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Factoring Matrices into the Product of Circulant and Diagonal Matrices

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Abstract A generic matrix $A \in \mathbb{C}^{n \times n}$ is shown to be the product of circulant and diagonal matrices with the number of factors being $2n - 1$ at most. The demonstration is constructive, relying on first factoring matrix subspaces equivalent to polynomials in a permutation matrix over diagonal matrices into linear factors. For the linear factors, the sum of two scaled permutations is factored into the product of a circulant matrix and two diagonal matrices. Extending the monomial group, both low degree and sparse polynomials in a permutation matrix over diagonal matrices, together with their permutation equivalences, constitute a fundamental sparse matrix structure. Matrix analysis gets largely done polynomially, in terms of permutations only.

Keywords Circulant matrix · Diagonal matrix · Sparsity structure · Matrix factoring · Polynomial factoring · Multiplicative Fourier compression

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1 Introduction

There exists an elegant result, motivated by applications in optical image processing, stating that any matrix $A \in \mathbb{C}^{n \times n}$ is the product of circulant and diagonal matrices [15,18].[1] In this paper it is shown that, generically, $2n - 1$ factors suffice. (For various aspects of matrix factoring, see [13].) The demonstration is constructive, relying on first factoring matrix subspaces equivalent to polynomials in a permutation matrix over diagonal matrices into linear factors. This is achieved by solving structured systems of polynomial equations. Located on the borderline between commutative and non-commutative algebra, such subspaces are shown to constitute a fundamental sparse matrix structure of polynomial type extending, e.g., band matrices. In particular, now matrix analysis gets largely done entirely polynomially. Then for the linear factors, a factorization for the sum of two PD matrices into the product of a circulant matrix and two diagonal matrices is derived.

A scaled permutation, also called a PD matrix, is the product of a permutation and a diagonal matrix. In the invertible case we are dealing with the monomial group, giving rise to the sparsest possible nonsingular matrix structure. A way to generalize this is to allow more nonzero entries per line by considering sums of PD matrices. The sum of two PD matrices can be analyzed in terms of permutation equivalence which turns out to be instrumental for extending the structure. Although the notion of permutation equivalence is graph theoretically nonstandard, combinatorial linear algebraically it is perfectly natural [2, p. 4]. There arises a natural concept of cycles which can be used to show that the inverse of a nonsingular sum of two PD matrices carries a very special structure and can be inexpensively computed.

To extend the set of sums of two PD matrices in a way which admits factoring, a polynomial structure in permutations is suggested. That is, let P be a permutation matrix and denote by p a polynomial over diagonal matrices. Define matrix subspaces of $\mathbb{C}^{n \times n}$ as

$$P_1 \left\{ p(P) \,\middle|\, \deg(p) \leq j \right\} P_2 \tag{1.1}$$

with fixed permutations P_1 and P_2. This provides a natural extension by the fact that the case $j = 0$ corresponds to PD matrices while $j = 1$ yields the sums of two PD matrices. The case $j = 2$ covers, e.g., finite difference matrices, including periodic problems. In this manner, whenever $j \ll n$, the sparsity pattern of such a matrix subspace carries an intrinsic polynomial structure which can be used to analyze sparsity more generally in terms of the so-called polynomial permutation degree. For an equally natural option, the notion of sparse polynomial can be analogously adapted to (1.1), i.e., allow j to be large but require most of the coefficients to be zeros. (For sparse polynomials, see [17] and references therein.) In any case, now matrix analysis gets largely done polynomially, in terms of powers of a permutation. Namely, completely analogously to univariate complex polynomials, these subspaces admit

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[1] In particular, any unitary matrix $A \in \mathbb{C}^{n \times n}$ can be interpreted as being a diffractive optical system. See [15] how products of discrete Fourier transforms and diagonal matrices model diffractive optical elements.

factoring. To factor (1.1) into linear factors, it turns out that it suffices to consider the problem of factoring polynomials in the cyclic shift[2] over diagonal matrices.

Let P thus be the cyclic shift and set $P_1 = P_2 = I$. Then for any $A \in \mathbb{C}^{n \times n}$ there exists a unique polynomial p over diagonal matrices of degree $n - 1$ at most such that $p(P) = A$. With this representation, the problem of factoring A into the product of circulant and diagonal matrices converts into the problem of factoring p into linear factors. For a generic matrix this is possible (see Theorem 4.3) through consecutively solving systems of polynomial equations. Quite intriguingly, this allows regarding matrices as polynomials which have been factored. In particular, a linear factor is, generically, the product of two diagonal matrices and a circulant matrix. Consequently, once this factoring process has been completed, we have

$$A = D_1 C_2 D_3 \cdots D_{2n-3} C_{2n-2} D_{2n-1} \qquad (1.2)$$

with diagonal and circulant matrices D_{2j-1} and C_{2j} for $j = 1, \ldots, n$. Or, alternatively, purely Fourier analytically one can view this as a factorization involving discrete Fourier transforms and diagonal matrices.[3]

