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Duality for Frames

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Abstract The subject of this article is the duality principle, which, well beyond its stand at the heart of Gabor analysis, is a universal principle in frame theory that gives insight into many phenomena. Its fiber matrix formulation for Gabor systems is the driving principle behind seemingly different results. We show how the classical duality identities, operator representations and constructions for dual Gabor frames are in fact aspects of the dual Gramian matrix fiberization and its sole duality principle, giving a unified view to all of them. We show that the same duality principle, via dual Gramian matrix analysis, holds for dual (or bi-) systems in abstract Hilbert spaces. The essence of the duality principle is the unitary equivalence of the frame operator and the Gramian of certain adjoint systems. An immediate consequence is, for example, that, even on this level of generality, dual frames are characterized in terms of biorthogonality relations of adjoint systems. We formulate the duality principle for irregular Gabor systems which have no structure whatsoever to the sampling of the shifts and modulations of the generating window. In case the shifts and modulations are sampled from lattices we show how the abstract matrices can be reduced to the simple structured fiber matrices of shift-invariant systems, thus arriving back in the well understood territory. Moreover, in the arena of multiresolution analysis

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(MRA)-wavelet frames, the mixed unitary extension principle can be viewed as the duality principle in a sequence space. This perspective leads to a construction scheme for dual wavelet frames which is strikingly simple in the sense that it only needs the completion of an invertible constant matrix. Under minimal conditions on the MRA, our construction guarantees the existence and easy constructability of non-separable multivariate dual MRA-wavelet frames. The wavelets have compact support and we show examples for multivariate interpolatory refinable functions. Finally, we generalize the duality principle to the case of transforms that are no longer defined by discrete systems, but may have discrete adjoint systems.

Keywords Dual frame · Adjoint system · Duality principle · Wavelets · Gabor frame · Dual Gramian analysis · Filter bank

Mathematics Subject Classification 42C15 · 42C40 · 42C30 · 65T60

1 Introduction

Duality for frames is about the characterization and construction of dual frames of a given frame and the representation of the elements of a Hilbert space by the dual pairs. All these are implied by the duality principle, which stems from the truism that knowing the columns of a matrix is as good as knowing its rows. Frames are systems of vectors in Hilbert spaces that satisfy certain stability requirements. They can be used for robust and sparse linear representations. If one associates a system in some way with the columns of a matrix, or collection of matrices, one can study it via the rows. The simple idea is to consider the rows as another, an adjoint, system in a potentially different Hilbert space. The duality principle thus arises between systems which can be associated with the columns, respectively rows, of the same matrix or collection of matrices and implies, for example, that one is a frame if and only if the other is a Riesz sequence. For this strategy to be meaningful, in particular in infinite dimensional Hilbert spaces, the (infinite) matrices in some way have to represent the analysis and synthesis operators of the systems and their properties should be linked to the properties of those operators. The column–row relationship is thus ultimately linked to an adjoint relationship of the respective operators, but depending on the kind of representation in different ways. This shows the strengths of the matrix viewpoint. While it might be hard to derive an adjoint system from considering an adjoint operator, inspection of the rows of the matrix readily reveals an adjoint system and different matrix representations can yield a wealth of different adjoint systems.

Matrix representations have long been a well established tool for the study and construction of frames in Hilbert spaces of square-integrable functions. In the series of papers [98,100–103] the structure of shift-invariant or generalized shift-invariant systems of functions is used to represent the Fourier transform of the synthesis operator through a continuum of so called fiber matrices built from the generators of the system. The duality principle for Gabor frames [102], derived from column–row relationships, and the unitary and mixed unitary extension principles for wavelet frames [100,101] are but two central results that follow from this analysis. The class of Gabor and wavelet frames is widely used in practice to handle many signal processing problems. Tight wavelet frames have been implemented with excellent results in numerous image restoration algorithms such as inpainting [8,45], denoising [15,60,107], deblurring [11,12,14], demosaicing [85] and enhancement [71]. Wavelet frame related algorithms have been developed to solve medical and biological image processing problems, e.g. medical image segmentation [44,113], X-ray computer tomography (CT) image reconstruction [47], and protein molecule 3D reconstruction from electron microscopy images [83]. Frames provide large flexibility in designing filters with improved performance in applications. For example, the filters used for image restoration in [1,13,91] are learned from the image, resulting in filters that capture certain features of the image and lead to a transform that gives a better sparse representation. In [75] Gabor frame filter banks are designed to achieve high orientation selectivity that adapts to the geometry of image edges for sparse image approximation. Wavelet filters can be considered as discrete approximations of certain differential operators and thus the tight wavelet frame based approach for image processing has close ties with the PDE based approaches. Its connection to the total variation model is established in [9], to the Mumford–Shah model in [10], and to the nonlinear evolution PDE models in [46].

Since separable Hilbert spaces are sequence spaces with respect to some orthonormal basis, bounded linear operators between them are always given by infinite matrices. Dual Gramian analysis and the duality principle therefore are available to study frames also on the abstract infinite dimensional level. In [52] systems in abstract Hilbert spaces are studied by means of the pre-Gramian matrix, representing the synthesis operator, and by the Gramian and dual Gramian matrices, which represent the composition of the synthesis operator and the analysis operator in different orders. Adjoint systems are introduced for a given system by considering an adjoint (column–row) relationship of the respective pre-Gramian matrices and complement the study of the original system through the duality principle in the exact spirit of the results for shift-invariant systems in the function spaces: The dual Gramian of the system is the Gramian of its adjoint system. This can be considered the core statement of the duality principle of abstract frame theory. It is underlying the abstract duality principle previoulsy formulated in [19] and offers applications well beyond Gabor analysis, for example a new perspective on the unitary extension principle, which leads to a simple construction scheme for MRA-based multivariate tight wavelet frames proposed in [52]. In this paper, we move on from the study of the frame properties of a single system as in [52], to the study of dual pairs of systems in separable Hilbert spaces. While for a given frame one always has its pre-image under its frame operator as a canonical dual frame at one's disposal, in concrete situations it can be desirable to consider other dual frame pairs. To name only two instances, the canonical dual frame of a Gabor frame of compactly supported generators will in general not be generated by compactly supported functions [102], and the canonical dual frame of a wavelet frame will in general not have a wavelet structure at all [6,39]. There are on the other hand infinitely many alternative duals which do have wavelet structure [100]. As for an example in finite dimensional Hilbert spaces, the canonical dual is in general not optimal in handling higher order Sigma-Delta quantization errors, instead Sobolev duals are best suited, see [2]. The analysis of the mixed frame operator of two systems, which we therefore turn to, differs from that of a single system, for instance, in that the mixed frame operator is no longer self-adjoint. We extend the matrix notions and results of [52] to cover the case of pairs of systems by introducing the mixed dual Gramian matrices. Our most central result remains the abstract duality principle, part of which characterizes dual frame pairs in terms of biorthogonality relations of the adjoint system counterparts.

The abstract (mixed) dual Gramian analysis can be applied to any system and may simplify the analysis as soon as the system exhibits some structure. As an example, we formulate the duality principle for irregular Gabor systems, making use of their shift and modulation structure. If those shifts and modulations are sampled from lattices, (regular) Gabor systems are a particularly well structured class of shift-invariant systems and, via Fourier transform, we show how the abstract pre-Gramian reduces to the fiber pre-Gramian matrices of [98,102]. The underlying principles of dual Gramian analysis and the duality principle are the same regardless of the particular matrix representation of the synthesis operator. We show how they furnish a unified and simple treatment of dual Gabor systems, naturally and straightforwardly leading to the classical Walnut and Janssen representations of the mixed frame operator and to the classical Wexler–Raz biorthogonality relations as part of the duality principle. As already observed in [102], the duality principle on each fiber also is the essence of numerous painless constructions of Gabor windows going back to [38] and we use it to explicitly construct dual Gabor windows. The dual window pairs, whose construction we outline, have coinciding support and arbitrary smoothness can be obtained. Most importantly, the method can be easily generalized to the multivariate case.

The dual Gramian analysis developed in [98] has been applied to investigate dual wavelet (or affine) systems, which by nature are not shift-invariant, by extending them to quasi-affine systems, which are shift-invariant and still share the same frame properties [100,101]. If the wavelet system is generated from an MRA, the mixed unitary extension principle for the construction of dual wavelet frames has been derived from this analysis. The mixed unitary extension principle is the discrete dual frame condition in $\ell_2(\mathbb{Z}^d)$ of certain filter bank systems formed by the wavelet masks, which in turn can be viewed in the general framework of the mixed dual Gramian analysis in abstract Hilbert spaces developed in this paper. An application of the abstract duality principle at this point, leads to a new and simple way of constructing dual wavelet frames, which, in contrast to existing constructions that involve factoring polynomials, only requires completing and inverting a constant matrix. Especially in the multidimensional case, this greatly simplifies the task of finding dual wavelet frames, and moreover ensures the existence of dual MRA-wavelet frames for $L_2(\mathbb{R}^d)$ with compactly supported generators, under very weak condition on the refinement mask. The construction scheme can, for example, start from any mask of a compactly supported stable refinable function whose integer shifts form a partition of unity. In particular, the mask may contain negative entries. This flexibility cannot be facilitated by the related tight wavelet frame construction proposed in [52]. We outline several examples of multivariate dual wavelet frames derived from our construction.

In summary, this paper has the following contributions: We develop and apply the mixed dual Gramian analysis in the general Hilbert space setting. We use it to study dual frames rather than restricting attention to tight frames, which we review as a special case along the way. Through the mixed dual Gramian viewpoint we derive new results on dual frames. Applying it to Gabor systems, we show a duality principle for systems with no structure at all to the shifts and modulations of the generating window. For regular Gabor systems the duality principle has been a topic of intensive research for years, with many well known results and constructions. We show a unified approach to those classical results, which, if one looks at them from the right angle, can be understood as consequences of the mixed dual Gramian analysis. For example, the Wexler–Raz biorthonormality relations, the Janssen (or Wexler–Raz) representation and the Walnut representation can be viewed and proven directly once the dual Gramian analysis is established as shown in this paper. In the sequence space, we use the duality principle to get a filter bank construction which leads us to a construction scheme that guarantees dual MRA-wavelet frames in multidimensions for practically any usable refinable function. Finally, we go beyond systems to translation-invariant transforms. We provide a limiting case of the duality principle for Gabor systems, now between the translation-invariant Gabor transform and a shift-invariant system of a single generating function. Moreover, we provide a link between translation-invariant wavelet transforms and filter banks.

The paper is organized as follows: In Sect. 2, we review basic facts surrounding the notions Bessel system, (dual) frame and Riesz basis defined by the synthesis and analysis operators and their compositions, and discuss the duality principle that naturally arises from the definitions. Moreover, we present some new results on dual frames and mixed operators. Here we stay on the operator level, not yet considering matrix representations. In Sect. 3, we extend the (infinite) matrix analysis of given systems and their adjoint systems, as considered in [52], to cover the case of two systems, with the duality principle as the central result. We consider abstract dual frame characterizations which are in the spirit of known results in more concrete spaces. Moreover we apply the duality principle to filter banks and give a very flexible construction, inspired from [52], which we later use in the dual wavelet frame construction. In Sect. 4, we formulate the duality principle for irregular Gabor frames. For the case of regular Gabor systems, we reduce the abstract pre-Gramian to the fiber pre-Gramian matrices for shift-invariant systems defined in [98] and put the classical duality identities in the perspective of the dual Gramian analysis. We further consider the construction of dual Gabor frames of high smoothness and in high dimensions. Our attention in this section is on the dual system case, rather than the tight frame case focused on in [102]. In Sect. 5, the mixed unitary extension principle for dual wavelet frames is put into the perspective of the duality principle and we present the construction of dual wavelet frames based on it, along with several examples. This construction yields the very strong existence results, guaranteeing dual MRA-wavelet frames for a very large class of multiresolution analyses, independent of the dimension. The duality principle is ultimately a statement about operators, namely the analysis and synthesis operators, and in Sect. 6 we consider translation-invariant transforms which have a similar structure but are not strictly related to systems as in the previous sections.

2 Duality and Dual Frames

A dual frame of a frame provides the linear functionals which yield the coefficients for series expansions in terms of the frame elements. Frames and their dual frames are best treated in terms of bounded linear operators and their adjoint operators. The duality principle is based on this duality notion of functional analysis. Frames first have been introduced in [50] and for further developments on frames see e.g. [18,25].

2.1 Frames, Riesz Sequences and the Duality Principle

Let H be a separable Hilbert space and let X be a *system* in H, i.e. a countable indexed family $X = \{x_i\}_{i\in\mathcal{I}}$ of elements in H. The system may contain elements multiple times. In concrete applications the index set $\mathcal{I}$ reflects the organization of the system, e.g. shifts and modulations for Gabor systems or shifts and dilations for wavelet systems. To simplify our notation, we will omit writing out the index set $\mathcal{I}$. In this case we let the vectors index themselves, i.e. we use X as system and index set $\mathcal{I}$ simultaneously. Whenever X is considered as its indexing set, all vectors are identified as different indices for the purpuse of indexing (even identical ones that may appear multiple times). The *synthesis operator* of X is defined by

$$T_X : \ell_0(X) \to H : c \mapsto \sum_{x\in X} c(x)x$$

on the dense subspace $\ell_0(X)$ consisting of all finitely supported sequences in $\ell_2(X)$, the Hilbert space of all square summable sequences indexed by X. The system X is called a *Bessel system* if T_X is bounded, in which case we consider T_X as its unique continuous extension to a bounded operator on $\ell_2(X)$. The norm $\|T_X\|$ is called the *Bessel bound* of X. A Bessel system X is called ℓ_2-*independent* if T_X is injective, and *fundamental* if T_X has dense range in H. If and only if X is a Bessel system, the *analysis operator*

$$T_X^* : H \to \ell_2(X) \colon h \mapsto \{\langle h, x\rangle\}_{x\in X}$$

is bounded (otherwise it is only formally defined and may not map into $\ell_2(X)$). Bessel systems and their properties might therefore as well be characterized by the analysis operator of the system. In the Bessel case, T_X^* is the adjoint operator of T_X. Numerically stable reconstruction, from coefficients given by application of the analysis operator of a system, requires the analysis operator to provide an isomorphic embedding of H into $\ell_2(X)$. In this case, i.e. if there exist $0 < A \leq B$ such that

$$A\|h\| \leq \|T_X^* h\| \leq B\|h\| \quad \text{for all} \quad h \in H, \tag{2.1}$$

the system X is called a *frame* for H, and in particular a *tight frame* if $A = B = 1$. The system X is called a *frame sequence* if it is a frame for a closed subspace of H. If, on the other hand, the synthesis operator provides an isomorphic embedding of $\ell_2(X)$ into H, i.e. if there exist $0 < A \leq B$ such that

$$A\|c\| \leq \|T_X c\| \leq B\|c\| \quad \text{for all} \quad c \in \ell_2(X), \tag{2.2}$$

then X is called a *Riesz sequence* and if this holds for $A = B = 1$, then the elements of X are orthonormal. A Riesz sequence is called a *Riesz basis* if the closed linear span of X is H.

Due to the dual character of (2.1) and (2.2), one may consider two systems X and X^* as adjoint to each other if their synthesis operators are the adjoint operators of each other, i.e. in case $T_X^* = T_{X^*}$, perhaps up to unitary equivalence. It is then immediate that X is a (fundamental) Bessel system if and only if X^* is an (ℓ_2-independent) Bessel system; and that X is a (tight) frame if and only if X^* is a (orthonormal) Riesz sequence. This duality principle, $T_X^* = T_{X^*}$, on the level of operators is rather crude and has plenty of room for adjustment when taking into account the specifics of the system and the underlying Hilbert space in concrete situations. It is not immediately clear how to find an adjoint system on the level of operators. If, however, one has a matrix representation of the operator, adjoint systems appear as the rows and columns. The simple principle around which the results in this paper revolve is thus provided by the duality between adjoint operators and by their matrix representation.

Duality Principle *The systems X and X^* are adjoint to each other if for some matrix representation of the synthesis operator of X, the columns can be associated with X while the rows can be associated with X^*. Analysis (resp. synthesis) properties of X are characterized by synthesis (resp. analysis) properties of X^*.*

In a nutshell, the duality principle is thus the matrix perspective on adjoint operators, no matter how technical the specifics of meaningful matrix representations and inference on the systems in certain infinite dimensional situations may be. The trivial case is of course the one of finite systems in finite dimensions. If $X = \{x_n\}_{n=1}^N \subset \mathbb{C}^M$, then, with respect to the standard orthonormal bases, T_X is given by the matrix

$$J_X = \begin{pmatrix} x_1(1) & \cdots & x_N(1) \\ \vdots & \ddots & \vdots \\ x_1(M) & \cdots & x_N(M) \end{pmatrix}$$

and T_X^* by its adjoint matrix J_X^*. Thus, a possible adjoint system of X is given by

$$X^* = \{(\overline{x_n(m)})_{n=1,\dots,N} : m = 1, \dots, M\} \subset \mathbb{C}^N,$$

i.e. by the complex conjugates of the rows of J_X. The adjoint relationship between X and X^* is now extensively used in the study of finite frames in finite dimensional spaces, see e.g. [7,17]. Similarly, adjoint systems in abstract infinite dimensional Hilbert spaces have been defined in [52] using infinite matrix representations. The system then has to meet some weak condition depending on the basis chosen for the matrix representation. We will expand on this ideas in Sect. 3 and, for example in Sect. 4, will see how the freedom to choose the basis for the representation can make general matrix representations useful as soon as the given system exhibits some structure. Even better tailored representations can be achieved if in addition the specifics of the Hilbert space are as well being exploited. One may study unitarily equivalent versions of the relevant operators if they have simpler matrix representation. All this has been done in the series of papers [98–102] for general and particular shift-invariant systems in $L_2(\mathbb{R}^d)$. The technique used there still goes one step further. It analyzes the properties of the Fourier transform of the synthesis and analysis operators by representing them by a whole continuum of matrices, the so called *fibers*, instead of just one matrix. In compensation for moving to a whole family of matrices, those matrices are simply composed of the Fourier transforms of the generators of the systems and have a very regular structure. Properties of the analysis and synthesis operators can then be characterized by properties of the fibers, which have to hold in a uniform way, and adjoint systems are introduced via the rows and columns of the fiber matrices, see Sect. 4. In Sect. 6 we will return to the duality principle on the operator level when discussing translation-invariant transforms. Though they do not arise as analysis operators of discrete systems, in certain cases they do have discrete adjoint systems.

It is trivial to observe that $T_X^* = T_{X^*}$ implies $T_X T_X^* = T_{X^*}^* T_{X^*}$. This observation is important since the frame and Riesz properties of the system X can also be characterized using the self-adjoint operators $T_X^* T_X$ and $T_X T_X^*$. Indeed, X is a Bessel system if and only if one of those operators, and therefore both, are bounded. In this case X is ℓ_2-independent if and only if $T_X^* T_X$ is invertible; X is a Riesz sequence if and only if $T_X^* T_X$ has a bounded inverse; and X is an orthonormal sequence if and only if $T_X^* T_X = I$, where I denotes the identity on H. A Bessel system X is fundamental if and only if $T_X T_X^*$ is injective; it is a frame for H if and only if $T_X T_X^*$ has a bounded inverse; and it is a tight frame if and only if $T_X T_X^* = I$. Matrix representations of the analysis and synthesis operators yield matrix representations for their compositions, the *Gramian* representing $T_X^* T_X$ and the *dual Gramian* representing $T_X T_X^*$. This motivates to refer to the matrix representation of T_X as the *pre-Gramian*. If X is a tight frame T_X^* may have many left inverses besides T_X. For this, let $Y = RX$ where $R : X \to H$ is a map. More precisely, we consider two systems $X = \{x_i\}_{i\in\mathcal{I}}$ and $Y = \{y_i\}_{i\in\mathcal{I}'}$ with index sets $\mathcal{I}$ and $\mathcal{I}'$ between which there is a bijection $R\colon \mathcal{I} \to \mathcal{I}'$. Then, essentially, X and Y have the same index set. To simplify our notation, and in drawing on our viewing of X and Y as indexing themselves, we will say that we consider a map $R : X \to H$ and let $Y = RX$. Here, again in slight abuse of notation, we are actually establishing a map between the index sets of the systems X and Y. If now X and $Y = RX$ are Bessel systems in H, then one can consider $T_Y T_X^*$ and if this operator is the identity on H then X and Y are called a pair of *dual frames* for H (in particular, both are frames). The *mixed operators* $T_Y^* T_X$ and $T_Y T_X^*$ are then represented by *mixed dual Gramians* and *mixed Gramians*. The duality principle now can be formulated as a matrix equality (up to unitary transforms or complex conjugations):

The mixed dual Gramian of two systems is the mixed Gramian of their adjoint systems.

A major aspect of this article is the construction of dual/tight frames. The duality principle can often facilitate constructions since the adjoint systems and their biorthogonality properties are usually more accessible than the original systems.

We end this subsection discussing a few more relevant preliminaries about frames. While (2.1) and (2.2) are in perfect duality, the arising definitions are of different quality. Every Riesz basis is in fact a frame, with the optimal bounds in (2.2) and

(2.1) coinciding, but the converse is not true. Notice that a Bessel system X is frame sequence if and only if any of the following equivalent conditions is satisfied: (i) $\operatorname{ran} T_X$ is closed, (ii) T_X is bounded below on $(\ker T_X)^\perp$, (iii) $\operatorname{ran} T_X^* = (\ker T_X)^\perp$, (iv) T_X^* is bounded below on $(\ker T_X^*)^\perp$. A frame sequence is a frame for H whenever $\operatorname{ran} T_X = H$. A Bessel system X is a Riesz sequence if and only if T_X is injective and T_X^* is bounded below on $(\ker T_X^*)^\perp$. That is, a frame X for H is a Riesz basis for H if and only if T_X is injective, or equivalently if $\operatorname{ran} T_X^* = \ell_2(X)$. If X is a frame, we refer to the optimal bounds $A = \|T_X^\dagger\|^{-1} = \|(T_X^*)^\dagger\|^{-1}$ and $B = \|T_X\| = \|T_X^*\|$ in (2.1) as the *lower* and *upper frame bounds*[1] of X. Here, given some bounded operator T between two Hilbert spaces, $T^\dagger$ denotes its partial inverse, i.e. the inverse of the restriction of T to $(\ker T)^\perp$. The injectivity imposed on T_X by (2.2) makes series expansions in terms of elements of Riesz sequences unique. This is not the case for series expansions in terms of the elements of frame sequences and is usually referred to as frames being *redundant*. A question about reducing the redundancy of a frame is whether every frame contains a Riesz basis. The answer of course is negative, since the vectors of a Riesz basis are necessarily bounded and bounded below away from zero in norm. Thus, if $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis, then $\{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \ldots\}$ (where $\frac{1}{\sqrt{n}}e_n$ appears n times) is a tight frame which does not contain a Riesz basis. Counterexamples still exist if one does not allow frames which contain a subsequence converging to zero in norm [16,106]. In other words, it is in general not possible to choose a coordinate subspace of the coefficient space $\ell_2(X)$ of a tight frame X, such that the restriction of T_X to this subspace becomes injective while still being onto. Here, by a coordinate subspace of $\ell_2(X)$ we mean any subspace of the form $\overline{\operatorname{span}}\{e_x\}_{x\in Y}$ where Y is a subsystem of X and $\{e_x\}_{x\in X} \subset \ell_2(X)$ are the standard unit vectors. That is, the coordinate subspaces are those subspaces that can be identified with $\ell_2(Y)$ for some subsystem Y of X. The situation changes if one considers arbitrary subspaces of the coefficient space. Given a Bessel system X, the question becomes whether there is some subspace H' of $\ell_2(X)$ such that $T_X|_{H'}$ is bounded below, i.e. such that

$$\|(T_X|_{H'})^\dagger\|^{-1}\|c\| \leq \|T_X c\| \leq \|T_X|_{H'}\|\|c\| \quad \text{for all} \quad c \in H'. \tag{2.3}$$

Note that X is a frame sequence if (2.3) holds for $H' = (\ker T_X)^\perp$, that X contains a Riesz sequence if (2.3) holds for a coordinate subspace $\ell_2(X)$, and that X is a Riesz sequence if (2.3) holds for $H' = \ell_2(X)$. If X is a frame for H, then $T_X T_X^*$ is bounded below and onto, thus $T_X|_{\operatorname{ran} T_X^*}$ is bounded below and onto and one can choose $H' = \operatorname{ran} T_X^*$. Moreover, $\operatorname{ran} T_X^*$ is exactly the space of coefficients needed for X to span H, so we really may restrict our attention to precisely this subspace of $\ell_2(X)$. In effect, in this view the distinction between frame and Riesz property vanishes and in this sense one can always make a redundant system non-redundant by considering it on a smaller coefficient space. The more redundant a system is, the fewer coefficients one needs to represent H since $\ker T_X$ gets larger while $\operatorname{ran} T_X^*$ gets smaller.

