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Generalized Prolate Spheroidal Wave Functions: Spectral Analysis and Approximation of Almost Band-Limited Functions

Abderrazek Karoui[1] · Ahmed Souabni[1]

Abstract In this work, we first give various explicit and local estimates of the eigenfunctions of a perturbed Jacobi differential operator. These eigenfunctions generalize the famous classical prolate spheroidal wave functions (PSWFs), founded in 1960s by Slepian and his co-authors and corresponding to the case $\alpha = \beta = 0$. They also generalize the new PSWFs introduced and studied recently in Wang and Zhang (Appl Comput Harmon Anal 29:303–329, 2010), denoted by GPSWFs and corresponding to the case $\alpha = \beta$. The main content of this work is devoted to the previous interesting special case $\alpha = \beta > -1$. In particular, we give further computational improvements, as well as some useful explicit and local estimates of the GPSWFs. More importantly, by using the concept of a restricted Paley–Wiener space, we relate the GPSWFs to the solutions of a generalized energy maximisation problem. As a consequence, many desirable spectral properties of the self-adjoint compact integral operator associated with the GPSWFs are deduced from the rich literature of the PSWFs. In particular, we show that the GPSWFs are well adapted for the spectral approximation of the classical c-band-limited as well as almost c-band-limited functions. Finally, we provide the reader with some numerical examples that illustrate the different results of this work.

✉ Abderrazek Karoui
   abkaroui@gmail.com; abderrazek.karoui@fsb.rnu.tn

[1] Department of Mathematics, Faculty of Sciences of Bizerte,
   University of Carthage, Carthage, Tunisia

🔶 Birkhäuser

Mathematics Subject Classification   42C10 · 33E10 · 34L10 · 41A30

1 Introduction

We first recall that for a bandwidth $c > 0$, the infinite countable set of the eigenfunctions of the finite Fourier transform $\mathcal{F}_c$, defined on $L^2(-1, 1)$ by $\mathcal{F}_c f(x) = \int_{-1}^{1} e^{icxy} f(y)\, dy$, are known as the prolate spheroidal wave functions (PSWFs). They have been extensively studied in the literature, since the pioneer work on the subject of Slepian and co-workers, see [11,15,16]. The interest in the PSWFs is essentially due to their wide range of applications in different scientific research area such as signal processing, physics, applied mathematics, see for example [7–9,18]. Recently, in [19], the authors have given a generalization of the PSWFs by considering the eigenfunctions of the special case of the weighted Fourier transform $\mathcal{F}_c^{(\alpha)}$, defined by $\mathcal{F}_c^{(\alpha)} f(x) = \int_{-1}^{1} e^{icxy} f(y) (1 - y^2)^\alpha\, dy$, $\alpha > -1$. Note that although, the extra weight function $\omega_\alpha(x) = (1 - x^2)^\alpha$, generates new computational complications, the resulting eigenfunctions have some advantages over the classical PSWFs. In this paper, we first give some useful analytic and local estimates of a more general Jacobi-type PSWFs. This is done by using special spectral techniques from the theory of Sturm–Liouville operators applied to the following Jacobi perturbed differential operator, defined on $C^2([-1, 1])$ by

$$\begin{aligned}
\mathcal{L}_c^{(\alpha,\beta)} \varphi(x) &= (1 - x^2)\varphi''(x) + ((\beta - \alpha) - (\alpha + \beta + 2)x)\varphi'(x) - c^2 x^2 \varphi(x) \\
&= \mathcal{L}_0 \varphi(x) - c^2 x^2 \varphi(x).
\end{aligned} \quad (1)$$

Note that in the limiting case $c = 0$, the eigenfunctions of the previous differential operator are reduced to the Jacobi polynomials $P_k^{(\alpha,\beta)}$. Moreover, in the special case $c > 0$, $\alpha = \beta = 0$, these eigenfunctions correspond to the classical Slepian PSWFs. Moreover, in the case where $\alpha = \beta > -1$, it has been shown in [19] that the finite weighted Fourier transform $\mathcal{F}_c^{(\alpha)}$ commutes with $\mathcal{L}_c^{(\alpha,\alpha)}$. Hence, both operators have the same eigenfunctions, called generalized prolate spheroidal wave functions (GPSWFs) and simply denoted by $\psi_{n,c}^{(\alpha)}$, $n \geq 0$. They are solutions of the following integral equation

$$\mathcal{F}_c^{(\alpha)} \psi_{n,c}^{(\alpha)}(x) = \int_{-1}^{1} e^{icxy} \psi_{n,c}^{(\alpha)}(y) \omega_\alpha(y)\, dy = \mu_n^{(\alpha)}(c) \psi_{n,c}^{(\alpha)}(x), \quad |x| \leq 1. \quad (2)$$

Here, $\mu_n^{(\alpha)}(c)$ is the eigenvalue of the integral operator $\mathcal{F}_c^{(\alpha)}$, associated with the eigenfunction $\psi_{n,c}^{(\alpha)}$.

The important part of this work is devoted to the study of the GPSWFs, that is $\alpha = \beta$. In particular, we give their analytic extensions to the whole real line. As a consequence, we obtain an explicit and practical formula (in terms of a ratio of two fast converging series) for computing the eigenvalues $\mu_n^{(\alpha)}(c)$. Note that the behaviour as

well as the decay rate of these eigenvalues, play a crucial role in most applications of the GPSWFs. In [19], by using an heuristic asymptotic analysis, the authors have given an asymptotic super-exponential decay rate of the $|\mu_n^{(\alpha)}(c)|$. In the second part of this work, we prove that for any $\alpha \geq 0$, the super-exponential decay rate of $|\mu_n^{(\alpha)}(c)|$ starts holding from the plunge region around $n = \dfrac{2c}{\pi}$. The proof of this result is based on the characterization of the GPSWFs as solutions of a generalized energy maximization problem, over a restricted Paley–Wiener space $B_c^{(\alpha)}$, given by

$$B_c^{(\alpha)} = \{ f \in L^2(\mathbb{R}), \ \text{Support } \widehat{f} \subset [-c, c], \ \widehat{f} \in L^2((-c, c), \omega_{-\alpha} \left(\dfrac{\cdot}{c} \right)).\}$$

Here, $\widehat{f}$ denotes the Fourier transform of $f \in L^2(\mathbb{R})$, defined by

$$\widehat{f}(\xi) = \lim_{A \to +\infty} \int_{-A}^{A} e^{-ix\xi} f(x)\, dx, \quad \xi \in \mathbb{R}.$$

More precisely, for a real number $\alpha > -1$, let J_α denote the Bessel function of the first type and order α and consider the self-adjoint compact operator $\mathcal{Q}_c^\alpha = \dfrac{c}{2\pi} \mathcal{F}_c^{\alpha *} \circ \mathcal{F}_c^\alpha$, defined on $L^2(I, \omega_\alpha)$, $I = [-1, 1]$ by

$$\mathcal{Q}_c^\alpha g(x) = \int_{-1}^{1} \dfrac{c}{2\pi} \mathcal{K}_\alpha(c(x - y)) g(y) \omega_\alpha(y)\, dy,$$

$$\mathcal{K}_\alpha(x) = \sqrt{\pi} 2^{\alpha+1/2} \Gamma(\alpha + 1) \dfrac{J_{\alpha+1/2}(x)}{x^{\alpha+1/2}}. \tag{3}$$

By rewriting the energy maximization problem in term of the previous integral operator, one gets a characterization of the eigenvalues $\lambda_n^{(\alpha)}(c) = \dfrac{c}{2\pi} |\mu_n^{(\alpha)}(c)|^2$ of $\mathcal{Q}_c^\alpha$ as a countable sequence generated by the energy problem. From this, we conclude that the $\lambda_n^{(\alpha)}(c)$ decay with respect to the parameter α, that is for $c > 0$, and any $n \in \mathbb{N}$,

$$0 < \lambda_n^{(\alpha)}(c) \leq \lambda_n^{(\alpha')}(c) < 1, \quad \forall \alpha \geq \alpha' \geq 0.$$

Hence, by using the precise behaviour as well as the sharp decay rate of the $\lambda_n^{(0)}(c)$, given in [6], one gets a similar behaviour and decay rate for the $(\lambda_n^{(\alpha)}(c))_n$, for any $\alpha \geq 0$.

This work is organized as follows. In Sect. 2, we give some mathematical preliminaries on Jacobi polynomials and their finite weighted Fourier transform. Also, we describe the Bouwkamp method for the Jacobi series expansion of the eigenfunctions $\psi_{n,c}^{(\alpha,\beta)}$ of the differential operator $\mathcal{L}_c^{(\alpha,\beta)}$. Then, we give some explicit and local estimates of these eigenfunctions. These estimates will be particularly useful in the subsequent study of the GPSWFs, given by Sects. 3 and 4. In Sect. 3, we first give an improvement as well as a new kind of decay rate of the Gegenbauer's series expansion coefficients of the GPSWFs $\psi_{n,c}^{(\alpha)}$. Then, we give the analytic extension of the

GPSWFs to the whole real line, as well as an explicit and practical formula (as a ratio of two fast converging series) for the accurate computation of the eigenvalues $\mu_n^{(\alpha)}(c)$ and consequently of $\lambda_n^{(\alpha)}(c)$. In Sect. 4, we characterize the GPSWFs as solutions of a generalized energy maximization problem and we prove the monotonicity of the $\lambda_n^{(\alpha)}(c)$ with respect to the parameter α. Moreover, we show that the GPSWFs are well adapted for the approximation of the classical c-band-limited as well as almost c-band-limited functions. Finally, in Sect. 5, we provide the reader with numerical examples that illustrate the different results of this work.

Notations and normalizations: The following notations will be frequently used in this work,

$$\omega_{\alpha,\beta}(y) = (1 - y)^\alpha (1 + y)^\beta, \quad \omega_\alpha(y) = (1 - y^2)^\alpha, \;\; I = [-1, 1].$$

Moreover, the eigenfunctions $\psi_{n,c}^{(\alpha,\beta)}$ and the GPSWFs $\psi_{n,c}^{(\alpha)}$ are normalized so that

$$\int_{-1}^1 \left(\psi_{n,c}^{(\alpha,\beta)}(t)\right)^2 \omega_{\alpha,\beta}(t)\, dt = 1, \quad \int_{-1}^1 \left(\psi_{n,c}^{(\alpha)}(t)\right)^2 \omega_\alpha(t)\, dt = 1.$$

2 Eigenfunctions of a Perturbed Jacobi Differential Operator

In this section, we give a description of the series expansion of the eigenfunctions $\psi_{n,c}^{(\alpha,\beta)}(x)$ of $\mathcal{L}_c^{(\alpha,\beta)}$, given by (1) and with respect to the basis of normalized Jacobi polynomials $\mathcal{B} = \{\widetilde{P}_k^{(\alpha,\beta)}, \; k \geq 0\}$. Also, we give some properties as well as local estimates of $\psi_{n,c}^{(\alpha,\beta)}(x)$, generalizing some of those given in [3,5] in the special case $\alpha = \beta = 0$. For this purpose, we need the following mathematical preliminaries.

