admits expansions in terms which are homogeneous with respect to x, respectively ξ, satisfying suitable compatibility relations, see, e.g., [18,19]. In particular, φ admits a principal symbol, given by a triple $(\varphi_1, \varphi_2, \varphi_3)$, that is, it can be written as

$$\begin{aligned} \varphi(x, \xi) = \chi(\xi)\, \varphi_1(x, \xi) + \chi(x)(\varphi_2(x, \xi) - \chi(\xi)\, \varphi_3(x, \xi)) \\ \mod \ \mathrm{SG}^{0,0}_{1,1}(\mathbf{R}^{2d}). \end{aligned} \tag{5.7}$$

In (5.7), χ is a 0-excision function, while $\chi(\xi)\, \varphi_1(x, \xi)$, $\chi(x)\, \varphi_2(x, \xi)$, $\chi(\xi)\, \chi(x)\, \varphi_3(x, \xi) \in \mathrm{SG}^{1,1}_{1,1}(\mathbf{R}^{2d})$, where φ_1 is 1-homogeneous with respect to the variable ξ, φ_2 is 1-homogeneous with respect to the variable x, and φ_3 is 1-homogeneous with respect to each one of the variables x, ξ. Observe that then

$$\begin{aligned} \varphi(x, \xi) &= \chi(\xi)\, \varphi_1(x, \xi) \quad \mod \ \mathrm{SG}^{1,0}_{1,1}(\mathbf{R}^{2d}), \\ \varphi(x, \xi) &= \chi(x)\, \varphi_2(x, \xi) \quad \mod \ \mathrm{SG}^{0,1}_{1,1}(\mathbf{R}^{2d}), \\ \varphi(x, \xi) &= \chi(x)\, \chi(\xi)\, \varphi_3(x, \xi) \quad \mod \ \mathrm{SG}^{0,1}_{1,1}(\mathbf{R}^{2d}) + \mathrm{SG}^{1,0}_{1,1}(\mathbf{R}^{2d}). \end{aligned} \tag{5.8}$$

The homogeneity of the leading terms in (5.8) implies, in particular,

$$\begin{aligned} \varphi'_x(x, \xi) &= |\xi| \left[\varphi'_{1,x}\left(x, \frac{\xi}{|\xi|}\right) + |\xi|^{-1} r_1(x, \xi) \right] \text{ for } |\xi| > R, \\ \varphi'_\xi(x, \xi) &= |x| \left[\varphi'_{2,\xi}\left(\frac{x}{|x|}, \xi\right) + |x|^{-1} r_2(x, \xi) \right] \text{ for } |x| > R, \\ \varphi'_x(x, \xi) &= |\xi| \left[\varphi'_{3,x}\left(\frac{x}{|x|}, \frac{\xi}{|\xi|}\right) + |\xi|^{-1} (r_{31}(x, \xi) + s_{31}(x, \xi)) \right] \\ &\qquad \text{for } |x|, |\xi| > R, \\ \varphi'_\xi(x, \xi) &= |x| \left[\varphi'_{3,\xi}\left(\frac{x}{|x|}, \frac{\xi}{|\xi|}\right) + |x|^{-1} (r_{32}(x, \xi) + s_{32}(x, \xi)) \right] \\ &\qquad \text{for } |x|, |\xi| > R, \end{aligned} \tag{5.9}$$

with $r_1, r_2, r_{31}, r_{32} \in \mathrm{SG}^{0,0}_{1,1}(\mathbf{R}^{2d})$, $s_{31} \in \mathrm{SG}^{-1,1}_{1,1}(\mathbf{R}^{2d})$, $s_{32} \in \mathrm{SG}^{1,-1}_{1,1}(\mathbf{R}^{2d})$. By the properties of generalised SG symbols and (5.9) it is possible to prove that all the conditions in Definition 5.10 are fulfilled.

We can now state the first of our main results concerning the propagation of (global) singularities under the action of the generalised SG FIOs.

Theorem 5.14 *Let $\varphi \in \mathfrak{F}^r$ be m-admissible, $m \in \{1, 2, 3\}$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$, supported outside $\mathbf{R}^d \times \Omega$, $\Omega \subseteq \mathbf{R}^d$ open. Assume that ω_0 is ϕ-invariant, where ϕ is as in Theorem 3.16. Assume also that a is* SG*-elliptic and $(\mathcal{B}, \mathcal{C})$ is a weakly-I* SG*-ordered pair with respect to $(r, \rho, \omega_0, \varphi, \Omega)$. Then*

$$\mathrm{WF}^m_{\mathcal{C}}(\mathrm{Op}_\varphi(a) f) = \phi(\mathrm{WF}^m_{\mathcal{B}}(f)), \quad f \in \mathscr{S}'(\mathbf{R}^d), \tag{5.10}$$

where ϕ is the canonical transformation (5.6), generated by φ.

Proof We only prove the result for $m = 3$. The other cases follow by similar arguments and are left for the reader. Let $(y_0, \eta_0) = \phi^{-1}(x_0, \xi_0) \in \Theta^m_{\mathcal{B}}(f)$, $m \in \{1, 2, 3\}$, and let $c_m \in \mathrm{SG}^{0,0}_{1,1}$ be a symbol as in (5.5) and Remark 5.4 such that $\mathrm{Op}(c_m)u \in \mathcal{B}$. Recalling Remark 2.14, the weight $\omega(x, \xi) = \vartheta_{0,0}(x, \xi) = 1 \in \mathscr{P}_{1,1}$ is invariant with respect to any SG diffeomorphism with SG^0 parameter dependence. Let $A = \mathrm{Op}_\varphi(a)$, $C_m = \mathrm{Op}(c_m)$, and let B be a parametrix for A. Then for some q_m we have

$$Q_m = A \circ C_m \circ B, \quad Q_m = \mathrm{Op}(q_m),$$

or equivalently,

$$Q_m \circ A = A \circ C_m \mod \mathrm{Op}(\mathrm{SG}^{-\infty,-\infty}).$$

By Theorem 3.18 and (5.6), we have $q_m = c_m \circ \phi^{-1} \mod \mathrm{SG}^{-1,-1}_{1,1}$, which implies $q_m \in \mathrm{SG}^{0,0}_{1,1}$. Then $(x_0, \xi_0) \in \Theta^m_{\mathcal{C}}(Af)$, since $Q_m(Af) \equiv A(C_m f) \in \mathcal{C}$ by the hypotheses on $(\mathcal{B}, \mathcal{C})$. This means that

$$\phi(\Theta^m_{\mathcal{B}}(f)) \subseteq \Theta^m_{\mathcal{C}}(Af). \tag{5.11}$$

Complementing (5.11) with respect to Ω_m, repeating a similar argument starting from Af, recalling Remark 5.12 and that ϕ is a diffeomorphism, we finally obtain (5.10). □

The next result is proved in a similar fashion. In fact, with notation analogous to the one used in the proof of Theorem 5.14, denoting $B = \mathrm{Op}^*_\varphi(b)$, we have that $Q_m = B \circ C_m \circ B^{-1}$ satisfies $\mathrm{Sym}\,(Q_m) = c_m \circ \phi$ modulo lower order terms. It is then enough to recall Remark 5.11. Evidently, when one deals with SG-ordered spaces $(\mathcal{B}_1, \mathcal{C}_1, \mathcal{B}_2, \mathcal{C}_2)$, both (5.10) and (5.12) hold, as stated in Corollary 5.16.

Corollary 5.15 *Let $\varphi \in \mathfrak{F}^r$ be m-admissible, $m \in \{1,2,3\}$, $\omega_0 \in \mathscr{P}_{1,1}(\mathbf{R}^{2d})$, $b \in \mathrm{SG}^{(\omega_0)}_{1,1}(\mathbf{R}^{2d})$, supported outside $\Omega \times \mathbf{R}^d$, $\Omega \subseteq \mathbf{R}^d$ open. Assume that ω_0 is ϕ^{-1}-invariant, where ϕ is as in Theorem 3.16. Assume also that b is* SG*-elliptic and that $(\mathcal{B},\mathcal{C})$ is a weakly-II* SG*-ordered pair with respect to $(r,\rho,\omega_0,\varphi,\Omega)$. Then*

$$\mathrm{WF}^m_{\mathcal{B}}(\mathrm{Op}^*_\varphi(b)f) = \phi^{-1}(\mathrm{WF}^m_{\mathcal{C}}(f)), \quad f \in \mathscr{S}'(\mathbf{R}^d), \tag{5.12}$$

with the inverse ϕ^{-1} of the canonical transformation (5.6).