[1] Often in the literature the squares of those numbers are called lower and upper frame bounds.

2.2 Dual Frames and Mixed Operators

This article emphasizes dual frame pairs, tight frames being a special case. Among the dual frames of a frame, one enjoys special prominence. If X is a frame for H, then X and $\mathcal{S}^{-1}X$, where $\mathcal{S} = T_X T_X^*$, are dual frames since $\mathcal{S}^{-1}$ is self-adjoint and therefore $T_{\mathcal{S}^{-1}X}^* = T_X^*\mathcal{S}^{-1}$. The system $\mathcal{S}^{-1}X$ is called the *canonical dual frame* of X. In particular, a tight frame has itself as canonical dual frame. The canonical dual frame $\mathcal{S}^{-1}X$ is distinguished from any other dual frame RX by several properties. For example, $\mathcal{S}^{-1}X$ is the unique dual frame to make the projector $T_{RX}^* T_X$ an orthogonal projector, see [100]. Also, $\|T_{\mathcal{S}^{-1}X}^* f\| \le \|T_{RX}^* f\|$ for any $f \in H$, see [37]. Moreover, $\mathcal{S}^{-1}$ is the only operator among all dual frame maps R that is self-adjoint, i.e. if X and RX are dual frames, then RX is the canonical dual frame of X if and only if $\langle x, Rx'\rangle = \langle Rx, x'\rangle$ for all $x, x' \in X$, see [100]. The canonical dual frame can also be used to verify the independence properties of the system. Specifically, see [100], if X is a frame for H, then $\langle x, \mathcal{S}^{-1}x\rangle \le 1$ for all $x \in X$ and X is a Riesz basis if and only if $\langle x, \mathcal{S}^{-1}x\rangle = 1$ for all $x \in X$. In particular, a tight frame is an orthonormal basis if and only if all its elements have unit norm.

We now prove some facts about the mixed operators $T_Y T_X^*$ and $T_Y^* T_X$. The first is in the spirit of the canonical dual frame.

Proposition 2.1 *Let X and $Y = RX$ be frames for H such that* $\operatorname{ran} T_X^* = \operatorname{ran} T_Y^*$. *Then $T_Y T_X^*$ is boundedly invertible and $(T_Y T_X^*)^{-1}Y$ and X are dual frames.*

Proof We have

$$\operatorname{ran} T_Y T_X^* = \operatorname{ran}(T_Y|_{\operatorname{ran} T_X^*}) = \operatorname{ran}(T_Y|_{\operatorname{ran} T_Y^*}) = \operatorname{ran}(T_Y|_{(\ker T_Y)^\perp}) = \operatorname{ran} T_Y = H.$$

To show the injectivity of $T_Y T_X^*$, let $f \in H$ such that $T_Y T_X^* f = 0$. Then $T_X^* f \in \ker T_Y = (\operatorname{ran} T_Y^*)^\perp = (\operatorname{ran} T_X^*)^\perp$ which gives $T_X^* f = 0$. Since X is a frame, T_X^* is injective, and thus $f = 0$, showing that $T_Y T_X^*$ is injective. A similar proof shows that $T_X T_Y^*$ is also invertible. Thus, by the open mapping theorem, $T_Y T_X^*$ and $T_X T_Y^*$ are boundedly invertible on H and, denoting $Q = (T_Y T_X^*)^{-1}R$, we have

$$T_{QX}^* h = \{\langle h, (T_Y T_X^*)^{-1}Rx\rangle\}_{x\in X}$$

for any $h \in H$, i.e. $T_{QX}^* = T_Y^*(T_X T_Y^*)^{-1}$. Therefore, $T_X T_{QX}^*$ is the identity on H. □

Proposition 2.2 *Let X and $Y = RX$ be Bessel systems in H such that* $\operatorname{ran} T_X^* = \operatorname{ran} T_Y^*$. *Then the following are equivalent:*

(i) X and Y are frames.
(ii) $T_Y T_X^*$ and $T_X T_Y^*$ are bounded below.

Proof Suppose X and Y are frames for H. Then T_X is bounded below on $(\ker T_X)^\perp = \operatorname{ran} T_X^* = \operatorname{ran} T_Y^*$ and T_Y^* is bounded below on H. Thus, for every $h \in H$ we have

$$\|T_X^\dagger\|^{-1}\|(T_Y^*)^\dagger\|^{-1}\|h\| \le \|T_X^\dagger\|^{-1}\|T_Y^* h\| \le \|T_X T_Y^* h\|.$$

Hence $T_X T_Y^*$ is bounded below on H and similarly is $T_Y T_X^*$. On the other hand, suppose $T_X T_Y^*$ is bounded below. Since T_X is bounded, T_Y^* is bounded below. Thus Y is a frame. Similarly, X is a frame. □

Note that the assumption $\operatorname{ran} T_X^* = \operatorname{ran} T_Y^*$ in Propositions 2.1 and 2.2 is essential. There do exist *orthogonal frames* X and $Y = RX$ such that $T_Y T_X^* = T_X T_Y^* = 0$, see [77]. Moreover, note that $\operatorname{ran} T_X^* = \operatorname{ran} T_Y^*$ is not implied by Proposition 2.2(ii). Indeed, the canonical dual of a frame X is the only Bessel system $R'X$ in H for which $T_{R'X} T_X^* = I$ and $\operatorname{ran} T_X^* = \operatorname{ran} T_{R'X}^*$, see [98]. The condition $\operatorname{ran} T_X^* = \operatorname{ran} T_Y^*$ is not needed for (ii) to imply (i) in Proposition 2.2.

Proposition 2.3 *Let X and $Y = RX$ be Bessel systems in H such that* $\operatorname{ran} T_X = \operatorname{ran} T_Y$. *Then the following are equivalent:*

(i) X and Y are Riesz sequences.
(ii) $T_Y^* T_X$ and $T_X^* T_Y$ are bounded below.

Proof If $T_Y^* T_X$ is bounded below and T_Y^* is bounded, then T_X is bounded below. Thus X is a Riesz sequence. On the other hand, if X and Y are Riesz sequences, then T_X is bounded below and T_Y^* is bounded below on $(\ker T_Y^*)^\perp = (\ker T_X^*)^\perp$. Therefore $T_Y^* T_X$ is bounded below. □

We now turn to the restriction of the coefficient space to some subspace of $\ell_2(X)$ through an orthogonal projection. That is, consider the operator $T_X P_{H'} T_X^*$ where X is a Bessel system in H and $P_{H'}$ is the orthogonal projection of $\ell_2(X)$ onto a subspace H' of $\ell_2(X)$. If $\operatorname{ran} T_X^* \subset H'$, then $T_X P_{H'} T_X^* = T_X T_X^*$, which is the classical situation with T_X acting on the whole coefficient space $\operatorname{ran} T_X^*$. In general, decompose H' into the orthogonal direct sum $H' = H_1' \oplus H_2'$, where $H_1' = H' \cap \operatorname{ran} T_X^*$ and H_2' is the orthogonal complement of H_1' in H'. Then $H_1' \subset \operatorname{ran} T_X^*$ and $H_2' \subset \ker T_X$, i.e. $T_X P_{H_2'} T_X^*$ is identical zero.

Operators of the form $T_X P_{H'} T_X^*$ arise in many contexts. Let, say, X be a tight frame for H, let Y be a subsystem of X and $M(Y) = \sum_{x\in Y} \langle \cdot, x\rangle x$. Then the mapping M, defined on the family of all subsystems of X, is a simple example of a positive operator valued measure, a notion playing a major role in quantum information theory, describing generalized measurements [104]. In general, $M(Y)$ is not an orthogonal projection. Indeed, letting $P(Y)$ be the orthogonal projection of $\ell_2(X)$ onto the subspace $\overline{\operatorname{span}}\{e_x\}_{x\in Y}$, then $M(Y) = T_X P(Y) T_X^*$, which is an instance of Naimark's dilation theorem (see e.g. [90]). The mapping P defines a projection valued measure, describing the standard measurements in quantum theory. We now show that $T_X P_{H'} T_X^*$ is an orthogonal projection whenever X is a tight frame for certain subspaces related to H'.

Proposition 2.4 *Let X be a fundamental Bessel system in H and let H' be a subspace of $\ell_2(X)$. Then the following are equivalent:*

(i) $T_X T_X^*$ is the identity on $E_1 = \{f \in H : T_X^* f \in H'\}$.
(ii) $T_X T_X^*$ is the identity on $E_2 = \operatorname{ran} T_X|_{H'}$.

If (i) (or (ii)) holds, then $T_X P_{H'} T_X^*$ is the orthogonal projection onto E_1 and $E_1 = E_2$.

Proof With H' decomposed as above, notice that $T_X P_{H'} T_X^* = T_X P_{H'_1} T_X^* + T_X P_{H'_2} T_X^* = T_X P_{H'_1} T_X^*$. Hence, $\operatorname{ran} T_X|_{H'} = \operatorname{ran} T_X|_{H'_1}$ and $T_X T_X^*$ is the identity on E_1 if and only if it is the identity on $\{f \in H : T_X^* f \in H'_1\}$. Therefore, assume as we may, that $H' \subset \operatorname{ran} T_X^*$. (i) $\Rightarrow$ (ii): Note that $T_X T_X^*$ maps E_1 onto E_2 and is the identity on E_1, thus these two sets coincide. (ii) $\Rightarrow$ (i): If $T_X^* h \in H'$, then $(T_X T_X^*) T_X T_X^* h = (T_X T_X^*) h$, i.e. $T_X T_X^* h - h \in \ker T_X T_X^*$. Since X is fundamental, $H = \overline{\operatorname{ran} T_X} = (\ker T_X^*)^\perp = (\ker T_X T_X^*)^\perp$, which yields $\ker T_X T_X^* = \{0\}$.

Now assume (i). As $T_X P_{H'} T_X^*$ is self-adjoint it remains to show that it is the identity on its range. If $h \in E_1$, then $T_X P_{H'} T_X^* h = T_X T_X^* h = h$ and thus $E_1 \subset \operatorname{ran} T_X P_{H'} T_X^*$. It therefore remains to show $\operatorname{ran} T_X P_{H'} T_X^* \subset E_1$. To this end, let $h \in \operatorname{ran} T_X P_{H'} T_X^*$, say $h = T_X P_{H'} T_X^* g$. Since $H' \subset \operatorname{ran} T_X^*$, we have $P_{H'} T_X^* g = T_X^* f$ for some $f \in E_1$. Therefore, by (ii), $T_X^* h = T_X^* T_X P_{H'} T_X^* g = T_X^* T_X T_X^* f = T_X^* f \in H'$, i.e. $h \in E_1$. □

Without the assumption on X to be fundamental, (ii) does not imply (i). Take, say, $H = \mathbb{R}^3$ with the standard orthonormal basis $\{e_1, e_2, e_3\}$ and let $X = \{e_1, e_2\}$. Let $H' = \operatorname{ran} T_X^* = \mathbb{R}^2$. Then $T_X T_X^*$ is the identity on E_2 but not on $E_1 = \mathbb{R}^3$. Merely under the assumption on X being fundamental, E_1 in general does not coincide with E_2 (or its closure). Take $H = \mathbb{R}^2$ and $X = \sqrt{2/5}\{(0,1)^\top, (1,0)^\top, (1,1)^\top\}$. Let $H' = \operatorname{span}\{(0,1,1)^\top\} \subset \operatorname{span}\{(1,0,1)^\top, (0,1,1)^\top\} = \operatorname{ran} T_X^*$. Then $E_1 = \operatorname{span}\{(1,0)^\top\}$ but $E_2 = \operatorname{span}\{(2,1)^\top\}$. Note also that in this example $T_X P_{H'} T_X^*$ is the orthogonal projection onto E_2, i.e. the assumption on $T_X P_{H'} T_X^*$ to be an orthogonal projection in the above result does not imply (i) (nor (ii)).

According to Proposition 2.4, the operator $T_X P_{H'} T_X^*$ is a projection whenever $T_X T_X^*$ is the identity on certain subspaces of H, i.e. when X is a tight frame for certain subspaces of H. A weaker condition is the assumption that $T_X T_X^*$ is bounded below on the aforementioned subspaces. However, note that if H' is a proper subspace of $\operatorname{ran} T_X^*$, we can choose a nonzero $h \in H$ such that $T_X^* h \in (\operatorname{ran} T_X^*) \ominus H'$. Then $T_X P_{H'} T_X^* h = 0$. Thus, in this case $T_X P_{H'} T_X^*$ cannot be bounded below. Nevertheless, if X is a fundamental Bessel system in H and $H' \subset \operatorname{ran} T_X^*$, then $T_X T_X^*$ is bounded below on E_1 if and only if $(T_X T_X^*|_{E_1})^{-1}$ is bounded on E_2.

3 Mixed Dual Gramian Analysis

Bounded linear operators between sequence spaces have natural representations as infinite matrices. If T is a bounded linear operator between two separable Hilbert spaces with orthonormal bases $\mathcal{O}'$ and $\mathcal{O}$, the matrix $(a_{e,e'})_{e\in\mathcal{O}, e'\in\mathcal{O}'}$ corresponding to T is defined by the relation $Te' = \sum_{e\in\mathcal{O}} a_{e,e'} e$, i.e. its entries are $a_{e,e'} = \langle Te', e\rangle$. The synthesis operator T_X of a given system X in H is a (potentially unbounded) densely defined linear operator from $\ell_2(X)$ to H and the key to dual Gramian analysis and the duality principle in abstract Hilbert spaces is to study it through an infinite matrix. In [52], the standard unit vectors $\{e_x\}_{x\in X}$ are chosen as an orthonormal basis for $\ell_2(X)$ and the *pre-Gramian* J_X of X associated with an orthonormal basis $\mathcal{O}$ of H is defined as the (infinite) matrix

$$J_X := (\langle T_X e_x, e\rangle)_{e\in\mathcal{O}, x\in X} = (\langle x, e\rangle)_{e\in\mathcal{O}, x\in X}. \tag{3.4}$$

Assuming X to be a Bessel system, or under the weaker assumption[2]

$$\sum_{x \in X} |\langle x, e\rangle|^2 < \infty \quad \text{for all} \quad e \in \mathcal{O}, \tag{3.5}$$

the pre-Gramian matrix and its adjoint represent T_X and T_X^* in the following way, where U denotes the synthesis operator of $\mathcal{O}$:

$$U J_X c = T_X c \quad \text{for all} \quad c \in \ell_0(X),$$

and

$$T_X^* U d = J_X^* d \quad \text{for all} \quad d \in \ell_0(\mathcal{O}).$$

Consequently, one can study Bessel, frame, and Riesz properties of X by considering the pre-Gramian matrix, its adjoint, and two Hermitian matrices: the Gramian $G_X = J_X^* J_X$ and the dual Gramian $\widetilde{G}_X = J_X J_X^*$. This has been done in [52] and we now extend those results to the analysis of two systems.

3.1 Mixed Dual Gramian Matrices

Let X be a system in H and $R : X \to H$ a map. Assuming (3.5) for X and RX with respect to an orthonormal basis $\mathcal{O}$ of H, define their *mixed Gramian matrix* as

$$G_{RX,X} := J_{RX}^* J_X = \left(\sum_{e \in \mathcal{O}} \langle x', e\rangle \langle e, Rx\rangle \right)_{x \in X, x' \in X} = (\langle x', Rx\rangle)_{x \in X, x' \in X}, \tag{3.6}$$

and their *mixed dual Gramian matrix* as

$$\widetilde{G}_{RX,X} := J_{RX} J_X^* = \left(\sum_{x \in X} \langle Rx, e\rangle \langle e', x\rangle \right)_{e \in \mathcal{O}, e' \in \mathcal{O}}. \tag{3.7}$$

These matrices represent $T_{RX}^* T_X$ and $T_{RX} T_X^*$ as follows.

Proposition 3.1 *Let X and RX be systems in H which satisfy (3.5) with respect to an orthonormal basis $\mathcal{O}$. Suppose the mixed Gramian $G_{RX,X}$ and mixed dual Gramian $\widetilde{G}_{RX,X}$ are both defined with respect to $\mathcal{O}$ and let U be the synthesis operator of $\mathcal{O}$. Then*

$$\langle T_X c, T_{RX} d\rangle = d^* G_{RX,X} c \quad \textit{for all} \quad c, d \in \ell_0(X),$$

[2] Consider, e.g., $\ell_2(\mathbb{N})$ with the standard unit vector basis $\mathcal{O} = \{e_n\}_{n \in \mathbb{N}}$. Then $X = \{n e_n\}_{n \in \mathbb{N}}$ satisfies (3.5) but is not a Bessel sequence.

and

$$\langle T_X^* Uc, T_{RX}^* Ud\rangle = d^* \widetilde{G}_{RX,X} c \quad \textit{for all} \quad c, d \in \ell_0(\mathcal{O}).$$

If X and RX are Bessel systems, then $G_{RX,X}$ defines a bounded operator on $\ell_2(X)$ and $\widetilde{G}_{RX,X}$ defines a bounded operator on $\ell_2(\mathcal{O})$. Further, in this case

$$T_{RX}^* T_X c = G_{RX,X} c \quad \textit{for all} \quad c \in \ell_2(X), \tag{3.8}$$

and

$$U^* T_{RX} T_X^* Uc = \widetilde{G}_{RX,X} c \quad \textit{for all} \quad c \in \ell_2(\mathcal{O}). \tag{3.9}$$

Proof Let $c, d \in \ell_0(X)$. Then

$$\begin{aligned}\langle T_X c, T_{RX} d\rangle &= \left\langle \sum_{x'\in X} c(x')x', \sum_{x\in X} d(x)Rx \right\rangle \\ &= \sum_{x\in X} \overline{d(x)} \sum_{x'\in X} c(x')\langle x', Rx\rangle = d^* G_{RX,X} c.\end{aligned}$$

For $c, d \in \ell_0(\mathcal{O})$ we get

$$\begin{aligned}\langle T_X^* Uc, T_{RX}^* Ud\rangle &= \sum_{x\in X} \langle Uc, x\rangle\langle Rx, Ud\rangle \\ &= \sum_{e\in\mathcal{O}} \overline{d(e)} \sum_{e'\in\mathcal{O}} c(e') \sum_{x\in X} \langle e', x\rangle\langle Rx, e\rangle = d^* \widetilde{G}_{RX,X} c.\end{aligned}$$

□

In the following two results, $\mathcal{O}$ is the orthonormal basis corresponding to the considered pre-Gramian and U is its synthesis operator. The relationship (3.9) implies the following characterization of dual frames X and RX.

Corollary 3.2 *Let X and RX be Bessel systems for H. Then X and RX are dual frames for H if and only if $\widetilde{G}_{RX,X} = I$ on $\ell_2(\mathcal{O})$.*

In case $RX = X$ the mixed dual Gramian matrix becomes the dual Gramian matrix. Using the dual Gramian matrix, the canonical dual frame can be found by matrix inversion and the property of being a Riesz basis can be verified by evaluating the inner products of the elements of X.

Corollary 3.3 *Let X be a frame in H. Then $\langle U^* x, \widetilde{G}_{X,X}^{-1} U^* x\rangle \le 1$ for all $x \in X$. Moreover, X is a Riesz basis if and only if $\langle U^* x, \widetilde{G}_{X,X}^{-1} U^* x\rangle = 1$ for all $x \in X$.*

3.2 Adjoint Systems and Dual Frames

The adjoint system, introduced in [52], is a useful tool for the study of frame systems. In this section we use it for the study of dual frame pairs.

3.2.1 Adjoint Systems and Duality Principle

Using the pre-Gramian, one can define adjoint systems of a given system by linking the columns and rows of J_X. The columns of J_X represent the vectors of the system X against an orthonormal basis $\mathcal{O}$, whereas the rows of J_X represent the vectors of some adjoint system in $\ell_2(X)$. The definition in [52] is as follows.

Definition 3.4 Let X be a system and J_X its pre-Gramian with respect to an orthonormal basis $\mathcal{O}$ such that (3.5) holds. A system X^* (in a potentially different Hilbert space) is called an *adjoint system* of X, if there is an orthonormal basis $\mathcal{O}'$ such that X^* and $\mathcal{O}'$ satisfy (3.5), and such that the pre-Gramian J_{X^*} of X^* with respect to $\mathcal{O}'$ satisfies

$$J_{X^*} = U J_X^* V \tag{3.10}$$

for some unitary operators U and V.[3]

The most important application of the adjoint system is the duality principle, a relationship between the Gramian and dual Gramian matrices of systems and their adjoint systems that has been observed in [52]. The duality principle holds in completely analogous fashion for the more general case of two systems, saying that the mixed dual Gramian matrix of two systems X and RX is (unitarily equivalent to) the mixed Gramian matrix of their adjoint systems X^* and $(RX)^*$, and vice versa. For this, note that, whenever we consider several systems and their corresponding adjoint systems we will always assume that all of them are satisfying the respective adjoint relationship (3.10) with respect to the same orthonormal bases and unitaries. Then, indeed,

$$\widetilde{G}_{RX,X} = J_{RX} J_X^* = V J_{(RX)^*}^* U U^* J_{X^*} V^* = V G_{(RX)^*,X^*} V^*$$

and

$$G_{RX,X} = J_{RX}^* J_X = U^* J_{(RX)^*} V^* V J_{X^*}^* U = U^* \widetilde{G}_{(RX)^*,X^*} U.$$

We therefore have the following abstract duality principle.

Theorem 3.5 *Let X, RX be systems in H and X^*, resp. $(RX)^*$, be adjoint systems of X, resp. RX (with respect to the same orthonormal bases and unitaries). Then, up to unitary equivalence, $\widetilde{G}_{RX,X}$ is equal to $G_{(RX)^*,X^*}$ and $G_{RX,X}$ is equal to $\widetilde{G}_{(RX)^*,X^*}$.*

[3] Note that one might also consider complex conjugations of the entries of the pre-Gramian in (3.10) without introducing essential changes to the discussion that follows.

Consequently, if X and RX are Bessel systems, then X and RX are dual frames if and only if X^* is biorthonormal to $(RX)^*$.

The assumption on the existence of the adjoint systems in the duality principle puts a restriction on the systems for which it can be applied. Both have to satisfy (3.5) (for the same orthonormal basis), for which it is sufficient that both systems are Bessel systems. Note further, that the above biorthonormality is with respect to the natural indexing of the vectors of X^* and $(RX)^*$ by means of the orthonormal basis chosen for the representation of the pre-Gramian matrices J_X and J_{RX}.