2.1 Mathematical Preliminaries

We first recall that for two real numbers $\alpha, \beta > -1$, the Jacobi polynomials $P_k^{(\alpha,\beta)}$ are given by the following three term recursion formula

$$P_{k+1}^{(\alpha,\beta)}(x) = (A_k x + B_k) P_k^{(\alpha,\beta)}(x) - C_k P_{k-1}^{(\alpha,\beta)}(x), \quad x \in [-1, 1]. \tag{4}$$

with $P_0^{(\alpha,\beta)}(x) = 1, \quad P_1^{(\alpha,\beta)}(x) = \frac{1}{2}(\alpha + \beta + 2)x + \frac{1}{2}(\alpha + \beta)$. Here,

$$
\begin{aligned}
A_k &= \frac{(2k + \alpha + \beta + 1)(2k + \alpha + \beta + 2)}{2(k + 1)(k + \alpha + \beta + 1)}, \\
B_k &= \frac{(\alpha^2 - \beta^2)(2k + \alpha + \beta + 1)}{2(k + 1)(k + \alpha + \beta + 1)(2k + \alpha + \beta)} \\
C_k &= \frac{(k + \alpha)(k + \beta)(2k + \alpha + \beta + 2)}{(k + 1)(k + \alpha + \beta + 1)(2k + \alpha + \beta)}.
\end{aligned}
\tag{5}
$$

In the sequel, we let $\widetilde{P}_k^{(\alpha,\beta)}$ denote the normalized Jacobi polynomial of degree k so that

$$\|\widetilde{P}_k^{(\alpha,\beta)}\|_{L^2_{\omega_{\alpha,\beta}(I)}}^2 = \int_{-1}^{1} (\widetilde{P}_k^{(\alpha,\beta)}(y))^2 \omega_{\alpha,\beta}(y)\,dy = 1.$$

It is well known that in this case, we have

$$\widetilde{P}_k^{(\alpha,\beta)}(x) = \frac{1}{\sqrt{h_k}} P_k^{(\alpha,\beta)}(x), \quad h_k = \frac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{k!(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}. \quad (6)$$

Straightforward computations give us the following useful identities

$$\widetilde{P}_{k+1}^{(\alpha,\beta)}(x) = (a_k x + b_k)\widetilde{P}_k^{(\alpha,\beta)}(x) - c_k \widetilde{P}_{k-1}^{(\alpha,\beta)}(x), \quad (7)$$

$$a_k = \sqrt{\frac{h_k}{h_{k+1}}} A_k, \quad b_k = \sqrt{\frac{h_k}{h_{k+1}}} B_k, \quad c_k = \sqrt{\frac{h_{k-1}}{h_{k+1}}} C_k. \quad (8)$$

$$\begin{aligned}
x^2 \widetilde{P}_k^{(\alpha,\beta)}(x) &= \frac{1}{a_k a_{k+1}} \widetilde{P}_{k+2}^{(\alpha,\beta)}(x) - \left(\frac{b_{k+1}}{a_k a_{k+1}} + \frac{b_k}{a_k^2}\right) \widetilde{P}_{k+1}^{(\alpha,\beta)}(x) \\
&\quad + \left(\frac{c_{k+1}}{a_k a_{k+1}} + \frac{b_k^2}{a_k^2} + \frac{c_k}{a_k a_{k-1}}\right) \widetilde{P}_k^{(\alpha,\beta)}(x) \\
&\quad - \left(\frac{c_k b_k}{a_k^2} + \frac{c_k b_{k-1}}{a_k a_{k-1}}\right) \widetilde{P}_{k-1}^{(\alpha,\beta)}(x) + \frac{c_k c_{k-1}}{a_k a_{k-1}} \widetilde{P}_{k-2}^{(\alpha,\beta)}(x) \quad (9)
\end{aligned}$$

The explicit expressions and bounds of the different moments of the weight function $\omega_{\alpha,\beta}$ as well as of the Jacobi polynomials $\widetilde{P}_k^{(\alpha,\beta)}$ will be frequently needed in this work. For this purpose, we first recall the following useful inequalities for the Gamma function, see [2],

$$\sqrt{2e}\left(\frac{x+1/2}{e}\right)^{x+1/2} \leq \Gamma(x+1) \leq \sqrt{2\pi}\left(\frac{x+1/2}{e}\right)^{x+1/2}, \quad x > 0. \quad (10)$$

Next, for an integer $k \geq 0$, let

$$I_k^{\alpha,\beta} = \int_{-1}^{1} y^k \omega_{\alpha,\beta}(y)\,dy \quad (11)$$

be the kth moment of $\omega_{\alpha,\beta}$. To get an upper bound for $I_k^{\alpha,\beta}$, we may assume that $\alpha \geq \beta$. In this case, we have

$$I_k^{\alpha,\beta} = \int_0^1 y^k (1-y)^\alpha (1+y)^\beta \, dy + \int_0^1 y^k (1-y)^\beta (1+y)^\alpha \, dy$$

$$\leq \int_0^1 y^k (1-y^2)^\alpha \, dy + 2^{\alpha-\beta} \int_0^1 y^k (1-y^2)^\beta \, dy \leq \frac{1}{2} \left(B(k/2 + 1/2, \alpha+1) \right.$$

$$\left. + 2^{\alpha-\beta} B(k/2 + 1/2, \beta+1) \right). \tag{12}$$

Here, $B(\cdot, \cdot)$ is the Beta function given by $B(x, y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $x, y > 0$. Moreover, by using (10), taking into account that for any real number $a > -1$, the function

$$\varphi(x) = \left(1 + \frac{a}{x}\right)^{a+x}, \quad x \geq 1 \tag{13}$$

is decreasing on $[1, \infty)$ to e^a and by using some straightforward computations, one gets

$$I_k^{\alpha,\beta} \leq \frac{1}{2} \sqrt{\frac{\pi}{e}} \frac{2^{1+\beta}}{k^{1+\beta}} \left(\Gamma(1+\alpha) + 2^{\alpha-\beta} \Gamma(1+\beta) \right), \quad \alpha \geq \beta.$$

Consequently, for any real numbers $\alpha, \beta > -1$, we have

$$I_k^{\alpha,\beta} \leq \frac{C_{\alpha,\beta}}{k^{1+\min(\alpha,\beta)}}, \quad \forall \alpha, \beta > -1, \quad C_{\alpha,\beta} = (2^\alpha + 2^\beta) \sqrt{\frac{\pi}{e}} \Gamma(1 + \max(\alpha, \beta)). \tag{14}$$

In the special case where $\alpha = \beta$, and by using the parity of $\omega_\alpha(y) = (1 - y^2)^\alpha$ as well as the previous bound, one gets

$$I_{2k+1}^{\alpha,\alpha} = 0, \quad I_{2k}^{\alpha,\alpha} = B(k + 1/2, \alpha+1) \leq \sqrt{\frac{\pi}{e}} \frac{\Gamma(\alpha+1)}{k^{1+\alpha}}. \tag{15}$$

Also, note that for given integers $k \geq n \geq 0$, and by using the Rodrigues formula for the Jacobi polynomials, one gets the following formula for the kth moments of $\widetilde{P}_n^{(\alpha,\beta)}$, with $k \geq n$,

$$M_{k,n} = \int_{-1}^1 x^k \widetilde{P}_n^{(\alpha,\beta)}(x) \omega_{\alpha,\beta}(x) \, dx = \frac{1}{2^n \sqrt{h_n}} \binom{k}{n} \int_{-1}^1 x^{k-n} (1-x)^{n+\alpha} (1+x)^{n+\beta} \, dx. \tag{16}$$

In particular, if $\alpha = \beta$, one gets

$$M_{k,n} = \int_{-1}^1 x^k \widetilde{P}_n^{(\alpha,\alpha)}(x) \omega_\alpha(x) \, dx$$

$$= \begin{cases} 0 & \text{if } k < n \text{ or } k - n \text{ is odd} \\ \frac{\binom{k}{n}}{2^n \sqrt{h_n}} B\left(\frac{k-n+1}{2}, n + \alpha + 1\right) & \text{otherwise.} \end{cases} \tag{17}$$

On the other hand, it is interesting to note that the weighted finite Fourier transform of Jacobi polynomial is given by the following explicit expression, see [13, p. 456],

$$
\int_{-1}^{1} e^{ixy} P_k^{(\alpha,\beta)}(y)\omega_{\alpha,\beta}(y)\,dy = \frac{(ix)^k e^{ix}}{k!} 2^{k+\alpha+\beta+1} B(k+\alpha+1, k+\beta+1)_1
$$
$$
\times F_1(k+\alpha+1, 2k+\alpha+\beta+2, -2ix), \quad (18)
$$

where $B(x,y)$ is the Beta function and $_1F_1(a,b,c)$ is the Kummer's function. It is well known, see [13, p. 326] that the Kummer's function has the following integral representation

$$
_1F_1(a,b;z) = \frac{1}{B(a,b)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt, \quad z \in \mathbb{C}, \quad \mathcal{R}e(b) > \mathcal{R}e(a) > 0.
$$
$$(19)$$

2.2 Computation and First Properties of the Eigenfunctions of $\mathcal{L}_c^{(\alpha,\beta)}$

In this paragraph, we first describe the Bouwkamp method for the computation of the bounded eigenfunctions and the corresponding eigenvalues of the operator $\mathcal{L}_c^{(\alpha,\beta)}$, given by (1). Then, we give some general properties of these eigenfunctions. Note that Bouwkamp method can be briefly described as the representation of a perturbed version of classical orthogonal polynomials differential operator. This representation is done by the use of the original classical orthogonal polynomials. In our case, we consider the Jacobi orthonormal basis of $L^2(I, \omega_{\alpha,\beta})$, given by $\mathcal{B}^{\alpha,\beta} = \{\widetilde{P}_k^{(\alpha,\beta)}(x), k \geq 0\}$. Then, thanks to this method, the computation of the bounded eigenfunctions $\psi_{n,c}^{(\alpha,\beta)}$ of $\mathcal{L}_c^{(\alpha,\beta)}$ and their associated eigenvalues $\chi_n(c)$ is reduced to the computation of the eigenvectors and the associated eigenvalues of the infinite order matrix representation of $\mathcal{L}_c^{(\alpha,\beta)}$ with respect to the basis $\mathcal{B}^{\alpha,\beta}$. It is interesting to note that only a finite number of the main diagonals of this representation matrix are not identically zeros. To the best of our knowledge, Niven, was the first to use this method in the early 1880s, see [12].

Note that since $\psi_{n,c}^{(\alpha,\beta)} \in L^2(I, \omega_{\alpha,\beta})$, then its series expansion with respect to the basis $\mathcal{B}^{\alpha,\beta}$ is given by

$$
\psi_{n,c}^{(\alpha,\beta)}(x) = \sum_{k \geq 0} \beta_k^n \widetilde{P}_k^{(\alpha,\beta)}(x), \quad x \in [-1, 1]. \tag{20}
$$

By combining (20) and the facts that

$$
-\mathcal{L}_c^{(\alpha,\beta)} \psi_{n,c}^{(\alpha,\beta)}(x) = \chi_n(c)\psi_{n,c}^{(\alpha,\beta)}(x), \quad -\mathcal{L}_0^{(\alpha,\beta)} \widetilde{P}_k^{(\alpha,\beta)}(x) = \chi_k(0)\widetilde{P}_k^{(\alpha,\beta)}(x),
$$
$$
\chi_k(0) = k(k+\alpha+\beta+1),
$$

one can easily check that the expansion coefficients $(\beta_k^n)_{k\geq 0}$, $n \geq 0$ and the eigenvalues $(\chi_n(c))_{n\geq 0}$ are given by the following infinite order eigensystem

$$\mathbf{D}^{\alpha,\beta} \cdot \mathbf{B}_n = \chi_n(c)\,\mathbf{B}_n, \quad \mathbf{D}^{\alpha,\beta} = \left[d_{i,j}\right]_{i,j \geq 0}, \quad \mathbf{B}_n = [\beta_k^n, k \geq 0]^T. \quad (21)$$

Here, $\mathbf{D}^{\alpha,\beta}$ is a 5-diagonals matrix representation of the operator $-\mathcal{L}_c^{(\alpha,\beta)}$ with coefficients given by

$$d_{i,i-2} = d_{i,i-2} = c^2 \frac{1}{a_{i-1}a_{i-2}}, \quad d_{i,i-1} = -c^2 \left(\frac{b_i}{a_i a_{i-1}} + \frac{b_{i-1}}{a_{i-1}^2} \right)$$

$$d_{i,i} = i(i + \alpha + \beta + 1) + c^2 \left(\frac{c_{i+1}}{a_i a_{i+1}} - \frac{b_i^2}{a_i^2} + \frac{c_i}{a_i a_{i-1}} \right),$$

$$d_{i,i+1} = c^2 \left(\frac{c_{i+1}b_{i+1}}{a_{i+1}^2} + \frac{c_{i+1}b_i}{a_{i+1}a_i} \right), \quad d_{i,j} = 0, \quad \text{if } |j - i| \geq 3. \quad (22)$$

We recall that the coefficients a_i, b_i, c_i are given by (5) and (8). In the special case where $\alpha = \beta$, we have $b_i = 0$, so that the previous eigensystem is reduced to a symmetric tri-diagonal system. In this case, for a fixed integer $n \geq 0$, the sequence $(\beta_k^n)_{k \geq 0}$ satisfies the following eigensystem

$$c^2 \frac{1}{a_{k+2}a_{k+1}} \beta_{k+2}^n + \left(k(k + 2\alpha + 1) + c^2 \left(\frac{c_{k+1}}{a_k a_{k+1}} + \frac{c_k}{a_k a_{k-1}} \right) \right) \beta_k^n$$

$$+ c^2 \frac{1}{a_k a_{k-1}} \beta_{k-2}^n = \chi_n(c)\beta_k^n. \quad (23)$$

An expanded form of this system is given by

$$\frac{\sqrt{(k+1)(k+2)(k+2\alpha+1)(k+2\alpha+2)}}{(2k+2\alpha+3)\sqrt{(2k+2\alpha+5)(2k+2\alpha+1)}} c^2 \beta_{k+2}^n + \left(k(k+2\alpha+1) \right.$$

$$+ c^2 \frac{2k(k+2\alpha+1) + 2\alpha - 1}{(2k+2\alpha+3)(2k+2\alpha-1)} \Big) \beta_k^n$$

$$+ \frac{\sqrt{k(k-1)(k+2\alpha)(k+2\alpha-1)}}{(2k+2\alpha-1)\sqrt{(2k+2\alpha+1)(2k+2\alpha-3)}} c^2 \beta_{k-2}^n = \chi_n(c)\beta_k^n, \quad k \geq 0. \quad (24)$$

The following proposition provides us with some properties of the eigenfunctions $\psi_{n,c}^{(\alpha,\beta)}(x)$ and eigenvalues $\chi_n(c)$, generalizing some known properties for Jacobi polynomials.