Corollary 5.16 *Let $\varphi \in \mathfrak{F}^r$ be m-admissible, $m \in \{1,2,3\}$, $\omega_1, \omega_2 \in \mathscr{P}_{1,1}(\mathbf{R}^{2d})$. Moreover, let $a \in \mathrm{SG}^{(\omega_1)}_{r,\rho}(\mathbf{R}^{2d})$, $b \in \mathrm{SG}^{(\omega_2)}_{r,\rho}(\mathbf{R}^{2d})$, with a supported outside $\mathbf{R}^d \times \Omega$, b supported outside $\Omega \times \mathbf{R}^d$, respectively, where $\Omega \subseteq \mathbf{R}^d$ is open. Assume that a and b are* SG*-elliptic and that $(\mathcal{B}_1,\mathcal{C}_1,\mathcal{B}_2,\mathcal{C}_2)$ are* SG*-ordered with respect to*

$$r,\ \rho,\ \omega_1,\ \omega_2,\ \varphi \text{ and } \Omega.$$

If $f \in \mathscr{S}'(\mathbf{R}^d)$, then

$$\mathrm{WF}^m_{\mathcal{C}_1}(\mathrm{Op}_\varphi(a)f) = \phi(\mathrm{WF}^m_{\mathcal{B}_1}(f)),$$

and

$$\mathrm{WF}^m_{\mathcal{B}_2}(\mathrm{Op}^*_\varphi(b)f) = \phi^{-1}(\mathrm{WF}^m_{\mathcal{C}_2}(f)),$$

with the canonical transformation ϕ in (5.6) and its inverse ϕ^{-1}, provided that ω_1 and ω_2 satisfy the invariance properties required in Theorem 5.14 and Corollary 5.15, respectively.

The next result generalizes Theorem 5.14 and Corollaries 5.15 and 5.16 to the case where the involved amplitudes are not SG-elliptic. In such a situation, the set of admissible phase functions needs to be slightly restricted, similarly to the calculus of Fourier integral operators developed in [36]. Such restriction is not very harmful, since the phase functions we are mostly interested in are those appearing in the next Sect. 5.3, and it can be proved that they fulfill (5.14) below if a sufficiently small "time interval" $J' = [-T', T']$ is chosen. This can be easily verified by checking the technique of solution of the involved eikonal equations, see, e.g., [15,17,18,36]. Here the symbols satisfy

$$\operatorname{supp} a \subseteq \mathbf{R}^d \times \Omega^{\complement}, \quad \operatorname{supp} b \subseteq \Omega^{\complement} \times \mathbf{R}^d \tag{5.13}$$

for suitable open set Ω, where $\Omega^{\complement}$ equals $\mathbf{R}^d \setminus \Omega$, and the phase function satisfies

$$\begin{aligned} &|\langle x\rangle^{-1+|\alpha|}\langle\xi\rangle^{-1+|\beta|} D^\alpha_x D^\beta_\xi \kappa(x,\xi)| \le \tau, \quad x,\xi \in \mathbf{R}^d,\ |\alpha+\beta| \le 2, \\ &\text{where} \quad \kappa(x,\xi) = \varphi(x,\xi) - \langle x,\xi\rangle \in \mathrm{SG}^{1,1}_{1,1}(\mathbf{R}^{2d}). \end{aligned} \tag{5.14}$$

Theorem 5.17 *Let $\varphi \in \mathfrak{F}^r$ be m-admissible, $m \in \{1, 2, 3\}$, and fulfill* (5.14) *for a fixed $\tau \in (0, 1)$. Also let $\omega_1 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, $r, \rho \geq 1/2$, and let $a \in \mathrm{SG}^{(\omega_1)}_{r,\rho}(\mathbf{R}^{2d})$ satisfy* (5.13). *Finally assume that $(\mathcal{B}_1, \mathcal{C}_1)$ is a weakly-I* SG*-ordered pair with respect to $(r, \rho, \omega_1, \varphi, \Omega)$. Then*

$$\begin{aligned}&\mathrm{WF}^m_{\mathcal{C}_1}(\mathrm{Op}_\varphi(a) f) \subseteq \Lambda^m_{\mathcal{B}_1}(f),\\&\Lambda^m_{\mathcal{B}_1}(f) = \{(x, \xi) = \phi(y, \eta);\, (y, \eta) \in \mathrm{WF}^m_{\mathcal{B}_1}(f)\}^{\mathrm{con}_m}, \quad f \in \mathscr{S}'(\mathbf{R}^d),\end{aligned} \tag{5.15}$$

where ϕ is the canonical transformation generated by φ in (5.6).

Theorem 5.18 *Let $\varphi \in \mathfrak{F}^r$ be m-admissible, $m \in \{1, 2, 3\}$, and fulfill* (5.14) *for a fixed $\tau \in (0, 1)$. Also let $\omega_2 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, $r, \rho \geq 1/2$, and let $b \in \mathrm{SG}^{(\omega_2)}_{r,\rho}(\mathbf{R}^{2d})$ satisfy* (5.13). *Finally assume that $(\mathcal{B}_2, \mathcal{C}_2)$ is a weakly-II* SG*-ordered pair with respect to $(r, \rho, \omega_2, \varphi, \Omega)$ Then*

$$\begin{aligned}&\mathrm{WF}^m_{\mathcal{B}_2}(\mathrm{Op}^*_\varphi(b) f) \subseteq \Lambda^m_{\mathcal{C}_2}(f)^*,\\&\Lambda^m_{\mathcal{C}_2}(f)^* = \{(y, \eta) = \phi^{-1}(x, \xi);\, (x, \xi) \in \mathrm{WF}^m_{\mathcal{C}_2}(f)\}^{\mathrm{con}_m}, \quad f \in \mathscr{S}'(\mathbf{R}^d),\end{aligned} \tag{5.16}$$

where ϕ^{-1} is the inverse of the canonical transformation ϕ in (5.6).

Corollary 5.19 *Let $\varphi \in \mathfrak{F}^r$ be m-admissible, $m \in \{1, 2, 3\}$, and fulfill* (5.14) *for a fixed $\tau \in (0, 1)$. Also let $\omega_1, \omega_2 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, $r, \rho \geq 1/2$, and $a \in \mathrm{SG}^{(\omega_1)}_{r,\rho}(\mathbf{R}^{2d})$ and $b \in \mathrm{SG}^{(\omega_2)}_{r,\rho}(\mathbf{R}^{2d})$ satisfy* (5.13). *Assume also that $(\mathcal{B}_1, \mathcal{C}_1, \mathcal{B}_2, \mathcal{C}_2)$ are* SG*-ordered with respect to*

$$r,\ \rho,\ \omega_1, \omega_2,\ \varphi \text{ and } \Omega.$$

Then both (5.15) *and* (5.16) *hold.*

In the results above, V^{con_m} for $V \subseteq \Omega_m$, is the smallest m-conical subset of Ω_m which includes V, $m \in \{1, 2, 3\}$.

We prove only Theorem 5.17. Theorem 5.18 follows by similar arguments and is left for the reader.

Proof of Theorem 5.17 Since here we are dropping the ellipticity hypothesis on the amplitude a, we use only the composition results between generalised SG pseudo-differential operators and Fourier integral operators given in Sect. 3.3. That is, the proofs of the theorem again rely on the generalised SG asymptotic expansions discussed in [22], and on the properties of the admissible phase functions. We now prove (5.15) in detail for the case $m = 3$, by showing the opposite inclusion between the complements of the involved sets with respect to Ω_3. In the sequel, we write $\mathcal{B}$ and $\mathcal{C}$ in place of $\mathcal{B}_1$ and $\mathcal{C}_1$, respectively.