The paper is organized as follows. Section 2 is concerned with the set of sums of two PD matrices. Their inversion is analyzed. A link with the so-called $\mathcal{DCD}$ matrices in Fourier optics is established. In Sect. 3, to extend the set of sums of two PD matrices, polynomials in a permutation matrix over diagonal matrices are considered. Sparsity of matrices is then measured in terms this polynomial structure. Section 4 is concerned with factoring polynomials in a permutation over diagonal matrices into first degree factors. Factorization algorithms are devised. A solution to the problem of factoring into the product of circulant and diagonal matrices is provided. A conjecture on the optimal number of factors is made together with related Fourier compression problems.

2 The Sum of Two PD Matrices

This section is concerned with extending diagonal matrices to PD matrices, the set of scaled permutations $\mathcal{PD}$. Once done, we consider matrices consisting of the sum of two PD matrices. Here $\mathcal{P}$ denotes the set of permutations and $\mathcal{D}$ the set of diagonal matrices. In the invertible case we are dealing with the following classical matrix group.

Definition 2.1 By monomial matrices is meant the group consisting of matrix products of permutation matrices with nonsingular diagonal matrices.

The group property is based on the fact that if P is a permutation and D a diagonal matrix, then

$$DP = PD^P, \qquad (2.1)$$

where $D^P = P^T D P$ is a diagonal matrix as well. It turns out that this "structural" commutativity allows doing practically everything the usual commutativity does.

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[2] The cyclic shift of size n-by-n has ones below the main diagonal and at the position $(1, n)$.

[3] This is a more operator theoretic formulation admitting an extension to infinite dimensions [12].

Regarding applications, monomial matrices appear in representation theory [5,19] and in numerical analysis of scaling and reordering linear systems of equations [9]. See also [6, Chapter 5.3] for a link with circulant matrices. It is noteworthy that the monomial group is maximal in the general linear group of $\mathbb{C}^{n \times n}$ [8].

The following underscores that PD matrices provide a natural extension of diagonal matrices.

Definition 2.2 [1] A generalized diagonal of $A \in \mathbb{C}^{n \times n}$ is obtained by retaining exactly one entry from each row and each column of A.

To put this into perspective in view of normality, observe that $\mathcal{PD}$ is closed under taking the Hermitian transpose. Thereby, conforming with Definition 2.2, its unitary orbit

$$\{U\mathcal{PD}U^* \,|\, UU^* = I\} \tag{2.2}$$

leads to the respective notion of extended normality. This is supported by the fact that, like for normal matrices, the eigenvalue problem for PD matrices can be regarded as being completely understood; see [6, Chapter 5.3]. To actually recover whether a given matrix $A \in \mathbb{C}^{n \times n}$ belongs to (2.2), compute the singular value decomposition $A = U\Sigma V^*$ of A and look at V^*U.[4]

PD matrices can be regarded as belonging to a more general sparse matrix hierarchy defined as follows.

Definition 2.3 A matrix subspace $\mathcal{V}$ of $\mathbb{C}^{n \times n}$ is said to be standard if it has a basis of consisting standard basis matrices.[5]

There is a link with graph theory. That is, standard matrix subspaces of $\mathbb{C}^{n \times n}$ are naturally associated with the adjacency matrices of digraphs with n vertices. In particular, the following bears close resemblance to complete matching, underscoring the importance of PD matrices in linear algebra more generally through the determinant. A matrix subspace is said to be nonsingular if it contains invertible elements.

Proposition 2.4 *A matrix subspace $\mathcal{V}$ of $\mathbb{C}^{n \times n}$ is nonsingular only if its sparsity pattern contains a monomial matrix.*

Proof If $A \in \mathbb{C}^{n \times n}$ is invertible, then by expanding the determinant using the Leibniz formula, one term in the sum is necessarily nonzero. The term corresponds to a monomial matrix.

Let us now focus on the sum of two PD matrices. A monomial matrix is readily inverted by separately inverting the factors of the product. For the sum of two PD matrices, a rapid application of the inverse is also possible, albeit with different standard techniques.

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[4] This approach certainly works in the generic case of D having differing diagonal entries in the absolute value. In this paper we do not consider the numerical recovering of whether A belongs to (2.2) in general.

[5] A standard basis matrix of $\mathbb{C}^{n \times n}$ has exactly one entry equaling one while its other entries are zeros.