Proposition 1 *For given real numbers $c > 0$, $\alpha, \beta > -1$, let $\psi_{n,c}^{(\alpha,\beta)}$ be the nth eigenfunction associated with $\mathcal{L}_c^{(\alpha,\beta)}$, and normalized so that $\|\psi_{n,c}^{(\alpha,\beta)}\|_{L^2(I,\omega_{\alpha,\beta})} = 1$. Then we have*

(P_1) *The set $\mathcal{B} = \{\psi_{n,c}^{(\alpha,\beta)}, n \geq 0\}$ is an orthonormal basis of $L^2(I, \omega_{\alpha,\beta})$.*
(P_2) *If $\psi_{n,c}^{(\beta,\alpha)}$ is the nth normalized eigenfunction of $\mathcal{L}_c^{(\beta,\alpha)}$, then $\psi_{n,c}^{(\alpha,\beta)}$ and $\psi_{n,c}^{(\beta,\alpha)}$ are associated to the same eigenvalue $\chi_n(c)$. Moreover, they are related to each others by the following rule*

$$\psi_{n,c}^{(\alpha,\beta)}(-x) = (-1)^n \psi_{n,c}^{(\beta,\alpha)}(x), \quad x \in \mathbb{R}. \tag{25}$$

Proof Property (P_1) follows from the general spectral theory of Sturm–Liouville operators. To prove (P_2), we use the following well known property for Jacobi polynomials, see [17, p. 59]

$$P_k^{(\alpha,\beta)}(-x) = (-1)^k P_k^{(\beta,\alpha)}(x). \tag{26}$$

Let $\mathbf{D}^{(\alpha,\beta)} = [d_{i,j}]_{i,j\geq 0}$ and $\mathbf{D}^{(\beta,\alpha)} = [\widetilde{d}_{i,j}]_{i,j\geq 0}$ be the matrix representation of $-\mathcal{L}_c^{(\alpha,\beta)}$ and $-\mathcal{L}_c^{(\beta,\alpha)}$ with respect to the basis of Jacobi polynomials $\widetilde{P}_k^{(\alpha,\beta)}$ and $\widetilde{P}_k^{(\beta,\alpha)}$, respectively. Then from (4) and (22), one gets

$$\widetilde{d}_{i,i+k} = (-1)^d_{i,i+k}, \quad -2 \leq k \leq 2, \quad i \geq 0 \quad \text{and} \quad \widetilde{d}_{i,j} = 0, \quad \text{if } |i-j| \geq 3. \tag{27}$$

Let $\mathbf{B}_n = [\beta_k^n, k \geq 0]^T$ and $\widetilde{\mathbf{B}}_n = [\widetilde{\beta}_k^n, k \geq 0]^T = [(-1)^k \beta_k^n, k \geq 0]^T$. Since $\mathbf{D}^{(\alpha,\beta)}\mathbf{B}_n = \chi_n(c)\mathbf{B}_n$, then by using (27), it is easy to see that $\mathbf{D}^{(\beta,\alpha)}\widetilde{\mathbf{B}}_n = \chi_n(c)\widetilde{\mathbf{B}}_n$. This means that $\psi_{n,c}^{(\alpha,\beta)}$ and $\psi_{n,c}^{(\beta,\alpha)}$ are associated to the same eigenvalue $\chi_n(c)$. Moreover, the series expansion of $\psi_{n,c}^{(\beta,\alpha)}$ in the basis $\{\widetilde{P}_k^{(\beta,\alpha)}, \ k \geq 0\}$ is obtained from the series expansion of $\psi_{n,c}^{(\alpha,\beta)}$, as follows

$$\psi_{n,c}^{(\beta,\alpha)}(x) = c_n \sum_{k=0}^{\infty} (-1)^k \beta_k^n \widetilde{P}_k^{(\beta,\alpha)}(x), \tag{28}$$

for some constant c_n. By combining (26) and the previous equality, one concludes that $\psi_{n,c}^{(\alpha,\beta)}(-x) = c_n \psi_{n,c}^{(\beta,\alpha)}(x)$. Moreover, since $\|\psi_{n,c}^{(\alpha,\beta)}\|_{L^2(I,\omega_{\alpha,\beta})} = \|\psi_{n,c}^{(\beta,\alpha)}\|_{L^2(I,\omega_{\beta,\alpha})} = 1$ and since $\psi_{n,c}^{(\alpha,\alpha)}$ has the same parity as n, see [19], then $c_n = (-1)^n$. This concludes the proof of (25). $\qquad\square$

Also, we should mention that the $(n+1)$th eigenvalue $\chi_n(c)$ satisfies the following classical inequalities,

$$n(n+\alpha+\beta+1) \leq \chi_n(c) \leq n(n+\alpha+\beta+1) + c^2, \quad \forall n \geq 0. \tag{29}$$

To get the previous upper bound, we consider the following Sturm–Liouville form of $-\mathcal{L}_c^{(\alpha,\beta)}$,

$$-\mathcal{L}_c^{(\alpha,\beta)}(f)(x) = -\frac{d}{dx}\left[\omega_{\alpha,\beta}(x)(1-x^2)f'(x)\right] + c^2 x^2 \omega_{\alpha,\beta}(x) \tag{30}$$
$$= -\mathcal{L}_0^{(\alpha,\beta)}(f)(x) + c^2 x^2 \omega_{\alpha,\beta}(x).$$

Then, from the well known Poincaré Min–Max characterization of the eigenvalue of a self-adjoint operator, applied to the operator $-\mathcal{L}_c^{(\alpha,\beta)}$, one gets

$$
\chi_n(c) = \min_{\dim H=n} \max_{u \in H, \|u\|=1} \int_{-1}^{1} \left(-\mathcal{L}_0^{(\alpha,\beta)}(u)(x)u(x) + c^2 x^2 (u(x))^2 \right) \omega_{\alpha,\beta}(x)\, dx
$$

$$
\leq \min_{\dim H=n} \max_{u \in H, \|u\|=1} \int_{-1}^{1} -\mathcal{L}_0^{(\alpha,\beta)}(u)(x)u(x)\omega_{\alpha,\beta}(x)\, dx + c^2 \|u\|^2
$$

$$
\leq \chi_n(0) + c^2 = n(n+\alpha+\beta+1) + c^2.
$$

Next, to get a lower bound, it suffices to see that the self-adjoint operator $-\mathcal{L}_c^{(\alpha,\beta)} - (-\mathcal{L}_0^{(\alpha,\beta)}) = c^2 x^2$ is a positive operator, which implies that $\chi_n(c) \geq \chi_n(0)$.

2.3 Local Estimates of the Eigenfunctions of $\mathcal{L}_c^{(\alpha,\beta)}$

In this paragraph, we give various explicit and local estimates of the $\psi_{n,c}^{(\alpha,\beta)}$. These estimates will be needed to prove some of the results of Sect. 3 and 4 of this work. We should mention that in the literature, only few references have studied the problem of the explicit estimates of the classical PSWFs and their eigenvalues $\chi_n(c)$, see [3,5,14]. The following proposition provides us with explicit local bounds of $\psi_{n,c}^{(\alpha,\beta)}$, generalizing a similar result given in [3] for the special case $\alpha = \beta = 0$.

Proposition 2 *For real numbers $c > 0$, $\alpha, \beta > -1$, with $\alpha + \beta + 1 \geq 0$. Let $n \in \mathbb{N}$ be such $q = c^2/\chi_n(c) < 1$. Then we have*

$$
\sup_{t \in [0,1]} (1-t^2)\omega_{\alpha,\beta}(t) \left(\left| \psi_{n,c}^{(\alpha,\beta)}(t) \right|^2 + \frac{1-t^2}{(1-qt^2)\chi_n(c)} \left| (\psi_{n,c}^{(\alpha,\beta)})'(t) \right|^2 \right)
$$
$$
\leq 2(1+\max(\alpha,\beta)). \tag{31}
$$

Moreover, if $\alpha = \beta$, then we have

$$
\sup_{t \in [-1,1]} (1-t^2)\omega_{\alpha}(t) \left(\left| \psi_{n,c}^{(\alpha)}(t) \right|^2 + \frac{1-t^2}{(1-qt^2)\chi_n(c)} \left| (\psi_{n,c}^{(\alpha)})'(t) \right|^2 \right) \leq 1+\alpha. \tag{32}
$$

Proof The proof uses a classical technique for the local estimates of the eigenfunctions of a Sturm–Liouville operator. In our case, we first note that by using property (P_2) of Proposition 1, it suffices to consider the case $\alpha \geq \beta$, since the case $\beta \geq \alpha$, follows from the equality (25). Next, consider the auxiliary function, defined on [0, 1] by

$$
Z_n(t) = \left(\psi_{n,c}^{(\alpha,\beta)}(t) \right)^2 + \frac{1-t^2}{\chi_n(c)(1-qt^2)} \left(\left(\psi_{n,c}^{(\alpha,\beta)} \right)' \right)^2 (t).
$$

Since $\psi_{n,c}^{(\alpha,\beta)}$ is the eigenfunction of the operator $-\mathcal{L}_c^{(\alpha,\beta)}$ associated with the eigenvalue $\chi_n(c)$, then straightforward computations give us

$$
Z_n'(t) = \frac{2((\psi_{n,c}^{(\alpha,\beta)})')^2(t)}{\chi_n(c)(1-qt^2)} \left((\alpha-\beta) + (1+\alpha+\beta)t + qt\frac{1-t^2}{1-qt^2} \right). \tag{33}
$$

Since $0 \leq q \leq 1$, $\alpha - \beta \geq 0$ and $\alpha + \beta + 1 \geq 0$, then it is easy to see that

$$Z_n'(t) \geq 0, \quad \forall\, t \in [0, 1].$$

Next, we consider a second auxiliary function, given by

$$K_n(t) = (1 - t^2)\omega_{\alpha,\beta}(t)Z_n(t), \quad t \in [0, 1].$$

Then, by using (33), one can easily check that there exists a positive valued function $A(\cdot)$ on $[-1, 1]$ with

$$
\begin{aligned}
K_n'(t) &= \omega_{\alpha,\beta}(t)\left(-2t + (\beta - \alpha) - (\alpha + \beta)t\right)Z_n(t) + (1 - t^2)\omega_{\alpha,\beta}(t)Z_n'(t) \\
&= \left(-2t + (\beta - \alpha) - (\alpha + \beta)t\right)\omega_{\alpha,\beta}(t)\left(\psi_{n,c}^{(\alpha,\beta)}\right)^2(t) + A(t)\left(\psi_{n,c}^{(\alpha,\beta)}{}'\right)^2(t) \\
&\geq \left((\beta - \alpha) - (2 + \alpha + \beta)t\right)\omega_{\alpha,\beta}(t)\left(\psi_{n,c}^{(\alpha,\beta)}\right)^2(t).
\end{aligned}
$$

Finally, since $K_n(1) = 0$ and since $\displaystyle\int_{-1}^{1}\left(\psi_{n,c}^{(\alpha,\beta)}\right)^2(t)\,\omega_{\alpha,\beta}(t)\,dt = 1$, then by using the last inequality, one gets

$$
\begin{aligned}
K_n(t) - K_n(1) = K_n(t) &\leq \max_{t\in[0,1]}\left((\alpha - \beta) + (2 + \alpha + \beta)t\right) \\
&\quad \times \int_{0}^{1}\left(\psi_{n,c}^{(\alpha,\beta)}\right)^2(t)\,\omega_{\alpha,\beta}(t)\,dt \leq 2(1 + \alpha).
\end{aligned}
$$

Finally, if $\beta = \alpha$, then from the parity of $\psi_{n,c}^{(\alpha)}(t)$, we have $\displaystyle\int_{0}^{1}\left(\psi_{n,c}^{(\alpha)}\right)^2(t)\,\omega_{\alpha}(t)\,dt = 1/2$, which means that the previous upper bound is replaced by $1 + \alpha$. This concludes the proof of the proposition. $\qquad\square$

The following proposition provides us with an estimate of the maximum of the $\psi_{n,c}^{(\alpha,\beta)}$ inside the interval I.