Let $(x_0, \xi_0) \notin \Lambda^3_{\mathcal{B}}(f)$ for $f \in \mathcal{B}$, and set $2N = \min\{|x_0|, |\xi_0|\}{>}0$. By its definition in (5.15), $\Lambda^3_{\mathcal{B}}(f)$ is a closed 3-conical set. Then, choosing $\varepsilon > 0$ sufficiently small, it is possible to find a 3-conical set of the form

$$\Gamma^{4\varepsilon,4\varepsilon,N/4}_{3,x_0,\xi_0} = \left\{(x,\xi)\in\mathbf{R}^{2d};\ \left|\frac{x}{|x|}-\frac{x_0}{|x_0|}\right|<4\varepsilon,\right. \left.\left|\frac{\xi}{|\xi|}-\frac{\xi_0}{|\xi_0|}\right|<4\varepsilon,\ |x|,|\xi|\geq\frac{N}{4}\right\}$$

such that $\Gamma^{4\varepsilon,4\varepsilon,N/4}_{3,x_0,\xi_0}\cap\Lambda^3_{\mathcal{B}}(f)=\emptyset$. Then, as it is also possible (see Sect. 5.1 above and [14]), pick $q\in \mathrm{SG}^{0,0}_{1,1,}$ such that

$$\operatorname{supp} q\subseteq\Gamma^{2\varepsilon,2\varepsilon,N/2}_{3,x_0,\xi_0} \text{ and } (x,\xi)\in\Gamma^{\varepsilon,\varepsilon,N}_{3,x_0,\xi_0}\Rightarrow q(x,\xi)=1.$$

We now observe that $(y_0,\eta_0)=\phi^{-1}(x_0,\xi_0)\notin \mathrm{WF}^3_{\mathcal{B}}(f)$, in view of the definition of $\Lambda^3_{\mathcal{B}}(f)$. Setting $2\widetilde{N}=\min\{|y_0|,|\eta_0|\}$, we can consider the subset of Ω_3 given by

$$W=\mathrm{WF}^3_{\mathcal{B}}(f)\cap\Omega_3^{\widetilde{N}}.$$

W is closed, and, by Remark 5.7 it is 3-conical. Then there exist two 3-conical neighborhoods U,V of W such that $W\subset V\subset U\subset\Omega_3$. For instance, for an arbitrarily small $\tilde\delta>0$, one can consider the coverings of W given by

$$\widetilde{U}=\bigcup_{(z_0,\zeta_0)\in W}\Gamma^{4\tilde\delta,4\tilde\delta,\widetilde{N}/4}_{3,z_0,\zeta_0},\quad \widetilde{V}=\bigcup_{(z_0,\zeta_0)\in W}\Gamma^{2\tilde\delta,2\tilde\delta,\widetilde{N}/2}_{3,z_0,\zeta_0}.$$

By a standard compactness argument on the unit sphere of $\mathbf{R}^d$, define V and U as suitable finite subcoverings extracted from $\widetilde{V}$ and $\widetilde{U}$, respectively. Since $\Gamma^{2\tilde\delta,2\tilde\delta,\widetilde{N}/4}_{3,z_0,\zeta_0}\subset\Gamma^{4\tilde\delta,4\tilde\delta,\widetilde{N}/4}_{3,z_0,\zeta_0}$, we get $W\subset V\subset U$, as desired. Then take a symbol $\chi\in\mathrm{SG}^{0,0}_{1,1}$ such that

$$\begin{aligned}&\operatorname{supp}\chi\subset U,\ (y,\eta)\in V\Rightarrow\chi(y,\eta)=1,\\&\operatorname{supp} q\cap\{(x,\xi)=\phi(y,\eta):(y,\eta)\in\operatorname{supp}\chi\}=\emptyset,\end{aligned}$$

which is possible by choosing $\tilde\delta$ small enough, in view of the hypotheses and of (3) in Definition 5.10. Indeed, we can start from a 3-conical neighbourhood $Z\supset\Lambda^3_{\mathcal{B}}(f)\cap\Omega_3^N$, obtained as a finite union of sets of the form $\Gamma^{2\varepsilon,2\varepsilon,N/2}_{3,t_0,\tau_0}$, $(t_0,\tau_0)\in\Lambda^3_{\mathcal{B}}(f)$, disjoint from $\Gamma^{4\varepsilon,4\varepsilon,N/4}_{3,x_0,\xi_0}$, by choosing $\varepsilon>0$ suitably small. Observing that all the involved sets are 3-conical, it is then possible to choose $\tilde\delta$ small enough such that $\phi(U)\subset Z$, and χ with the desired properties.

Let us now consider

$$\begin{aligned}[\mathrm{Op}(q)\circ\mathrm{Op}_\varphi(a)](f)&=[\mathrm{Op}(q)\circ\mathrm{Op}_\varphi(a)\circ\mathrm{Op}(1-\chi)]f\\&\quad+[\mathrm{Op}(q)\circ\mathrm{Op}_\varphi(a)\circ\mathrm{Op}(\chi)]f. \qquad (5.17)\end{aligned}$$

By Remark 2.14, the weight $\vartheta_{0,0}(x,\xi)=1$ is invariant with respect to any SG diffeomorphism with SG^0 parameter dependence.

For $C=\mathrm{Op}(q)\circ\mathrm{Op}_\varphi(a)\circ\mathrm{Op}(\chi)$, we apply Theorems 3.7 and 3.8, and find that $C=\mathrm{Op}_\varphi(c)\mathrm{Mod}\,\mathrm{Op}(\mathscr{S})$ with

$$c(x,\eta)\sim\sum p_{\alpha\beta kl}(x,\eta)\cdot(\partial_\xi^k q)(x,\varphi'_x(x,\eta))\cdot(\partial_\xi^\alpha\partial_x^\beta a)(x,\eta)\\ \cdot(\partial_x^l\chi)(\varphi'_\xi(x,\eta),\eta)\sim 0,$$

which implies that $C\colon\mathscr{S}'\to\mathscr{S}$. In fact, setting $\xi=\varphi'_x(x,\eta)$, $y=\varphi'_\xi(x,\eta)$, by (5.6) we have $(x,\xi)=\phi(y,\eta)$, and, by construction, $\mathrm{supp}\,\partial_\xi^k q\cap\phi(\mathrm{supp}\,\partial_x^l\chi)=\emptyset$. Now, setting

$$\Sigma=\{(y,\eta)\in\mathrm{supp}(1-\chi)\colon |y|,|\eta|\geq\widetilde{N}/2\},$$

again by construction we have $\Sigma\cap\mathrm{WF}^3_{\mathcal{B}}(f)=\emptyset$. Then there exist $p\in\mathrm{SG}^{0,0}_{1,1}$ such that $\mathrm{Op}(p)f\in\mathcal{B}$ and $p(x,\xi)\geq C>0$ on Σ, and $r,s\in\mathrm{SG}^{0,0}_{1,1}$ such that

$$\mathrm{Op}(r)-\mathrm{Op}(s)\circ\mathrm{Op}(p)\colon\mathscr{S}'\to\mathscr{S}, \tag{5.18}$$

with $r(x,\xi)\equiv 1$ for $|x|,|\xi|\geq\frac{N}{2}$ belonging to a 3-conical neighborhood of Σ. This can be proved by relying on the concept of SG-ellipticity with respect to a symbol (or local md-ellipticity, cf. [14], Ch. 2, §3). We can write

$$\begin{aligned}\mathrm{Op}(1-\chi)f&=[\mathrm{Op}(1-\chi)\circ\mathrm{Op}(1-r)]f\\ &\quad+[\mathrm{Op}(1-\chi)\circ[\mathrm{Op}(r)-\mathrm{Op}(s)\circ\mathrm{Op}(p))]f\\ &\quad+[\mathrm{Op}(1-\chi)\circ\mathrm{Op}(s)][\mathrm{Op}(p)f].\end{aligned}$$

The first term is in $\mathscr{S}$, since the symbols of the two operators in the composition have, by construction, disjoint supports. The second term is in $\mathscr{S}$ as well, by (5.18). The third term is in $\mathcal{B}$, since this is true for $\mathrm{Op}(p)f$, $\mathrm{Op}(1-\chi)\circ\mathrm{Op}(s)=\mathrm{Op}(\lambda)$, with $\lambda\in\mathrm{SG}^{0,0}_{1,1}$, and $\mathcal{B}$ is SG-admissible.