In the case of single systems, i.e. R being the identity, the duality principle reduces to the following: Up to unitaries, the dual Gramian of the system X is the Gramian of its adjoint system X^*, and vice versa, see [52]. Considering quadratic forms, the Bessel property of X can also be characterized by the (dual) Gramian matrix. Therefore, X is Bessel if and only if X^* is Bessel; X is Bessel and fundamental if and only if X^* is Bessel and ℓ_2-independent; X is a frame if and only if X^* is a Riesz sequence; X is a tight frame if and only if X^* is an orthonormal sequence. The roles of X and X^* in the previous statements are interchangeable, since X is an adjoint system of X^*.

In later sections we will apply the duality principle to Gabor systems and wavelet systems. Here we only mention two short illustrating examples.

Example 3.6 The finite dimensional case is the simplest. Let, say, $X = \{x_n\}_{n=1}^N$ and $Y = \{y_n\}_{n=1}^N$ be spanning sets in $\mathbb{C}^M$. The pre-Gramians J_X and J_Y with respect to the standard orthonormal basis of $\mathbb{C}^M$ are the matrices which have the vectors of X, respectively Y, as columns. The adjoint systems are given in $\mathbb{C}^N$ by the complex conjugate rows of those matrices. Hence the duality principle implies that the columns of J_X and J_Y, i.e. X and Y, are dual frames, if and only if the rows of J_X and J_Y are biorthonormal, i.e. $\sum_{n=1}^N x_n(i)\overline{y_n(j)} = \delta_{ij}$ for $i, j = 1, 2, \ldots, M$.

Example 3.7 Let X be a system in H, satisfying (3.5) with respect to an orthonormal basis $\mathcal{O}$ of H. Similarly to the finite dimensional case, an adjoint system in $\ell_2(X)$ is given by the sequences which make up the rows of J_X:

$$\left\{(\langle x, e\rangle)_{x\in X} : e \in \mathcal{O}\right\}.$$

As in Definition 3.4, this adjoint system may be unitarily mapped from the sequence space $\ell_2(X)$ into a different Hilbert space. If, say, $\{v_x\}_{x\in X}$ is an orthonormal basis of some Hilbert space, then

$$X^* = \left\{\sum_{x\in X} \langle x, e\rangle v_x : e \in \mathcal{O}\right\} \tag{3.11}$$

is an adjoint system of X. If X and $\mathcal{O}$ have the same cardinality, $\{v_x\}_{x\in X}$ can be choosen to be some orthonormal basis for H. Then X^* is an adjoint system of X in H, first defined in [19] as the Riesz-dual sequence of X (see also [20,28,30]). Note that in this case $\mathcal{O}$ might as well be indexed by X, i.e. $\mathcal{O} = \{e_x\}_{x\in X}$ and

$$X^* = \left\{ x^* = \sum_{x' \in X} \langle x', e_x \rangle v_{x'} : x \in X \right\}.$$

Remarkably, in [19] the Riesz dual sequence is defined and its dualities with X are shown without reference to any matrix representations. The matrix viewpoint directly reveals all those dualities via the duality principle. If, for example, X is a frame and $\mathcal{S} = T_X T_X^*$, then

$$\left\{ (\mathcal{S}^{-1}x)^* = \sum_{x' \in X} \langle \mathcal{S}^{-1}x', e_x \rangle v_{x'} : x \in X \right\}$$

defines an adjoint system of $\mathcal{S}^{-1}X$ and therefore the biorthogonality $\langle x^*, (\mathcal{S}^{-1}x')^* \rangle = \delta_{x,x'}$ for all $x, x' \in X$ follows from Theorem 3.5, since $\mathcal{S}^{-1}X$ is the canonical dual frame of X.

As another application of Definition 3.4 we observe a relationship between the mixed frame operators of systems and their adjoint systems. Let $X, Y = RX$ and Z be Bessel systems in H, and $\mathcal{O}$ be an orthonormal basis of H. Then the obvious matrix relationship

$$J_X J_Y^* J_Z = (J_Z^* J_Y J_X^*)^*$$

holds, where all pre-Gramians are being considered with respect to $\mathcal{O}$. If X^*, Y^*, Z^* are adjoint systems of X, Y, Z, respectively, all with respect to the same orthonormal basis $\mathcal{O}'$ and unitaries U, V, then

$$J_X J_Y^* J_Z = V (J_{Z^*} J_{Y^*}^* J_{X^*})^* U. \tag{3.12}$$

Note that $J_X J_Y^* J_Z$ is the pre-Gramian matrix of the system $X' = T_X T_Y^* Z$ while $J_{Z^*} J_{Y^*}^* J_{X^*}$ is the pre-Gramian matrix of the system $Y' = T_{Z^*} T_{Y^*}^* X^*$. Equation (3.12) implies that X' and Y' are adjoint systems as in Definition 3.4.

Theorem 3.8 *If $X, Y = RX, Z$ are Bessel systems in H, where $R : X \to H$ is a map, and X^*, Y^*, Z^* are their respective adjoint systems (with respect to the same orthonormal bases and unitaries), then the system $T_X T_Y^* Z$ is an adjoint system of $T_{Z^*} T_{Y^*}^* X^*$.*

3.2.2 Other Dual Frame Characterizations

The family of all dual frames of a given frame can be parametrized in terms of the adjoint systems.

Proposition 3.9 *Let X be a frame for H and $R : X \to H$ a map such that RX is a Bessel system. Denote $\mathcal{S} = T_X T_X^*$. Let X^*, $(RX)^*$ and $(\mathcal{S}^{-1}X)^*$ be adjoint systems of X, RX and $\mathcal{S}^{-1}X$, respectively (with respect to the same orthonormal bases and unitaries). Then* *(i)* $(\mathcal{S}^{-1}X)^*$ *is a system in* $\operatorname{ran} T_{X^*}$*,*

(ii) RX *is a dual frame of* X *if and only if* $(RX)^*$ *is an additive perturbation of* $(\mathcal{S}^{-1}X)^*$ *by vectors from* $(\operatorname{ran} T_{X^*})^\perp$ *whose collection is a Bessel system.*

Proof (i) We have $\ker T_X = \ker T_{\mathcal{S}^{-1}X}$, so in particular $\ker J_X = \ker J_{\mathcal{S}^{-1}X}$, i.e. $(\operatorname{ran} J_X^*)^\perp = (\operatorname{ran} J_{\mathcal{S}^{-1}X}^*)^\perp$ and in turn $\operatorname{ran} J_X^* = \operatorname{ran} J_{\mathcal{S}^{-1}X}^*$. Therefore, $\operatorname{ran} U^* J_{X^*} V^* = \operatorname{ran} U^* J_{(\mathcal{S}^{-1}X)^*} V^*$ with unitaries U and V and thus $\operatorname{ran} J_{X^*} = \operatorname{ran} J_{(\mathcal{S}^{-1}X)^*}$.

(ii) Suppose RX is a dual frame for X, i.e. $J_{RX} J_X^* = I$. Then $(J_{RX} - J_{\mathcal{S}^{-1}X}) J_X^* = 0$ since $J_{\mathcal{S}^{-1}X} J_X^* = I$. Taking adjoint yields

$$\operatorname{ran}\left(J_{RX}^* - J_{\mathcal{S}^{-1}X}^*\right) \subset \ker J_X = \left(\operatorname{ran} J_X^*\right)^\perp,$$

and thus

$$\operatorname{ran}\left(U^*\left(J_{(RX)^*} - J_{(\mathcal{S}^{-1}X)^*}\right)V^*\right) \subset (\operatorname{ran} U^* J_{X^*} V^*)^\perp,$$

i.e.

$$\operatorname{ran}\left(J_{(RX)^*} - J_{(\mathcal{S}^{-1}X)^*}\right) \subset (\operatorname{ran} J_{X^*})^\perp.$$

Conversely, suppose the vectors of $(RX)^*$ are perturbations of the vectors of $(\mathcal{S}^{-1}X)^*$ by certain elements of $(\operatorname{ran} T_{X^*})^\perp$ whose collection is a Bessel system. If $y \in (\operatorname{ran} T_{X^*})^\perp$, then y is orthogonal to every vector of the system X^*. Since $(\mathcal{S}^{-1}X)^*$ is biorthonormal to X^*, by the duality principle Theorem 3.5, this implies that $(RX)^*$ is biorthonormal to X^*. In turn, RX and X are dual frames. □

A characterization in a similar spirit has been shown in [19, Theorem 4.21] using Riesz-dual sequences and in [62, Lemma 7.6.1] for all dual frames of a given Gabor frame. In the finite dimensional setting a similar classification of all dual frames in terms of duality can be found in [18, Proposition 1.17, Corollary 1.9]. Another interesting characterization of all dual frames has been given in [84]. The canonical dual frame $\mathcal{S}^{-1}X$ of X can now be characterized by norm minimization properties analogue to the Gabor frame case, in which the canonical dual window is characterized among all dual windows as the one of minimal norm and as the window that among all dual windows is closest, in L_2-norm, to the primary window, see [62, Proposition 7.6.2]. The proof is precisely as for this special case (see also [19, Proposition 20]).

Proposition 3.10 *Let* X *be a frame for* H *and* $R : X \to H$ *a map such that* RX *is a dual frame of* X*. Denote* $\mathcal{S} = T_X T_X^*$*. Index all adjoints by the orthonormal basis* $\mathcal{O}$ *with respect to which the pre-Gramians* J_X *and* J_{RX} *are represented, for example* $X^* = \{X_e^*\}_{e \in \mathcal{O}}$*. Then the following are equivalent:*

(i) $RX = \mathcal{S}^{-1}X$*.*

(ii) If $R' : X \to H$ *is a map such that* $R'X$ *is a dual frame of* X*, then*

$$\|(RX)_e^*\| < \|(R'X)_e^*\|$$

whenever $(RX)^*_e \neq (R'X)^*_e$.

(iii) If $R' : X \to H$ *is a map such that* $R'X$ *is a dual frame of* X*, then*

$$\left\| \frac{(RX)^*_e}{\|(RX)^*_e\|} - \frac{X^*_e}{\|X^*_e\|} \right\| < \left\| \frac{(R'X)^*_e}{\|(R'X)^*_e\|} - \frac{X^*_e}{\|X^*_e\|} \right\|$$

whenever $(RX)^*_e \neq (R'X)^*_e$.

Proof For the first equivalence note that by Proposition 3.9 every vector of $(R'X)^*$ is the sum of two orthogonal vectors, say $(R'X)^*_e = (\mathcal{S}^{-1}X)^*_e + y_e$, and thus

$$\|(R'X)^*_e\|^2 = \|(\mathcal{S}^{-1}X)^*_e\|^2 + \|y_e\|^2 \geq \|(\mathcal{S}^{-1}X)^*_e\|^2.$$

The second equivalence can be established by noting that, by the biorthonormality of $(RX)^*$ and X^*, we have

$$\left\| \frac{(RX)^*_e}{\|(RX)^*_e\|} - \frac{X^*_e}{\|X^*_e\|} \right\|^2 = 2 - \frac{2}{\|(RX)^*_e\|\|X^*_e\|}.$$

□

With the notion of the adjoint system, the properties of the dual systems characterized by the mixed frame operator can now be phrased in terms of the mixed dual Gramian matrix. For example, finding a dual frame can be done by finding a matrix inverse.

Corollary 3.11 *Let* X *and* $Y = RX$ *be frames for* H *such that* $\overline{span}\{X^*\} = \overline{span}\{Y^*\}$*. Let* $\mathcal{O}$ *be an orthonormal basis for* H *and* U *be the synthesis operator of* $\mathcal{O}$*. Then* $\widetilde{G}_{Y,X}$ *is boundedly invertible and* $U\widetilde{G}^{-1}_{Y,X}U^*Y$ *and* X *are dual frames.*

In the following corollary the verification of the property of the mixed operators to be bounded below in Proposition 2.2 is transferred to the mixed dual Gramian matrices.

Corollary 3.12 *Let* X *and* $Y = RX$ *be Bessel systems in* H *such that* $\overline{span}\{X^*\} = \overline{span}\{Y^*\}$*. Then the following are equivalent:*

(i) X *and* Y *are frames.*
(ii) $\widetilde{G}_{X,Y}$ *and* $\widetilde{G}_{Y,X}$ *are bounded below.*

By the duality principle, two Bessel systems X and $Y = RX$ are frames if and only if X^* and Y^* are Riesz sequences. Now $\widetilde{G}_{X,Y} = G_{X^*,Y^*}$ and $\widetilde{G}_{X,Y} = G_{Y^*,X^*}$ up to unitaries. Thus, by Corollary 3.12, if $\overline{span}\{X^*\} = \overline{span}\{Y^*\}$, then X^* and Y^* are Riesz sequences if and only if G_{X^*,Y^*} and G_{Y^*,X^*} are bounded below. Replacing X^* by X and Y^* by Y yields the following. If $\overline{span}\{X\} = \overline{span}\{Y\}$, then X and Y are Riesz sequences if and only if $G_{X,Y}$ and $G_{Y,X}$ are bounded below. This is Proposition 2.3 stated in terms of Gramian matrices. In this sense Propositions 2.2 and 2.3 can be considered as one.

3.3 Duality for Filter Bank Frames

Filter banks are a central tool in signal processing, with a vast literature on their connections to Gabor systems (see e.g. [3–5,34]) and wavelet systems (see e.g. [33,35, 68,86,108,109,114]). Filter banks are the implementation of N-shift-invariant systems in $\ell_2(\mathbb{Z}^d)$ of the form

$$X = X(a, N) := \Big\{(a_l(n - kN))_{n\in\mathbb{Z}^d} : l \in \mathbb{Z}_r, k \in \mathbb{Z}^d\Big\}, \tag{3.13}$$

where $a = \{a_l\}_{l\in\mathbb{Z}_r}$ are *filters* in $\ell_2(\mathbb{Z}^d)$, $N \in \mathbb{N}$ is the *(sub)sampling rate*, $r \in \mathbb{N}$ is the number of *channels* and $\mathbb{Z}_r := \mathbb{Z}/r\mathbb{Z}$.

The analysis operator

$$T_X^* : \ell_2(\mathbb{Z}^d) \to \ell_2(\mathbb{Z}_r \times \mathbb{Z}^d) : c \mapsto \Big(\downarrow_N \ (c * \overline{a_l(-\cdot)}\Big)(k))_{(l,k)\in\mathbb{Z}_r\times\mathbb{Z}^d}$$

is composed of discrete convolutions followed by downsampling by the factor N, i.e. $\downarrow_N d(k) = d(kN)$ for $k \in \mathbb{Z}^d$. This transform is called an *analysis filter bank*. The synthesis operator

$$T_X : \ell_2(\mathbb{Z}_r \times \mathbb{Z}^d) \to \ell_2(\mathbb{Z}^d) : c \mapsto \sum_{l\in\mathbb{Z}_r} (\uparrow_N c(l, \cdot)) * a_l$$

is given by upsampling followed by discrete convolutions. Here, for fixed $l \in \mathbb{Z}_r$, $\uparrow_N c(l, k)$ is equal to $c(l, N^{-1}k)$ if N divides all entries of $k \in \mathbb{Z}^d$ and is equal to 0 otherwise. The transform given by the synthesis operator of X is called a *synthesis filter bank*. A *filter bank* consists of an analysis and synthesis filter bank with equal number of channels and the same sampling rate but, in general, with respect to different filters. If $Y = X(b, N)$ for filters $b = \{b_l\}_{l\in\mathbb{Z}_r}$ in $\ell_2(\mathbb{Z}^d)$, then the pair X and Y, or more precisely the mixed operator $T_Y T_X^*$, is called a *perfect reconstruction filter bank* whenever X and Y are dual frames in $\ell_2(\mathbb{Z}^d)$. A *tight frame filter bank* is a perfect reconstruction filter bank with coinciding analysis and synthesis filters, i.e. $X = Y$.

We now use the abstract pre-Gramian analysis to study the frame properties of filter banks. Due to the finite number of filters, X satisfies (3.5) with respect to the canonical orthonormal basis of $\ell_2(\mathbb{Z}^d)$. The corresponding pre-Gramian matrix of X is

$$J_X = (a_l(n - kN))_{n\in\mathbb{Z}^d,(l,k)\in\mathbb{Z}_r\times\mathbb{Z}^d}.$$

A crucial observation, first made in [52], is the highly regular structure of the adjoint system X^* given by the rows of J_X. Indeed,

$$X^* = \{(a_l(n))_{(l,n)\in\mathbb{Z}_r\times\Omega_j} : j \in \mathbb{Z}^d\}$$

with $\Omega_j := j + N\mathbb{Z}^d$, i.e. each element of the adjoint system is the concatenation of the filters entries indexed by the $N\mathbb{Z}^d$-coset of an index. By the duality principle, X is a Bessel system if and only if X^* is a Bessel system, and, in this case, X is a (tight) frame if and only if X^* is an (orthonormal) Riesz sequence. Moving on to the two systems case, the mixed dual Gramian matrix of X and Y is

$$\widetilde{G}_{Y,X} = J_Y J_X^* = \left(\sum_{l=0}^{r-1} \sum_{k \in \mathbb{Z}^d} \overline{a_l(n' - kN)} b_l(n - kN) \right)_{n,n' \in \mathbb{Z}^d}$$

and $\widetilde{G}_{Y,X} = G_{Y^*,X^*}$ is the duality principle. In particular, two Bessel systems X and Y are dual frames if and only if $\widetilde{G}_{Y,X} = G_{Y^*,X^*} = I$ on $\ell_2(\mathbb{Z}^d)$. As it happens, the duality principle provides a significant simplification. Due to their highly regular structure, the orthonormality of the adjoint systems is more accessible than the dual frame condition of the original systems. The fact that $\widetilde{G}_{Y,X} = I$ if and only if $G_{Y^*,X^*} = I$, is spelled out for the current situation in the following result about X and Y and their respective adjoint systems.

Theorem 3.13 *Let $X = X(a, N)$ and $Y = X(b, N)$, for filters $a = \{a_l\}_{l=0}^{r-1}$ and $b = \{b_l\}_{l=0}^{r-1}$ in $\ell_2(\mathbb{Z}^d)$ and $N \in \mathbb{N}$. Then $\widetilde{G}_{X,Y}$ is the identity if and only if*

$$\sum_{l=0}^{r-1} \sum_{n \in \Omega_j} \overline{a_l(n)} b_l(n + k) = \delta_{k,0}$$

for all $j, k \in \mathbb{Z}^d / N\mathbb{Z}^d$.

We now show how the duality principle can be used for a perfect reconstruction filter bank construction. Its generality and simplicity makes it flexible enough to be useful in design problems that require many requirements on the constructed filters. For example, in Sect. 5.2 we will refine it to meet additional constraints and to result in a simple dual MRA-wavelet frame construction. As reasonable for the design problem, we now restrict ourselves to finitely supported filters, also referred to as finite impulse response (FIR) filters. Besides making the filter bank systems automatically Bessel, all information on the systems can now be written in well structured finite matrices in terms of the filters. Indeed, Corollary 3.14 can be easily visualized in the FIR case as follows. Let

$$A = \begin{pmatrix} a_0(n_1) & \cdots & a_0(n_m) \\ \vdots & \ddots & \vdots \\ a_{r-1}(n_1) & \cdots & a_{r-1}(n_m) \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_0(n_1) & \cdots & b_0(n_m) \\ \vdots & \ddots & \vdots \\ b_{r-1}(n_1) & \cdots & b_{r-1}(n_m) \end{pmatrix},$$

where $\{n_1, \ldots, n_m\} = \mathcal{B} \cap \mathbb{Z}^d$ and $\mathcal{B}$ is some box, i.e. d-dimensional interval, containing the support of the filters $\{a_l, b_l\}_{l=0}^{r-1}$. The concatenations of the columns of A and B indexed by the same coset give precisely the different sequences of the adjoint systems X^* and Y^* and the infinite dual Gramian condition is reduced to the condition on the finite order matrix A^*B to be diagonal with diagonal entries indexed by the different cosets summing to one. The following sufficient condition is a direct consequence of the duality principle Theorem 3.13.

Corollary 3.14 *Let $X = X(a, N)$ and $Y = X(b, N)$, for FIR filters $a = \{a_l\}_{l=0}^{r-1}$ and $b = \{b_l\}_{l=0}^{r-1}$ in $\ell_2(\mathbb{Z}^d)$ and $N \in \mathbb{N}$. Then X and Y are dual frames in $\ell_2(\mathbb{Z}^d)$ provided that*

$$\sum_{l=0}^{r-1} \overline{a_l(n)} b_l(n') = 0 \quad \text{and} \quad \sum_{l=0}^{r-1} \sum_{n \in \Omega_j} \overline{a_l(n)} b_l(n) = 1$$

for all $n, n' \in \mathbb{Z}^d$ with $n \neq n'$ and all $j \in \mathbb{Z}^d / N\mathbb{Z}^d$.

This condition leads to the following result on the construction of perfect reconstruction filter banks. Given any linearly independent FIR filters defining some analysis filter bank, it provides large degrees of freedom in designing a synthesis filter bank via a matrix inversion.

Construction 3.15 *Let $A = (a_l(n_j))_{l,j \in \mathbb{Z}_r} \in \mathbb{C}^{r \times r}$ be invertible and $M \in \mathbb{C}^{r \times r}$ be a diagonal matrix with diagonal $(d(n_0), \ldots, d(n_{r-1}))$ such that*

$$\sum_{n \in \Omega_j} d(n) = 1$$

for every $j \in \mathbb{Z}^d / N\mathbb{Z}^d$. Let $B = (b_l(n_j))_{l,j \in \mathbb{Z}_r} = (A^*)^{-1} M$. Then the filters $a = \{a_l\}_{l=0}^{r-1} \subset \ell_2(\mathbb{Z}^d)$ and $b = \{b_l\}_{l=0}^{r-1} \subset \ell_2(\mathbb{Z}^d)$ defined by A and B generate dual frames $X(a, N)$ and $X(b, N)$ in $\ell_2(\mathbb{Z}^d)$.

Proof By construction, $A^* B = M$. Thus the n-th column of A is orthogonal to the m-th column of B whenever $n \neq m$. Moreover, $\sum_{l=0}^{r} \sum_{n \in \Omega_j} \overline{a_l(n)} b_l(n) = \sum_{n \in \Omega_j} d(n) = 1$ for every $j \in \mathbb{Z}^d / N\mathbb{Z}^d$. The claim therefore follows from Corollary 3.14, since the systems are Bessel due to the finite support of the filters. □

Under stronger conditions on the matrices one can derive tight frame filter banks. If in Construction 3.15 one starts out with a unitary matrix A and a diagonal matrix D which in addition has only nonnegative entries, then the entries of the matrix $AD^{1/2}$ define a tight frame filter bank.

4 Gabor Systems

The Fourier transform on $L_2(\mathbb{R}^d)$, which for $f \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ is defined by

$$\hat{f}(\omega) := \int_{\mathbb{R}^d} f(x) e^{-i\omega \cdot x} \, dx, \quad \omega \in \mathbb{R}^d,$$

reveals the frequency content of the signal f in a non-localized manner. Local changes of the signal will in general have a global effect on its Fourier transform. A way to counter this is the introduction of a compactly supported or fast decaying *window* $\phi \in L_2(\mathbb{R}^d)$, resulting in the *windowed (or short-time) Fourier transform*

$$V_\phi f(\omega, t) := \langle f, M^\omega E^t \phi \rangle = \int_{\mathbb{R}^d} f(x) e^{-i\omega \cdot x} \overline{\phi(x-t)}\, dx, \quad (\omega, t) \in \mathbb{R}^{2d}. \quad (4.14)$$

Here, E^t denotes the translation and M^t the modulation operator on $L_2(\mathbb{R}^d)$, i.e. $E^t f = f(\cdot - t)$ and $M^t f = e_t f$, where $e_t \colon x \mapsto e^{it\cdot x}$ and $x, t \in \mathbb{R}^d$. The goal to analyze and/or numerically stable reconstruct a signal from discrete samples of its continuous time-frequency representation (4.14) leads to considering the properties of the *irregular Gabor system (or Weyl-Heisenberg system)*

$$X = \{E^\gamma M^\eta \phi \colon (\gamma, \eta) \in \Lambda\},$$

where $\Lambda \subset \mathbb{R}^{2d}$ is some discrete set. In addition to a good localization of the window, a good simultaneous frequency localization of the window is important. This makes windows desirable, which in addition are smooth, i.e. have fast decaying Fourier transform. Here the Balian-Low theorem, see e.g. [36,55,62,63], sets some theoretical boundaries. If, say, the shifts and modulations are sampled from lattices, then there do not exist windows with good joint time and frequency localization that generate orthonormal bases. However, there exist windows with excellent time-frequency localization, that generate frames and even tight frames, thus ensuring numerically stable and even perfect reconstruction. This makes Gabor systems an example of systems for which it becomes imperative to oversample, i.e. to move beyond orthonormal bases into the realm of frames.