Proposition 3 *Let $c > 0$, and $\alpha \geq \beta$ with $\alpha + \beta \geq -1$, then for any positive integer n with $q = c^2/\chi_n(c) \leq 1$, we have*

$$
\sup_{x\in[0,1]}|\psi_{n,c}^{(\alpha,\beta)}(x)| = |\psi_{n,c}^{(\alpha,\beta)}(1)| \leq C_\alpha(\chi_n(c))^{\frac{1+\alpha}{2}}, \quad C_\alpha = \frac{2^{1+\max(\alpha,0)}}{\sqrt{3+\alpha}}\left(\frac{1+\alpha}{3+\alpha}\right)^{1+\frac{\alpha}{2}}
\tag{34}
$$

Moreover, if $\alpha = \beta$, then we have

$$
\sup_{x\in[-1,1]}|\psi_{n,c}^{(\alpha)}(x)| = |\psi_{n,c}^{(\alpha)}(1)| \leq \frac{C_\alpha}{\sqrt{2}}(\chi_n(c))^{\frac{1+\alpha}{2}}.
\tag{35}
$$

Proof We first recall that the auxiliary function Z_n given by :

$$Z_n(x) = \left(\psi_{n,c}^{(\alpha,\beta)}(x)\right)^2 + \frac{1-x^2}{\chi_n(c)(1-qx^2)}\left(\left(\psi_{n,c}^{(\alpha,\beta)}\right)'\right)^2(x)$$

is increasing over [0,1] whenever $q = c^2/\chi_n(c) \leq 1$, $\alpha \geq \beta$ and $\alpha + \beta + 1 \geq 0$. Hence, we have

$$\sup_{x\in[0,1]} Z_n(x) = Z_n(1) = \left(\psi_{n,c}^{(\alpha,\beta)}(x)\right)^2$$

which implies that

$$\sup_{x\in[0,1]} |\psi_{n,c}^{(\alpha,\beta)}(x)| = |\psi_{n,c}^{(\alpha,\beta)}(1)|, \quad \forall n \in \mathbb{N}, \text{ with } q \leq 1. \tag{36}$$

Moreover, if $\alpha = \beta$ then from the parity of $\psi_{n,c}^{(\alpha)}$, one gets

$$\sup_{x\in[-1,1]} |\psi_{n,c}^{(\alpha)}(x)| = |\psi_{n,c}^{(\alpha)}(1)|, \quad \forall n \in \mathbb{N}, \text{ with } q \leq 1. \tag{37}$$

Next, we show how to get the upper bounds of $|\psi_{n,c}^{(\alpha,\beta)}(1)|$ and $|\psi_{n,c}^{(\alpha)}(1)|$. To alleviate notation, we simply denote $\psi_{n,c}^{(\alpha,\beta)}$ by $\psi_{n,c}$ and $\chi_n(c)$ by χ_n. Also, without loss of generality, we may assume that $\psi_{n,c}(1) > 0$. Since

$$(\psi_{n,c}'(x)(1-x^2)\omega_{\alpha,\beta}(x))' = -\chi_n(c)\omega_{\alpha,\beta}(x)(1-qx^2)\psi_{n,c}(x), \tag{38}$$

then

$$\psi_{n,c}'(x) = \frac{\chi_n}{(1-x^2)\omega_{\alpha,\beta}(x)} \int_x^1 \omega_{\alpha,\beta}(t)(1-qt^2)\psi_{n,c}(t)dt$$

$$\leq \frac{\chi_n}{(1-x^2)}(1-qx^2)(1-x)\psi_{n,c}(1) = \chi_n(1-qx^2)\psi_{n,c}(1).$$

Hence,

$$\psi_{n,c}(1) - \psi_{n,c}(x) \leq \chi_n Q_q(x)\psi_{n,c}(1), \qquad Q_q(x) = (1-qx^2)(1-x^2). \tag{39}$$

Next, let $x_n \in [0,1]$ be such that $Q_q(x_n) = \frac{a}{\chi_n}$, where the constant a to be fixed later on. By substituting x with x_n in (39) and by using (31), one gets

$$\psi_{n,c}(1) \leq \frac{1}{1-a}\psi_{n,c}(x_n) \leq \frac{1}{(1-a)}\sqrt{2(1+\alpha)}\left(\frac{1}{Q_q(x_n)\omega_{\alpha,\beta}(x_n)}\right)^{1/2}.$$

That is

$$\psi_{n,c}(1) \leq \frac{\sqrt{2(1+\alpha)}}{a^{1/2}(1-a)} \frac{\chi_n^{1/2}}{\sqrt{\omega_{\alpha,\beta}(x_n)}} \tag{40}$$

Note that the admissible solution of $Q_q(x_n) = \frac{a}{\chi_n}$ is given by $x_n = \left(\frac{(q+1)-\sqrt{(q-1)^2+\frac{4aq}{\chi_n}}}{2q}\right)^{1/2}$. Consequently,

$$\frac{1-\frac{a}{\chi_n}}{1+\sqrt{\frac{aq}{\chi_n}}} \leq x_n = \left(\frac{2(1-\frac{a}{\chi_n})}{q+1+(1-q)\sqrt{1+\frac{4aq}{\chi_n(q-1)^2}}}\right)^{\frac{1}{2}} \leq \left(1-\frac{a}{\chi_n}\right)^{\frac{1}{2}} \leq 1-\frac{a}{2\chi_n}.$$

It is easy to see that in this case, we have

$$\frac{a}{2\chi_n} \leq 1-x_n \leq \frac{a}{\chi_n}\left(1+\sqrt{\frac{\chi_n}{a}}\right).$$

Consequently, by using the first inequality when $\alpha \geq 0$ and the second inequality when $-1/2 \leq \alpha < 0$, one gets

$$\frac{1}{\sqrt{\omega_{\alpha,\beta}(x_n)}} \leq \begin{cases} \left(\frac{\chi_n}{2a}\right)^{\alpha/2} 2^{-\beta/2} \leq \sqrt{2}\left(\frac{\chi_n}{a}\right)^{\alpha/2} & \text{if } -1/2 \leq \alpha < 0 \\ \left(\frac{2\chi_n}{a}\right)^{\alpha/2} \frac{1}{(1+x_n)^{\beta/2}} \leq 2^{\alpha+1/2}\left(\frac{\chi_n}{a}\right)^{\alpha/2} & \text{if } \alpha \geq 0. \end{cases} \tag{41}$$

Hence, by combining (40) and (41), one gets

$$\psi_{n,c}^{(\alpha,\beta)}(1) \leq \frac{2^{1+\max(\alpha,0)}\sqrt{(1+\alpha)}}{a^{(1+\alpha)/2}(1-a)} \chi_n^{\frac{1+\alpha}{2}}. \tag{42}$$

Since the maximum of $a^\gamma(1-a)$ is attained at $a = \frac{\gamma}{1+\gamma}$ then for $\gamma = \frac{1+\alpha}{2}$, one gets (34). Finally, (35) follows from the parity of $\psi_{n,c}^{(\alpha)}$ and (32). □

Remark 1 The techniques used for the proof of inequality (40) are similar to those used in [3] to prove a similar inequality, restricted to the special case $\alpha = \beta = 0$. Nonetheless, the general setting of the previous proposition requires handling the new quantity $\sqrt{\omega_{\alpha,\beta}(x_n)}$ that generates extra difficulties to obtain the local estimates (34) and (35).

3 Generalized Prolate Spheroidal Wave Functions: Computations and Analytic Extension

In the sequel, we restrict ourselves to the case $\alpha = \beta > -1$. In the first part of this section, we further improve the super-exponential decay rate of the GPSWFs

expansion coefficients $(\beta_k^n)_k$, that has been given in given in [19]. Then, we show that for sufficiently large values of n and up a certain order K_n, all the coefficients β_k^n, $0 \leq k \leq K_n$ are positive. As a consequence of this positivity result and the previous fast decay of the β_k^n, we show that in the case where $\alpha = \beta$, the expansion coefficients $(\beta_k^n)_k$ are essentially concentrated around $k = n$. In the second part of this section, we give the analytic extension of the GPSWFs, together with an explicit expression for the eigenvalues $\mu_n^{(\alpha)}(c)$ as a ratio of two fast convergent series.

3.1 Computation and Analytic Extension of the GPSWFs

We first note that in the interesting special case where $\alpha = \beta$, formula (18) is simplified in a significant manner. This is given by the following lemma.

Proposition 4 *Let $\alpha > -1$, then we have*

$$\int_{-1}^{1} e^{ixy} P_k^{(\alpha,\alpha)}(y)\omega_\alpha(y)\,dy = i^k \sqrt{\pi} \left(\frac{2}{x}\right)^{\alpha+1/2} \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)} J_{k+\alpha+1/2}(x), \quad x \in \mathbb{R}. \tag{43}$$

Here, J_a denotes the Bessel function of the first kind and order α.

Proof It is well known, see for example [1, p. 200], that if $a > -1$ and $z = -2ix$, $x \in \mathbb{R}$, then we have,

$$_1F_1(a + 1/2, 2a + 1; -2ix) = \Gamma(a + 1) \left(\frac{2}{x}\right)^a e^{-ix} J_a(x). \tag{44}$$

By combining (18) and the previous equality with $a = k + \alpha + 1/2$, one gets

$$\mathcal{F}_1^\alpha(P_k^{(\alpha,\alpha)})(x) = \int_{-1}^{1} e^{ixy} P_k^{(\alpha,\alpha)}(y)\omega_\alpha(y)\,dy$$
$$= \frac{(ix)^k 2^{2k+3\alpha+3/2}}{x^{k+\alpha+1/2}} \frac{\Gamma(k+\alpha+3/2)}{\Gamma(k+1)} B(k+\alpha+1, k+\alpha+1) J_{k+\alpha+1/2}(x). \tag{45}$$

Moreover, by using the following identities of Beta and Gamma functions,

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \Gamma(x+1) = x\Gamma(x), \quad \Gamma(b)\Gamma(b+1/2) = \sqrt{\pi}\frac{\Gamma(2b)}{2^{2b-1}},$$

one gets

$$\mathcal{F}_1^\alpha P_k^{(\alpha,\alpha)}(x)$$
$$= i^k \frac{2^{2k+3\alpha+3/2}}{(x)^{\alpha+1/2}} (k+\alpha+1/2) \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)} \frac{\Gamma(k+\alpha+1)\Gamma(k+\alpha+1/2)}{\Gamma(2k+2\alpha+2)} J_{k+\alpha+1/2}(x)$$

$$= i^k \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)} \frac{2^{2k+3\alpha+3/2}}{(cx)^{\alpha+1/2}} \frac{k+\alpha+1/2}{2k+2\alpha+1} \frac{\Gamma(k+\alpha+1)\Gamma(k+\alpha+1/2)}{\Gamma(2k+2\alpha+1)} J_{k+\alpha+1/2}(x)$$

$$= i^k \sqrt{\pi} \left(\frac{2}{x}\right)^{\alpha+1/2} \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)} J_{k+\alpha+1/2}(x), \quad x \in \mathbb{R}.$$

$\square$

Remark 2 In the special case $\alpha = 0$, the equality (43) is reduced to the well known classical finite Fourier transform of Legendre function, see for example [1, p. 343].

As a first consequence of the previous proposition, one gets a simple and straightforward proof of the following result that has been already given by Lemma 3.4 in [19] and with different kind of proof.

Corollary 1 *Under the above notations, for any real numbers $c > 0$ and $\alpha > -1$, we have*

$$\beta_k^n = \int_{-1}^1 \widetilde{P}_k^{(\alpha,\alpha)}(y) \psi_{n,c}^{(\alpha)}(y) \omega_\alpha(y) \, dy$$

$$= \frac{2\sqrt{\pi} i^k}{\mu_n^{(\alpha)}(c)\sqrt{h_k}} \left(1 + (-1)^{n+k}\right) \int_0^1 \frac{J_{k+\alpha+\frac{1}{2}}(ct)}{(ct)^{\alpha+\frac{1}{2}}} \psi_{n,c}^{(\alpha)}(t) \omega_\alpha(t) \, dt. \quad (46)$$

Proof Just write

$$\beta_k^n = \frac{1}{\mu_n^{(\alpha)}(c)} \int_{-1}^1 \widetilde{P}_k^{(\alpha,\alpha)}(y) \left(\int_{-1}^1 e^{icty} \psi_{n,c}^{(\alpha)}(t) \omega_\alpha(t) \, dt\right) \omega_\alpha(y) \, dy$$

$$= \frac{\sqrt{\pi} i^k}{\mu_n^{(\alpha)}(c)\sqrt{h_k}} \int_{-1}^1 \frac{J_{k+\alpha+\frac{1}{2}}(ct)}{(ct)^{\alpha+\frac{1}{2}}} \psi_{n,c}^{(\alpha)}(t) \omega_\alpha(t) \, dt.$$

To conclude, it suffices to write the previous integral as $\int_{-1}^1 = \int_0^1 + \int_{-1}^0$ and use the facts that the function $\psi_{n,c}^{(\alpha)}$ and $t \mapsto \dfrac{J_{k+\alpha+\frac{1}{2}}(ct)}{(ct)^{\alpha+\frac{1}{2}}}$ has the same parity as n and k, respectively. $\square$

A decay rate of the expansion coefficients is given by the following proposition that improves the result given by Theorem 3.4 in [19].