Proposition 2.5 *Suppose a nonsingular $A \in \mathbb{C}^{n \times n}$ is the sum of two PD matrices. Computing a partially pivoted LU factorization of A costs $O(n)$ operations and requires $O(n)$ storage.*

Proof Any row operation in the Gaussian elimination removes one and brings one element to the row which is being operated. Performing a permutation of rows does not change this fact. Thus, in U there are two elements in each row at most. By the symmetry, there are at most two elements in each column of L.

Monomial matrices have a block analogue. By a block monomial matrix we mean a nonsingular matrix consisting of a permutation matrix which has in place of ones nonsingular matrices of the same size. Zeros are replaced with block zero matrices of the same size. By similar arguments, Proposition 2.5 has an analogue for the sum of two block PD matrices.[6]

The set of sums of two PD matrices, denoted by $\mathcal{PD} + \mathcal{PD}$, is no longer a group. We argue that is has many fundamental properties, though.

Proposition 2.6 $\mathcal{PD} + \mathcal{PD}$ *is closed in $\mathbb{C}^{n \times n}$. Moreover, any $A \in \mathbb{C}^{n \times n}$ is similar to an element of $\mathcal{PD} + \mathcal{PD}$.*

Proof With fixed permutations P_1 and P_2, the matrix subspace

$$\mathcal{V} = \mathcal{D}P_1 + \mathcal{D}P_2. \tag{2.3}$$

is closed. Being a finite union of closed sets (when P_1 and P_2 vary among permutations), the set $\mathcal{PD} + \mathcal{PD}$ is closed as well.

For the claim concerning similarity, it suffices to observe that $\mathcal{PD} + \mathcal{PD}$ contains Jordan matrices.

Suppose $A \in \mathbb{C}^{n \times n}$ is large and sparse. The problem of approximating A with an element of $\mathcal{PD} + \mathcal{PD}$ is connected with preprocessing. In preprocessing the aim is to find two monomial matrices to make $D_1 P_1 A D_2 P_2$ more banded than A; see, e.g., [4,7] and [3, p. 441].[7] Now the permutations P_1 and P_2 in should be picked in such a way that a good approximation to A in (2.3) exists. The reason for this becomes apparent in connection with Theorem 2.7 below.

We have a good understanding of the singular elements of the matrix subspace (2.3). To see this, recall that two matrix subspaces $\mathcal{V}$ and $\mathcal{W}$ are said to be equivalent if there exist nonsingular matrices $X, Y \in \mathbb{C}^{n \times n}$ such that $\mathcal{W} = X \mathcal{V} Y^{-1}$. This is a fundamental notion. In particular, if X and Y can be chosen among permutations, then $\mathcal{V}$ and $\mathcal{W}$ are said to be permutation equivalent. In what follows, by the cyclic shift is meant the permutation

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[6] Block diagonal matrices are used, e.g., in preconditioning. Thereby the sum of two block PD matrices is certainly of interest by providing a more flexible preconditioning structure.

[7] The aim of preprocessing depends, to some degree, on whether one uses iterative methods or sparse direct methods; see [3, p. 438].

$$
S = \begin{bmatrix}
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix}
\tag{2.4}
$$

of unspecified size. When $n = 1$ we agree that $S = I$. The following result, which turns out to be of central relevance in extending $\mathcal{PD} + \mathcal{PD}$, should be contrasted with $(0, 1)$-matrices whose line sum equals 2; see [2, Chapter 1]. Observe that, due to (2.1), $\mathcal{PD} + \mathcal{PD}$ is invariant under permutation equivalence.

Theorem 2.7 *Let $\mathcal{V}$ be the matrix subspace defined in* (2.3). *Then*

$$
\mathcal{V} = \hat{P}_1 (\mathcal{D} + \mathcal{D}P) \hat{P}_2
\tag{2.5}
$$

for permutations $\hat{P}_1$, $\hat{P}_2$ and $P = S_1 \oplus \cdots \oplus S_k$, where S_j denotes a cyclic shift of unspecified size for $j = 1, \ldots, k$.

Proof Start by performing the permutation equivalence

$$
\mathcal{V}P_2^T = \{\mathcal{D}P_1 P_2^T + \mathcal{D}\}.
$$

Then there are cycles associated with the matrix subspace $\mathcal{V}P_2^T$ once we represent $P_1 P_2^T$ by its cycles as $P_1 P_2^T = QPQ^T$ with a permutation Q. Thereby $\mathcal{V} = Q\{\mathcal{D}P + \mathcal{D}\}Q^T P_2$.

Regarding preprocessing, observe that $\mathcal{D} + \mathcal{D}P$ in (2.5) can be regarded as essentially possessing a banded structure.