We use the structure of irregular Gabor systems to analyze them via the abstract dual Gramian and characterize Bessel, frame and Riesz properties by the duality principle after introducing adjoint systems. While there exists a vast literature on irregular Gabor systems, to this point there has not been any duality principle. Most results concern perturbation and density theorems for the sampling set, see e.g. [26,53,54,61,80,105, 111].

The irregular Gabor system contains the time-frequency shifts of the window, as modulations become shifts under the Fourier transform. Ideally those time and frequency shifts are to be treated equally, taking samples in some regular fashion, say, for $\Lambda = K \times L$ where $K, L \subset \mathbb{R}^d$ are lattices. Such (regular) Gabor systems are special instances of shift-invariant systems and we show how the Fourier transform of the abstract pre-Gramian, for a particular choice of basis, decomposes into the fiber pre-Gramian matrices of [98,102], allowing a representation of the synthesis operator by a family of simple structured infinite matrices composed of the Fourier transform of the generator of the system. This so called fiberization technique has been used to characterize Bessel, frame and Riesz properties of shift-invariant and Gabor systems, and to estimate the corresponding bounds, again see [98,102]. The fiber dual Gramian analysis, by which these results are achieved, is in the same spirit as its abstract counterpart described above and in [52]. While in the abstract setting properties of an operator are transferred to properties of its representing matrix, fiberization refers to transferring the properties of the operators to analogous properties of a whole family of (simpler structured) matrices, the so called fiber matrices, for which those properties have to hold, often in a uniform way. Emphasis in [102] is on the fiber dual Gramian analysis of single systems, e.g. on characterizing tight frames. Here we present a unified fiber mixed dual Gramian analysis of dual Gabor systems, and show how the classical identies of Gabor analysis can be seen as consequences of the fiber dual Gramian analysis and its duality principle. Duality results have been shown for the case that Λ is a lattice in [56,58], using techniques different from matrix representations.

The orthonormal basis used to study irregular Gabor systems in the next subsection is composed by shifts of the local Fourier orthonormal basis and is thus itself a regular Gabor system. Before proceeding, we recall the relevant notation and facts about lattices needed for the treatment of regular Gabor systems in the remainder of this section. Let $K \subset \mathbb{R}^d$ be a lattice, i.e. the image of $\mathbb{Z}^d$ under some invertible linear map $A_K : \mathbb{R}^d \to \mathbb{R}^d$. The *volume* of K is $|K| := |\det A_K|$ and its *dual lattice* is $\widetilde{K} := \{\tilde{k} \in \mathbb{R}^d : \tilde{k} \cdot k \in 2\pi\mathbb{Z}, \forall k \in K\}$, which implies $|K||\widetilde{K}| = (2\pi)^d$. A motivation of this definition is that $e^{ik\cdot\tilde{k}} = 1$ whenever $k \in K, \tilde{k} \in \tilde{K}$. Such complex exponential factors appear, say, in the commutator relations of the translation and modulation operators. The dual lattice allows the matrix representation for shift-invariant systems ((4.23) below). For further motivation of the dual lattice see e.g. [56,57]. By Ω_K we denote a subset of $\mathbb{R}^d$, whose K-shifts essentially partition $\mathbb{R}^d$. The Lebesgue measure of Ω_K is $|K|$. The *density* of a pair (K, L) of lattices in $\mathbb{R}^d$ is $\mathrm{den}\,(K, L) := \frac{(2\pi)^d}{|K||L|}$ and the *adjoint* of (K, L) is $(\widetilde{L}, \widetilde{K})$. Then $\mathrm{den}\,(K, L)\mathrm{den}\,(\widetilde{L}, \widetilde{K}) = 1$.

4.1 Irregular Gabor Systems

We discuss the duality principle for irregular Gabor systems with no assumptions on the structure of the sampling set. The abstract pre-Gramian matrix (3.4) can be defined for any system. It may be simpler in case the system exhibits a structure that can be exploited by choosing an appropriate orthonormal basis. Given an irregular Gabor system

$$X = \{E^\gamma M^\eta \phi \colon (\gamma, \eta) \in \Lambda\},$$

generated by $\phi \in L_2(\mathbb{R}^d)$ through some unstructured countable set $\Lambda \subset \mathbb{R}^{2d}$, one might draw on the shift-modulation structure of X by choosing the orthonormal basis

$$\left\{|\widetilde{K}|^{-1/2} M^k E^{\tilde{k}} \chi_{\Omega_{\widetilde{K}}} : k \in K, \tilde{k} \in \widetilde{K}\right\} \tag{4.15}$$

for some lattice $K \subset \mathbb{R}^d$. The resulting pre-Gramian

$$J_X = |\widetilde{K}|^{-1/2} \Big(e^{i\eta\cdot(\tilde{k}-\gamma)} \Big\langle M^{\eta-k} E^{\gamma-\tilde{k}} \phi, \chi_{\Omega_{\widetilde{K}}} \Big\rangle\Big)_{(k,\tilde{k})\in K\times\widetilde{K},(\gamma,\eta)\in\Lambda},$$

though complicating at first glance, possesses a lot of structure. For example, for fixed $\tilde{k}$ and γ, the submatrix indexed by (k, η) analyzes the frequency components of

$E^{\gamma-\tilde{k}}\phi$ on $\Omega_{\widetilde{K}}$. Moreover, if ϕ has, say, compact support, then J_X is a sparse matrix, composed of infinitely many infinite band-matrix blocks. In case the rows of J_X are square summable, i.e. if

$$\sum_{(\gamma,\eta)\in\Lambda} \left|\left\langle M^{\eta-k}E^{\gamma-\tilde{k}}\phi, \chi_{\Omega_{\widetilde{K}}}\right\rangle\right|^2 < \infty \quad \text{for all} \quad k \in K, \tilde{k} \in \widetilde{K} \tag{4.16}$$

which for example is the case whenever X is Bessel, the entries of the dual Gramian

$$\begin{aligned}\widetilde{G}_{X,X} &= J_X J_X^* \\ &= |\widetilde{K}|^{-1}\left(\sum_{(\gamma,\eta)\in\Lambda}\left\langle M^{\eta-k}E^{\gamma-\tilde{k}}\phi, \chi_{\Omega_{\widetilde{K}}}\right\rangle\left\langle \chi_{\Omega_{\widetilde{K}}}, M^{\eta-k'}E^{\gamma-\tilde{k}'}\phi\right\rangle\right)_{(k,\tilde{k}),(k',\tilde{k}')\in K\times\widetilde{K}}\end{aligned}$$

of X are well-defined and the rows of J_X define a system

$$X^* = \left\{\left(|\widetilde{K}|^{-1/2}e^{i\eta\cdot(\tilde{k}-\gamma)}\left\langle M^{\eta-k}E^{\gamma-\tilde{k}}\phi, \chi_{\Omega_{\widetilde{K}}}\right\rangle\right)_{(\gamma,\eta)\in\Lambda} : k \in K, \tilde{k} \in \widetilde{K}\right\} \tag{4.17}$$

in $\ell_2(\Lambda)$. Under the condition that also the columns of J_X are square summable, i.e. that

$$\sum_{(k,\tilde{k})\in K\times\widetilde{K}} \left|\left\langle M^{\eta-k}E^{\gamma-\tilde{k}}\phi, \chi_{\Omega_{\widetilde{K}}}\right\rangle\right|^2 < \infty \quad \text{for all} \quad (\gamma,\eta) \in \Lambda, \tag{4.18}$$

as seen in the previous section, the systems X and X^* are adjoint systems of each other and the dual Gramian of X is equal to the Gramian of X^*, i.e.

$$\widetilde{G}_{X,X} = G_{X^*,X^*}, \tag{4.19}$$

and vice versa. Before we formally state the consequences of the duality principle for irregular Gabor systems following from this identity, note that, in accordance with Definition 3.4, any image of X^* under a unitary map into a separable Hilbert space is as well an adjoint system of X. To get an adjoint system in $L_2(\mathbb{R}^d)$, one may for example use the orthonormal basis (4.15) to unitarily map X^* into $L_2(\mathbb{R}^d)$. That is, identifying Λ with $K \times \widetilde{K}$ via a bijection $\Lambda \to K \times \widetilde{K}$: $(\gamma, \eta) \mapsto (R\gamma, R\eta)$, one may consider the adjoint system $\{f_{k,\tilde{k}}\}_{(k,\tilde{k})\in K\times\widetilde{K}} \subset L_2(\mathbb{R}^d)$, where

$$f_{k,\tilde{k}} := |\widetilde{K}|^{-1}\sum_{(\gamma,\eta)\in\Lambda} e^{i\eta\cdot(\tilde{k}-\gamma)}\left\langle M^{\eta-k}E^{\gamma-\tilde{k}}\phi, \chi_{\Omega_{\widetilde{K}}}\right\rangle M^{R\gamma}E^{R\eta}\chi_{\Omega_{\widetilde{K}}}. \tag{4.20}$$

In other words, the orthonormal basis (4.15) is used for reconstruction of the coefficient sequence (4.17). Using this orthonormal basis, it is in fact easy to see that the pre-Gramian matrix of (4.20) is the same as that of (4.17), namely the adjoint of J_X. One may of course choose other orthonormal bases, e.g. wavelet bases or Wilson bases (see

[42]) for better time-frequency localization, to get a different structured adjoint system. Nevertheless, the duality between the two systems X and X^* in its core happens on the level of sequences. The system X is being considered in $\ell_2(K \times \widetilde{K})$ via its coefficient sequences with respect to the chosen orthonormal basis. Its adjoint system X^* is being considered in $\ell_2(\Lambda)$, i.e. as a system of sequences indexed by the original system X itself. This picture is obvious considering the pre-Gramian matrix. The duality principle (4.19) has the following consequence for irregular Gabor systems.

Theorem 4.1 *Let $X = \{E^\gamma M^\eta \phi\colon (\gamma, \eta) \in \Lambda\}$ be an irregular Gabor system in $L_2(\mathbb{R}^d)$ and $K \subset \mathbb{R}^d$ be a lattice such that (4.16) and (4.18) hold. Further, let X^* be the $\ell_2(\Lambda)$-system (4.17) or its image under some unitary map in some Hilbert space. Then:*

(i) The system X is a Bessel system if and only if X^* is a Bessel system, in which case the Bessel bounds coincide.
(ii) If X is Bessel, then it is fundamental if and only if X^* is ℓ_2-independent.
(iii) The system X is a frame if and only if X^* is a Riesz sequence, in which case the frame bounds and Riesz bounds coincide. In particular, X is a tight frame if and only if X^* is an orthonormal sequence.

Given a second window, i.e. a second system $Y = \{E^\gamma M^\eta \psi\colon (\gamma, \eta) \in \Lambda\}$, and considering J_Y and Y^* with respect to the same orthonormal basis (4.15) (and under the same potential unitary map as for X^*), the essence of the duality principle Theorem 3.5 is

$$\widetilde{G}_{X,Y} = G_{X^*,Y^*} \tag{4.21}$$

provided the rows and columns of J_X and J_Y are square summable. The square summability is guaranteed by the stronger assumption on X and Y to be Bessel and the duality principle yields the following characterization for two irregular Gabor systems to be dual frames.

Theorem 4.2 *Suppose $X = \{E^\gamma M^\eta \phi\colon (\gamma, \eta) \in \Lambda\}$ and $Y = \{E^\gamma M^\eta \psi\colon (\gamma, \eta) \in \Lambda\}$ are Bessel systems in $L_2(\mathbb{R}^d)$. Let X^* and Y^* be their respective adjoint systems with respect to (4.15) (see (4.17)) or their images under the same unitary map (see (4.20)). Then X and Y are dual frames if and only if X^* and Y^* are biorthonormal.*

The duality principle is stated for the particular orthonormal basis (4.15). While any other orthonormal basis is possible, we will see in the following subsection how in case the shifts and modulations of the Gabor system are sampled from lattices, the abstract pre-Gramian of essentially this orthonormal basis becomes significantly simpler and can be expressed through the fiber matrices of shift-invariant systems. Moreover, this allows for adjoint systems, which not only have a closed form expression in $L_2(\mathbb{R}^d)$, but have Gabor structure closely tied to that of the original system. Gabor structured systems induced by an *adjoint* lattice have also been constructed in the case that shifts and modulations of the primary Gabor system are sampled from a joint (non-separable) lattice, see [56,58]. There the dualities between the two systems are shown without reference to matrix representations.

4.2 Gabor Systems and Fiberization

If the Gabor system is generated by regular lattices, the pre-Gramian matrix, for a suitable basis, exhibits a strong blockwise structure and can be simplified to the fiber pre-Gramian matrices of shift-invariant systems introduced in [98].

Given $\phi \in L_2(\mathbb{R}^d)$ and two lattices $K, L \subset \mathbb{R}^d$, the set

$$X = (K, L)_\phi := \{E^k M^l \phi \colon k \in K, l \in L\}$$

is called the *(regular) Gabor system* generated by ϕ. The system X is K-shift-invariant, being the collection of all K-shifts of the set $\{M^l \phi : l \in L\}$. Considering the structure of X, choose

$$\left\{((2\pi)^d |\widetilde{K}|)^{-1/2} E^k M^{-\tilde{k}} \widehat{\chi}_{\Omega_{\widetilde{K}}}(-\cdot) : k \in K, \tilde{k} \in \widetilde{K}\right\}$$

as the orthonormal basis for the abstract pre-Gramian matrix of X, which essentially is the Fourier transform of the orthonormal basis (4.15). Then

$$\begin{aligned} J_X &= ((2\pi)^d |\widetilde{K}|)^{-1/2} \left(\left\langle E^{k'} M^l \phi, E^k M^{-\tilde{k}} \widehat{\chi}_{\Omega_{\widetilde{K}}}(-\cdot)\right\rangle\right)_{(k,\tilde{k}) \in K \times \widetilde{K}, (k',l) \in K \times L} \\ &= ((2\pi)^d |\widetilde{K}|)^{-1/2} \left(\left\langle E^{\tilde{k}+l} \hat{\phi}, M^{k'-k} \chi_{\Omega_{\widetilde{K}}}\right\rangle\right)_{(k,\tilde{k}) \in K \times \widetilde{K}, (k',l) \in K \times L}. \end{aligned}$$

For $\tilde{k} \in \widetilde{K}$ and $l \in L$ fixed, J_X consists of repeated blocks of the Fourier sequence of $\hat{\phi}(\cdot - \tilde{k} - l)$ on $\Omega_{\widetilde{K}}$. The abstract pre-Gramian is therefore linked to the *fiber pre-Gramian matrices* of the Gabor system X, which have been introduced in [102] as the infinite matrices

$$\mathcal{J}_X(\omega) = |K|^{-1/2} \left(\hat{\phi}(\omega - \tilde{k} - l)\right)_{\tilde{k} \in \widetilde{K}, l \in L} \tag{4.22}$$

indexed by $\omega \in \mathbb{R}^d$. To formulate the connection, recall that the Fourier transform of $c \in \ell_2(K \times L)$ is defined as $\hat{c} := (\widehat{c_l})_{l \in L}$, where $\widehat{c_l} := \sum_{k \in K} c_l[k] e_{-k}$ is the Fourier series of the restriction c_l of c to $K \times \{l\}$.

Theorem 4.3 *Let $K, L \subset \mathbb{R}^d$ be lattices. Let $X = (K, L)_\phi$ and J_X be the pre-Gramian with respect to $\{((2\pi)^d |\widetilde{K}|)^{-1/2} E^k M^{-\tilde{k}} \widehat{\chi}_{\Omega_{\widetilde{K}}}(-\cdot) : k \in K, \tilde{k} \in \widetilde{K}\}$. Then*

$$(J_X c)^\wedge(\omega) = \mathcal{J}_X(\omega) \hat{c}(\omega)$$

for every $c \in \ell_0(X) = \ell_0(K \times L)$ and a.e. $\omega \in \Omega_{\widetilde{K}}$.

Proof Let $c \in \ell_0(X)$. For a.e. $\omega \in \Omega_{\widetilde{K}}$ we have

$$\mathcal{J}_X(\omega) \widehat{c}(\omega) = |K|^{-1/2} \left(\sum_{l \in L} \hat{\phi}(\omega - \tilde{k} - l) \widehat{c_l}(\omega)\right)_{\tilde{k} \in \widetilde{K}}$$

$$= |K|^{-1/2}|\widetilde{K}|^{-1}\left(\sum_{l\in L}\sum_{k'\in K} c_l[k']e^{-ik'\cdot\omega}\sum_{k\in K}\left\langle E^{\tilde{k}+l}\hat{\phi}, M^k\chi_{\Omega_{\widetilde{K}}}\right\rangle e^{ik\cdot\omega}\right)_{\tilde{k}\in\widetilde{K}}$$
$$= ((2\pi)^d|\widetilde{K}|)^{-1/2}\left(\sum_{k\in K}\sum_{l\in L}\sum_{k'\in K} c_l[k']\left\langle E^{\tilde{k}+l}\hat{\phi}, M^{k'-k}\chi_{\Omega_{\widetilde{K}}}\right\rangle e^{-ik\cdot\omega}\right)_{\tilde{k}\in\widetilde{K}}.$$

The last term is the Fourier transform of the $\ell_2(K\times\widetilde{K})$-sequence $J_X c$. □

The fiber pre-Gramian matrices $\mathcal{J}_X(\omega)$ have first been introduced in [98], where they are used to study shift-invariant systems, i.e. systems of the form $X=\{E^k\phi\colon k\in K, \phi\in\Phi\}$ for some system Φ in $L_2(\mathbb{R}^d)$ and some lattice $K\subset\mathbb{R}^d$, through the representation

$$((T_X c)^\wedge(\omega-\tilde{k}))_{\tilde{k}\in\widetilde{K}} = |K|^{1/2}\mathcal{J}_X(\omega)\hat{c}(\omega), \tag{4.23}$$

which holds for all $c\in\ell_0(X)$ and a.e. $\omega\in\Omega_{\widetilde{K}}$ and where

$$\mathcal{J}_X(\omega) = |K|^{-1/2}(\hat{\phi}(\omega-\tilde{k}))_{\tilde{k}\in\widetilde{K},\phi\in\Phi} \tag{4.24}$$

is the fiber pre-Gramian matrix of the shift-invariant system. Here, as before for X, we use Φ as system and index set, i.e. if a function appears twice in Φ, for purposes of indexing $\mathcal{J}_X(\omega)$ those are considered as different indices. Regular Gabor systems are the special case of Φ being modulations of the window $\phi\in L_2(\mathbb{R}^d)$ and Theorem 4.3 holds for the general case of shift-invariant systems with the obvious modifications. By (4.23), the Fourier transform of $T_X c$ at ω is a.e. equal to the 0th entry of $\mathcal{J}_X(\omega)\hat{c}(\omega)$ and questions about the operator T_X are transfered to questions about a continuum of simple structured matrices. This starting point was used for the analysis of shift-invariant systems in [98], Gabor systems in [102], affine systems in [100,101] and generalized shift-invariant systems in [103]. The analysis of Gabor systems in [102] mainly focuses on single Gabor systems X. The properties of $T_X T_X^*$ are transfered to the family of simpler matrices $\widetilde{\mathcal{G}}_{X,X}(\omega)=\mathcal{J}_X(\omega)\mathcal{J}_X^*(\omega)$, the fiber dual Gramians, for which the properties have to hold in a uniform way in $\omega\in\mathbb{R}^d$. Here $\mathcal{J}_X^*(\omega)$ denotes the matrix adjoint of each fiber matrix. Precisely, with $\delta(\omega):=\|\widetilde{\mathcal{G}}_{X,X}(\omega)\|$ and $\delta^-(\omega):=\|\widetilde{\mathcal{G}}_{X,X}(\omega)^{-1}\|$ for $\omega\in\mathbb{R}^d$ (where $\delta^-(\omega)=\infty$ if $\widetilde{\mathcal{G}}_{X,X}(\omega)$ is not invertible), X is a Bessel system if and only if $\delta\in L_\infty$. The Bessel bound then is $\|\delta\|_{L_\infty}^{1/2}$. If X is a Bessel system, it is a frame if and only if $\delta^-\in L_\infty$. The lower frame bound then is $\|\delta^-\|_{L_\infty}^{-1/2}$. In particular, X is a tight frame if and only if $\widetilde{\mathcal{G}}_{X,X}(\omega)$ is the identity for a.e. $\omega\in\mathbb{R}^d$. In the next subsections we present the mixed dual Gramian analysis for dual Gabor systems.

4.3 Mixed Dual Gramian Analysis

Given Gabor systems $X=(K,L)_\phi$ and $Y=(K,L)_\psi$, the pre-Gramian fibers $\mathcal{J}_X(\omega)$ can be used to represent the synthesis operator T_X, while their adjoint matrices, denoted by $\mathcal{J}_X^*(\omega)$, can be used to represent the analysis operator T_X^*, see [98,102]. Consequently the mixed operator $T_Y T_X^*$ can be represented using $\mathcal{J}_Y(\omega)\mathcal{J}_X^*(\omega)$. Precisely (see [102] for further details), if $\phi, \psi \in L_2(\mathbb{R}^d)$ are such that $X = (K, L)_\phi$ and $Y = (K, L)_\psi$ are Bessel systems, the *fiber mixed dual Gramian matrices* are

$$\widetilde{\mathcal{G}}_{Y,X}(\omega) := \mathcal{J}_Y(\omega)\mathcal{J}_X^*(\omega) = |K|^{-1}\left(\sum_{l\in L}\hat{\psi}(\omega-\tilde{k}-l)\overline{\hat{\phi}(\omega-\tilde{k}'-l)}\right)_{\tilde{k},\tilde{k}'\in\widetilde{K}}, \tag{4.25}$$

where $\omega \in \mathbb{R}^d$. Then

$$((T_Y T_X^* f)^\wedge(\omega-\tilde{k}))_{\tilde{k}\in\widetilde{K}} = \widetilde{\mathcal{G}}_{Y,X}(\omega)(\hat{f}(\omega-\tilde{k}))_{\tilde{k}\in\widetilde{K}} \tag{4.26}$$

for all $f \in L_2(\mathbb{R}^d)$ and a.e. $\omega \in \mathbb{R}^d$. Note that for any $\phi, \psi \in L_2(\mathbb{R}^d)$ the entries of the mixed dual Gramians are locally integrable, and therefore a.e. finite, regardless of the Bessel assumption on X and Y. This assumption, however, ensures that the matrix product $\widetilde{\mathcal{G}}_{Y,X}(\omega)(\hat{f}(\omega-\tilde{k}))_{\tilde{k}\in\widetilde{K}}$ for any $f \in L_2(\mathbb{R}^d)$ has a.e. finite values, since the relevant series a.e. converge absolutely.