Proposition 5 *For given real numbers $c > 0$, $\alpha > -1$ and integers $k, n \in \mathbb{N}$, let*

$$\beta_k^n = \int_{-1}^1 \widetilde{P}_k^{(\alpha,\alpha)}(x) \, \psi_{n,c}^{(\alpha)}(x) \, \omega_\alpha(x) dx.$$

Then, we have

$$|\beta_k^n| \leq \frac{C_\alpha}{|\mu_n^{(\alpha)}(c)|} \frac{1}{k^{1+\alpha/2}} \frac{1}{2^k} \left(\frac{ec}{2k+1}\right)^k, \quad (47)$$

where $C_\alpha = \dfrac{\pi^{7/4}\sqrt{\Gamma(1+\alpha)}(3/2)^{3/4}(3/2+2\alpha)^{3/4+\alpha}}{2^{\alpha+1}e^{\alpha+5/4}}$.

Proof By using the expression of β_k^n and by combining (2) and (18), one gets

$$
\begin{aligned}
\beta_k^n &= \frac{1}{\mu_n^{(\alpha)}(c)} \int_{-1}^1 \int_{-1}^1 \widetilde{P}_k^{(\alpha,\beta)}(x) \left(\int_{-1}^1 e^{icxy} \psi_{n,c}(y) \omega_\alpha(y)\, dy \right) \omega_\alpha(x)\, dx \\
&= \frac{1}{\mu_n(c)} \int_{-1}^1 \left(\int_{-1}^1 e^{icxy} \widetilde{P}_k^{(\alpha,\beta)}(x) \omega_\alpha(x)\, dx \right) \psi_{n,c}^{(\alpha)}(y) \omega_\alpha(y)\, dy \\
&= \frac{1}{\mu_n^{(\alpha)}(c)} \int_{-1}^1 \frac{(icy)^k e^{icy}}{k! \sqrt{h_k}} 2^{k+2\alpha+1} B(k+\alpha+1, k+\alpha+1) \cdot \\
&\qquad {}_1F_1(k+\alpha+1, 2k+2\alpha+2; -2icy) \psi_n(y)\, \omega_\alpha(y)\, dy.
\end{aligned}
$$

On the other hand, from the integral representation of Kummer's function given by (19), one gets

$$
\begin{aligned}
&|{}_1F_1(k+\alpha+1, 2k+2\alpha+2; -2icy)| \\
&= \frac{\Gamma(2k+2\alpha+2)}{\Gamma(k+\alpha+1)^2} \left| \int_0^1 e^{-2icyt} t^{k+\alpha}(1-t)^{k+\alpha} dt \right| \\
&\leq \frac{1}{B(k+\alpha+1, k+\alpha+1)} \left| \int_0^1 e^{-2icyt} t^{k+\alpha}(1-t)^{k+\alpha} dt \right| \\
&\leq \frac{B(k+\alpha+1, k+\alpha+1)}{B(k+\alpha+1, k+\alpha+1)} = 1.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
|\beta_k^n| &\leq \frac{B(k+\alpha+1, k+\alpha+1)}{|\mu_n^{(\alpha)}(c)|} \frac{c^k 2^{k+2\alpha+1}}{k! \sqrt{h_k}} \left(\int_{-1}^1 (\psi_{n,c}^{(\alpha)})^2(y) \omega_\alpha(y)\, dy \right)^{1/2} \\
&\quad \times \left(\int_{-1}^1 y^{2k} \omega_\alpha(y)\, dy \right)^{1/2} \\
&\leq \frac{B(k+\alpha+1, k+\alpha+1)}{|\mu_n^{(\alpha)}(c)|} \frac{c^k 2^{k+3/2(\alpha+\alpha+1)}}{k! \sqrt{h_k}} \sqrt{I_{2k}^{\alpha,\alpha}}.
\end{aligned} \tag{48}
$$

Note that from (10), we have

$$
\frac{c^k}{k!} = \frac{c^k}{\Gamma(k+1)} \leq \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2k+1}} \left(\frac{ec}{2k+1} \right)^k. \tag{49}
$$

In a similar manner, we get the following upper bound and lower bound of the quantity $B(k+\alpha+1, k+\alpha+1)$ and the normalization constant h_k, given as follows.

$$
\begin{aligned}
B(k+\alpha+1, k+\alpha+1) &= \frac{\Gamma(k+\alpha+1)^2}{\Gamma(2k+2\alpha+2)} \leq \frac{\sqrt{2\pi}}{2^{2k+2\alpha+1}} \frac{(2k+2\alpha+1)^{2k+2\alpha+1}}{(2k+2\alpha+3/2)^{2k+2\alpha+3/2}} \\
&\leq \frac{\sqrt{2\pi}}{2^{2k+2\alpha+1}} \frac{1}{\sqrt{2k+2\alpha+3/2}}
\end{aligned} \tag{50}
$$

Also, by using (10), the decay of the function φ, given by (13) as well some straight-forward computations, one gets

$$\frac{e}{\pi} \frac{(2e)^{2\alpha+1}}{(3/2)^{3/2}(3/2+2\alpha)^{3/2+2\alpha}} \frac{1}{2k+2\alpha+1} \leq h_k$$
$$\leq \frac{\pi}{e} \left(\frac{2}{e}\right)^{2\alpha+1} \frac{(3/2)^{3/2}(3/2+2\alpha)^{3/2+2\alpha}}{2k+2\alpha+1}. \tag{51}$$

Finally, by combining (15), (48), (49)–(51), one gets the desired result (47). □

Remark 3 By using our notation, the decay rate of the $(\beta_k^n)_k$, given by Theorem 3.4 of [19] can be written as $\dfrac{C_\alpha''}{|\mu_n^{(\alpha)}(c)|} \dfrac{1}{k^{1+\alpha/2}} \left(\dfrac{ec}{2k+1}\right)^k$, for some constant C_α''. The previous proposition ensures that this decay is further improved by a factor of $1/2^k$.

The following theorem provides us with a second decay rate of the $(\beta_k^n)_{k\geq0}$, valid for sufficiently large values of n and the values of $0 \leq k < n$ not too close to n. We should mention that the techniques of the proof of this theorem, given in "Appendix: Proof of Theorem 1", are inspired from those developed for the special case $\alpha = 0$ and given in a joint work of one of us [4].

Theorem 1 *Let $c > 0$, be a fixed positive real number. Then, for all positive integers n, k such that $q = c^2/\chi_n \leq 1$ and $k(k + 2\alpha + 1) + C_\alpha c^2 \leq \chi_n(c)$, we have*

$$|\beta_0^n| \leq \sqrt{\frac{\Gamma(\alpha + 3/2)}{\sqrt{\pi}\Gamma(\alpha + 1)}} \sqrt{1+\alpha}|\mu_n^{(\alpha)}(c)| \quad \text{and} \quad |\beta_k^n| \leq C_\alpha' \left(\frac{2}{q}\right)^k |\mu_n^{(\alpha)}(c)|. \tag{52}$$

Here, $C_\alpha' = \dfrac{2^\alpha (3/2)^{3/4}(3/2+2\alpha)^{3/4+\alpha}}{e^{2\alpha+3/2}}\sqrt{1+\alpha}$ *and* $C_\alpha = 2M_\alpha + N_\alpha$ *with*

$$M_\alpha = \max\left(1/4, \sqrt{\frac{2(2\alpha+2)}{(2\alpha+5)(2\alpha+3)^2}}\right),$$
$$N_\alpha = \max\left(\frac{3}{2\alpha+5}, \frac{1}{2} + \frac{|4\alpha^2-1|}{(2\alpha+3)(2\alpha+7)}\right). \tag{53}$$

3.2 Analytic Extension of the GPSWFs

In this paragraph, we give explicit formulae for the analytic extension of the GPSWFs to the whole real line, as well for computing the eigenvalues $\mu_n^{(\alpha)}(c)$ associated with the weighted finite Fourier transform $\mathcal{F}_c^\alpha$. We first note that due to Eq. (2), the GPSWFs have analytic extension to $\mathbb{R}$. In fact, it is well known, see [17, p. 168], that if $\alpha > -1$, then

$$\sup_{y\in[-1,1]} |\widetilde{P}_k^{(\alpha,\beta)}(y)| =\leq \frac{1}{\sqrt{h_k} B(k+1, \alpha+1)} \leq M_\alpha k^{\alpha+1/2},$$

for some constant M_α. Moreover, by using the super-exponential decay rate of the expansion coefficients $(\beta_k^n)_k$ combined with (2), (20) and (43), one gets

$$\psi_{n,c}^{(\alpha)}(x) = \frac{\sqrt{\pi}2^{\alpha+1/2}}{\mu_n^{(\alpha)}(c)} \sum_{k\geq 0} i^k \beta_k^n \frac{\Gamma(k+\alpha+1)}{\sqrt{h_k}k!} \frac{J_{k+\alpha+1/2}(cx)}{(cx)^{\alpha+1/2}}, \quad \forall x \neq 0.$$

Moreover, from the parity of the $\psi_{n,c}^{(\alpha)}$, it is easy to see that $\mu_n^{(\alpha)}(c) = i^n|\mu_n^{(\alpha)}(c)|$, $n \geq 0$. Hence, by using the fact that the previous expansion coincides with the expansion (20) at $x = 1$, one obtains the following analytic extension of the GPSWFs as well as an explicit formula for their associated eigenvalues $\mu_n^{(\alpha)}(c)$,

$$\psi_{n,c}^{(\alpha)}(x) = \frac{\sqrt{\pi}2^{\alpha+1/2}}{|\mu_n^{(\alpha)}(c)|} \sum_{k\geq 0} i^{k-n} \beta_k^n \frac{\Gamma(k+\alpha+1)}{\sqrt{h_k}k!} \frac{J_{k+\alpha+1/2}(cx)}{(cx)^{\alpha+1/2}}, \quad \forall x \neq 0 \quad (54)$$

with

$$\mu_n^{(\alpha)}(c) = i^n \sqrt{\pi} \left(\frac{2}{c}\right)^{\alpha+1/2} \frac{\sum_{k\geq 0} i^{k-n} \beta_k^n \frac{\Gamma(k+\alpha+1)}{\sqrt{h_k}k!} J_{k+\alpha+1/2}(c)}{\sum_{k\geq 0} \frac{\beta_k^n}{\sqrt{h_k} B(k+\alpha,k)}}, \quad n \geq 0. \quad (55)$$

We should mention that due to the facts that the coefficients $(\beta_k^n)_k$ are concentrated around $k = n$ and decay super-exponentially, the previous formula is accurate and practical for computing the $\mu_n^{(\alpha)}(c)$. Also, note that in [19], the authors have given some properties of the eigenvalues $\mu_n^{(\alpha)}(c)$ (denoted by $\lambda_n^{(\alpha)}(c)$ in [19]). In particular, by considering the operator $\mathcal{F}_c^\alpha$ as a Hilbert–Schmidt operator acting on $L^2(I, \omega_\alpha)$, it has been shown that

$$\sum_{n\geq 0} |\mu_n^{(\alpha)}(c)|^2 = \|\mathcal{F}_c^\alpha\|_{HS}^2 = \left(\int_{-1}^1 \omega_\alpha(x)\,dx\right)^2 = \frac{\pi\Gamma^2(1+\alpha)}{\Gamma^2(\alpha+3/2)}.$$

Here, $\|\cdot\|_{HS}$ denotes the Hilbert–Schmidt norm. More importantly, in [19], the authors have noted that the $\mu_n^{(\alpha)}(c)$ has an asymptotic super-exponential decay rate given by

$$|\mu_n^{(\alpha)}(c)| \approx \frac{e^\alpha}{4^\alpha} \sqrt{\frac{\pi e}{2n+2\alpha+3}} \left(\frac{ec}{4n+4\alpha+2}\right)^n, \quad n \gg 1. \quad (56)$$

4 GPSWFs as Solutions of an Energy Maximization Problem and Quality of Approximation

In the first part of this section, we show that in the case where $\alpha \geq 0$, the GPSWFs are solutions of an energy maximization problem over a generalized Paley–Wiener space and with respect to certain weighted norms. As important consequences of this characterization, we get a monotonicity result the sequence $\lambda_n^{(\alpha)}(c) = \frac{c}{2\pi}|\mu_n^{(\alpha)}(c)|^2$

with respect to the parameter α. Moreover, by using the results of [6], one gets a better understanding of the behaviour and the super-exponential decay rate of the $(\lambda_n^{(\alpha)}(c))_{n\geq 0}$. In the second part, we show that the GPSWFs are well adapted for the approximation of functions from the classical Paley–Wiener space B_c as well as of almost c-band-limited functions.