By all the considerations above, the mapping properties of $\mathrm{Op}_\varphi(a)$, the fact that also $\mathcal{C}$ is SG-admissible and that $q\in\mathrm{SG}^{0,0}_{1,1}$, we get

$$\begin{aligned}[\mathrm{Op}(q)\circ\mathrm{Op}_\varphi(a)]f&=[\mathrm{Op}(q)\circ\mathrm{Op}_\varphi(a)\circ\mathrm{Op}(1-\chi)]f\quad\mathrm{mod}\ \mathscr{S}\\ &=[\mathrm{Op}(q)\circ\mathrm{Op}_\varphi(a)]\underbrace{[\mathrm{Op}(1-\chi)f]}_{\in\mathcal{B}}\quad\mathrm{mod}\ \mathscr{S}\end{aligned}$$

giving that

$$[\mathrm{Op}(q)\circ\mathrm{Op}_\varphi(a)]f\in\mathcal{C},$$

which proves $(x_0, \xi_0) \notin \mathrm{WF}^3_{\mathcal{C}}(\mathrm{Op}_\varphi(a) f)$, and the claim. We observe that (5.15) for the case $m = 1$ can be proved by the same argument used, e.g., in [36], Chap. 10, Sect. 3. The case $m = 2$ of (5.15) can then be obtained in a completely similar fashion, by exchanging the role of variable and covariable. The details are left for the reader. □

Remark 5.20 As it was observed in [26], there is a simple and useful relation between the global wave-front set of f and of $\widehat{f}$. Namely, with $m, n \in \{1, 2, 3\}$ such that n equals 2, 1 and 3, when m equals 1, 2 and 3, respectively, we have

$$T\big(\mathrm{WF}^m_{\mathcal{B}}(f)\big) = \mathrm{WF}^n_{\mathcal{B}_T}(\widehat{f}),$$

where $\mathcal{B} = M(\omega, \mathscr{B})$, the torsion T is given by $T(x, \xi) = (-\xi, x)$, and $\mathscr{B}_T = \{\, F \circ T = T^*F\, ;\ F \in \mathscr{B}\,\}$, $\omega_T = \omega \circ T$, $\mathcal{B}_T = M(\omega_T, \mathscr{B}_T)$. Notice that $\mathscr{F}$ is bijective and continuous, together with its inverse, from $\mathcal{B}_T$ onto $\mathcal{B}$.

It is also immediate to obtain a similar relation among the wave-front sets of f and $\check{f}$, where $\check{f} = f \circ R$ is the pull-back of f under the action of the reflection $R(y) = -y$. Indeed, since obviously, for any $a \in \mathrm{SG}^{m,\mu}_{r,\rho}$ and $f \in \mathscr{S}'$,

$$\mathrm{Op}(a)\check{f} = [\mathrm{Op}(\check{a}) f]\check{\ },$$

it follows, for $m \in \{1, 2, 3\}$,

$$R(\mathrm{WF}^m_{\mathcal{B}}(f)) = \mathrm{WF}^m_{\check{\mathcal{B}}}(\check{f}),$$

where $\check{\mathscr{B}} = \{\, R^*F\, ;\ F \in \mathscr{B}\,\}$, $\check{\mathcal{B}} = M(\omega_R, \check{\mathscr{B}})$, where $\omega_R = \omega \circ R$. Notice that, in many cases, $\check{\mathcal{B}}$=$\mathcal{B}$. For instance, this is true for all the functional spaces considered in Sect. 4, and, in general, for all $M(\omega, \mathscr{B})$ such that $\check{\mathscr{B}} = \mathscr{B}$ and ω is even. Similarly to the above, R^* is bijective and continuous, together with its inverse, from $\mathcal{B}$ onto $\check{\mathcal{B}}$.

By (3.6), rewritten as

$$\mathrm{Op}^*_{-\varphi^*}(a^*) f = (\mathscr{F} \circ \mathrm{Op}_\varphi(a) \circ \mathscr{F}^{-1} f)\check{\ }, \quad f \in \mathscr{S}',$$

and the above definitions of $\mathcal{B}_T$ and $\check{\mathcal{B}}$, it also follow that, if $(\mathcal{B}, \mathcal{C})$ is weakly-I SG-ordered with respect to $(r, \rho, \omega_0, \varphi, \Omega)$, we find that, for any $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}$, supported outside $\mathbf{R}^d \times \Omega$, $\mathrm{Op}^*_{-\varphi^*}(a^*)\colon \mathcal{B}_T \to \check{\mathcal{C}}_T$ continuously, that is, $(\check{\mathcal{C}}_T, \mathcal{B}_T)$ is weakly-II SG-ordered with respect to $(r, \rho, \omega_0, \varphi, \Omega)$.

5.3 Applications to SG-Hyperbolic Problems

In this subsection we apply the results obtained above to the SG-hyperbolic problems considered in [18,19], to which we refer for the details omitted here. We show how, under natural conditions, the singularities described by the generalised wave-front sets $\mathrm{WF}^m_{\mathcal{B}}(g)$, $m = 1, 2, 3$, for a scalar- or vector-valued initial data $g \in \mathcal{B}$, propagate to the solution $f(t) = f(t, \cdot) \in \mathcal{B}$, $t \in [-T, T]$. More precisely, the points of $\mathrm{WF}^m_{\mathcal{C}}(f(t))$

lie on bicharacteristics curves determined by $\mathrm{WF}^m_{\mathcal{B}}(g)$, $m = 1, 2, 3$, and by the phase functions of the Fourier operators $\mathrm{Op}_{\varphi_k(t)}(a_k(t))$, $k = 1, \ldots, \mu$, such that, modulo smooth remainders (see below),

$$f(t) = (\mathrm{Op}_{\varphi_1(t)}(a_1(t))(t) + \cdots + \mathrm{Op}_{\varphi_\mu(t)}(a_\mu(t))(t))g.$$

Notice that the hyperbolic operators involved in such Cauchy problems arise naturally as local representations of (modified) wave operators of the form $L = \Box_{\mathfrak{g}} - V$, with a suitable potential V and the D'Alembert operator $\Box_{\mathfrak{g}}$, on manifolds of the form $\mathbf{R}_t \times M_x$, equipped with a hyperbolic metric $\mathfrak{g} = \mathrm{diag}(-1, \mathfrak{h})$, where $\mathfrak{h}$ is a suitable Riemannian metric on the manifold with ends M. In this way,

$$L = \Box_{\mathfrak{g}} - V = -\partial_t^2 + \Delta_{\mathfrak{h}} - V = -\partial_t^2 + P,$$

where $\Delta_{\mathfrak{h}}$ is the Laplace-Beltrami operator on M associated with the metric $\mathfrak{h}$ and we have set $P = \Delta_{\mathfrak{h}} - V$. In the following Example 5.21, we show that this indeed occurs, considering a rather simple situation with $\dim M = 2$.

Example 5.21 Assume $\dim M = 2$ and consider, as local model of one of the "ends" of M, the cylinder in $\mathbf{R}^3$ given by $u^2 + v^2 = 1$, $z > 1$, that is, the manifold $M_\infty = S^1 \times (1, +\infty)$. First, we have to equip M_∞ with a $\mathscr{S}$-structure, namely, an SG-compatible atlas (see [14,40]). This can be easily accomplished here, by choosing a standard product atlas on $S^1 \times (1, +\infty)$, identifying S^1 with the unit circle in $\mathbf{R}^2$ centred at the origin, as we now explain. With coordinates (u, v) on $\mathbf{R}^2$, set

$$\Omega'_1 := S^1 \backslash \{(0, 1)\}, \qquad \Omega'_2 := S^1 \backslash \{(0, -1)\},$$
$$\nu'_1 : \Omega'_1 \to \mathbf{R} \colon (u, v) \mapsto \frac{u}{1 - v},$$
$$\nu'_2 : \Omega'_2 \to \mathbf{R} \colon (u, v) \mapsto \frac{u}{1 + v}$$

It is immediate to show that $(\nu'_1)^{-1} : \nu'_1(\Omega'_1) \to \Omega'_1 \subset S^1$ is

$$t \mapsto \left(\frac{2t}{1 + t^2}, -\frac{1 - t^2}{1 + t^2} \right),$$

and $(\nu'_2)^{-1} : \nu'_2(\Omega'_2) \to \Omega'_2 \subset S^1$ is

$$t \mapsto \left(\frac{2t}{1 + t^2}, \frac{1 - t^2}{1 + t^2} \right),$$

so that, for $t \in \nu'_2(\Omega'_1 \bigcap \Omega'_2) = (-\infty, 0) \cup (0, +\infty)$, we find

$$\nu'_{12}(t) = \nu'_1((\nu'_2)^{-1}(t)) = \nu'_1 \left(\frac{2t}{1 + t^2}, \frac{1 - t^2}{1 + t^2} \right) = \frac{1}{t}.$$