The image of a Gabor system under the Fourier transform is again a Gabor system, with the roles of the shift and modulation lattices interchanged. Thus the Bessel, Riesz, frame and dual frame properties can equivalently be studied through the Fourier transform counterparts $\widehat{X} = (L, K)_{\widehat{\phi}}$ and $\widehat{Y} = (L, K)_{\widehat{\psi}}$ of the Bessel systems X and Y. Their *fiber mixed dual Gramian matrices* are

$$\begin{aligned}\widetilde{\mathcal{G}}_{\widehat{Y},\widehat{X}}(\omega) &= \mathcal{J}_{\widehat{Y}}(\omega)\mathcal{J}_{\widehat{X}}^*(\omega)\\ &= (2\pi)^{2d}|L|^{-1}\left(\sum_{k\in K}\psi(-\omega+\tilde{l}+k)\overline{\phi(-\omega+\tilde{l}'+k)}\right)_{\tilde{l},\tilde{l}'\in\widetilde{L}}\end{aligned} \tag{4.27}$$

and

$$((T_{\widehat{Y}} T_{\widehat{X}}^* \hat{f})^\wedge(\omega-\tilde{l}))_{\tilde{l}\in\widetilde{L}} = (2\pi)^d \widetilde{\mathcal{G}}_{\widehat{Y},\widehat{X}}(\omega)(f(-\omega+\tilde{l}))_{\tilde{l}\in\widetilde{L}}$$

holds for all $f \in L_2(\mathbb{R}^d)$ and a.e. $\omega \in \mathbb{R}^d$. On the other hand

$$\begin{aligned}((T_{\widehat{Y}} T_{\widehat{X}}^* \hat{f})^\wedge(\omega-\tilde{l}))_{\tilde{l}\in\widetilde{L}} &= (2\pi)^d((T_Y T_X^* f)^{\wedge\wedge}(\omega-\tilde{l}))_{\tilde{l}\in\widetilde{L}}\\ &= (2\pi)^{2d}((T_Y T_X^* f)(-\omega+\tilde{l}))_{\tilde{l}\in\widetilde{L}}\end{aligned}$$

for all $f \in L_2(\mathbb{R}^d)$ and a.e. $\omega \in \mathbb{R}^d$ and therefore

$$((T_Y T_X^* f)(-\omega+\tilde{l}))_{\tilde{l}\in\widetilde{L}} = (2\pi)^{-d}\widetilde{\mathcal{G}}_{\widehat{Y},\widehat{X}}(\omega)(f(-\omega+\tilde{l}))_{\tilde{l}\in\widetilde{L}} \tag{4.28}$$

for all $f \in L_2(\mathbb{R}^d)$ and a.e. $\omega \in \mathbb{R}^d$. We refer to (4.26) and (4.28) as the representation of the Gabor frame operator in Fourier and time domain, respectively. Evaluating (4.26) at $\tilde{k} = 0$ and (4.28) at $\tilde{l} = 0$, yields two different representations of the frame operator in both Fourier and time domain.

Switching the order of multiplication of the fiber pre-Gramian matrices leads to the *fiber mixed Gramian matrices*

$$\mathcal{G}_{Y,X}(\omega) := \mathcal{J}_Y^*(\omega)\mathcal{J}_X(\omega) = |K|^{-1}\left(\sum_{\tilde{k}\in\widetilde{K}} \overline{\hat{\psi}(\omega - \tilde{k} - l)}\hat{\phi}(\omega - \tilde{k} - l')\right)_{l,l'\in L} \tag{4.29}$$

and

$$\begin{aligned}\mathcal{G}_{\widehat{Y},\widehat{X}}(\omega) &= \mathcal{J}_{\widehat{Y}}^*(\omega)\mathcal{J}_{\widehat{X}}(\omega)\\ &= (2\pi)^{2d}|L|^{-1}\left(\sum_{\tilde{l}\in\widetilde{L}} \overline{\psi(-\omega + k + \tilde{l})}\phi(-\omega + k' + \tilde{l})\right)_{k,k'\in K}\end{aligned} \tag{4.30}$$

for $\omega \in \mathbb{R}^d$, which can be used for fiberized representations of the mixed operator $T_Y^* T_X$. If X, Y are Bessel and $c \in \ell_2(K \times L)$, then

$$(T_Y^* T_X c)^\wedge(\omega) = \mathcal{G}_{Y,X}(\omega)\hat{c}(\omega) \tag{4.31}$$

for a.e. $\omega \in \mathbb{R}^d$. (For the definition of the Fourier transform of c, see the remark preceding Theorem 4.3.)

The matrices $\mathcal{G}_{Y,X}(\omega)$ and $\widetilde{\mathcal{G}}_{Y,X}(\omega)$ only differ in switching the lattices from (K, L) to $(\widetilde{L}, \widetilde{K})$. As a result, the mixed dual Gramian matrices of the original systems now become the mixed Gramian matrices of two new systems with lattices $(\widetilde{L}, \widetilde{K})$. This is essentially the duality principle. The new systems with lattices $(\widetilde{L}, \widetilde{K})$ are also the adjoint systems defined in [102]. We review those fiber adjoint systems in the following subsection.

4.4 Adjoint System and Duality Principle

Definition 3.4 for adjoint systems is most general in the sense that one can always consider the rows of the pre-Gramian of the original system as a new system and draw to the duality between the two systems whenever those rows and columns are square summable. In general, as observed for shift-invariant (in particular Gabor) systems in [52] and above, it might however not be possible to express the adjoint system explicitly as a system of the same structure. Similar observations apply for the fiber representations. Denoting the columns of the fiber pre-Gramian matrix $\mathcal{J}_X(\omega)$ of a shift-invariant system X as a system X_ω at each $\omega \in \Omega_{\widetilde{K}}$, the fiberization technique transfers the properties of X to properties of the collection of systems $\{X_\omega\}_{\omega\in\Omega_{\widetilde{K}}}$, see [98]. On each fiber the adjoint system of X_ω can be introduced as the collection of rows of $\mathcal{J}_X(\omega)$ and the duality principle can be discussed. In particular, the collection of all adjoint systems of the systems $\{X_\omega\}_{\omega\in\Omega_{\widetilde{K}}}$ can be considered as the adjoint system of the given shift-invariant system. In case of a given Gabor shift-invariant system X it is possible, as we will review below, to find another Gabor system Y, such that all of its fiber systems Y_ω are adjoint systems of the respective fiber systems X_ω. In other words, $\mathcal{J}_X(\omega) = \mathcal{J}_Y^*(\omega)$ on each fiber (modulo a complex conjugation). This may not be possible for a general shift-invariant system. However, the fiber systems and the corresponding duality principle discussed on each fiber is still of some interest. This weak formulation of the duality principle is very convenient for the construction of dual Gabor windows detailed in the Sect. 4.6. It also is the foundation for the dual wavelet frame construction through the MEP, see (5.48) below.

As discussed in the last subsection, by changing the lattices from (K, L) to $(\widetilde{L}, \widetilde{K})$ the fiber dual Gramian matrices of the original systems become the Gramian matrices of the new systems. With this in mind, the *adjoint system* of the Gabor system $X = (K, L)_\phi$ has been defined in [102] (up to the scalar factor) as

$$X^* = (\mathrm{den}\,(K, L))^{1/2}(\widetilde{L}, \widetilde{K})_\phi. \tag{4.32}$$

The $(l, \tilde{k}) \in (L, \widetilde{K})$ entry of $\mathcal{J}_{X^*}(\omega)$ is $|K|^{-1/2}\hat{\phi}(\omega - l - \tilde{k})$. Inspection of matrix entries directly yields

$$\mathcal{J}_{X^*}(\omega) = \overline{\mathcal{J}_X^*(\omega)} \tag{4.33}$$

for a.e. $\omega \in \mathbb{R}^d$. On each fiber level, this is the analogy to the abstract Definition 3.4. Given another Gabor systems $Y = (K, L)_\psi$, one observes

$$\widetilde{\mathcal{G}}_{X,Y}(\omega) = \mathcal{J}_X(\omega)\mathcal{J}_Y^*(\omega) = \overline{\mathcal{J}_{X^*}(\omega)\mathcal{J}_{Y^*}(\omega)} = \overline{\mathcal{G}_{X^*,Y^*}(\omega)}$$

for a.e. $\omega \in \mathbb{R}^d$; the duality principle for Gabor systems.

Theorem 4.4 *Let $X = (K, L)_\phi$ and $Y = (K, L)_\psi$ be Bessel systems and X^*, Y^* their respective adjoint systems defined in (4.32). Then $\widetilde{\mathcal{G}}_{X,Y}(\omega) = \overline{\mathcal{G}_{X^*,Y^*}(\omega)}$ for a.e. $\omega \in \mathbb{R}^d$. Consequently, X and Y are dual frames if and only if X^* and Y^* are biorthonormal.*

The roles of X and X^* are interchangeable since $X^{**} = X$. If $X = Y$, the duality principle says that the dual Gramian matrices of X are (up to complex conjugation) equal to the Gramian matrices of its adjoint system X^*. This is the essence of the duality principle derived in [102] for single systems. In the case of single systems the (dual) Gramian matrices are Hermitian and, by considering quadratic forms, the boundedness of the relevant operators, i.e. the Bessel property, can also be characterized through the fiber (dual) Gramian matrices. The duality principle for single systems therefore implies (see [102, Theorem 2.2]): The system X is Bessel if and only if X^* is Bessel, in which case the Bessel bounds coincide; If X is Bessel, then it is fundamental if and only if X^* is ℓ_2-independent; The system X is a frame if and only if X^* is a Riesz sequence, in which case the frame bounds of X coincide with the Riesz bounds of X^*. In particular, X is a tight frame if and only if X^* is an orthonormal sequence.

One can consider a limiting case of the duality principle. Loosely speaking, if one lattice is sampled with infinite density, i.e. zero volume, its dual lattice has density zero. This is the case for the translation-invariant Gabor transform and the duality principle can still be observed. The transform has a shift-invariant system as adjoint system. No modulations appear in the adjoint system, as its modulation lattice is of zero density, see Sect. 6.2.

In the abstract setting a simple matrix relation revealed the relationship Theorem 3.8 between the mixed frame operator of systems and their adjoints. For three Gabor Bessel systems $X=(K,L)_\phi$, $Y=(K,L)_\psi$ and $Z=(K,L)_g$, both $X'=T_Y T_Z^* X$ and $Y'=T_{X^*}T_{Z^*}^* Y^*$ are Gabor systems with windows $T_Y T_Z^*\phi$ and $(\operatorname{den}(K,L))^{1/2}T_{X^*}T_{Z^*}^*\psi$, respectively. As in Theorem 3.8, the system X' is an adjoint system of Y'.

Proposition 4.5 *Suppose $X=(K,L)_\phi$, $Y=(K,L)_\psi$ and $Z=(K,L)_g$ are Bessel systems with adjoint systems X^*, Y^*, Z^* defined in (4.32). Then $T_Y T_Z^* X$ is an adjoint system of $T_{X^*}T_{Z^*}^* Y^*$. In particular, the windows of the two systems coincide, i.e.*

$$T_Y T_Z^*\phi = T_{X^*}T_{Z^*}^*\psi. \tag{4.34}$$

As observed in [102], (4.34) follows from the matrix identity

$$\mathcal{J}_Y(\omega)\mathcal{J}_Z^*(\omega)\mathcal{J}_X(\omega) = (\mathcal{J}_X^*(\omega)\mathcal{J}_Z(\omega)\mathcal{J}_Y^*(\omega))^* = (\overline{\mathcal{J}_{X^*}(\omega)\mathcal{J}_{Z^*}^*(\omega)\mathcal{J}_{Y^*}(\omega)})^*$$

for $\omega\in\mathbb{R}^d$. By (4.26) the $(0,0)$th entry of $\mathcal{J}_Y(\omega)\mathcal{J}_Z^*(\omega)\mathcal{J}_X(\omega)$ is equal to $(T_Y T_Z^*\phi)^\wedge(\omega)$ for a.e. ω. On the other hand, the $(0,0)$th entry of $(\overline{\mathcal{J}_{X^*}(\omega)\mathcal{J}_{Z^*}^*(\omega)\mathcal{J}_{Y^*}(\omega)})^*$ and of $\mathcal{J}_{X^*}(\omega)\mathcal{J}_{Z^*}^*(\omega)\mathcal{J}_{Y^*}(\omega)$ are equal and the latter is equal to $(T_{X^*}T_{Z^*}^*\psi)^\wedge(\omega)$ for a.e. ω.

Remark In [19], Casazza et al. introduce the Riesz-dual sequence (see Example 3.7 above) as an adjoint system of a given system. They then ask the question whether one can always choose orthonormal bases, with respect to which the Riesz-dual sequence of a Gabor system $(K,L)_\phi$ coincides with the adjoint system $(\widetilde{L},\widetilde{K})_\phi$. This question does not necessarily arise if one considers the column–row relationship of any matrix representation of the synthesis operator as underlying principle of the adjoint relationship. A system and its Riesz-dual sequence are adjoint via the abstract pre-Gramian representation, while $(K,L)_\phi$ and $(\widetilde{L},\widetilde{K})_\phi$ are adjoint via the fiber pre-Gramian representation. Different matrix representations can be particularly taylored to make the best use of a specific given setting and structure of the system. The above question is then asking whether one can realize a certain adjoint relationship via a specific different matrix representation. Concerning this problem, note that adjoint systems defined through the abstract pre-Gramian directly yield a statement about the synthesis operators of the respective systems to be adjoint operators (up to a unitary transform), while the adjoint relation on each fiber pre-Gramian makes finding connections between these operators much more involved, as questions about domain and target spaces become more subtle, see [102, Sect. 3.2].

4.5 The Classical Duality Identities Via Dual Gramian Analysis

The essence of several important classical identities for dual Gabor systems is the duality principle. We briefly show that the Walnut representation, the Wexler–Raz biorthogonality relations and the Janssen (or Wexler–Raz) representation, are merely reformulations and certain aspects of the fiberized dual Gramian matrix representation of the Gabor frame operator and the duality principle.

Throughout this subsection, suppose the Gabor systems $X = (K, L)_\phi$ and $Y = (K, L)_\psi$ are Bessel. The Walnut representation is the fiberized representation of the Gabor frame operator in time domain (4.28), evaluated at $\tilde{l} = 0$. It says that

$$T_Y T_X^* f = |\widetilde{L}| \sum_{\tilde{l} \in \widetilde{L}} \sum_{k \in K} E^k \psi \overline{E^{\tilde{l}+k} \phi} E^{\tilde{l}} f$$

for all $f \in L_2(\mathbb{R}^d)$. This representation has first been proposed for rectangular lattices in [115], where it is shown under the technical condition on the windows ϕ, ψ to belong to the Wiener space

$$\left\{ g \in L_\infty(\mathbb{R}^d) \colon \sum_{n \in \mathbb{Z}^d} \|g E^n \chi_{[0,1]^d}\|_\infty < \infty \right\}.$$

This condition implies that X and Y are Bessel systems (see [102, Corollary 3.26]), which is the natural and weaker condition for the fiberized representation (4.28) to hold.

The duality principle Theorem 4.4 establishes that

$$\widetilde{\mathcal{G}}_{X,Y}(\omega) = \overline{\mathcal{G}_{X^*,Y^*}(\omega)} \tag{4.35}$$

for a.e. $\omega \in \mathbb{R}^d$, where X^* and Y^* are the respective adjoint systems as defined in (4.32). If X and Y are Bessel systems, using the dual-Gramian representation of the mixed frame operator, (4.35) in particular implies that the two Gabor Bessel systems X and Y are dual frames if and only if

$$(\mathrm{den}\,(K, L))\langle \psi, E^{\tilde{l}} M^{\tilde{k}} \phi \rangle = \delta_{\tilde{l},0} \delta_{\tilde{k},0} \quad \text{for all} \quad (\tilde{l}, \tilde{k}) \in \widetilde{L} \times \widetilde{K}. \tag{4.36}$$

The characterization (4.36) is known as Wexler–Raz biorthogonality relations. It is picking out one case of the more general duality identity (4.35), the case that the dual Gramian fibers are the identity if and only if the Gramian fibers of their adjoint systems are the identity. The statement that those matrices are actually equal, however, implies many other interesting dualities. For instance, it cannot be deduced from (4.36) that X is a frame if and only if its adjoint system X^* is a Riesz basis. This, on the other hand, is one of the direct consequences of the duality principle (4.35) for the case $X = Y$.

The discrete version of the Wexler–Raz biorthogonality relations has first been observed in [116]. The Wexler–Raz biorthogonality relations in the continuous case have first been announced in [97], with the proof appearing in [102]. They have independently been proved in various places. In [56], they are proven in the non-separable lattice case without reference to matrices. The approach in [43,72] for the one dimensional case is via the Janssen or Wexler–Raz representation

$$T_Y T_Z^* \phi = T_{X^*} T_{Z^*}^* \psi, \tag{4.37}$$

which holds for Bessel systems X, Y and $Z = (K, L)_g$ and their respective adjoints X^*, Y^* and Z^* as defined in (4.32). We reviewed (4.37) in Proposition 4.5, as part of $T_Y T_Z^* X$ and $T_{X^*} T_{Z^*}^* Y^*$ being adjoint systems of each other.

4.6 Construction of Dual Gabor Windows

We explicitly construct windows for dual Gabor frames based on the duality principle. For two Gabor systems to be dual frames one is confronted with the task to verify that analysis and synthesis with respect to the two systems amounts to the identity operator. Checking this from the generators of the systems for all functions of the space is nontrivial. The duality principle can simplify this problem to checking conditions only involving the two windows. As a consequence of the duality principle two Bessel Gabor systems $X = (K, L)_\phi$ and $Y = (K, L)_\psi$ are dual frames if and only if the rows of the respective pre-Gramians in (4.22) are biorthogonal (for a.e. ω). This immediately yields the first characterization of the following result, while the second follows from the same argument used on the Fourier transform images $\widehat{X}$ and $\widehat{Y}$.

Theorem 4.6 [102] *If $\phi, \psi \in L_2(\mathbb{R}^d)$ are such that the Gabor systems $(K, L)_\phi$ and $(K, L)_\psi$ are Bessel systems, then $(K, L)_\phi$ and $(K, L)_\psi$ are dual frames if and only if one (and therefore both) of the following conditions holds:*

$$\sum_{l \in L} E^l \hat{\phi} E^{\tilde{k}+l} \overline{\hat{\psi}} = |K| \delta_{\tilde{k},0} \quad \textit{for all} \quad \tilde{k} \in \widetilde{K}, \tag{4.38}$$

or

$$\sum_{k \in K} E^k \phi E^{k+\tilde{l}} \overline{\psi} = |\widetilde{L}|^{-1} \delta_{\tilde{l},0} \quad \textit{for all} \quad \tilde{l} \in \widetilde{L}. \tag{4.39}$$

In case the two windows ϕ and ψ coincide, the Gabor system $(K, L)_\phi$ is a tight frame and the following construction is already sketched in [102]. It has also been observed in [38] as the "painless" way to construct tight univariate Gabor frames by guaranteeing orthogonality through disjointness of support. Constructions of this type, that work by starting from a partition of unity, have since produced an extensive literature.

Proposition 4.7 *Let $\phi, \psi \in L_2(\mathbb{R}^d)$ be compactly supported, bounded and suppose $\sum_{k \in K} E^k (\phi \overline{\psi}) = 1$. Choose a lattice L such that $\widetilde{L}$ is sparse enough to ensure that the support of $E^{\tilde{l}} \psi$ is disjoint from the support of ϕ for all $\tilde{l} \in \widetilde{L} \backslash \{0\}$. Then the Gabor systems $(K, L)_{c\phi}$ and $(K, L)_{c\psi}$ are dual frames, where $c = |\widetilde{L}|^{-1/2}$.*

Proof Since ϕ and ψ are compactly supported and bounded, both Gabor systems are Bessel systems, see [98, Corollary 1.6.3]. By the choice of L, condition (4.39) reduces to one condition, namely for the case $\tilde{l} = 0$. This condition is met after scaling, since the K-shifts of $\phi\overline{\psi}$ form a partition of unity. □

Note that the shift lattice K is determined by the shift size in the partition of unity. One could however use some dilation to restate the results for arbitrary sizes of the shift lattice. An analogue result can be formulated starting from (4.38) instead of (4.39), i.e., by changing the roles of the lattices, the partition of unity in Proposition 4.7 might also be used to construct bandlimited dual windows. We now illustrate the construction, starting from different classes of partitions of unity.

Partitions of unity characterize orthonormal refinable functions for multiresolution analyses and one may draw from this arsenal for Gabor window constructions.

Example 4.8 The bandlimited refinable function defined by Meyer in [89] is given in Fourier domain by

$$h(\omega) = \cos\left[\frac{\pi\beta}{2}\left(\frac{3|\omega|}{2\pi} - 1\right)\right]$$

for $\omega \in \mathbb{R}$, where β is some C^k or C^∞-function for which $\beta(x) = 0$ if $x \le 0$ and $\beta(x) = 1$ if $x \ge 1$ and

$$\beta(x) + \beta(1 - x) = 1 \tag{4.40}$$

for all $x \in \mathbb{R}$. Note that this implies $h(\omega) = 1$ for $|\omega| \le 2\pi/3$ and $h(\omega) = 0$ for $|\omega| \ge 4\pi/3$. Moreover, the regularity of h is the same as the regularity of β. Since β satisfies (4.40), one gets

$$\sum_{k\in 2\pi\mathbb{Z}} |h(w + k)|^2 = 1.$$

Therefore, $g = \sqrt{3/(8\pi)}h$ is the window of a tight Gabor frame with respect to $(2\pi\mathbb{Z}, \frac{3}{4}\mathbb{Z})$. For a factorization of $|g|^2$ into two different functions one can get dual Gabor windows.

A more systematic approach is to factor certain classes of (piecewise) polynomials.

4.6.1 Piecewise Polynomials

One family of functions whose shifts form a partition of unity, and which thus may be used for constructing dual windows, are B-splines. The B-spline B_m of order $m \in \mathbb{N}$ is inductively given by $B_1 = \chi_{[0,1]}$ and $B_{m+1} = B_m * B_1$. Since $\sum_{k\in\mathbb{Z}} B_m(x + k) = 1$, if one factors B_m into two bounded functions supported in $[0, m]$, say $B_m = \phi\psi$, then for any $a \ge m$ the functions $a^{-1/2}\phi$ and $a^{-1/2}\psi$ are dual Gabor windows with respect to $(\mathbb{Z}, 2\pi a^{-1}\mathbb{Z})$. Since B-splines are nonnegative, one may also take their square root as window function for a tight frame. This idea has been used in [75] to construct complex-valued discrete tight Gabor frames, which exhibit good orientation selectivity and are useful in various image processing problems.