4.1 GPSWFs as Solutions of an Energy Maximization Problem and Consequences

We recall that the starting point of the theory of the classical PSWFs (corresponding to the GPSWFs with $\alpha = 0$) is the solution of the following energy maximization problem, see [16]

$$f = \arg\max_{f\in B_c} \frac{\|f\|^2_{L^2(I)}}{\|f\|^2_{L^2(\mathbb{R})}} = \arg\max_{f\in B_c} 2\pi \frac{\|f\|^2_{L^2(I)}}{\|\widehat{f}\|^2_{L^2(\mathbb{R})}}, \tag{57}$$

where B_c is the Paley–Wiener of c-band-limited functions given by

$$B_c = \{f \in L^2(\mathbb{R}), \ \text{Support } \widehat{f} \subseteq [-c, c]\}. \tag{58}$$

More precisely, it has been shown in [16] that from B_c, $\psi_{0,c}^{(0)}$ is the most concentrated function in $I = [-1, 1]$ with the largest energy concentration ratio $0 < \lambda_0^{(0)}(c) < 1$. Moreover, for any integer $n \geq 1$, $\psi_{n,c}^{(0)}$ is the most concentrated function from B_c which is orthogonal to the previous $\psi_{i,c}^{(0)}$, $0 \leq i \leq n - 1$. The orthogonality is with respect to the two usual inner products of $L^2(I)$ and $L^2(\mathbb{R})$. As it will be seen, the extension of the previous characterization of the PSWFs to the more general case of the GPSWFs provides us with a better understanding of the behaviour and the super-exponential decay rate of the eigenvalues $(\lambda_n^{(\alpha)}(c))_{n\geq 0}$. For $\alpha > 0$, we define the restricted Paley–Wiener space of weighted c-band-limited functions by

$$B_c^{(\alpha)} = \{f \in L^2(\mathbb{R}), \ \text{Support } \widehat{f} \subseteq [-c, c], \ \widehat{f} \in L^2\big((-c, c), \omega_{-\alpha}(\tfrac{\cdot}{c})\big)\}. \tag{59}$$

Here, $L^2\big((-c, c), \omega_{-\alpha}(\tfrac{\cdot}{c})\big)$ is the weighted $L^2(-c, c)$-space with norm given by

$$\|f\|^2_{L^2\big((-c,c),\omega_{-\alpha}(\tfrac{\cdot}{c})\big)} = \int_{-c}^{c} |f(t)|^2 \omega_{-\alpha}\left(\frac{t}{c}\right) dt.$$

Note that when $\alpha = 0$, the restricted Paley–Wiener space $B_c^{(0)}$ is reduced to the usual space B_c. Also, since for any $\alpha \geq \alpha'$, $\widehat{f} \in L^2\big((-c, c), \omega_{-\alpha'}(\tfrac{\cdot}{c})\big)$ implies that $\widehat{f} \in L^2\big((-c, c), \omega_{-\alpha}(\tfrac{\cdot}{c})\big)$ then one gets

$$B_c^{(\alpha)} \subseteq B_c^{(\alpha')}, \qquad \forall \alpha \geq \alpha' \geq 0. \tag{60}$$

Remark 4 We give an example of a function from a restricted Paley–Wiener space. If $c > 1$, then it has been shown in [10] that the function

$$\eta(x) = \frac{\sin(cx - x/2)}{x^4} \left(2\sin\left(\frac{x}{2}\right) - x\cos\left(\frac{x}{2}\right) \right), \quad x \in \mathbb{R}$$

is a *c*-band-limited function. Moreover, its Fourier transform is an even function given by

$$\widehat{\eta}(-\xi) = \widehat{\eta}(\xi) = \begin{cases} 1 & \text{if } \xi \in [0, c-1] \\ f(\xi + 2 - c) & \text{if } \xi \in [c-1, c] \\ 0 & \text{if } \xi \geq c. \end{cases}$$

where

$$f(-x) = f(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \\ -4 + 12x - 9x^2 + 2x^3 & \text{if } x \in [1, 2] \\ 0 & \text{if } x \geq 2. \end{cases}$$

Since for $\xi \in [c-1, c]$, $\widehat{\eta}(\xi) = (2\xi - 2c + 3)(\xi - c)^2$, then it is easy to see that η belongs to the restricted Paley–Wiener space B_c^α for any $0 \leq \alpha < 5$.

The generalized maximization problem is formulated as follows. We note that from (43) with $k = 0$, one gets the finite Fourier transform of the weight function ω_α, given by

$$\int_{-1}^{1} e^{ixy} \omega_\alpha(y) \, dy = \sqrt{\pi} 2^{\alpha+1/2} \Gamma(\alpha + 1) \frac{J_{\alpha+1/2}(x)}{x^{\alpha+1/2}} = \mathcal{K}_\alpha(x), \quad x \in \mathbb{R}. \quad (61)$$

Next, if $f \in B_c^{(\alpha)}$, then $\widehat{f}(x) = g(x)\omega_\alpha\left(\frac{x}{c}\right)$ with some $g \in L^2\left((-c, c), \omega_{-\alpha}(\frac{\cdot}{c})\right)$. By using the inverse Fourier transform, one gets

$$\frac{\|f\|_{L^2(I,\omega_\alpha)}^2}{\|\widehat{f}\|_{L^2(\omega_{-\alpha}(\frac{\cdot}{c}))}^2} = \frac{1}{\|\widehat{f}\|_{L^2(\omega_{-\alpha}(\frac{\cdot}{c}))}^2} \int_{-1}^{1} f(t) \cdot \overline{f(t)} \omega_\alpha(t) \, dt$$

$$= \frac{1}{\|\widehat{f}\|_{L^2(\omega_{-\alpha}(\frac{\cdot}{c}))}^2} \frac{1}{4\pi^2} \int_{-1}^{1} \int_{-c}^{c} e^{ity} \widehat{f}(y) \, dy \cdot \int_{-c}^{c} e^{-itx} \overline{\widehat{f}(x)} \, dx \omega_\alpha(t) \, dt$$

$$= \frac{1}{\|\widehat{f}\|_{L^2(\omega_{-\alpha}(\frac{\cdot}{c}))}^2} \frac{1}{4\pi^2} \int_{-c}^{c} \int_{-c}^{c} \left(\int_{-1}^{1} e^{it(y-x)} \omega_\alpha(t) \, dt \right) \widehat{f}(y) \, dy$$

$$\cdot \int_{-c}^{c} e^{-itx} \overline{\widehat{f}(x)} \, dx \omega_\alpha(t) \, dt$$

$$= \frac{1}{\|\widehat{f}\|^2_{L^2(\omega_{-\alpha}(\frac{\cdot}{c}))}} \frac{1}{4\pi^2} \int_{-c}^{c} \int_{-c}^{c} \mathcal{K}_\alpha(y-x)g(y)\omega_\alpha\left(\frac{y}{c}\right) dy \cdot \overline{g(x)}\omega_\alpha\left(\frac{x}{c}\right) dx$$

$$= \frac{1}{4\pi^2} \frac{\int_{-c}^{c} \left[\int_{-c}^{c} \mathcal{K}_\alpha(y-x)g(y)\omega_\alpha\left(\frac{y}{c}\right) dy\right] \cdot \overline{g(x)}\omega_\alpha\left(\frac{x}{c}\right) dx}{\int_{-c}^{c} g(x)\overline{g(x)}\omega_\alpha\left(\frac{x}{c}\right) dx}. \tag{62}$$

Here, $\mathcal{K}_\alpha$ is as given by (61). Note that since the compact integral operator $\mathcal{Q}^\alpha$ defined on $L^2(\omega_\alpha(\frac{\cdot}{c}))$ by

$$\mathcal{Q}^\alpha g(x) = \frac{1}{4\pi^2} \int_{-c}^{c} \mathcal{K}_\alpha(y-x)g(y)\omega_\alpha\left(\frac{y}{c}\right) dy$$

has a symmetric kernel, then it is well known that in this case, $\max_{f \in B_c^\alpha} 2\pi \dfrac{\|f\|^2_{L^2_{\omega_\alpha}(I)}}{\|\widehat{f}\|^2_{L^2(\omega_{-\alpha}(\frac{\cdot}{c}))}}$ is attained at the eigenfunction of $2\pi \mathcal{Q}^\alpha$, associated with the largest eigenvalue. Hence, by using a trivial change of variable and functions, the generalized energy maximization problem is reduced to the solution of the following eigenproblem

$$\mathcal{Q}_c^{(\alpha)} G(x) = \int_{-1}^{1} \frac{c}{2\pi} \mathcal{K}_\alpha(c(x-y))G(y)\omega_\alpha(y) dy = \lambda G(x), \quad x \in [-1, 1]. \tag{63}$$

On the other hand, it has been shown in [19] that the kernel $\mathcal{K}_\alpha(c(x-y))$ is nothing but the kernel of the composition operators $\mathcal{F}_c^{\alpha*} \circ \mathcal{F}_c^\alpha$. Hence, the operators $\mathcal{Q}_c^\alpha$ and $\mathcal{F}_c^\alpha$ have the same eigenfunctions, given by the GPSWFs, $\psi_{n,c}^{(\alpha)}$, and associated to the respective eigenvalues $\lambda_n^{(\alpha)}$, $\mu_n^{(\alpha)}(c)$. These eigenvalues are related to each others by the following rule

$$\lambda_n^{(\alpha)}(c) = \frac{c}{2\pi}|\mu_n^{(\alpha)}(c)|^2, \quad n \geq 0. \tag{64}$$

Since from (60), if $0 \leq \alpha' \leq \alpha$, then we have $B_c^{(\alpha)} \subseteq B_c^{(\alpha')}$ and since for $f \in B_c^{(\alpha)}$, then we have

$$\|f\|^2_{L^2(I, \omega_\alpha)} \leq \|f\|^2_{L^2(I, \omega_{\alpha'})}, \quad \|\widehat{f}\|^2_{L^2(\omega_{-\alpha}(\frac{\cdot}{c}))} \geq \|\widehat{f}\|^2_{L^2(\omega_{-\alpha'}(\frac{\cdot}{c}))}.$$

Hence, we have

$$\lambda_0^{(\alpha)}(c) = \sup_{f \in B_c^{(\alpha)}} \frac{\|f\|^2_{L^2(I, \omega_\alpha)}}{\|\widehat{f}\|^2_{L^2(\omega_{-\alpha}(\frac{\cdot}{c}))}} \leq \sup_{f \in B_c^{(\alpha)}} \frac{\|f\|^2_{L^2(I, \omega_{\alpha'})}}{\|\widehat{f}\|^2_{L^2(\omega_{-\alpha'}(\frac{\cdot}{c}))}} \leq \sup_{f \in B_c^{(\alpha')}} \frac{\|f\|^2_{L^2(I, \omega_{\alpha'})}}{\|\widehat{f}\|^2_{L^2(\omega_{-\alpha'}(\frac{\cdot}{c}))}}$$

$$= \lambda_0^{(\alpha')}(c).$$

More generally, for an integer $n \geq 1$, let $\psi_{0,c}^{(\alpha')}, \ldots, \psi_{n-1,c}^{(\alpha')}$ be the first most concentrated GPSWFs, associated with the respective eigenvalues $\lambda_0^{(\alpha')}(c) > \lambda_1^{(\alpha')}(c) >$

$\cdots > \lambda_{n-1}^{(\alpha')}(c)$. Note that the previous strict inequalities are due to the fact that these eigenvalues are simple, see [19]. By combining the previous formulation of the energy maximization problem and the well known Min–Max principle for eigenvalues of compact operator, one concludes that if S_n, H_n stand for an arbitrary subspace of dimension n of $B_c^{(\alpha)}$, and $B_c^{(\alpha')}$, respectively, then we have

$$\lambda_n^{(\alpha)}(c) = \max_{S_n \subset B_c^{(\alpha)}} \min_{\psi \in S_n} \frac{\|\psi\|_{L^2(I,\omega_\alpha)}^2}{\|\widehat{\psi}\|_{L^2(\omega_{-\alpha}(\frac{\cdot}{c}))}^2} \leq \max_{S_n \subset B_c^{(\alpha')}} \min_{\psi \in S_n} \frac{\|\psi\|_{L^2(I,\omega_{\alpha'}))}^2}{\|\widehat{\psi}\|_{L^2(\omega_{-\alpha'}(\frac{\cdot}{c}))}^2}$$

$$\leq \max_{H_n \subset B_c^{(\alpha')}} \min_{\psi \in H_n} \frac{\|\psi\|_{L^2(I,\omega_{\alpha'})}^2}{\|\widehat{\psi}\|_{L^2(\omega_{-\alpha'}(\frac{\cdot}{c}))}^2} = \lambda_n^{(\alpha')}(c).$$

We have just proved the following theorem giving the monotony of the eigenvalues $\lambda_n^{(\alpha)}(c)$ with respect to the parameter α.

Theorem 2 *For a given real number $c > 0$, and an integer $n \geq 0$, we have*

$$\lambda_n^{(\alpha)}(c) \leq \lambda_n^{(\alpha')}(c), \quad \forall \alpha \geq \alpha' \geq 0. \tag{65}$$

It is important to mention that a super-exponential decay rate of the sequence $(\lambda_n^{(\alpha)}(c))_n$ as well as an estimate of the location of the plunge region, where the fast decay starts are important consequences of the previous proposition. These two results follow directly from the results given in [6], where an explicit formula for estimating the $\lambda_n^{(0)}(c)$ has been developed. This explicit formula enjoys with a surprising accuracy as soon as n reaches or goes beyond the plunge region around the value $n_c = \frac{2c}{\pi}$. Also, it proves that the exact asymptotic super-exponential decay rate is given by the quantity $e^{-2n\log\left(\frac{4n}{ec}\right)}$. From the previous theorem with $\alpha' = 0$, one concludes that for any $\alpha > 0$, the sequence $(\lambda_n^{(\alpha)}(c))_n$ has a super-exponential decay rate, bounded above by the decay rate of $(\lambda_n^{(0)}(c))_n$. Moreover, the fast decay of these $(\lambda_n^{(\alpha)}(c))_n$ starts around $n_c = \frac{2c}{\pi}$. In the numerical results section, we give different tests that illustrate these precise behaviours of $(\lambda_n^{(\alpha)}(c))_n$.