Now set

$$\Omega_1 := \Omega_1' \times (1, +\infty), \qquad \Omega_2 := \Omega_2' \times (1, +\infty),$$

define $\nu_1 : \Omega_1 \to U_1 \subset \mathbf{R}^2$ by

$$(u, v, z) \mapsto (\nu_1'(u, v), 1) \frac{z}{\sqrt{1 + (\nu_1'(u, v))^2}} = \left(\frac{u}{1 - v}, 1 \right) \frac{z}{\sqrt{1 + \dfrac{u^2}{(1 - v)^2}}},$$

and $\nu_2 : \Omega_2 \to U_2 \subset \mathbf{R}^2$ by

$$(u, v, z) \mapsto (\nu_2'(u, v), 1) \frac{z}{\sqrt{1 + (\nu_2'(u, v))^2}} = \left(\frac{u}{1 + v}, 1 \right) \frac{z}{\sqrt{1 + \dfrac{u^2}{(1 + v)^2}}}.$$

Again, it is easy to obtain the expressions of $\nu_1^{-1} : U_1 \to \Omega_1$ and $\nu_2^{-1} : U_2 \to \Omega_2$, and to prove that, with coordinates $x = (x_1, x_2)$ on $\mathbf{R}^2$,

$$\nu_{12}(x_1, x_2) = \nu_1((\nu_2)^{-1}(x_1, x_2)) = \left(\tfrac{x_2}{x_1}, 1 \right) x_1 = (x_2, x_1),$$

which shows that the atlas $\{(\Omega_j, \nu_j), j = 1, 2\}$ defines a $\mathscr{S}$-structure on M_∞, since $\langle \nu_{12}(x) \rangle = \langle x \rangle$ (see again, e.g., [14,40]). Next, for any $\mu > 0$, define a metric $\mathfrak{h}'$ on $\{(u, v, z) \in \mathbf{R}^3 : z > 1\}$ by

$$(\mathfrak{h}'_{ij}) := \begin{pmatrix} \dfrac{z^2}{4\langle z \rangle^\mu} & 0 & 0 \\ 0 & \dfrac{z^2}{4\langle z \rangle^\mu} & 0 \\ 0 & 0 & \dfrac{1}{\langle z \rangle^\mu} \end{pmatrix}.$$

With $x \in U_1$, and denoting by J_1 the Jacobian matrix of ν_1^{-1}, it turns out that the pull-back metric $\mathfrak{h} := (\nu_1^{-1})^* \mathfrak{h}'$ on M_∞ is given by

$$(\mathfrak{h}_{ij}) = J_1 \left((\nu_1^{-1})^* \mathfrak{h}'_{ij} \right)|_{U_1} J_1^t = \begin{pmatrix} \dfrac{1}{\langle x \rangle^\mu} & 0 \\ 0 & \dfrac{1}{\langle x \rangle^\mu} \end{pmatrix}.$$

In the same way, one can show that the metric $\mathfrak{h}$ has the same local expression for $x \in U_2$. Finally, let us compute the Laplace-Beltrami operator on M_∞ associated with

$\mathfrak{h}$ in the chosen local coordinates. We have, of course, $(\mathfrak{h}^{ij}) = \mathrm{diag}(\langle x\rangle^{\mu}, \langle x\rangle^{\mu})$ and $\sqrt{|\det \mathfrak{h}|} = \langle x\rangle^{-\mu}$, thus, for any $f \in C^\infty(M_\infty)$,

$$\begin{aligned}\Delta_{\mathfrak{h}} f &= \tfrac{1}{\langle x\rangle^{-\mu}} \textstyle\sum_{i,j=1}^{2} \tfrac{\partial}{\partial x^j} \left(\langle x\rangle^{-\mu} \mathfrak{h}^{ij} \tfrac{\partial f}{\partial x^i}\right) \\ &= \langle x\rangle^{\mu} \textstyle\sum_{i,j=1}^{2} \tfrac{\partial}{\partial x^j} \left(\delta^{ij} \tfrac{\partial f}{\partial x^i}\right) \\ &= \langle x\rangle^{\mu} \textstyle\sum_{i=1}^{2} \tfrac{\partial^2 f}{\partial x_i^2} = \langle x\rangle^{\mu} \left(\tfrac{\partial^2}{\partial x_1^2} + \tfrac{\partial^2}{\partial x_2^2}\right) f,\end{aligned}$$

that is

$$\Delta_{\mathfrak{h}} = \langle x\rangle^{\mu} \Delta,$$

where Δ is the standard Laplacian on $\mathbf{R}^2$. Choosing $V(x) = \langle x\rangle^{\mu}$, the local symbol of $P = \Delta_{\mathfrak{h}} - V$ is

$$p(x,\xi) = -\langle x\rangle^{\mu}\langle \xi\rangle^2 = -(1 + x_1^2 + x_2^2)^{\frac{\mu}{2}}(1 + \xi_1^2 + \xi_2^2), \tag{5.19}$$

which obviously belongs to $\mathrm{SG}^{2,\mu}_{1,1}(\mathbf{R}^2 \times \mathbf{R}^2)$ and is SG-elliptic. In [20], the spectral theory for elliptic self-adjoint operators, generated by local symbols with (different) orders $m, \mu > 0$, has been considered. On the other hand, the case $\mu = 2$ is of special interest in the context of the SG-hyperbolic operators (see below), since then we have that L, in local coordinates, is given by

$$\begin{aligned} L &= \Box_{\mathfrak{g}} - V = -\partial_t^2 + \Delta_{\mathfrak{h}} - V \\ &= D_t^2 - P = D_t^2 - \langle\cdot\rangle^2 \langle D\rangle^2. \end{aligned} \tag{5.20}$$

In the sequel of this subsection, the subscript “cl” denotes the subclasses of SG symbols which are classical, see [18]. Notice that the symbol (5.19) actually belongs to $\mathrm{SG}^{2,\mu}_{1,1,\mathrm{cl}}(\mathbf{R}^2 \times \mathbf{R}^2)$. We first need to recall some definitions and results, mainly taken from [15,16,18].

Definition 5.22 Let $J = [-T, T] \subset \mathbf{R}$, $T > 0$, and consider the linear operator

$$L = D_t^{\nu} + \mathrm{Op}(p_1(t))\, D_t^{\nu-1} + \cdots + \mathrm{Op}(p_\nu(t)), \tag{5.21}$$

with $p_j = p_j(t,x,\xi) \in C^\infty(J, \mathrm{SG}^{1,1}_{1,1,\mathrm{cl}}(\mathbf{R}^{2d}))$. Let

$$l(x,\xi,t,\tau) = \tau^{\nu} + q_1(t,x,\xi)\tau^{\nu-1} + \cdots + q_\nu(t,x,\xi)$$

be the principal symbol of L, with $q_j \in C^\infty(J, \mathrm{SG}^{j,j}_{1,1,\mathrm{cl}}(\mathbf{R}^{2d}))$ such that $q_j(t) = q_j(t,\cdot)$ is the principal symbol of $p_j(t) = p_j(t,\cdot)$, in the sense of (5.7). L is called SG-classical hyperbolic with constant multiplicities if the characteristic equation

$$\tau^{\nu} + q_1(t,x,\xi)\tau^{\nu-1} + \cdots + q_\nu(t,x,\xi) = 0 \tag{5.22}$$

has $\mu \leq \nu$ distinct real roots $\tau_j = \tau_j(t, x, \xi) \in C^\infty(J, \mathrm{SG}^{1,1}_{1,1,\mathrm{cl}}(\mathbf{R}^{2d}))$ with multiplicities l_j, $1 \leq l_j \leq \nu$, $j = 1, \dots, \mu$, which satisfy, for a suitable $C > 0$ and all $t \in J$, $x, \xi \in \mathbf{R}^d$, $|x| + |\xi| \geq R > 0$,

$$\tau_{j+1}(t, x, \xi) - \tau_j(t, x, \xi) \geq C\langle\xi\rangle\langle x\rangle, \ \ j = 1, \dots, \mu - 1. \tag{5.23}$$

L is called strictly hyperbolic if it is hyperbolic with constant multiplicities and the multiplicity of all the τ_j, $j = 1, \dots, \nu = \mu$, is equal to 1.