Example 4.9 Starting from the linear B-spline B_2, we get that $2^{-1/2}B_2$ and $2^{-1/2}\chi_{[0,2]}$ are dual Gabor windows with respect to $(\mathbb{Z}, \pi\mathbb{Z})$. The cubic B-spline

$$B_4(x) = \frac{1}{6}\begin{cases} x^3 & \text{if } \ 0 \le x < 1 \\ -3x^3 + 12x^2 - 12x + 4 & \text{if } \ 1 \le x < 2 \\ 3x^3 - 24x^2 + 60x - 44 & \text{if } \ 2 \le x < 3 \\ (4-x)^3 & \text{if } \ 3 \le x < 4 \\ 0 & \text{else} \end{cases}$$

can be factored into a piecewise linear and a piecewise quadratic polynomial:

$$\tilde{\phi}(x) = \frac{1}{6}\begin{cases} (a-1)^{-1}x^2 & \text{if } \ 0 \le x < 1 \\ x^2 + (3a-12)x + 4a^{-1} & \text{if } \ 1 \le x < 2 \\ x^2 + (3b-24)x + 44b^{-1} & \text{if } \ 2 \le x < 3 \\ (3-b)^{-1}(4-x)^2 & \text{if } \ 3 \le x < 4 \\ 0 & \text{else} \end{cases},$$

$$\tilde{\psi}(x) = \begin{cases} (a-1)x & \text{if } \ 0 \le x < 1 \\ -x + a & \text{if } \ 1 \le x < 2 \\ x - b & \text{if } \ 2 \le x < 3 \\ (3-b)(4-x) & \text{if } \ 3 \le x \le 4 \\ 0 & \text{else} \end{cases},$$

where a (resp. b) is the (only) real solution of the second (resp. third) cubic equation in B_4. Thus $2^{-1}\tilde{\phi}$ and $2^{-1}\tilde{\psi}$ are dual Gabor windows with respect to $(\mathbb{Z}, \frac{\pi}{2}\mathbb{Z})$.

In principle this method can be used on B-splines of higher order by solving higher order polynomial equations in order to get factorizations. In general however, those solutions will be numerical in nature and have no explicit closed form. Moreover, using this method one can only guarantee continuity but no higher order smoothness of the windows. In order to construct smoother windows, we now turn to alternative constructions.

4.6.2 Trigonometric Polynomials

Starting from the identity $\cos^2 x + \sin^2 x = 1$ and restricting ourselves to functions of compact support, let $h = \cos^2(\cdot\,\pi/2)\chi_{[-1,1]}$. Then $h(x) + h(x-1) = 1$ for all $x \in [0, 1]$ and the integer shifts of h are a partition of unity. Factoring h into two functions, e.g. $\cos(\cdot\,\pi/2)\chi_{[-1,1]}$ and $\cos(\cdot\,\pi/2)\chi_{[-1,1]}$, we can get two symmetric and continuous dual Gabor windows with respect to $(\mathbb{Z}, \pi\mathbb{Z})$. In order to get windows with higher smoothness, we now improve this construction by leveraging on the idea for the construction of pseudo-spline wavelet masks, see [48], namely that higher powers can result in higher smoothness. That is, we now start from the identity

$$1 = \left(\cos^2\left(\frac{\pi x}{2}\right) + \sin^2\left(\frac{\pi x}{2}\right)\right)^{2m-1}, \tag{4.41}$$

for some nonnegative integer m. Define h in a similar fashion as above by cutting off the first m-terms of the binomial expansion of (4.41), i.e. let

$$h(x) = \cos^{2m}\left(\frac{\pi x}{2}\right) \sum_{j=0}^{m-1} \binom{2m-1}{j} \cos^{2(m-1-j)}\left(\frac{\pi x}{2}\right) \sin^{2j}\left(\frac{\pi x}{2}\right) \chi_{[-1,1]}(x). \tag{4.42}$$

As above, the integer shifts of h are a partition of unity. Taking, say, $l, \tilde{l} \in \mathbb{N}$ such that $l + \tilde{l} = m$, h can be factored into the two functions

$$\phi(x) = \cos^{2l}\left(\frac{\pi x}{2}\right) \chi_{[-1,1]}(x),$$

and

$$\psi(x) = \cos^{2\tilde{l}}\left(\frac{\pi x}{2}\right) \sum_{j=0}^{m-1} \binom{2m-1}{j} \cos^{2(m-j-1)}\left(\frac{\pi x}{2}\right) \sin^{2j}\left(\frac{\pi x}{2}\right) \chi_{[-1,1]}(x).$$

Then for any $a \geq |\operatorname{supp} h| = 2$ the functions $a^{-1/2}\phi$ and $a^{-1/2}\psi$ are dual Gabor windows with respect to $(\mathbb{Z}, 2\pi a^{-1}\mathbb{Z})$. Note that, the larger m is chosen, the smoother one can make the two functions.

The idea to use higher powers in order to improve smoothness has already been used in [74] to construct wavelet masks satisfying interpolatory conditions and higher order smoothness properties. A generalization to multidimensions has for example been considered in [73]. There, one starts as well from a partition of unity, namely from $a \in \ell_0(\mathbb{Z}^d)$ satisfying the *interpolatory condition*

$$\sum_{\nu \in \mathbb{Z}_2^d} \hat{a}(\omega + \pi\nu) = 1 \quad \text{for all } \omega \in \mathbb{R}^d, \tag{4.43}$$

where $\mathbb{Z}_2^d := \mathbb{Z}^d/2\mathbb{Z}^d$ and $\hat{a}(\omega) = \sum_{n\in\mathbb{Z}^d} a(n)e^{-in\cdot\omega}$. After raising to some positive integer power, this interpolatory condition is being factored in [73] to construct higher order smoothness interpolatory functions in high dimensions. Here, we will use the factoring to derive dual windows. The technique of [73] is as follows. The formal Laurent polynomial P corresponding to the mask $\hat{a}$ is

$$P(z) = \sum_{n\in\mathbb{Z}^d} a(n)z^n.$$

Defining

$$P_\nu(z) = P(z\exp(-\pi i\nu)),\ \nu \in \mathbb{Z}_2^d,\ |z| = 1,$$

the interpolatory condition (4.43) is equivalent to

$$\sum_{\nu\in\mathbb{Z}_2^d} P_\nu(z) = 1 \quad \text{for all } |z| = 1. \tag{4.44}$$

This in turn implies

$$\left(\sum_{\nu\in\mathbb{Z}_2^d} P_\nu(z)\right)^{mN} = \sum_{|\gamma|=mN}\left(C_{mN}^{\gamma}\prod_{\mu\in\mathbb{Z}_2^d} P_\mu^{\gamma_\mu}(z)\right) = 1$$

for all $|z| = 1$ and all $m, N \in \mathbb{N}$, where C_{mN}^{γ} are the multinomial coefficients. Letting $m = 2^d$ and $N \in \mathbb{N}$, this interpolatory condition for P is being factored in [73, Theorem 2.3] as follows.[4] Define

$$G_0 = \left\{\gamma \in \mathbb{N}_0^m : |\gamma| = mN, \gamma_0 > N \text{ and } \gamma_0 > \gamma_\nu, \nu \in \mathbb{Z}_2^d\backslash\{0\}\right\},$$

$$G_j = \left\{\gamma \in \mathbb{N}_0^m : |\gamma| = mN, \gamma_0 > N, \gamma_0 \geq \gamma_\nu, \nu \in \mathbb{Z}_2^d\backslash\{0\},\ \text{with exactly } j \text{ equalities}\right\},$$

for $j = 1, \ldots, m-2$, and

$$H = \sum_{j=0}^{m-2}\frac{1}{j+1}\left(\sum_{\gamma\in G_j} C_{mN}^{\gamma} P^{\gamma_0 - 1}\prod_{\nu\in\mathbb{Z}_2^d\backslash\{0\}} P_\nu^{\gamma_\nu}\right) + C_{mN}^{(N,\ldots,N)}\prod_{\nu\in\mathbb{Z}_2^d} P_\nu^N.$$

Then [73, Theorem 2.3] proves that the product PH satisfies the interpolatory condition (4.44). The following example shows one particular construction of dual Gabor frames based on this result.

Example 4.10 A possible $a \in \ell_0(\mathbb{Z}^d)$ to satisfy the interpolatory condition (4.43) is given by

$$\begin{aligned}\hat{a}(\omega) = \frac{1}{2}&\left(\cos\left(\frac{\omega_1}{2}\right)\cos\left(\frac{\omega_2}{2}\right)\cos\left(\frac{\omega_1+\omega_2}{2}\right)\right)^2\\ &\times\left(5 - \cos(\omega_1) - \cos(\omega_2) - \cos(\omega_1+\omega_2)\right),\end{aligned}$$

[4] The factor 2^d is an artifact of the dyadic dilations we use. The construction in [73] works for general dilation matrices and 2^d is being replaced by the determinant of the dilation matrix.

see [73]. The corresponding Laurant polynomial is

$$P(z) = \frac{1}{128}\left((z_1 + z_1^{-1})(z_2 + z_2^{-1})(z_1 z_2 + (z_1 z_2)^{-1})\right)^2 \times\left(5 - \frac{z_1^2 + z_1^{-2}}{2} - \frac{z_2^2 + z_2^{-2}}{2} - \frac{(z_1 z_2)^2 + (z_1 z_2)^{-2}}{2}\right),$$

where $z_1 = e^{-i\omega_1/2}$ and $z_2 = e^{-i\omega_2/2}$. Applying [73, Theorem2.3] with $m = 4$ and $N = 1$ yields

$$H = P(P^2 + 4P(P_{\nu_1} + P_{\nu_2} + P_{\nu_3}) + 12(P_{\nu_1}P_{\nu_2} + P_{\nu_2}P_{\nu_3} + P_{\nu_1}P_{\nu_3}) + 3(P_{\nu_1}^2 + P_{\nu_2}^2 + P_{\nu_3}^2) + 24P_{\nu_1}P_{\nu_2}P_{\nu_3}).$$

where $\nu_1, \nu_2, \nu_3 \in \mathbb{Z}_2^d \backslash \{0\}$ are the 3 coset elements. Therefore, defining

$$\phi(x) = \begin{cases} P(e^{-i\pi x/2}) & \text{if} \quad x \in [-1, 1]^2 \\ 0 & \text{else} \end{cases},$$

$$\psi(x) = \begin{cases} H(e^{-i\pi x/2}) & \text{if} \quad x \in [-1, 1]^2 \\ 0 & \text{else} \end{cases},$$

and choosing the lattices $(\mathbb{Z}^2, \pi\mathbb{Z}^2)$, the conditions of Theorem 4.6 are satisfied for the windows $2^{-1}\phi$ and $2^{-1}\psi$. The graphs of the dual windows are shown in Fig. 1.

4.6.3 Related Work

A considerable body of literature on the construction of dual Gabor windows already exists. In [24], a construction for a dual window of a given compactly supported window, in particular a given B-spline, is presented. The support of the dual window is twice as large as of the primary window. Moreover, the density of the modulation lattice depends on the support size of the primary window. Larger support, i.e. in the B-spline case higher smoothness of the primary window, forces a denser modulation lattice. While we have similar constraints on the lattice in our B-spline example, the supports of the two dual windows are the same. In our pseudo-spline example the order of smoothness can be increased independent of the support size and therefore of the modulation lattice. In [23,27,78,81] the authors construct dual windows that overcome the problem of support in [24]. The paper [81] gives several constructions of Gabor windows using spline functions and discusses the smoothness of the constructed windows and choice of lattices. However, it involves complicated symbolic computations, especially when the smoothness of the window is increased. In [23,78], the authors are motivated from the solution of

$$\hat{a}(\pi\omega)\overline{\hat{b}(\pi\omega)} + \hat{a}(\pi(\omega+1))\overline{\hat{b}(\pi(\omega+1))} = 1 \tag{4.45}$$

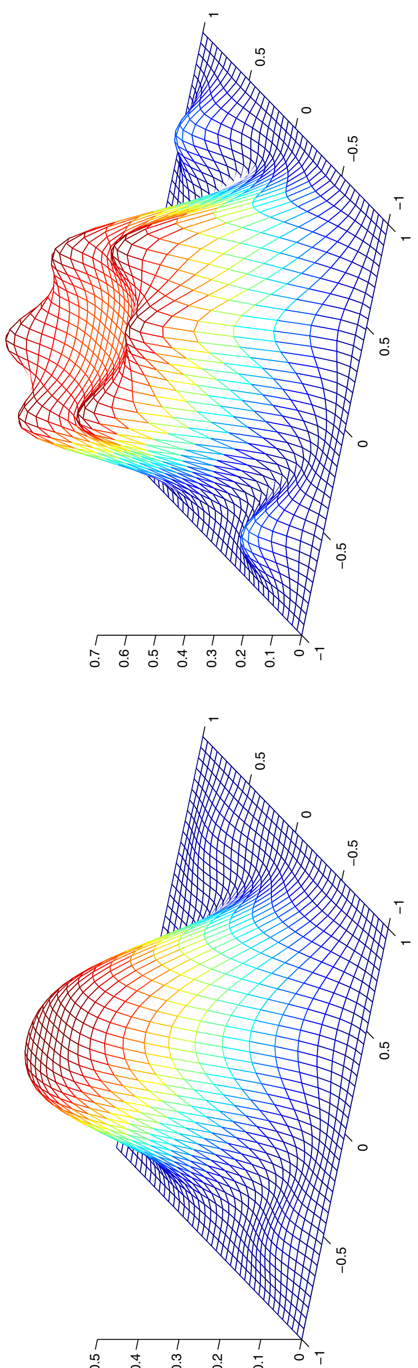

Fig. 1 Primary and dual Gabor windows of Example 4.10

to construct dual windows of equal support size. Several of their windows coincide with ours, which is not surprising since the trigonometric polynomials we construct from (4.42) solve (4.45). However, our method is easier than solving the polynomial equation (4.45) directly and moreover can be easily generalized to higher dimensions. The idea of [24] is generalized to higher dimensions in [29]. Similar to the one dimensional case, the support size of the dual window gets larger when the smoothness of the primary window increases, resulting in a denser modulation lattice. Our construction in the multi dimensional case keeps the desirable property of the one dimensional case of not changing the support when the smoothness of the window is increased. Moreover, using (4.38), all the methods of constructing compactly supported windows can be used to construct bandlimited Gabor windows.

5 Wavelets Systems

The short-time Fourier transform (4.14) uses a window of fixed support size and cannot provide information on different levels of resolution. The next step in the evolution of the Fourier transform has been the wavelet transform

$$\begin{aligned} W_\Psi f(s,t,\psi) &:= \langle f, D^s E^t \psi \rangle \\ &= \int_{\mathbb{R}^d} f(x) 2^{sd/2} \overline{\psi(2^s x - t)}\, dx, \quad (s,t,\psi) \in \mathbb{R} \times \mathbb{R}^d \times \Psi, \end{aligned} \tag{5.46}$$

for $f \in L_2(\mathbb{R}^d)$. Here Ψ is some finite system of functions in $L_2(\mathbb{R}^d)$, which are called *wavelets*, and $D^s : f \mapsto 2^{sd/2} f(2^s \cdot)$ denotes the (dyadic) dilation operator on $L_2(\mathbb{R}^d)$ for adapting the size of the wavelets. The question of numerically stable analysis and/or synthesis of a signal from discrete samples of its wavelet transform leads to considering the frame properties of the *wavelet (or affine) system*

$$\mathcal{X} = \mathcal{X}(\Psi) := \{D^k E^j \psi : k \in \mathbb{Z}, j \in \mathbb{Z}^d, \psi \in \Psi\}.$$

The system $\mathcal{X}$ is not shift-invariant, thus the fiber dual Gramian analysis for shift-invariant systems of [98] does not directly apply. Due to the commutator relation $D^k E^j = E^{2^{-k} j} D^k$ for $(k, j) \in \mathbb{Z} \times \mathbb{Z}^d$, each $\{D^k E^j \psi\}_{j \in \mathbb{Z}^d, \psi \in \Psi}$ is $2^{-k}\mathbb{Z}^d$-shift-invariant. That is, while the system of nonnegative dilation levels $\mathcal{X}_0 := \{D^k E^j \psi\}_{k \geq 0, j \in \mathbb{Z}^d, \psi \in \Psi}$ is $\mathbb{Z}^d$-shift-invariant, the subsystem of negative dilation levels is not. The *quasi-affine system* $\mathcal{X}^q$ of $\mathcal{X}$, which has been introduced in [101] to be able to apply the dual Gramian analysis for shift-invariant systems of [98], is the $\mathbb{Z}^d$-shift-invariant system

$$\mathcal{X}^q := \mathcal{X}_0 \cup \left(\bigcup_{k<0} \left\{ 2^{dk/2} E^{2^j} D^k \psi : j \in \mathbb{Z}^d, \psi \in \Psi \right\} \right)$$

generated from $\mathcal{X}$ by adding in at each negative dilation level the functions that are missing to make the system $\mathbb{Z}^d$-shift-invariant and rescaling each of those dilation levels. A main result of [101] is the invariance of the Bessel and frame properties (and the corresponding bounds) between affine systems and their quasi-affine completions. Fiber dual Gramian analysis of quasi-affine systems can therefore be used to completely characterize those properties and the characterizations apply to the quasi-affine system and its affine counterpart alike. Similar results hold for dual systems via mixed dual Gramian analysis [100], where to compensate for the missing self-adjointness, as in the abstract case, one has to start from Bessel systems.

The enormous value of wavelet systems for signal processing applications lies in their potential to provide multiscale representations of different levels of resolution. The dilation levels of the wavelet system $\mathcal{X}$ are then linked to a multiresolution analysis (MRA), that is, to a sequence of subspaces $\{V_k := D^k V\}_{k\in\mathbb{Z}}$ of $L_2(\mathbb{R}^d)$, which are scaled versions of a shift-invariant space $V := \overline{\operatorname{span}}\{E^j\phi\colon j \in \mathbb{Z}^d\}$ generated by some $\phi \in L_2(\mathbb{R}^d)$, and which provide increasingly better discrete approximations of arbitrary precision for signals in $L_2(\mathbb{R}^d)$ in the sense that $V_{k-1} \subset V_k$ for $k \in \mathbb{Z}$ and $\overline{\cup_{k\in\mathbb{Z}} V_k} = L_2(\mathbb{R}^d)$. Here, the nestedness requirement on the subspaces can be ensured by choosing the generating function to be *refinable*, i.e. to be determined by some *refinement mask* $a_0 \in \ell_2(\mathbb{Z}^d)$ via $\hat{\phi}(2\cdot) = \hat{a}_0\hat{\phi}$, where $\hat{a}_0$ is the Fourier series of a_0. For the union of the subspaces to be dense, it is for example sufficient that ϕ is a compactly supported refinable function with $\hat{\phi}(0) = 1$, see e.g. [76]. If the wavelets $\Psi = \{\psi_l\}_{l=1}^r$ are connected to the MRA via

$$\hat{\psi}_l(2\cdot) = \hat{a}_l\hat{\phi} \tag{5.47}$$

for some set of *wavelet masks* $\{a_l\}_{l=1}^r$ in $\ell_2(\mathbb{Z}^d)$, then $\mathcal{X}(\Psi)$ is called the *MRA-wavelet system* generated by the masks $\{a_l\}_{l=0}^r$.

MRA based affine and quasi-affine tight frames, allow fast decomposition and reconstruction algorithms to compute the frame coefficients on each dilation level in a cascading way by discrete convolutions with the masks, making them ideal for applications, see e.g. [32,41]. They can be characterized in terms of their masks, resulting in the mixed unitary extension principle (MEP) and the unitary extension principle (UEP) of [100,101] for the construction of such dual and tight MRA-wavelet frames. This gives a direct connection to filter banks as discussed in Sect. 3.3. We review this connection in Sect. 5.1 (and further comment on it in Sect. 6.3). In Sect. 5.2 we adapt the duality principle based filter bank construction of Corollary 3.14 to the construction of multivariate dual MRA-wavelet frames. For detailed introductions to wavelets and their applications we refer to e.g. [37,49,87].

5.1 Mixed Unitary Extension Principle and Mixed Dual Gramian Analysis

Let Ψ be a finite system in $L_2(\mathbb{R}^d)$ and let $R : \Psi \to L_2(\mathbb{R}^d)$ be a map. (As before, Ψ is used both as system and index set, and R is a map between the index sets of the systems.) Then the fiber mixed dual Gramian matrix of the shift-invariant systems $\mathcal{X}^q$ and $(R\mathcal{X})^q$ at $\omega \in \mathbb{T}^d := [-\pi, \pi]^d$ is

$$\widetilde{\mathcal{G}}_{\mathcal{X}^q,(R\mathcal{X})^q}(\omega) = \left(\sum_{\psi\in\Psi} \sum_{k=\kappa(\alpha-\beta)}^{\infty} \hat{\psi}(2^k(\omega+\alpha))\overline{\widehat{R\psi}(2^k(\omega+\beta))} \right)_{\alpha,\beta\in 2\pi\mathbb{Z}^d}, \tag{5.48}$$

with the dyadic valuation $\kappa : \mathbb{R}^d \to \mathbb{Z}^d : \omega \mapsto \inf\{k \in \mathbb{Z} : 2^k\omega \in 2\pi\mathbb{Z}^d\}$. The fiber matrices $\widetilde{\mathcal{G}}_{\mathcal{X}^q,(R\mathcal{X})^q}(\omega)$ represent the mixed frame operator of $\mathcal{X}^q$ and $(R\mathcal{X})^q$ and it has been proved in [100], that two Bessel systems $\mathcal{X}$ and $R\mathcal{X}$ are dual frames if and only if $\widetilde{\mathcal{G}}_{\mathcal{X}^q,(R\mathcal{X})^q}(\omega)$ is the identity matrix for a.e. $\omega \in \mathbb{T}^d$.

When the wavelet system is generated by an MRA (and under some further mild assumptions), the condition on the mixed dual Gramian matrix $\widetilde{\mathcal{G}}_{\mathcal{X}^q,(R\mathcal{X})^q}(\omega)$ to be the identity matrix for a.e. $\omega \in \mathbb{T}^d$, can be reduced to a sufficient condition on finite order matrices in terms of the masks. If one associates with the wavelet system $\mathcal{X}$, derived from the masks $\{a_l\}_{l=0}^r$, the matrices

$$\mathcal{M}_{\mathcal{X}}(\omega) := \begin{pmatrix} \hat{a}_0(\omega+\nu_1) & \dots & \hat{a}_r(\omega+\nu_1) \\ \vdots & \ddots & \vdots \\ \hat{a}_0(\omega+\nu_{2^d}) & \dots & \hat{a}_r(\omega+\nu_{2^d}) \end{pmatrix}, \tag{5.49}$$

where $\omega \in \mathbb{T}^d$ and $\{\nu_i\}_{i=1}^{2^d} = \{0, \pi\}^d$, then the MEP states that, under the conditions of Theorem 5.1, $\mathcal{X}$ and $\mathcal{Y}$ are dual wavelet frames whenever they are Bessel systems and $\mathcal{M}_{\mathcal{X}}(\omega)\mathcal{M}^*_{\mathcal{Y}}(\omega)$ is the identity for a.e. $\omega \in \mathbb{T}^d$. In other words, under the MRA assumption and some additional mild conditions, the infinite mixed dual Gramian matrix defined in (5.48) can be factored to the family of finite order matrices $\mathcal{M}_{\mathcal{X}}(\omega)$ and $\mathcal{M}_{\mathcal{Y}}(\omega)$ and the dual wavelet frame property for two wavelet Bessel systems is reduced to a condition on $\mathcal{M}_{\mathcal{X}}\mathcal{M}^*_{\mathcal{Y}}$. The MEP provides the following practical sufficient condition for two wavelet systems to be dual frames, and we refer to [100] for further details and the MEP under weaker conditions on the refinable functions and masks.

Theorem 5.1 [100] *Let ϕ_a and ϕ_b be compactly supported refinable functions with $\hat{\phi}_a(0) = \hat{\phi}_b(0) = 1$ and finitely supported refinement masks a_0 and b_0. Let $\{a_l\}_{l=1}^r$, resp. $\{b_l\}_{l=1}^r$, be the masks of a wavelet system $\mathcal{X}$ derived from ϕ_a, resp. $\mathcal{Y}$ derived from ϕ_b. Then $\mathcal{X}$ and $\mathcal{Y}$ are dual frames provided they are Bessel systems and*

$$\sum_{l=0}^{r} \hat{a}_l(\omega)\overline{\hat{b}_l(\omega+\nu)} = \delta_{\nu,0} \tag{5.50}$$

for all $\nu \in \{0, \pi\}^d$ and a.e. $\omega \in \mathbb{T}^d$.