4.2 Approximation of Band-Limited Functions by the GPSWFs

In this paragraph, we first show that when restricted to the interval I, the GPSWFs $\psi_{n,c}^{(\alpha)}$ are well adapted for the approximation of functions from the usual Paley–Wiener space B_c. As a result, we check that the GPSWFs are also well adapted for the approximation of almost band-limited functions. This type of functions have been defined in [11] as follows.

Definition 1 Let $\Omega = [-c, c]$, then a function f is said to be ϵ_Ω-band-limited in Ω if

$$\frac{1}{2\pi} \int_{|\xi|>c} |\widehat{f}(\xi)|^2 \, d\xi \leq \epsilon_\Omega^2.$$

Proposition 6 *Let $c > 0$, $\alpha \geq 0$ be two real numbers and let $f \in B_c$. For any positive integer $N > \frac{2c}{\pi}$, let*

$$S_N(f)(x) = \sum_{k=0}^{N} < f, \psi_{k,c}^{(\alpha)} >_{L^2(I,\omega_\alpha)} \psi_{k,c}^{(\alpha)}(x).$$

Then, we have

$$\left(\int_{-1}^{1} |f(t) - S_N f(t)|^2 \omega_\alpha(t) dt \right)^{1/2} \leq C_1 \sqrt{\lambda_N^{(\alpha)}(c)} \, (\chi_N(c))^{(1+\alpha)/2} \, \|f\|_{L^2(\mathbb{R})},$$
$$(66)$$

and

$$\sup_{x \in [-1,1]} |f(x) - S_N f(x)| \leq C_1 \sqrt{\lambda_N^{(\alpha)}(c)} \, (\chi_N(c))^{1+\alpha/2} \, \|f\|_{L^2(\mathbb{R})}. \quad (67)$$

for some uniform constant C_1 depending only on α.

Proof We first note that since $\mathcal{B} = \{\psi_{n,c}^{(\alpha)}, \ n \geq 0\}$ is an orthonormal basis of $L^2(I, \omega_\alpha)$, and since $\chi_I f \in L^2(I)$, where χ_I denotes the characteristic function, then we have

$$f(x) = \sum_{k \geq 0} < f, \psi_{k,c}^{(\alpha)} >_{L^2(I,\omega_\alpha)} \psi_{k,c}^{(\alpha)}(x), \quad a.e. \ x \in I. \quad (68)$$

On the other hand, since $f \in B_c$, then $f \in \mathcal{C}^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$. In particular, from the inverse Fourier transform, and by using the fact that $f \in B_c$, we have

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ixy} \widehat{f}(y) dy = \frac{1}{2\pi} \int_{-c}^{c} e^{ixy} \widehat{f}(y) dy = \frac{c}{2\pi} \int_{-1}^{1} e^{ictx} \widehat{f}(ct) \, dt,$$
$$\forall x \in [-1, 1]. \quad (69)$$

Consequently, for any integer $k \geq 0$, we have

$$|\langle f, \psi_{k,c}^{(\alpha)} \rangle_{L^2(I,\omega_\alpha)}| = \left| \int_{-1}^{1} f(x) \psi_{k,c}^{(\alpha)}(x) \omega_\alpha(x) \, dx \right|$$

$$= \frac{c}{2\pi} \left| \int_{-1}^{1} \left(\int_{-1}^{1} e^{ictx} \widehat{f}(ct) \, dt \right) \psi_{n,c}^{(\alpha)}(x) \omega_\alpha(x) \, dx \right|$$

$$= \frac{c}{2\pi} \left| \int_{-1}^{1} \widehat{f}(ct) \left(\int_{-1}^{1} e^{ictx} \psi_{k,c}^{(\alpha)}(x) \omega_\alpha(x) \, dx \right) dt \right|$$

$$= \frac{c}{2\pi} |\mu_k^{(\alpha)}(c)| \left| \int_{-1}^{1} \widehat{f}(ct) \psi_{n,c}^{(\alpha)}(t) \, dt \right|$$

$$\leq |\mu_k^{(\alpha)}(c)| \frac{c}{2\pi} \sup_{t \in [-1,1]} |\psi_{n,c}^{(\alpha)}(t)| \left(\int_{-1}^{1} |\widehat{f}(ct)|^2 \, dt \right)^{1/2}$$

$$\leq \frac{C_\alpha}{\sqrt{2}} \sqrt{\frac{c}{2\pi}} |\mu_k^{(\alpha)}(c)| (\chi_k(c))^{(1+\alpha)/2} \|f\|_{L^2(\mathbb{R})}$$

$$= \frac{C_\alpha}{\sqrt{2}} \sqrt{\lambda_N^{(\alpha)}(c)} \, (\chi_N(c))^{(1+\alpha)/2} \|f\|_{L^2(\mathbb{R})}. \tag{70}$$

Here, C_α is as given by (34). The last inequality follows from Plancherel formula and the bound over I of $|\psi_{n,c}^{(\alpha)}(t)|$, we have given in (35). On the other hand, by using the previous inequality, together with the super-exponential decay rate of the $|\mu_n^{(\alpha)}(c)|$, given by (56), as well as the Parseval's equality

$$\|f - S_N f\|_{L^2(I,\omega_\alpha)}^2 = \sum_{k \geq N+1} |\langle f, \psi_{k,c}^{(\alpha)}\rangle_{L^2(I,\omega_\alpha)}|^2,$$

one can easily get (66). Finally, to get (67), it suffices to combine the previous inequality, (56) as well as the upper bound of $|\psi_{n,c}^{(\alpha)}(t)|$.   □

As a consequence of the previous result, we have the following corollary concerning the quality of approximation of almost band-limited functions by the GPSWFs.

Corollary 2 *Let $f \in L^2(\mathbb{R})$ be an ϵ_Ω-band-limited in $\Omega = [-c, +c]$ and let $\alpha \geq 0$, then for any positive integer $N \geq \frac{2c}{\pi}$, we have*

$$\left(\int_{-1}^1 |f(t) - S_N f(t)|^2 \omega_\alpha(t)dt\right)^{1/2} \leq \epsilon_\Omega + C_1 \sqrt{\lambda_N^{(\alpha)}(c)} \, (\chi_N(c))^{(1+\alpha)/2} \|f\|_{L^2(\mathbb{R})} \tag{71}$$
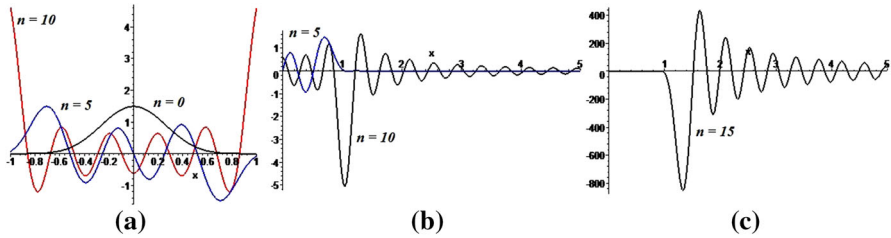
where the constant C_1 depends only on α.

Proof It suffices to consider the band-limiting operator π_Ω defined by:

$$\pi_\Omega(f)(x) = \frac{1}{2\pi} \int_\Omega e^{ix\omega} \widehat{f}(\omega) \, d\omega.$$

Since $\pi_\Omega f \in B_c$, $\|(f - \pi_\Omega f) - S_N(f - \pi_\Omega f)\|_{L^2(I)} \leq \|f - \pi_\Omega\|_{L^2(\mathbb{R})} \leq \epsilon_\Omega$ and $\|\pi_\Omega f\|_{L^2(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})}$, then by applying the result of the previous proposition to $\pi_\Omega f$, one gets

$$\|f - S_N(f)\|_{L^2(I,\omega_\alpha)} \leq \|(f - \pi_\Omega f) - S_N(f - \pi_\Omega f)\|_{L^2(I,\omega_\alpha)}$$
$$+ \|\pi_\Omega f - S_N(\pi_\Omega f)\|_{L^2_{\omega_\alpha(I)}}$$
$$\leq \epsilon_\Omega + C_1 \sqrt{\lambda_N^{(\alpha)}(c)} \, (\chi_N(c))^{(1+\alpha)/2} \|\pi_\Omega f\|_{L^2(\mathbb{R})}$$
$$\leq \epsilon_\Omega + C_1 \sqrt{\lambda_N^{(\alpha)}(c)} \, (\chi_N(c))^{(1+\alpha)/2} \|f\|_{L^2(\mathbb{R})}.$$

Fig. 1 **a** Graphs of $\psi_{n,c}^{\alpha}$ with $c = 5\pi$ and $\alpha = 0.5$, **b**, **c** graphs of some analytic extensions of the $\psi_{n,c}^{\alpha}$

5 Numerical Results

In this section, we give three examples that illustrate the different results of this work. The first example deals with the computation and the analytic extension of the GPSWFs.

Example 1 In this example, we give different numerical tests that illustrate the construction scheme of the GPSWFs $\psi_{n,c}^{(\alpha)}$. For this purpose, we have considered the values $\alpha = 0.5$ and $c = 5\pi$. Then, we have computed the different Jacobi expansion coefficients via the scheme of Sect. 2, by solving the eigensystem (21), truncated to the order $N = 90$. Figure 1a show the graphs of the $\psi_{n,c}^{(\alpha)}$ for the different values of $n = 0, 5, 15$. Note that these graphs illustrate some of the provided properties of the $\psi_{n,c}^{(\alpha)}$. Also, we have used formula (68) and computed the analytic extensions of the previous GPSWFs. The graphs of these extensions are given by Fig. 1b, c. Note that as predicted by the characterisation of the GPSWFs as solutions of the energy maximization problem, for the values of $n \leq 2c\pi$, the $\psi_{n,c}^{(\alpha)}$ are concentrated on I, whereas for $n > 2c/\pi$, they are concentrated on $\mathbb{R} \setminus I$.

Example 2 In this example, we illustrate the important result given by Theorem 2, concerning the monotonicity with respect to the parameter α of the sequence $\lambda_n^{(\alpha)}(c) = \dfrac{c}{2\pi}|\mu_n^{(\alpha)}(c)|^2$. For this purpose, we have used formula (55) and computed highly accurate values of $\mu_n^{(\alpha)}(c)$ and consequently of $\lambda_n^{(\alpha)}(c)$ with $c = 10\pi$ and with different values of $\alpha = 0, 0.5, 1.5$. Note that as predicted by Theorem 2, the sequence $\lambda_n^{(\alpha)}(c)$ is decreasing with respect to α. The graphs of the $\lambda_n^{(\alpha)}(c)$ as well as $\log(\lambda_n^{(\alpha)}(c))$ are given by Fig. 2a, b, respectively.

Example 3 In this last example, we illustrate the quality of approximation over I of band-limited and almost band-limited functions, by the GPSWFs. For this purpose, we have first considered the value of $\alpha = 0.5$ and the c-band-limited function $f(x) = \dfrac{\sin(cx)}{cx}$ with $c = 50$. By computing the projections $S_N(f)$, with $N = 32$ and $N = 40$, we found that
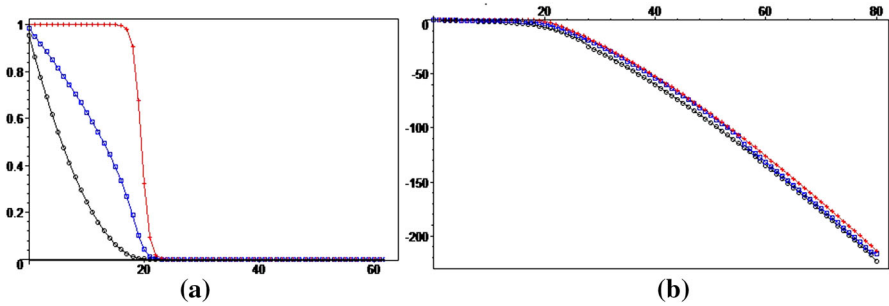
Fig. 2 **a** Graphs of the $\lambda_n^{(\alpha)}(c)$ for $c = 10\pi$, $\alpha = 1.5$ (*circles*), $\alpha = 0.5$ (*boxes*) and $\alpha = 0$ (*crosses*), **b** graphs of $\log(\lambda_n^{(\alpha)}(c))$

$$\sup_{x \in [-1,1]} |f(x) - S_{32}(f)(x)| \approx 2.22 \times 10^{-2},$$

$$\sup_{x \in [-1,1]} |f(x) - S_{40}(f)(x)| \approx 4.80 \times 10^{-6}.$$

As predicted by Proposition 6, the drastic improvement in the previous approximation errors is due to the fact that the second value of $N = 40$ lies after the plunge region of the eigenvalues $\lambda_n^{(\alpha)}(c)$, which is not the case for the first value of $N = 32$.