A standard strategy to solve the Cauchy problem

$$\begin{cases} Lf(t) = 0, & t \in J, \\ D_t^k f(0) = g_k, & k = 0, \dots, \nu - 1, \end{cases} \tag{5.24}$$

for L hyperbolic with constant multiplicities and initial data g_k, $k = 0, \dots, \nu - 1$, chosen in appropriate functional spaces, is to show that this is equivalent to solving, modulo smooth elements, a Cauchy problem for a first order system

$$\begin{cases} \dfrac{\partial F}{\partial t}(t) - iK(t)\,F(t) = 0, t \in J, \\ F(0) = G, \end{cases}$$

with a *coefficient matrix* K of special form. In our case, one obtains that $K = \mathrm{Op}((k_{ij}(t, x, D))_{i,j})$, is a $\mu\nu \times \mu\nu$ matrix of SG pseudo-differential operators with symbols $k_{ij} \in C^\infty(J, \mathrm{SG}^{1,1}_{1,1,\mathrm{cl}})$. Under suitable assumptions, see [18,19], the principal part k_1 of $k = k_1 + k_0$, $k_j \in C^\infty(J, \mathrm{SG}^{j,j}_{1,1,\mathrm{cl}})$, $j = 0, 1$, turns out to be diagonal, so that the system will be symmetric, cfr. [14–16]. This implies that the corresponding Cauchy problem is well-posed. One of the main advantages for using this algorithm is the following Proposition 5.23, which is an adapted version of the Mizohata Lemma of Perfect Factorization, proved in [17] for the general SG symbols (see also the references quoted therein).

Proposition 5.23 *Let L be an* SG*-classical hyperbolic linear operator with constant multiplicities l_j, $j = 1, \dots, \mu \leq \nu$, as in Definition 5.22. Then it is possible to factor L as*

$$L = L_\mu \cdots L_1 + \sum_{s=1}^{\nu} \mathrm{Op}(r_s(t))\, D_t^{\nu - s}$$

with $L_j = (D_t - \mathrm{Op}(\tau_j(t)))^{l_j} + \sum_{k=1}^{l_j} \mathrm{Op}(s_{jk}(t))\,(D_t - \mathrm{Op}(\tau_j(t)))^{l_j - k}$ *and*

$$s_{jk} \in C^\infty(J, \mathrm{SG}^{k-1,k-1}_{1,1,\mathrm{cl}}(\mathbf{R}^{2d})), r_s \in C^\infty(J, \mathscr{S}(\mathbf{R}^{2d})),$$
$$j = 1, \dots, \mu, k = 1, \dots, l_j, s = 1, \dots, \nu.$$

The following corollary, also obtained in [17], follows by means of a reordering of the roots τ_j of the principle symbol of L.

Corollary 5.24 *Let c_j, $j = 1, \dots, \mu$, denote the reorderings of the μ-tuple $(1, \dots, \mu)$ given by*

$$c_j(i) = \begin{cases} j+i & \text{for } j+i \le \mu \\ j+i-\mu & \text{for } j+i > \mu, \end{cases}$$
$$i, j = 1, \dots, \mu,$$

that is, $c_1 = (2, \dots, \mu, 1)$, ..., $c_\mu = (1, \dots, \mu)$. Then, under the same hypotheses of Proposition 5.23, we have

$$L = L^{(m)}_{c_m(\mu)} \cdots L^{(m)}_{c_m(1)} + \sum_{s=1}^{\nu} \mathrm{Op}(r_s^{(m)}(t)) D_t^{\nu-s}$$

with $L_j^{(m)} = (D_t - \mathrm{Op}(\tau_j(t)))^{l_j} + \sum_{k=1}^{l_j} \mathrm{Op}(s_{jk}^{(m)}(t))\, (D_t - \mathrm{Op}(\tau_j(t)))^{l_j-k}$ *and*

$$s_{jk}^{(m)} \in C^\infty(J, \mathrm{SG}^{k-1,k-1}_{1,1,\mathrm{cl}}(\mathbf{R}^{2d})),\ r_s^{(m)} \in C^\infty(J, \mathscr{S}(\mathbf{R}^{2d})),$$
$$m, j = 1, \dots, \mu, k = 1, \dots, l_j, s = 1, \dots, \nu.$$

Definition 5.25 We say that an SG-classical hyperbolic operator L is of Levi type if it satisfies the SG-Levi condition[1]

$$s_{jk}^{(m)} \in C^\infty(J, \mathrm{SG}^{0,0}_{1,1,\mathrm{cl}}(\mathbf{R}^{2d})), \quad m, j = 1, \dots, \mu, k = 1, \dots, l_j. \tag{5.25}$$

Theorem 5.26 below gives the well-posedness for the Cauchy problem (5.24) and the propagation results of the global wave-front sets $\mathrm{WF}^m_{\mathcal{C}}(f(t))$, $m = 1, 2, 3$, for the corresponding solution $f(t)$, under natural assumptions on the SG-admissible initial data spaces $\mathcal{B}_k$, $k = 0, \dots, \nu - 1$, and the SG-admissible solution space $\mathcal{C}$, see below for the precise statement. It immediately follows by the analysis of SG-classical hyperbolic Cauchy problems in [18], by Sect. 4 and by Theorem 5.14.

We here consider an SG-classical hyperbolic operator L with constant multiplicities and of Levi type, and denote by $l = \max\{l_1, \dots, l_\mu\}$ the maximum multiplicity of the distinct real roots τ_j, $j = 1, \dots, \mu$, of the characteristic equation (5.22). Then, as proved in [15,18], for any choice of initial data $g_k \in \mathscr{S}'(\mathbf{R}^d)$, $k = 0, \dots, \nu - 1$, the Cauchy problem (5.24) admits a unique solution $f \in C(J', \mathscr{S}'(\mathbf{R}^d))$, $J' = [-T', T']$, $0 < T' \le T$. Collecting the initial conditions in the vector

$$g = \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_{\nu-1} \end{pmatrix},$$

[1] Let us observe that (5.25) needs to be fulfilled only for a single value of m.

the solution f is given by

$$f(t) = (\mathrm{Op}_{\varphi_1(t)}(a_1(t)) + \cdots + \mathrm{Op}_{\varphi_\mu(t)}(a_\mu(t)))g,$$

where each $\mathrm{Op}_{\varphi_j(t)}(a_j(t))$ is a type I FIO with regular phase function $\varphi_j \in C^\infty(J', \mathfrak{F}^r) \cap C^\infty(J', \mathrm{SG}^{1,1}_{1,1,\mathrm{cl}}(\mathbf{R}^{2d}))$, solution of the eikonal equation associated with τ_j, and vector-valued amplitude functions $a_j = (a_{j0}, \ldots, a_{j\nu-1})$ with $a_{jk} \in C^\infty(J', \mathrm{SG}^{l-k-1,l-k-1}_{1,1,\mathrm{cl}}(\mathbf{R}^{2d}))$, $j = 1, \ldots, \mu$, $k = 0, \ldots, \nu - 1$.

Theorem 5.26 *Let L be as above, and let $g_k \in \mathcal{B}_k$, $k = 0, \ldots, \nu - 1$, with the ν-tuple of* SG*-admissible spaces $(\mathcal{B}_0, \ldots, \mathcal{B}_{\nu-1})$. Also assume that the* SG*-admissible space $\mathcal{C}$ is such that $(\mathcal{B}_k, \mathcal{C})$, $k = 0, \ldots, \nu - 1$, are weakly-I* SG*-ordered pairs with respect to*

$$1,\, 1,\, \langle x\rangle^{l-k-1}\langle \xi\rangle^{l-k-1},\, \varphi_k(t) \quad \textit{and} \quad \Omega.$$

Then the Cauchy problem (5.24) is well-posed with respect to $(\mathcal{B}_0, \ldots, \mathcal{B}_{\nu-1})$ and $\mathcal{C}$, $u \in C(J', \mathcal{C})$, and

$$\mathrm{WF}^m_{\mathcal{C}}(f(t)) \subseteq \bigcup_{j=1}^{\mu} \bigcup_{k=0}^{\nu-1} (\phi_j(t)(\mathrm{WF}^m_{\mathcal{B}_k}(g_k)))^{\mathrm{con}_m}, \quad m = 1, 2, 3, \tag{5.26}$$

where $\phi_j(t)$ is the canonical transformation (5.6) associated with the phase function $\varphi_j(t)$.