It is important to note that the matrix condition (5.50) is only sufficient for $\mathcal{X}$ and $\mathcal{Y}$ to be dual frames if they are Bessel systems to begin with. The Bessel condition is not required in the special case the case $\mathcal{X} = \mathcal{Y}$. It here again comes from considering nonself-adjoint operators. For simplicity, we assume in all of what follows that the refinable function ϕ is compactly supported with $\hat{\phi}(0) = 1$ and that all wavelet masks are finitely supported. If in addition all wavelet masks have *first order vanishing moments*, i.e. $\sum_{n\in\mathbb{Z}^d} a_l(n) = 0$ for all $l = 1, \dots, r$, then the MRA-wavelet system generated by the masks $\{a_l\}_{l=0}^r$ is a Bessel system (see e.g. [64]; or [99] for a dual Gramian argument under an additional mild smoothness condition on the refinable function).

The UEP is the special case of the MEP Theorem 5.1 for tight instead of dual frames, i.e. for $\mathcal{X} = \mathcal{Y}$. That is, if ϕ is a compactly supported refinable function with mask a_0 such that $\hat{\phi}(0) = 1$ and if $\{a_l\}_{l=1}^r$ are masks such that the matrices (5.49) have orthonormal rows for a.e. $\omega \in \mathbb{T}^d$, then the wavelet system derived from those masks is a tight frame. Condition (5.50) in this case implies that a.e. $\mathcal{M}_{\mathcal{X}}(\omega)$ can be extended to a unitary matrix. It is worth noticing that in contrast to the MEP, the UEP does not require the assumption that the wavelet system is Bessel. Also note that both, the MEP and the UEP, require $r \geq 2^d$.

Recall that the filter banks defined in (3.13) are N-shift-invariant systems. Therefore the fiber dual Gramian analysis of [98] for shift-invariant systems can be applied. The pre-Gramian fibers of the filter bank system $X = X(\{2^{d/2}a_l\}_{l=0}^r, 2)$ are precisely the matrices $\mathcal{M}_{\mathcal{X}}(\omega)$ and

$$((T_X c)^\wedge(\omega+\nu))_{\nu\in\{0,\pi\}^d} = 2^{d/2}\mathcal{M}_{\mathcal{X}}(\omega)(\hat{c}_l(2\omega))_{l\in\mathbb{Z}/r\mathbb{Z}} \quad \text{for} \quad c \in \ell_0(X),$$
$$(T_X^* c)^\wedge(\omega) = 2^{-d/2}\mathcal{M}_{\mathcal{X}}^*(\omega/2)(\hat{c}(\omega/2+\nu))_{\nu\in\{0,\pi\}^d} \quad \text{for} \quad c \in \ell_0(\mathbb{Z}/r\mathbb{Z}\times\mathbb{Z}^d),$$

for a.e. $\omega \in \mathbb{T}^d$. If $Y = X(\{2^{d/2}b_l\}_{l=0}^r, 2)$, then the mixed dual Gramian fibers of X and Y are $\widetilde{\mathcal{G}}_{Y,X}(\omega) := \mathcal{M}_{\mathcal{Y}}(\omega)\mathcal{M}_{\mathcal{X}}^*(\omega)$. Since the filters are FIR, X and Y are Bessel systems and thus $\widetilde{\mathcal{G}}_{Y,X}(\omega)$ is the identity for a.e. $\omega \in \mathbb{R}^d$ if and only if the (time-domain) dual Gramian matrix $\widetilde{G}_{Y,X}$ is the identity, i.e.

$$2^d \sum_{l=0}^{r} \sum_{k\in\mathbb{Z}^d} a_l(n+2k+\ell)\overline{b_l(2k+\ell)} = \delta_{n,0} \quad \text{for any} \quad n, \ell \in \mathbb{Z}^d. \tag{5.51}$$

Both are the necessary and sufficient conditions for the filter bank systems X and Y to be dual frames. Besides (5.50), the sufficiency condition provided by Theorem 5.1 contains the important second part that $\mathcal{X}$ and $\mathcal{Y}$ have to be Bessel systems. This however, as reviewed above, can be guaranteed by working with wavelet masks that have first order vanishing moments. In summary, if the wavelet masks $\{a_l, b_l\}_{l=1}^r$ have first order vanishing moments, then Theorem 5.1 leads to a much simpler sufficient condition for MRA-based wavelet systems to be dual wavelet frames. Namely, the generated wavelet systems $\mathcal{X}$ and $\mathcal{Y}$ are dual wavelet frames for $L_2(\mathbb{R}^d)$, whenever the filter bank systems $X = X(\{2^{d/2}a_l\}_{l=0}^r, 2)$ and $Y = X(\{2^{d/2}b_l\}_{l=0}^r, 2)$ are dual frames in $\ell_2(\mathbb{Z}^d)$.

In the next subsection we will use this connection and adapt the duality principle construction for perfect reconstruction filter banks to meet the vanishing moments requirements on the masks. In doing so, we give a simple matrix inversion scheme for constructing multivariate dual wavelet frames for a prescribed MRA as yet another application of the duality principle.

5.2 Dual Wavelet Frame Construction via Constant Matrix Inversion

Having established a link between the MEP and the mixed dual Gramian of filter banks, we propose a simple construction method for multivariate dual wavelet frames for a given MRA in terms of a constant matrix inversion scheme. This is done by fine tuning the filter bank Construction 3.15 to meet the extra requirement of first order vanishing moments. Our freedom to achieve this lies in the appropriate choice of the diagonal matrix involved. With this in mind, the construction starts from a real-valued refinement mask a_0 satisfying

$$2^d \sum_{n \in \Omega_j} a_0(n) = 1 \tag{5.52}$$

for all $j \in \mathbb{Z}^d/2\mathbb{Z}^d$, where $\Omega_j = (2\mathbb{Z}^d + j) \cap \text{supp}\,(a_0)$. Note that (5.52) is a rather mild requirement on a refinement mask. It is equivalent to $\hat{a}_0(0) = 1$ and $\hat{a}_0(j\pi) = 0$ for $j \in (\mathbb{Z}^d/2\mathbb{Z}^d)\backslash\{0\}$. If $\hat{a}_0(0) = 1$, then (5.52) holds provided that the cascade algorithm for a_0 converges in $L_2(\mathbb{R}^d)$ for any compactly supported initial function whose integer shifts are a partition of unity, see [82]. Recall that the cascade algorithm for the refinement mask a_0 is the sequence $\phi_n = 2^d \sum_{k \in \mathbb{Z}^d} a_0(k)\phi_{n-1}(2\cdot - k)$, $n \in \mathbb{N}$, where ϕ_0 is some compactly supported function. If, for example, a compactly supported refinable function is stable, i.e. its integer shifts form a Riesz sequence, and its integer shifts form a partition of unity, then its refinement mask satisfies (5.52). Refinement masks that satisfy (5.52) include masks of box splines, of certain butterfly subdivision schemes or of the interpolation function of [94]. We will use those in examples below.

If a_0 has, say, m nonzero entries, use those as the first row of the $m \times m$ matrix A in Construction 3.15. This defines a one-to-one correspondence between the support of a_0 and $\{1, \ldots, m\}$. The remaining $m - 1$ rows of A can now be completed under the premise that each of them has entries summing to zero. Via the same correspondence, these rows define $m-1$ finitely supported d-dimensional wavelet masks whose support is contained in the support of a_0. Choosing a_0 as the diagonal of the diagonal matrix M, the construction delivers a matrix B. We will show that the first row of B, via the same correspondence, defines a d-dimensional refinement mask (in fact it is equal to the first row of A) and its remaining rows define d-dimensional wavelet masks with first order vanishing moments and support contained in the support of a_0. The mask construction we propose is therefore as follows.

Construction 5.2 *Suppose the finitely supported real-valued refinement mask a_0 satisfies* (5.52).

- Step 1 (Initialization): *Define the first row of a matrix A by collecting the nonzero entries of a_0. Let M be the diagonal matrix with the first row of A as its diagonal.*
- Step 2 (Primary wavelet masks): *Complete the matrix A to be an invertible square matrix, each of whose remaining rows has entries summing to zero.*
- Step 3 (Dual wavelet masks): *Define $B = (A^*)^{-1}M$.*

That B indeed has the required properties is the content of the following lemma.

Lemma 5.3 *Let A and B be the matrices derived by Construction 5.2. Then the first rows of A and B coincide and each of their remaining rows has entries summing to zero.*

Proof That all but the first row of A sum to zero is a requirement in Construction 5.2. Every entry of the first row of $\tilde{A} = AM^{-1}$ is equal to 1 and $B\tilde{A}^* = I$. Therefore, the entries of the first row of B sum to 1 and the entries of each remaining row of B sum to 0. If $\tilde{B} = BM^{-1}$, then $\tilde{B}A^* = I$, where $\tilde{B}$, and therefore in particular the first row of $\tilde{B}$, is uniquely determined since A^* is invertible by construction. Since the entries of the first column of A^* sum to 1, while the entries of each remaining column of A^* sum to 0, it follows that each entry of the first row of $\tilde{B}$ is equal to 1. This implies that the first rows of A and B coincide. □

Before formulating our main result, we pause to summarize the mild conditions we have so far accumulated on the refinement mask and refinable function.

Assumption 5.4 *In the following results the refinement mask is assumed to be finitely supported, real-valued and to satisfy (5.52). The corresponding refinable function $\phi \in L_2(\mathbb{R}^d)$ is supposed to be compactly supported with $\widehat{\phi}(0) = 1$.*

Now let $\mathcal{X}$ and $\mathcal{Y}$ be the wavelet systems generated from the refinement mask and wavelet masks determined by A and B, respectively. Note that Lemma 5.3 in particular shows that both are derived from the same underlying MRA. Our main result is that $\mathcal{X}$ and $\mathcal{Y}$ are dual frames.

Theorem 5.5 *Suppose the refinement mask $a_0 \in \ell_2(\mathbb{Z}^d)$ satisfies Assumption 5.4. Then the masks derived by Construction 5.2 satisfy the MEP condition (5.50) and the wavelet systems $\mathcal{X}$ and $\mathcal{Y}$ generated by those masks are dual wavelet frames in $L_2(\mathbb{R}^d)$. Moreover, the support of the derived masks is no larger than the support of a_0 and the support of all wavelets is contained in any box containing the support of a_0.*

Proof Due to the finite support and first order vanishing moments of the wavelet masks guaranteed by Lemma 5.3, $\mathcal{X}$ and $\mathcal{Y}$ are Bessel systems. By Theorem 3.15 the MEP condition (5.50) is satisfied and Theorem 5.1 applies. The moreover part follows directly from the construction. □

One can find matrices satisfying Step 2 of Construction 5.2 for any finitely supported refinement mask. This implies the following existence result.

Theorem 5.6 *For any MRA of $L_2(\mathbb{R}^d)$ derived from a refinement mask satisfying Assumption 5.4 there exist dual wavelet frames with the following properties:*

(i) The number of primary and dual wavelets is one less than the size of the support of the refinement mask.
(ii) The support of all wavelet masks is contained in the support of the refinement mask.
(iii) The support of all wavelets is contained in any box containing the support of the refinable function.

Proof It only remains to note that Step 2 of Construction 5.2 can be executed for any finitely supported refinement mask. Indeed, if, as above, A is to be an $m \times m$ matrix, then its first row is not in $(\mathrm{span}\{\mathbf{1}\})^{\perp}$, where $\mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{R}^m$. Each of the remaining $m - 1$ rows must have entries summing to zero and one can choose any $m - 1$ linear independent vectors of the $(m - 1)$-dimensional space $(\mathrm{span}\{\mathbf{1}\})^{\perp}$ to complete A to be an invertible matrix. □

Remark Construction 5.2 is best possible in the sense that it cannot be improved to yield two different MRAs generated by real refinement masks for the primary and dual wavelets. Indeed, suppose a finitely supported mask satisfies (5.52) and its nonzero entries define the diagonal of a diagonal matrix M. The crux of the construction is to factor $M = A^* B$, with A and B such that their first rows $\mathbf{a}$ and $\mathbf{b}$ each have entries summing to one, while each of their remaining rows has entries summing to zero. Now, if A_0 and B_0 are the submatrices derived from A and B by deleting their first rows, then $A_0^* B_0 = M - \mathbf{a}^{\top}\mathbf{b}$. Letting $\mathbf{1}$ the constant one vector and $\mathbf{0}$ the constant zero vector, then $A_0^* B_0 \mathbf{1}^{\top} = A_0^* \mathbf{0}^{\top} = \mathbf{0}^{\top}$, while $(M - \mathbf{a}^{\top}\mathbf{b})\mathbf{1}^{\top} = \mathrm{diag}(M)^{\top} - \mathbf{a}^{\top}$. Thus $\mathrm{diag}(M) = \mathbf{a}$. Similarly, multiplying $\mathbf{1}$ from the left, $\mathrm{diag}(M) = \mathbf{b}$ follows. Consequently $\mathbf{a} = \mathbf{b}$, i.e. such a construction cannot produce different primary and dual refinement masks, no matter what the number of wavelets is.

As proposed in [52], if the refinement mask a_0 satisfies (5.52) and has nonnegative entries, then to construct a tight MRA-wavelet frame it suffices to find a matrix A such that $A^* A = M$. Letting $\tilde{A} = AM^{-1/2}$, this is equivalent to finding $\tilde{A}$ satisfying $\tilde{A}^* \tilde{A} = I$. Since the first row of $\tilde{A}$ has norm 1 this can always be done, implying the following result.

Theorem 5.7 *For any MRA of $L_2(\mathbb{R}^d)$ derived from a refinement mask with non-negative entries satisfying Assumption 5.4 there exist tight wavelet frames with the following properties:*

(i) The number of wavelets is one less than the size of the support of the refinement mask.
(ii) The support of all wavelet masks is contained in the support of the refinement mask.
(iii) The support of all wavelets is contained in any box containing the support of the refinable function.

5.3 Multivariate Dual Wavelet Frames from Interpolatory Refinable Functions

A particular example in the construction of dual wavelet frames are biorthogonal dual wavelets. They arise in the search for symmetric wavelets, since dyadic real orthonormal wavelet bases cannot contain of symmetric wavelets, except for the trivial Haar case [37]. Moreover, biorthogonal dual wavelets are easier to construct than orthonormal wavelet bases, since one "only" has to factor a polynomial rather than finding the square root of a polynomial. Several one dimensional dual wavelet frames have been constructed in [31,37,40,41,66,74]. The construction of multivariate dual frames, similar to the multivariate tight frame construction, becomes increasingly difficult, since it involves the completion of two matrices with polynomial entries. The constructions of non-separable tight wavelet frames by using refinable box splines first appeared in [92,93], where exponentially decaying orthogonal wavelets are constructed. Several biorthogonal wavelet constructions based on box splines have been proposed in [73,95,96]. Many multivariate biorthogonal wavelets with high order of vanishing moment are constructed in [22]. Also note that the lifting scheme, which is proposed in [112] and is essentially linked to biorthogonal wavelets, leads to several multivariate constructions, see e.g. [59,79,110]. Multivariate dual wavelet frame constructions via a projection method are proposed in [65]. Construction 5.2 for multivariate dual wavelet frames is simple since it only involves a constant matrix inversion.

Construction 5.2 can start from any given finitely supported real-valued refinement mask satisfying (5.52) and we now illustrate it by some examples. Example 5.8 is based on a piecewise linear box spline. The refinement mask used in Example 5.9 is derived from the butterfly subdivision scheme, see [51], while Example 5.10 starts from an interpolatory refinable function derived from a box spline, see [94]. Note that the latter two examples use interpolatory refinement masks containing negative entries, which cannot be used for the multivariate tight wavelet construction previously presented in [52]. In all the examples, the primary wavelet masks are defined based on discrete first or second order difference operators along certain directions.

Example 5.8 Starting from the box spline of the three directions $\{(1,0)^\top, (0,1)^\top, (1,1)^\top\}$ given by the mask

$$\frac{1}{8}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

we choose the 6 primary wavelet masks

$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\frac{1}{4}\begin{pmatrix} 0 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

These masks correspond to wavelets with certain directions. The dual refinement mask again is a_0 while the dual wavelets obtained from Construction 5.2 have the masks

$$\frac{1}{8}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \frac{1}{8}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \frac{1}{8}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\frac{1}{16}\begin{pmatrix} 0 & -3 & 1 \\ 1 & 2 & 1 \\ 1 & -3 & 0 \end{pmatrix}, \frac{1}{16}\begin{pmatrix} 0 & 1 & -3 \\ 1 & 2 & 1 \\ -3 & 1 & 0 \end{pmatrix}, \frac{1}{16}\begin{pmatrix} 0 & 1 & 1 \\ -3 & 2 & -3 \\ 1 & 1 & 0 \end{pmatrix}.$$

Example 5.9 The butterfly subdivision scheme, widely used in computer graphics, has first been proposed in [51]. If this subdivision scheme is applied on a regular grid with both coordinates indexed by integers, it corresponds to the refinement mask

$$\frac{1}{64}\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 2 & 8 & 8 & 2 & -1 \\ 0 & 0 & 8 & 16 & 8 & 0 & 0 \\ -1 & 2 & 8 & 8 & 2 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This mask satisfies condition (5.52) and hence can be used in Construction 5.2. In total we construct 24 primary wavelets masks, which can be found in Appendix 1. The dual wavelet masks can be computed from the primary masks by matrix inversion as described in Construction 5.2. Since the support of the refinement mask a_0 is large, the primary wavelets can cover a wide range of directions.

Example 5.10 Several interpolatory refinable functions have been constructed in [94] by using box splines. The mask of the interpolatory refinable function constructed using the box spline of the three directions $\{(1,0)^\top, (0,1)^\top, (1,1)^\top\}$ with multiplicity 2 is

$$\frac{1}{256}\begin{pmatrix} 0 & 0 & 0 & -1 & -3 & -3 & -1 \\ 0 & 0 & -3 & 0 & 6 & 0 & -3 \\ 0 & -3 & 6 & 33 & 33 & 6 & -3 \\ -1 & 0 & 33 & 64 & 33 & 0 & -1 \\ -3 & 6 & 33 & 33 & 6 & -3 & 0 \\ -3 & 0 & 6 & 0 & -3 & 0 & 0 \\ -1 & -3 & -3 & -1 & 0 & 0 & 0 \end{pmatrix},$$

which satisfies condition (5.52) and hence can be used in Construction 5.2. In total we construct 30 primary wavelets masks, see Appendix 2. The dual wavelet masks again can be computed via matrix inversion according to Construction 5.2. The support of the masks derived in this example is larger and hence more directions can be covered by the wavelets.

5.4 Filter Bank Revisted

In the wavelet literature one usually constructs tight or dual wavelet frames for given MRAs, i.e. the refinable function and its mask are already prescribed. We have for example done so in Sect. 5.2, using the connection between tight/dual MRA-wavelet frames and filter banks as described in Sect. 5.1. Here we consider a different perspective and ask whether for a given filter bank, regardless of how it is constructed, there is an underlying MRA-wavelet frame system in $L_2(\mathbb{R}^d)$ whose masks are the given filter bank. In general this is a hard question unless one likes to go to Sobolev spaces, see e.g. [67]. However, when the filter bank satisfies the UEP condition the answer is positive.

Let $\{a_l\}_{l=0}^r$ be a filter bank of finitely supported filters. Suppose this filter bank satisfies the UEP condition for subsampling rate 2, i.e.

$$2^d \sum_{l=0}^{r} \sum_{k\in\mathbb{Z}^d} a_l(n+2k+\ell)\overline{a_l(2k+\ell)} = \delta_{n,0} \quad \text{for any} \quad n, \ell \in \mathbb{Z}^d,$$

or equivalently in Fourier domain

$$\sum_{l=0}^{r} \hat{a}_l(\omega)\overline{\hat{a}_l(\omega+\nu)} = \delta_{\nu,0} \tag{5.53}$$

for all $\nu \in \{0,\pi\}^d$ and a.e. $\omega \in \mathbb{T}^d$. By (5.53), we have $r \geq 2^d$ and $\mathcal{M}_{\mathcal{X}}(\omega)$ in (5.49) can be extended to a unitary matrix for a.e. $\omega \in \mathbb{T}^d$. In particular, the norm of any column of this matrix is at most one, i.e.

$$\sum_{\nu\in\{0,\pi\}^d} |\hat{a}_l(\omega+\nu)|^2 \leq 1 \tag{5.54}$$

for a.e. $\omega \in \mathbb{T}^d$ and all $l = 0, \ldots, r$.

Assume one of the filters in the filter bank, say a_0, is a lowpass filter, i.e. $\hat{a}_0(0) = 1$. Then (5.53) automatically implies $\hat{a}_l(0) = 0$ for $l = 1, \ldots, r$ and (5.54) implies $\hat{a}_0(\nu) = 0$ for $\nu \in \{0,\pi\}^d \backslash \{0\}$. As long as we show that the lowpass filter a_0 defines a refinable function $\phi \in L_2(\mathbb{R}^d)$, then $\{\psi_l\}_{l=1}^r$, defined in (5.47) by the filters $\{a_l\}_{l=1}^r$ and this refinable function ϕ, generate a tight MRA-wavelet frame system in $L_2(\mathbb{R}^d)$. Define

$$\hat{\phi}(\omega) := \prod_{j=1}^{\infty} \hat{a}_0(2^{-j}\omega), \quad \omega \in \mathbb{R}^d.$$

It is clear that ϕ is a compactly supported refinable distribution. Using (5.54), one can prove that ϕ is a compactly supported refinable function in $L_2(\mathbb{R}^d)$ with refinement mask a_0. For completeness, we outline the proof which is contained in [21] for the univariate case.

Consider the cascade algorithm defined by

$$\hat{f}_n(\omega) = \hat{a}_0(2^{-1}\omega)\hat{f}_{n-1}(2^{-1}\omega) = \prod_{j=1}^{n} \hat{a}_0(2^{-j}\omega)\hat{f}_0(2^{-n}\omega), \quad \omega \in \mathbb{R}^d,$$

with $\hat{f}_0 = \chi_{\mathbb{T}^d}$. The pointwise limit $\hat{\phi}$ of $\{\hat{f}_n\}_{n\in\mathbb{N}}$ clearly satisfies the refinement equation $\hat{\phi}(2\cdot) = \hat{a}_0\hat{\phi}$. That ϕ is a function in $L_2(\mathbb{R}^d)$ is guaranteed by the UEP, more precisely by (5.54) which implies that $\{\hat{f}_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $L_2(\mathbb{R}^d)$. Indeed,

$$\|\hat{f}_n\|^2 = \int_{2^n(0,\pi)^d} \prod_{j=1}^{n-1} |\hat{a}_0(2^{-j}\omega)|^2 \sum_{\nu\in\{0,\pi\}^d} |\hat{a}_0(2^{-n}\omega+\nu)|^2\, d\omega \le \|\hat{f}_{n-1}\|^2$$

for all $n \ge 1$, thus $\|\hat{f}_n\| \le \|\hat{f}_0\| = (2\pi)^d$ for all $n \ge 1$ by induction. Since $\{\hat{f}_n\}_{n\in\mathbb{N}}$ converges pointwise to $\hat{\phi}$, Fatou's lemma implies $\|\hat{\phi}\| \le \liminf_{n\to\infty} \|\hat{f}_n\| < \infty$. Thus $\phi \in L_2(\mathbb{R}^d)$.