Next, to illustrate the approximation of almost band-limited functions by the GPSWFs, we have considered the Weierstrass function

$$W_s(x) = \sum_{k \geq 0} \frac{\cos(2^k x)}{2^{ks}}, \quad -1 \leq x \leq 1. \tag{72}$$

It is well known that $W_s \in H^{s-\epsilon}(I)$, $\forall \epsilon < s$, $s > 0$. One may consider W_s as a restriction over I of a function $W \in H^{s-\epsilon}(\mathbb{R})$. Note that if $f \in H^s(\mathbb{R})$ with $s > 0$, then

$$\int_{|\xi| > c} |\widehat{f}(\xi)|^2 \, d\xi \leq \int_{|\xi| > c} \frac{(1 + |\xi|)^{2s}}{(1 + |\xi|)^{2s}} |\widehat{f}(\xi)|^2 \, d\xi \leq \frac{1}{(1 + c)^{2s}} \|f\|_{H^s(\mathbb{R})}^2.$$

That is f is $\frac{1}{(1+c)^s} \|f\|_{H^s}$-almost band-limited to $[-c, c]$. Note that in [4], we have used the previous function to illustrate the quality of approximation by the classical PSWFs. In this example, we push forward this quality of approximation to the GPSWFs. For this purpose, we have considered the value of $\alpha = 0.5$ and the two couples of $(c, N) = (50, 60), (100, 90)$. Then we have computed the associated projection $S_N(W_s)$, for the value of $s = 1$. Note that thanks to (43), the different expansion coefficients $C_n(W_s) = \int_{-1}^{1} W_s(y) \psi_{n,c}^{(\alpha)}(y) \omega_\alpha(y) \, dy$ are computed exactly. In fact since W_s is an even function, and from the Jacobi series expansion of $\psi_{n,c}^{(\alpha)}$, the computation of the $C_n(W_s)$ is restricted to the even indexed coefficients and consequently to the
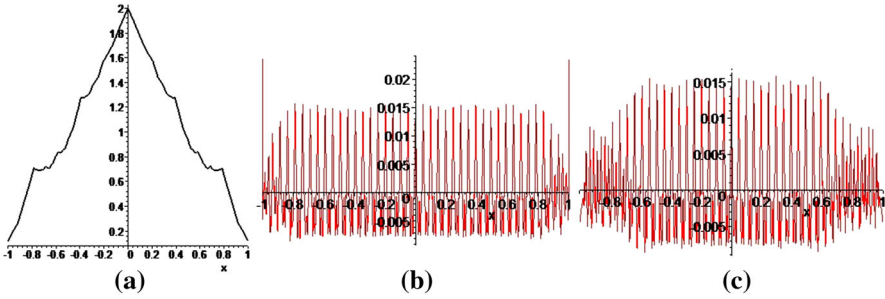
Fig. 3 **a** Graph of the Weierstarss function $W_1(x)$, **b** graph of the approximation error $W_1(x) - S_N(W_1)(x)$ with $\alpha = 0.5$, $(c, N) = (50, 60)$, **c** same as **b** with $(c, N) = (100, 90)$

computation of the different inner products with Jacobi polynomials of even degrees. More precisely, we have

$$C_{2m}(W_s) = \sum_{l \geq 0} \beta_{2l}^{2m} \sum_{k \geq 0} \frac{1}{2^{ks}} \int_{-1}^{1} \cos(2^k x) \, \widetilde{P}_{2l}^{(\alpha,\alpha)}(y) \, \omega_\alpha(y) \, dy$$

$$= \sqrt{\pi} 2^{\alpha+1/2} \sum_{l \geq 0} (-1)^l \beta_{2l}^{2m} \frac{\Gamma(2l + \alpha + 1)}{\sqrt{h_{2l}}(2l)!} \sum_{k \geq 0} 2^{-k(s+\alpha+1/2)} J_{2l+\alpha+1/2}(2^k).$$

The graph of W_1 is given by Fig. 3a, whereas the graphs of the approximation errors $W_1(x) - S_N(W_1)(x)$ corresponding to the two couples $(c, N) = (50, 60), (100, 90)$ are given by Fig. 3b, c, respectively. Note that as predicted by the theoretical results of Sect. 4, the approximation error decreases as c^{-s}, whenever the truncation order N lies beyond the plunge of the $(\lambda_n^{(\alpha)}(c))_n$. In this case, the extra error factor given by $\sqrt{\lambda_N^{(\alpha)}(c)} \chi_N^{1/2+\alpha}$ can be neglected comparing the factor c^{-s}.

Appendix: Proof of Theorem 1

The proof is divided into three steps. To alleviate notations of this proof, we will simply denote $\psi_{n,c}^{(\alpha)}$ and $\chi_n(c)$ by $\psi_{n,c}$ and χ_n, respectively.

First step: We prove that for for any positive integer j with $j(j + 2\alpha + 1) \leq \chi_n$, all moments $\int_{-1}^{1} y^j \psi_{n,c}(y) \, dy$ are non negative and

$$0 \leq \int_{-1}^{1} y^j \psi_{n,c}(y) \, \omega_\alpha(y) \, dy \leq \sqrt{1+\alpha} \left(\frac{1}{q}\right)^j |\mu_n^{(\alpha)}(c)|. \tag{73}$$

To this end, we first check that for any integer $k \geq 0$ satisfying $k(k + 2\alpha + 1) \leq \chi_n$, we have

$$\left|\psi_{n,c}^{(k)}(0)\right| \le (\sqrt{\chi_n})^k \sqrt{1+\alpha}. \tag{74}$$

It suffices to prove that $m_k = \frac{|\psi_{n,c}^{(k)}(0)|}{\sqrt{\chi_n}^k} \le \sqrt{1+\alpha}$. From the parity of $\psi_{n,c}$, we need only to consider derivatives of even or odd order. We assume that $n = 2l$ is even. The case where n is odd is done in a similar manner. Note that for a fixed n, $\psi_{n,c}^{(2l)}(0)$ has alternating signs, that is $\psi_{n,c}^{(k)}(0)\psi_{n,c}^{(k-2)}(0) < 0$ In fact, for $k = 0$, we have $\psi_{n,c}(0)\psi_{n,c}^{(2)}(0) = -\chi_n \psi_{n,c}(0)^2 < 0$. By induction, we assume that $\psi_{n,c}^{(k)}(0)\psi_{n,c}^{(k-2)} < 0$. As it is done in [6], we have

$$\psi_{n,c}^{(k+2)}(0)\psi_{n,c}^{(k)}(0) = \left(k(k+1+2\alpha) - \chi_n\right)\psi_{n,c}^{(k)}(0)^2 + k(k-1)c^2\psi_{n,c}^{(k-2)}(0)\psi_{n,c}^{(k)}(0). \tag{75}$$

By using the induction hypothesis as well as the fact that $k(k+1+\alpha+\beta) \le \chi_n$, one concludes that the induction assumption holds for the order k. Consequently, we have

$$|\psi_{n,c}^{(k+2)}(0)| = \left(\chi_n - k(k+1+2\alpha)\right)|\psi_{n,c}^{(k)}(0)| + k(k-1)c^2|\psi_{n,c}^{(k-2)}(0)|. \tag{76}$$

The previous equality implies that

$$m_{k+2} = \left(1 - \frac{k(k+1+2\alpha)}{\chi_n}\right)m_k + k(k-1)\frac{q}{\chi_n}m_{k-2}. \tag{77}$$

Hence, for any positive and even integer k with $k(k+2\alpha+1) \le \chi_n$, we have $m_k \le m_0 \le \sqrt{1+\alpha}$. This last inequality follows from (32) with $t = 0$. This proves the inequality (74). Moreover, by taking the jth derivative at zero on both sides of $\int_{-1}^{1} e^{icxy}\psi_{n,c}(y)\omega_\alpha(y)dy = \mu_n^{(\alpha)}(c)\psi_{n,c}(x)$, one gets

$$\int_{-1}^{1} y^j \psi_{n,c}(y)\,\omega_\alpha(y)dy = (-i)^j c^{-j}\mu_n^{(\alpha)}(c)\psi_n^{(j)}(0). \tag{78}$$

Since $\psi_n^{(j)}(0)$ and $\psi_n^{(j+2)}(0)$ have opposite signs, then the previous equation implies that all moments with even order j with $j(j+2\alpha+1) \le \chi_n$ have the same sign. The inequality (73) follows from (74).

Second step: We show that for all positive integers k, n with $k(k+2\alpha+1)+C_\alpha c^2 \le \chi_n(c)$, we have $\beta_k^n \ge 0$. Here C_α is as given by (53). The positivity of β_0^n (when n is even) and β_1^n (when n is odd) follow from the fact that

$$\beta_0^n = \sqrt{\frac{\Gamma(\alpha+3/2)}{\sqrt{\pi}\Gamma(\alpha+1)}}|\mu_n^{(\alpha)}(c)||\psi_{n,c}(0)|,$$

$$\beta_1^n = \sqrt{\frac{\Gamma(\alpha+3/2)}{\sqrt{\pi}\Gamma(\alpha+1)}}\sqrt{2\alpha+3}|\mu_n^{(\alpha)}(c)|\left|\frac{\psi_n'(0)}{c}\right|. \tag{79}$$

Since the β_k^n are given by (24), then by using the hypothesis of the theorem, we have

$$\beta_2^n = \frac{2\alpha + 3}{c} \sqrt{\frac{2(2\alpha + 5)}{2\alpha + 2}} \left(\chi_n - \frac{c^2}{2\alpha + 3} \right) \beta_0^n \geq \beta_0^n,$$

$$\beta_3^n = \frac{2\alpha + 5}{2c^2} \sqrt{\frac{3(\alpha + 1)}{2\alpha + 7}} \left((2\alpha + 2) + \frac{3c^2}{2\alpha + 5} \right) \beta_1^n \geq 0.$$

For $j \geq 2$ and by rearranging the system (24) and using the induction hypothesis $\beta_j^n \geq \beta_{j-2}^n \geq 0$, one gets

$$M_\alpha c^2 (\beta_{j+2}^n + \beta_{j-2}^n) \geq (\chi_n(c) - j(j + 2\alpha + 1) - N_\alpha c^2) \beta_j^n, \qquad (80)$$

where M_α and N_α are as given by (53). If we suppose that $\beta_{j+2} \leq \beta_j^n$, then from (80), one gets

$$2M_\alpha c^2 \beta_j^n \geq (\chi_n(c) - j(j + 2\alpha + 1) - N_\alpha c^2) \beta_j^n \qquad (81)$$

which contradicts the choice of C_α and the fact that $k(k + 2\alpha + 1) + C_\alpha c^2 \leq \chi_n(c)$. Hence, the induction hypothesis holds for β_{j+2}^n.

Third step: We prove (52). The first inequality follows from (79) and (74). To prove the second inequality, we recall that the moments $M_{j,k}$ of the normalized Jacobi polynomials $\widetilde{P}_k^{(\alpha,\alpha)}$ are given by (17) and they are non-negative. Moreover, since $x^j = \sum_{k=0}^j M_{jk} \widetilde{P}_k^{(\alpha,\alpha)}(x)$, then the moments of the $\psi_{n,c}$ are related to the GPSWFs series expansion coefficients by the following relation

$$\int_{-1}^1 x^j \psi_{n,c}(x) \, \omega_\alpha(x) \, dx = \sum_{k=0}^j M_{j,k} \beta_k^n.$$

Since from the previous step, we have $\beta_k^n \geq 0$, for any $0 \leq k \leq j$ and since the a_{jk} are non negative, then the previous equality implies that

$$\beta_j^n \leq \frac{1}{M_{j,j}} \int_{-1}^1 x^j \psi_{n,c}(x) \, \omega_\alpha(x) \, dx \leq \frac{1}{M_{j,j}} \sqrt{1+\alpha} \left(\frac{1}{q} \right)^j |\mu_n^{(\alpha)}(c)|. \qquad (82)$$

The last inequality follows from the result of the first step. Moreover, by using the explicit expression of $M_{j,j}$, given by (17), together with (10), (51), the decay of the function φ, given by (13), as well as some straightforward computations, one obtains

$$\frac{1}{M_{j,j}} \leq 2^j \frac{2^\alpha}{e^{2\alpha + 3/2}} \frac{(3/2)^{3/4} (3/2 + 2\alpha)^{3/4 + \alpha}}{\sqrt{3/2 + \alpha}}. \qquad (83)$$

Finally, by combining (82) and (83), one gets the second inequality of (52). □

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