Corollary 5.27 *Assume that the hypotheses of Theorem 5.26 hold. Then $\mathrm{WF}^m_{\mathcal{C}}(f(t))$, $t \in J'$, $m = 1, 2, 3$, consists of arcs of bicharacteristics, generated by the phase functions $\varphi_j(t)$ and emanating from points belonging to $\mathrm{WF}^m_{\mathcal{B}_k}(g_k)$, $k = 0, \ldots, \nu - 1$.*

5.4 Examples

We conclude with some examples where our propagation of singularities results can be applied. We initially look at the first order Cauchy problem

$$\begin{cases} (D_t + \mathrm{Op}(p_1(t)))f(t) = 0,\ t \in [-T, T], \\ u|_{t=0} = g. \end{cases} \tag{5.27}$$

In (5.27) we assume that $p_1(t)$ is a family of classical SG symbols of order (1,1) depending smoothly on t. The hyperbolicity condition means that its principal symbol $q_1(t)$ such that $p_1(t) - q_1(t) \in \mathrm{SG}^{0,0}_{1,1,\mathrm{cl}}(\mathbf{R}^{2d})$, is real-valued. We then have the representation of the solution to (5.27) in the form $f(t) = \mathrm{Op}_{\varphi(t)}(a(t))g$. Theorem 4.1, Remark 4.2 and Theorem 4.3 describe the loss of regularity and weight for the solutions in the corresponding functional settings.

Example 5.28 Let $1 < p < \infty$ and $g \in H^p_{\sigma_p,\sigma_p}(\mathbf{R}^d)$, where we have set $\sigma_p = (d-1)\left|\dfrac{1}{p} - \dfrac{1}{2}\right|$. Assume also that $p_1(t,x,\xi) = \langle c_1(t,x), \xi\rangle + c_0(t)$, so that $q_1(t,x,\xi) = \langle q_{01}(t,x), \xi\rangle$. We restrict here to large frequencies, as in [23], choosing a 0-excision function $\chi \in C^\infty(\mathbf{R}^d)$ such that $\chi(\xi) = 1$ for $|\xi| \geq 2\varepsilon$, for some sufficiently large $\varepsilon > 0$. Then Theorem 4.1 and Remark 4.2 imply that for each $t \in [-T', T']$, $0 < T' \leq T$, the solution f of the Cauchy problem (5.27) satisfies $\chi(D)f(t) \in L^p(\mathbf{R}^d)$. Moreover, for every $s, \sigma \in \mathbf{R}$, there are $C_T > 0$ and $0 < T' \leq T$ such that

$$\|\chi(D)f(t)\|_{H^p_{s,\sigma}(\mathbf{R}^d)} \leq C_T \|g\|_{H^p_{s+\sigma_p,\sigma+\sigma_p}(\mathbf{R}^d)},$$

for all $t \in [-T', T']$ and all $g \in H^p_{s+\sigma_p,\sigma+\sigma_p}(\mathbf{R}^d)$. Finally, since the hypotheses of Theorem 5.26 are satisfied, with $\mathcal{C} = H^p_{s,\sigma}(\mathbf{R}^d)$, $\mathcal{B}_0 = H^p_{s+\sigma_p,\sigma+\sigma_p}(\mathbf{R}^d)$, $r = \rho = 1$, $k = l = 1$, $\varphi(t)$, and $\Omega = B_\varepsilon(0)$, we have

$$\mathrm{WF}^m_{\mathcal{C}}(\chi(D)f(t)) \subseteq (\phi(t)(\mathrm{WF}^m_{\mathcal{B}_0}(g)))^{\mathrm{con}_m}, \quad m = 1, 2, 3,$$

where $\phi(t)$ is the canonical transformation (5.6) associated with the phase function $\varphi(t)$, which turns out to be (positively) homogeneous with respect to the covariable (since this is true for $q_1(t)$, see, e.g., [36]).

Example 5.29 Let $s, \sigma \in \mathbf{R}$ and $1 \leq p < \infty$ be such that

$$s \leq -d\left|\frac{1}{2} - \frac{1}{p}\right|, \quad \sigma \leq -d\left|\frac{1}{2} - \frac{1}{p}\right|,$$

and let $\omega_j \in \mathscr{P}_{1,1}(\mathbf{R}^{2d})$, $j = 1, 2$, be such that

$$\omega_2(x, \varphi'_x(x,\xi)) \lesssim \omega_1(\varphi'_\xi(x,\xi), \xi) \cdot \langle x\rangle^s \langle\xi\rangle^\sigma.$$

Also assume that ω_j is (ϕ_j, j)-invariant, $j = 1, 2$, with $\phi_1 \colon x \mapsto \varphi'_\xi(x,\xi)$ and $\phi_2 \colon \xi \mapsto \varphi'_x(x,\xi)$. This clearly holds true, in particular, for the trivial weights $\omega_j(x,\xi) = 1$ and for product type weights $\omega_j(x,\xi) = \langle x\rangle^{s_j}\langle\xi\rangle^{\sigma_j}$, with appropriate choices of $s_j, \sigma_j \in \mathbf{R}$, $j = 1, 2$.

Theorem 4.3 implies that, if $g \in M^p_{(\omega_1)}(\mathbf{R}^d)$. Then the solution $f(t)$ of the Cauchy problem (5.27) satisfies $f(t) \in M^p_{(\omega_2)}(\mathbf{R}^d)$, for each $t \in [-T', T']$, $0 < T' \leq T$. Moreover, there are $C_T > 0$, $0 < T' \leq T$ such that

$$\|f(t)\|_{M^p_{(\omega_2)}(\mathbf{R}^d)} \leq C_T \|g\|_{M^p_{(\omega_1)}(\mathbf{R}^d)},$$

for all $t \in [-T', T']$ and all $g \in M^p_{(\omega_1)}(\mathbf{R}^d)$. Finally, since the hypotheses of Theorem 5.26 are satisfied, with $\mathcal{C} = M^p_{(\omega_2)}(\mathbf{R}^d)$, $\mathcal{B}_0 = M^p_{(\omega_1)}(\mathbf{R}^d)$, $r = \rho = 1$, $k = l = 1$, $\varphi(t)$, and $\Omega = \emptyset$, we have

$$\mathrm{WF}^m_{\mathcal{C}}(f(t)) \subseteq (\phi(t)(\mathrm{WF}^m_{\mathcal{B}_0}(g)))^{\mathrm{con}_m}, \qquad m = 1, 2, 3,$$

with the canonical transformation $\phi(t)$ associated with $\varphi(t)$ in (5.6). Completely similar results hold true when $M^p_{(\omega_j)}(\mathbf{R}^d)$ is replaced by $M^\infty_{0,(\omega_j)}(\mathbf{R}^d)$, $j=1,2$.

As a first example involving second order SG-hyperbolic operators, we consider a variant of (5.20), choosing $V\equiv 0$ and focusing again on large frequencies, by adding a correction term of the form $\langle\cdot\rangle^2(1-\widetilde{\chi}(D)^2)|D|^2$, with a 0-excision function $\widetilde{\chi}$, that is

$$\begin{cases} L_{\widetilde{\chi}} f(t) = (D_t^2 - \langle\cdot\rangle^2\,(\widetilde{\chi}(D)|D|)^2) f(t) = 0,\ t\in[-T,T],\\ f|_{t=0} = g_0,\\ D_t f|_{t=0} = g_1.\end{cases} \tag{5.28}$$

In this case, the results on SG-hyperbolic operators stated in Sect. 5.3 cannot be applied directly, since (5.23) does not hold for the two roots $\tau_{1,2}(x,\xi) = \mp\langle x\rangle\widetilde{\chi}(\xi)|\xi|$. Nevertheless, we can anyway switch from the Cauchy problem (5.28) to an equivalent first order 2×2 system, setting $F_1(t)=f(t)$, $F_2=\langle D\rangle^{-1}\langle\cdot\rangle^{-1}D_t f(t) = \langle D\rangle^{-1}\langle\cdot\rangle^{-1}D_t F_1(t)$, namely,

$$\begin{cases} \dfrac{\partial F}{\partial t}(t) = i\left[\mathrm{Op}(k_1)+\mathrm{Op}(k_0)\right]F(t),\\ F(0)=G,\end{cases} \tag{5.29}$$

with

$$G = \begin{pmatrix} g_0 \\ \langle D\rangle^{-1}\langle\cdot\rangle^{-1}g_1\end{pmatrix},\quad k_1(x,\xi) = \begin{pmatrix} 0 & \langle x\rangle\langle\xi\rangle \\ \langle x\rangle(\widetilde{\chi}(\xi)|\xi|)^2)\langle\xi\rangle^{-1} & 0\end{pmatrix},$$

and k_0 a matrix of SG symbols of order $0,0$. The principal part k_1 of the coefficient matrix has, of course, real distinct eigenvalues $\tau_{1,2}$ in the region $|\xi|\geq c>0$. The theory developed in [14], Ch. 6, shows that the system (5.29) can be symmetrized and its principal part diagonalized, modulo order $0,0$ operators. By a variant of the computations in [15], in the region $|\xi|\geq c>0$ it can also be perfectly diagonalized, that is, the two equations can be decoupled, up to smoothing operators. Summing up, in the case of high frequencies, the Fourier integral operator method can be applied to (5.29), with the consequences described in the next Example 5.30.