6 Translation-Invariant Transforms

The Bessel, frame and Riesz properties of systems in Hilbert spaces are properties of the transforms they define, specifically of their analysis operators. In case of function spaces, say $L_2(\mathbb{R}^d)$, transforms that are translation-invariant, i.e. commute with translations, are of particular interest in signal processing applications. Any translation-invariant bounded linear operator on $L_2(\mathbb{R}^d)$ is given by a convolution with a tempered distribution whose Fourier transform is essentially bounded, with its essential bound being equal to the norm of the operator, see [70]. In this section we turn our attention to sequences of convolution operators and consider translation-invariant transforms of the form

$$T^*_{EX}\colon L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d, X)\colon f \mapsto (t \mapsto \langle f, E^t x\rangle)_{x\in X}, \tag{6.55}$$

where X is a system of functions in $L_2(\mathbb{R}^d)$ and

$$L_2(\mathbb{R}^d, X) := \{\tau = (\tau_x)_{x\in X}\colon \tau_x \in L_2(\mathbb{R}^d),\ (\|\tau_x\|)_{x\in X} \in \ell_2(X)\}$$

with inner product

$$\langle \tau, \tau'\rangle_{L_2(\mathbb{R}^d,X)} := \sum_{x\in X} \langle \tau_x, \tau'_x\rangle$$

and Fourier transform defined by $\hat{\tau} := (\widehat{\tau_x})_{x\in X}$. While $EX := \{E^t x\}_{x\in X, t\in\mathbb{R}^d}$ is not a system, i.e. not countable, we characterize the Bessel, frame and Riesz properties for the transform T^*_{EX} along the lines of the fiber dual Gramian analysis for shift-invariant systems. Again, a key observation is that applying a unitary transform significantly simplifies the analysis. After Fourier transform the operators have representations by families of simple matrices. We reviewed the fiber pre-Gramian matrices of shift-invariant systems in (4.24) and again refer to the original work [98] for the full picture. In the case of translation-invariant transforms the fiber dual Gramian analysis takes a rather simple form. The pre-Gramian degenerates to a row vector and thus the dual Gramian matrices to a Fourier multiplier, in the self-adjoint case, in fact, to the multiplier function guaranteed by the spectral theorem. This can be loosely interpreted as a limiting case of the dual Gramian analysis for shift-invariant systems in which the shift-lattice is of infinite density, i.e. zero volume, and therefore the dual lattice has zero density. For the special case of shift-invariant Gabor transforms, we show the duality principle between the transform and a shift-invariant system of a single generator. In case of the translation-invariant wavelet transform, we observe that the tight frame characterization is precisely the diagonal condition of the MEP matrix condition. This condition is sometimes erroneously used for the undecimated wavelet transform. However, the undecimated wavelet transform implicitly uses the quasi-affine system and so the perfect reconstruction condition needs the full MEP condition, see Sect. 6.3.

6.1 General

Let X be a system in $L_2(\mathbb{R}^d)$ and consider

$$T_{EX}\colon L_{c,0}(\mathbb{R}^d, X) \to L_2(\mathbb{R}^d)\colon \tau \mapsto \sum_{x\in X} \int_{\mathbb{R}^d} \tau_x(t) E^t x(\cdot)\, dt,$$

where

$$L_{c,0}(\mathbb{R}^d, X) := \{(\tau_x)_{x\in X} \subset L_2(\mathbb{R}^d)\colon \operatorname{supp}\tau_x \text{ is compact}, \ (\|\tau_x\|)_{x\in X} \in \ell_0(X)\}.$$

If T_{EX} is bounded, we consider it as its unique extension to a bounded operator on $L_2(\mathbb{R}^d, X)$. The operator T^*_{EX} in (6.55) is bounded if and only if T_{EX} is bounded. In this case, T^*_{EX} is the adjoint operator of T_{EX} and we call it a *Bessel transform*. If T^*_{EX} is a Bessel transform that is bounded below on $L_2(\mathbb{R}^d)$, we call it a *frame transform*, and in case the lower bound $\|(T^*_{EX})^\dagger\|^{-1}$ is equal to the bound $\|T^*_{EX}\|$, we call it a *tight frame transform*. Note that T^*_{EX} is a frame transform if and only if $T_{EX}T^*_{EX}$ is bounded and has a bounded inverse and that it is a tight frame transform if and only if $T_{EX}T^*_{EX}$ is the identity on $L_2(\mathbb{R}^d)$.

For $\tau \in L_{c,0}(\mathbb{R}^d, X)$, taking Fourier transform yields the representation

$$(T_{EX}\tau)^\wedge(\omega) = \sum_{x\in X} \widehat{x}(\omega)\widehat{\tau_x}(\omega) = \mathcal{J}_{EX}(\omega)\hat{\tau}(\omega) \tag{6.56}$$

for a.e. $\omega \in \mathbb{R}^d$, where the pre-Gramian matrix is defined as the row vector

$$\mathcal{J}_{EX}(\omega) := (\widehat{x}(\omega))_{x\in X}.$$

If T_{EX} is bounded, then (6.56) holds for all $\tau \in L_2(\mathbb{R}^d, X)$ and

$$(T^*_{EX} f)^\wedge(\omega) = \mathcal{J}^*_{EX}(\omega)\hat{f}(\omega)$$

for all $f \in L_2(\mathbb{R}^d)$ and a.e. $\omega \in \mathbb{R}^d$. Then

$$\|T^*_{EX} f\|^2 = \sum_{x\in X} \int_{\mathbb{R}^d} |\langle f, E^t x\rangle|^2\, dt = (2\pi)^{-d} \int_{\mathbb{R}^d} \sum_{x\in X} |\hat{x}(t)|^2 |\hat{f}(t)|^2\, dt \tag{6.57}$$

for all bandlimited $f \in L_2(\mathbb{R}^d)$, which might be equal to infinity. If, however, $\|T^*_{EX} f\| < \infty$ for all $f \in L_2(\mathbb{R}^d)$, then this implies that the dual Gramian

$$\widetilde{\mathcal{G}}_{EX,EX}(\omega) := \mathcal{J}_{EX}(\omega)\mathcal{J}^*_{EX}(\omega) = \sum_{x\in X} |\hat{x}(\omega)|^2$$

is finite for a.e. $\omega \in \mathbb{R}^d$. If T^*_{EX} is a Bessel transform, then

$$(T_{EX}T^*_{EX} f)^\wedge(\omega) = \widetilde{\mathcal{G}}_{EX,EX}(\omega)\hat{f}(\omega) \tag{6.58}$$

for all $f \in L_2(\mathbb{R}^d)$ and a.e. $\omega \in \mathbb{R}^d$. Further, if T^*_{EX} and T^*_{EY} are bounded, where $Y = RX$ and $R\colon X \to L_2(\mathbb{R}^d)$ is a map, then the mixed dual Gramian

$$\widetilde{\mathcal{G}}_{EY,EX}(\omega) := \mathcal{J}_{EY}(\omega)\mathcal{J}^*_{EX}(\omega) = \sum_{x\in X} \widehat{Rx}(\omega)\overline{\hat{x}(\omega)}$$

is finite for a.e. $\omega \in \mathbb{R}^d$ by the Cauchy–Schwarz inequality.

Theorem 6.1 *Let X be a system in $L_2(\mathbb{R}^d)$. Then the following holds.*

(i) T^*_{EX} is a Bessel transform if and only if $\widetilde{\mathcal{G}}_{EX,EX} \in L_\infty(\mathbb{R}^d)$. In this case $\|T^*_{EX}\|^2 = \|\widetilde{\mathcal{G}}_{EX,EX}\|_{L_\infty}$.

(ii) If T^*_{EX} is a Bessel transform, then it is a frame transform if and only if $1/\widetilde{\mathcal{G}}_{EX,EX} \in L_\infty(\mathbb{R}^d)$, where $1/\widetilde{\mathcal{G}}_{EX,EX}\colon \omega \mapsto 1/\widetilde{\mathcal{G}}_{EX,EX}(\omega)$ and with the conventions $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$. In this case $\|(T^*_{EX})^\dagger\|^{-2} = \|1/\widetilde{\mathcal{G}}_{EX,EX}\|_{L_\infty}$.

In particular, T^*_{EX} is a tight frame transform if and only if $\widetilde{\mathcal{G}}_{EX,EX}(\omega) = 1$ for a.e. $\omega \in \mathbb{R}^d$. If T^*_{EX} and T^*_{EY} are bounded, where $Y = RX$ and $R\colon X \to L_2(\mathbb{R}^d)$ is a map, then $T_{EY}T^*_{EX}$ is the identity on $L_2(\mathbb{R}^d)$ if and only if $\widetilde{\mathcal{G}}_{EY,EX}(\omega) = 1$ for a.e. $\omega \in \mathbb{R}^d$.

Proof If $\widetilde{\mathcal{G}}_{EX,EX} \in L_\infty(\mathbb{R}^d)$, then $\|T^*_{EX} f\|^2 \le \|\widetilde{\mathcal{G}}_{EX,EX}\|_{L_\infty}\|f\|^2_{L_2}$ for all bandlimited $f \in L_2(\mathbb{R}^d)$, thus T^*_{EX} is a Bessel transform and $\|T^*_{EX}\|^2 \le \|\widetilde{\mathcal{G}}_{EX,EX}\|_{L_\infty}$. If, on the other hand, T^*_{EX} is a Bessel transform, then (6.57) implies

$$\int_{\mathbb{R}^d} \left(\|T^*_{EX}\|^2 - \sum_{x\in X} |\hat{x}(t)|^2 \right) |\hat{f}(t)|^2\, dt \ge 0$$

for all bandlimited $f \in L_2(\mathbb{R}^d)$ and thus $\widetilde{\mathcal{G}}_{EX,EX} \in L_\infty(\mathbb{R}^d)$ with $\|\widetilde{\mathcal{G}}_{EX,EX}\|_{L_\infty} \le \|T^*_{EX}\|^2$. The remaining statements now follow from (6.58). □

In case T^*_{EX} is a frame transform, $Y = \{Rx\colon \widehat{Rx} = \hat{x}/\widetilde{\mathcal{G}}_{EX,EX}, x \in X\}$ is the canonical choice to yield $T_{EY}T^*_{EX} = I$ on $L_2(\mathbb{R}^d)$.

6.2 Translation-Invariant Gabor Transforms

Now let $K \subset \mathbb{R}^d$ be a lattice, $\phi \in L_2(\mathbb{R}^d)$ some window function and

$$X = \{M^k \phi : k \in K\}.$$

Then T^*_{EX} is the short-time Fourier transform after discretizing the modulations. The pre-Gramian

$$\mathcal{J}_{EX}(\omega) = (E^k \hat{\phi}(\omega))_{k \in K}$$

degenerates to a row vector composed by shifts of a single generator and thus can also be realized as the pre-Gramian of a shift-invariant system. Indeed

$$\mathcal{J}_{X^*}(\omega) = \overline{\mathcal{J}^*_{EX}}(\omega)$$

for the $\widetilde{K}$-shift-invariant system $X^* = \{|K|^{1/2} E^{\tilde{k}} \phi : \tilde{k} \in \tilde{K}\}$, which therefore is an adjoint system. This adjoint relationship can be interpreted as a limiting case of the situation for regular Gabor systems as follows. If T^*_{EX} is a translation-invariant transform, then it corresponds to a Gabor system with shift lattice of infinite density. Its dual lattice becomes the modulation lattice of the adjoint system. In this case, however, the dual lattice has density zero and thus no modulations appear in the adjoint system. The dual Gramian is given by

$$\widetilde{\mathcal{G}}_{EX,EX}(\omega) = \sum_{k \in K} |E^k \hat{\phi}(\omega)|^2,$$

which by the adjoint relationship is equal to the complex conjugate of the Gramian $\mathcal{J}^*_{X^*}(\omega)\mathcal{J}_{X^*}(\omega)$ for a.e. $\omega \in \mathbb{R}^d$. The duality principle implies the following.

Theorem 6.2 *Given* $X = \{M^k \phi : k \in K\}$ *and* $X^* = \{|K|^{1/2} E^{\tilde{k}} \phi : \widetilde{k} \in \widetilde{K}\}$*, where* $K \subset \mathbb{R}^d$ *is a lattice and* $\phi \in L_2(\mathbb{R}^d)$*, the following holds.*

(i) T^*_{EX} *is a Bessel transform if and only if* X^* *is a Bessel system.*
(ii) T^*_{EX} *is a frame transform if and only if* X^* *is a Riesz sequence. In this case the corresponding bounds coincide.*
(iii) In particular, T^*_{EX} *is a tight frame transform if and only if* X^* *is an orthonormal sequence.*

The analogue statement holds for the case of two windows.

Theorem 6.3 *Let* $X_\phi = \{M^k \phi : k \in K\}$ *and* $X_\psi = \{M^k \psi : k \in K\}$ *be Bessel systems, where* $K \subset \mathbb{R}^d$ *is a lattice and* $\phi, \psi \in L_2(\mathbb{R}^d)$*. Further, let* $X^*_\phi = \{|K|^{1/2} E^{\widetilde{k}} \phi : \widetilde{k} \in \widetilde{K}\}$ *and* $X^*_\psi = \{|K|^{1/2} E^{\widetilde{k}} \psi : \widetilde{k} \in \widetilde{K}\}$*. Then* $T_{EX_\phi} T_{EX_\psi}$ *is the identity if and only if* X^*_ϕ *and* X^*_ψ *are biorthonormal sequences.*

One might of course discretize the translation parameter in the short-time Fourier transform, while keeping the modulation parameter continuous. This however, after Fourier transformation, is again the above situation since time domain modulations are frequency shifts and vice versa.

6.3 Translation-Invariant Wavelet Transforms

Analysis and reconstruction from dilation samples of the continuous wavelet transform (5.46) leads to considering operators on $L_2(\mathbb{R}^d)$ which are of the form

$$f \mapsto \sum_{\psi\in\Psi}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^d} W_\Psi f(k,t) D^k E^t \psi \, dt = T_{EX} T_{EX}^* f,$$

where

$$X = X(\Psi) := \left\{ 2^{kd/2} D^k \psi \colon k \in \mathbb{Z}, \psi \in \Psi \right\}$$

and Ψ is a finite system in $L_2(\mathbb{R}^d)$. The properties of such operators can be addressed by considering the dual Gramian

$$\widetilde{\mathcal{G}}_{EX,EX}(\omega) = \sum_{\psi\in\Psi}\sum_{k\in\mathbb{Z}} |\hat{\psi}(2^{-k}\omega)|^2, \tag{6.59}$$

which is a.e. finite if, for example, all wavelets have first order vanishing moments, i.e. $|\hat{\psi}| = O(|\cdot|^{-1})$ for all $\psi \in \Psi$. More generally, if the wavelets in Ψ and $R\Psi$ have first order vanishing moments, where $R\colon \Psi \to L_2(\mathbb{R}^d)$ is a map, then the mixed dual Gramian

$$\widetilde{\mathcal{G}}_{EX(R\Psi),EX(\Psi)}(\omega) = \sum_{\psi\in\Psi}\sum_{k\in\mathbb{Z}} \widehat{R\psi}(2^{-k}\omega)\overline{\hat{\psi}(2^{-k}\omega)} \tag{6.60}$$

is a.e. well defined and the dual Gramian analysis of Theorem 6.1 on general translation-invariant transforms can be applied. The dual Gramians (6.59) and (6.60) appear as the diagonal of the dual Gramian and mixed dual Gramian of the quasi-affine system introduced in [100,101] (see also (5.48)). We will now follow that analysis.

Of particular interest is again the case of MRA-wavelets, say $\Psi = \{\psi_l\}_{l=1}^r$, generated from masks $\{a_l\}_{l=1}^r$ in $\ell_2(\mathbb{Z}^d)$ and a compactly supported refinable function $\phi \in L_2(\mathbb{R}^d)$ with mask $a_0 \in \ell_2(\mathbb{Z}^d)$. The dual Gramian in this case is

$$\widetilde{\mathcal{G}}_{EX,EX}(\omega) = \sum_{\psi\in\Psi}\sum_{k=-n}^{\infty} \left| \hat{\psi}\left(2^{-k}\omega)\right|^2 + \Theta(2^n\omega)|\hat{\phi}(2^n\omega)\right|^2$$

for any $n \in \mathbb{Z}$, with the *fundamental function*

$$\Theta(\omega) := \sum_{k=0}^{\infty}\sum_{l=1}^{r} |\hat{a}_l(2^k\omega)|^2 \prod_{j=0}^{k-1} |\hat{a}_0(2^j\omega)|^2$$

of $\{a_l\}_{l=0}^r$ as introduced in [101]. If $\lim_{\omega\to 0}\hat{\phi}(\omega) = 1$, then $\widetilde{\mathcal{G}}_{EX,EX} = 1$ if and only if $\lim_{n\to-\infty}\Theta(2^n\omega) = 1$ for a.e. ω. The latter in particular holds for masks satisfying $\sum_{l=0}^r |\hat{a}_l|^2 = 1$ since then $\Theta = 1$, see [101].

Theorem 6.4 *Let $\Psi = \{\psi_l\}_{l=1}^r$ be MRA-wavelets in $L_2(\mathbb{R}^d)$, with masks $\{a_l\}_{l=0}^r$ and compactly supported refinable function ϕ such that $\lim_{\omega\to 0}\hat{\phi}(\omega) = 1$. Then $T^*_{EX(\Psi)}$ is a tight frame transform if and only if $\lim_{n\to-\infty}\Theta(2^n\omega) = 1$ for a.e. ω. In particular, $T^*_{EX(\Psi)}$ is a tight frame transform whenever $\sum_{l=0}^r |\hat{a}_l|^2 = 1$.*

The fundamental function satisfies $\Theta = |\hat{a}_0|^2\Theta(2\cdot) + \sum_{l=1}^r |\hat{a}_l|^2$. Conversely, if Θ' is nonnegative, essentially bounded, 2π-periodic and continuous at the origin with $\Theta'(0) = 1$ such that $\Theta' = |\hat{a}_0|^2\Theta'(2\cdot) + \sum_{l=1}^r |\hat{a}_l|^2$, then Θ' is the fundamental function of $\{a_l\}_{l=0}^r$ and $\tilde{\mathcal{G}}_{EX} = 1$, see [41].

Theorem 6.5 *Let $\Psi = \{\psi_l\}_{l=1}^r$ be MRA-wavelets in $L_2(\mathbb{R}^d)$, with masks $\{a_l\}_{l=0}^r$ and compactly supported refinable function ϕ such that $\lim_{\omega\to 0}\hat{\phi}(\omega) = 1$. Then $T^*_{EX(\Psi)}$ is a tight frame transform whenever there exists a nonnegative, essentially bounded, 2π-periodic function Θ' which is continuous at the origin with $\Theta'(0) = 1$, and such that*

$$|\hat{a}_0(\omega)|^2\Theta'(2\omega) + \sum_{l=1}^r |\hat{a}_l(\omega)|^2 = \Theta'(\omega)$$

for a.e. $\omega \in \mathbb{T}^d$.

This result can be generalized to the case of *dual* transforms. The arguments are similar, working with the mixed dual Gramian and a mixed fundamental function. The familiar difference is that, for those functions to be a.e. well-defined, one now has to assume the transforms to be Bessel transforms to begin with, see [41].

Theorem 6.6 *Let $\Psi = \{\psi_l\}_{l=1}^r$ and $\tilde{\Psi} = \{\tilde{\psi}_l\}_{l=1}^r$ be MRA-wavelets in $L_2(\mathbb{R}^d)$ with masks $\{a_l\}_{l=0}^r$, resp. $\{b_l\}_{l=0}^r$, and compactly supported refinable functions ϕ, resp. $\tilde{\phi}$, such that $\lim_{\omega\to 0}\hat{\phi}(\omega) = \lim_{\omega\to 0}\hat{\tilde{\phi}}(\omega) = 1$. Assume that $T^*_{EX(\Psi)}$ and $T^*_{EX(\tilde{\Psi})}$ are Bessel transforms. Then $T_{EX(\Psi)}T^*_{EX(\tilde{\Psi})}$ is the identity on $L_2(\mathbb{R}^d)$ whenever there exists a nonnegative, essentially bounded, 2π-periodic function Θ' which is continuous at the origin with $\Theta'(0) = 1$, and such that*

$$\hat{a}_0(\omega)\overline{\hat{b}_0(\omega)}\Theta'(2\omega) + \sum_{l=1}^r \hat{a}_l(\omega)\overline{\hat{b}_l(\omega)} = \Theta'(\omega)$$

for a.e. $\omega \in \mathbb{T}^d$. Sufficient for $T_{EX(\Psi)}T^*_{EX(\tilde{\Psi})}$ to be the identity on $L_2(\mathbb{R}^d)$ is that $\sum_{l=0}^r \hat{a}_l\overline{\hat{b}_l} = 1$.

The condition $\sum_{l=0}^r \hat{a}_l\overline{\hat{b}_l} = 1$ in Theorem 6.6 is a part of the MEP condition (5.50), namely the equations for $\nu = 0$. It is, in other words, the diagonal condition on the mixed dual Gramian $\mathcal{M}_{\mathcal{X}}\mathcal{M}^*_{\mathcal{Y}}$. In spatial domain this is equivalent to

$$\sum_{l=0}^{r} \sum_{k\in\mathbb{Z}^d} a_l(n+k+\ell)\overline{b_l(k+\ell)} = \delta_{n,0} \quad \text{for any} \quad n, \ell \in \mathbb{Z}^d,$$

which, for finitely supported masks, is the dual Gramian characterization of the filter banks $X(\{a_l\}_{l=0}^r, 1)$ and $X(\{b_l\}_{l=0}^r, 1)$ to be dual frames, see Sect. 3.3. Recall that for filter banks with subsampling rate 2, the dual Gramian condition (5.51) is (under certain additional assumptions) associated to the dual frame property of the affine and quasi-affine system, i.e. of discrete wavelet systems. For higher subsampling rates it relates to different dilations of discrete wavelet systems. In contrast, the dual frame characterization for non-subsampled filter banks corresponds to the dual frame property of the dyadic translation-invariant wavelet transform, i.e. only the diagonal condition of the MEP condition. Masks that satisfy this weaker condition do not give rise to quasi-affine, i.e. discrete shift-invariant tight frame systems in $L_2(\mathbb{R}^d)$, but to dual frames in $\ell_2(\mathbb{Z}^d)$. This suffices in certain applications, e.g. in [46] those masks are linked to certain differential operators. However, the undecimated wavelet transform, i.e. the fast wavelet decomposition algorithm without downsampling as in e.g. [32,69,88,108], implicitly uses the quasi-affine system and the diagonal condition of the MEP condition is not enough to guarantee the tight or dual frame property of the quasi-affine systems.

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Appendix 1: Primary Wavelet Masks of Example 5.9

$$\frac{1}{2}\begin{pmatrix} 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0 \end{pmatrix},$$

$$\frac{1}{2}\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix},$$

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Appendix 2: Primary Wavelet Masks of Example 5.10

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\frac{1}{2}\begin{pmatrix}0&0&0&0&0&1&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&-1&0&0&0&0&0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\-1&0&0&0&0&0&0\end{pmatrix},$$

$$\frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},$$

$$\frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\-1&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&-1&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},$$

$$\frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&0&0&0\\0&0&0&-1&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},$$

$$\frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&0\\0&-1&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},$$

$$\frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\-1&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\1&0&0&0&0&0&-1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},$$

$$\frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&1&0&-1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0&0&0&-1&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&-1&0&0&0\end{pmatrix},$$

$$\frac{1}{4}\begin{pmatrix}0&0&0&0&-1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0&0&0&0&0&-1&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&-1&0&0&0&0&0\end{pmatrix},$$

$$\frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&-1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\-1&0&0&0&0&0&0\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&0&0\end{pmatrix},$$

$$\frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&-1\\0&0&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&0&0\\-1&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},$$

$$\frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&-1&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&-1&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},$$

$$\frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&-1&0&0&0\\0&0&0&2&0&0&0\\0&0&0&-1&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&2&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},$$

$$
\frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 2 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 2 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
\frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 2 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 2 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
$$

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