Example 5.30 Let $1<p<\infty$, $s,\sigma\in\mathbf{R}$, $g_0\in H^p_{s+1+\sigma_p,\sigma+1+\sigma_p}(\mathbf{R}^d)$, and $g_1\in H^p_{s+\sigma_p,\sigma+\sigma_p}(\mathbf{R}^d)$. Choose a 0-excision function χ as in Example 5.28 with $\varepsilon>0$ sufficiently large. Then the solution $f(t)$ of (5.28) satisfies $\chi(D)f(t)\in H^p_{s+1,\sigma+1}(\mathbf{R}^d)$, $t\in[-T',T']$, $0<T'\leq T$. Moreover, (5.26) holds with $\mu=2$, $\nu=1$, $\mathcal{C}=H^p_{s+1,\sigma+1}(\mathbf{R}^d)$, $\mathcal{B}_0=H^p_{s+1+\sigma_p,\sigma+1+\sigma_p}(\mathbf{R}^d)$ and $\mathcal{B}_1=H^p_{s+\sigma_p,\sigma+\sigma_p}(\mathbf{R}^d)$, namely

$$\mathrm{WF}^m_{\mathcal{C}}(\chi(D)f(t)) \subseteq \bigcup_{j=1}^{2}\Big[(\phi_j(t)(\mathrm{WF}^m_{\mathcal{B}_0}(g_0)))^{\mathrm{con}_m}\cup(\phi_j(t)(\mathrm{WF}^m_{\mathcal{B}_1}(g_1)))^{\mathrm{con}_m}\Big],$$

$m = 1, 2, 3$, where $\phi_j(t)$ is the canonical transformation (5.6) generated by the phase function $\varphi_j(t)$, solution to the eikonal equation associated with $\tau_{1,2}$, which turn out to be positively homogeneous with respect to the coverable (since this holds for $\tau_{1,2}$).

Consider now the Cauchy problem

$$\begin{cases} Lf(t) = (D_t^2 - \langle\cdot\rangle^2 \langle D\rangle^2) f(t) = 0, \ t \in [-T, T], \\ \quad f|_{t=0} = g_0, \\ D_t f|_{t=0} = g_1, \end{cases} \tag{5.30}$$

involving the operator (5.20). First of all, we prove that L is SG-classical strictly hyperbolic. Indeed, $p_2(x, \xi) = \langle x\rangle^2 \langle \xi\rangle^2 \in \mathrm{SG}^{2,2}_{1,1,\mathrm{cl}}(\mathbf{R}^{2d})$, with, for instance,

$$q_2(x, \xi) = \chi(\xi)|\xi|^2 \langle x\rangle^2 + \chi(x)(\langle\xi\rangle^2 |x|^2 - \chi(\xi)|x|^2 \cdot |\xi|^2) \geq 0,$$

for a fixed 0-excision function χ. The characteristic equation $\tau^2 - q_2(x, \xi) = 0$ has then two real distinct solutions, namely

$$\tau_{1,2}(x, \xi) = \mp\sqrt{q_2(x, \xi)} = \mp\tau_0(x, \xi) \in \mathrm{SG}^{1,1}_{1,1,\mathrm{cl}}(\mathbf{R}^{2d}), \tag{5.31}$$

such that

$$\tau_2(x, \xi) - \tau_1(x, \xi) = 2\tau_0(x, \xi) \gtrsim \langle x\rangle\langle\xi\rangle \text{ for } |x| + |\xi| \geq R,$$

with a suitable $R > 0$. Writing, as above, $L_j = D_t - \mathrm{Op}(\tau_j) + \mathrm{Op}(s_j)$, $j = 1, 2$, we find

$$\begin{aligned} L_2 L_1 &= D_t^2 - \mathrm{Op}(\tau_0)^2 + (\mathrm{Op}(s_1) + \mathrm{Op}(s_2))D_t + \mathrm{Op}(s_2)\,\mathrm{Op}(s_1) \\ &= D_t^2 - \mathrm{Op}(q_2) + \mathrm{Op}(a) + (\mathrm{Op}(s_1) + \mathrm{Op}(s_2))D_t + \mathrm{Op}(s_2)\,\mathrm{Op}(s_1), \end{aligned}$$

where

$$a(x, \xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha \tau_0(x, \xi) D_x^\alpha \tau_0(x, \xi) \in \mathrm{SG}^{0,0}_{1,1,\mathrm{cl}}(\mathbf{R}^{2d}).$$

Proposition 5.23 implies that there exist $s_1, s_2 \in \mathrm{SG}^{0,0}_{1,1,\mathrm{cl}}(\mathbf{R}^{2d})$, $s_1 = s_0 = -s_2 + r_1$, $r_1 \in \mathscr{S}(\mathbf{R}^{2d})$ such that

$$\mathrm{Op}(a) - \mathrm{Op}(s_0)^2 = \mathrm{Op}(q_2 - p_2) + \mathrm{Op}(r_2), \quad r_2 \in \mathscr{S}(\mathbf{R}^{2d}),$$

so that L also satisfies the Levi condition.

Example 5.31 Let $s, \sigma \in \mathbf{R}$ and $1 \leq p < \infty$ be such that

$$s \leq -d\left|\frac{1}{2} - \frac{1}{p}\right|, \quad \sigma \leq -d\left|\frac{1}{2} - \frac{1}{p}\right|,$$

and let $\omega_j \in \mathscr{P}_{1,1}(\mathbf{R}^{2d})$, $j = 0, 1, 2$, be such that, for $k = 1, 2$,

$$\begin{aligned} \omega_0(x, \varphi'_{kx}(x, \xi)) &\lesssim \omega_1(\varphi'_{k\xi}(x, \xi), \xi) \cdot \langle x \rangle^s \langle \xi \rangle^\sigma, \\ \omega_0(x, \varphi'_{kx}(x, \xi)) &\lesssim \omega_2(\varphi'_{k\xi}(x, \xi), \xi) \cdot \langle x \rangle^{s+1} \langle \xi \rangle^{\sigma+1}, \end{aligned} \tag{5.32}$$

Also assume that each ω_j, $j = 0, 1, 2$, is invariant with respect to the SG-diffeomorphims appearing in (5.32), generated by the phase functions $\varphi_j(t)$, solutions of the eikonal equations associated with τ_j, $j = 1, 2$, given by (5.31).

Theorem 4.3 implies that, if $g_0 \in M^p_{(\omega_1)}(\mathbf{R}^d)$, $g_1 \in M^p_{(\omega_2)}(\mathbf{R}^d)$, then the solution $f(t)$ of the Cauchy problem (5.27) satisfies $f(t) \in M^p_{(\omega_0)}(\mathbf{R}^d)$, for each $t \in [-T', T']$, $0 < T' \le T$. Moreover, setting $\mathcal{C} = M^p_{(\omega_0)}(\mathbf{R}^d)$, $\mathcal{B}_0 = M^p_{(\omega_1)}(\mathbf{R}^d)$, $\mathcal{B}_1 = M^p_{(\omega_2)}(\mathbf{R}^d)$, the inclusion (5.26) holds true with $\mu = 2$, $\nu = 1$, namely

$$\mathrm{WF}^m_{\mathcal{C}}(f(t)) \subseteq \bigcup_{j=1}^{2} \left[(\phi_j(t)(\mathrm{WF}^m_{\mathcal{B}_0}(g_0)))^{\mathrm{con}_m} \cup (\phi_j(t)(\mathrm{WF}^m_{\mathcal{B}_1}(g_1)))^{\mathrm{con}_m} \right],$$

$m = 1, 2, 3$, with the canonical transformations $\phi_j(t)$ generated by the phase functions $\varphi_j(t)$, $j = 1, 2$. A completely similar result holds true when $M^p_{(\omega_j)}(\mathbf{R}^d)$ is replaced by $M^\infty_{0,(\omega_j)}(\mathbf{R}^d)$, $j = 0, 1, 2$.

Finally, we recall that for certain types of choices of the symbol and the amplitude, we may avoid to have losses of regularity of the solution $f(t)$ compared to the initial data (cf. [2,12,42,45] and Remark 4.4).

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