The dimension of (2.5) is $2n$ if and only if all the cyclic shifts are of size larger than one. These matrix subspaces are sparse which is instrumental for large scale computations. In particular, consider the problem of establishing the number of permutations a matrix subspace with a given sparsity pattern contains. It reflects the minimum number of terms in the Leibniz formula for determinants; see Proposition 2.4. As two extremes, in $P\mathcal{D}$ with a fixed permutation P, there is just one. And, of course, in $\mathbb{C}^{n \times n}$ there are $n!$ permutations.

Corollary 2.8 *There are 2^l permutations in* (2.3)*, where l is the number of cyclic shifts in* (2.5) *of size larger than one.*

Proof The problem is invariant under a permutation equivalence, i.e., we may equally well consider $\mathcal{D} + \mathcal{D}P$. Let $\hat{P} \in \mathcal{W}$ be a permutation. When there is a cyclic shift of size one, $\hat{P}$ must have the corresponding diagonal entry. Consider the case when the cyclic shift S_j is of size larger than one. Each row and column of $\mathcal{W}$ contains exactly two nonzero elements, i.e., we must consider $\mathcal{D} + \mathcal{D}S_j$. There, by exclusion principle, $\hat{P}$ coincides either with S_j or the unit diagonal. Since $\hat{P}$ can be chosen either way, the claim follows.

In general, determining the singular elements of a matrix subspace is a tremendous challenge already when the dimension exceeds two [20].[8] By using the equivalence (2.5) and the Leibnitz formula, the singular elements of $\mathcal{V}$ can be readily determined as follows. If $D_1 = \mathrm{diag}(z_1, z_2, \ldots, z_{k_j})$ and $D_2 = \mathrm{diag}(z_{k_j+1}, z_{k_j+2}, \ldots, z_{2k_j})$, the task consists of finding the zeros of the multivariate polynomial

$$p_j(z_1, z_2, \ldots, z_{2k_j}) = \det(D_1 + D_2 S_j) = \prod_{l=1}^{k_j} z_l + (-1)^{k_j-1} \prod_{l=k_j+1}^{2k_j} z_l, \quad (2.6)$$

i.e., having $\prod_{l=1}^{k_j} z_l = (-1)^{k_j} \prod_{l=k_j+1}^{2k_j} z_l$ corresponds to a singular block.

Consider a nonsingular block $D_1 + D_2 S_j$ under the assumption that the first (equivalently, the second) term in (2.6) is nonzero. Then its inverse can be given in a closed form with the help of the following result.

Theorem 2.9 *Assume $S \in \mathbb{C}^{n \times n}$ is the cyclic shift and $D = \mathrm{diag}(d_1, \ldots, d_n)$. If $I + DS$ is nonsingular, then $(I + DS)^{-1} = \sum_{j=0}^{n-1} D_j S^j$ with the diagonal matrices $D_0 = \frac{1}{(-1)^{n-1} \prod_{j=1}^{n} d_j + 1} I$ and*

$$D_{j+1} = (-1)^{j+1} D_0 \prod_{k=0}^{j} D^{S^{kT}} \quad \text{for} \quad j = 0, \ldots, n-2. \quad (2.7)$$

Proof It is clear that the claimed expansion exists since any matrix $A \in \mathbb{C}^{n \times n}$ can be expressed uniquely as the sum

$$A = \sum_{j=0}^{n-1} D_j S^j, \quad (2.8)$$

i.e., the diagonal matrices D_j are uniquely determined. To recover the diagonal matrices of the claim for the inverse, consider the identity

$$(I + DS) \sum_{j=0}^{n-1} D_j S^j = \sum_{j=0}^{n-1} D_j S^j + \sum_{j=0}^{n-1} D D_j^{S^T} S^{j+1} = I,$$

where we denote $S D_j S^T$ by $D_j^{S^T}$ as in (2.1). The problem separates permutationwise, yielding $D_0 + D D_{n-1}^{S^T} = I$ for the main diagonal and the recursion

$$D_{j+1} + D D_j^{S^T} = 0 \quad \text{for} \quad j = 0, \ldots, n-2 \quad (2.9)$$

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[8] When the dimension is two, one essentially deals with a generalized eigenvalue problem. For solving generalized eigenvalue problems there are reliable numerical methods.

otherwise. This can be explicitly solved for $D_0 = ((-1)^{n-1}(DS)^n + I)^{-1}$. Thereby D_0 is the claimed translation of the identity matrix. Thereafter we may insert this into the recursion (2.9) to have the claim.

If actually both terms on the right-hand side in (2.6) are nonzero, then we are dealing with the sum of two monomial matrices. It can then be shown that we have a so-called $\mathcal{DCD}$ matrix, where $\mathcal{C}$ denotes the set of circulant matrices. (For applications, see [10,15] how such matrices appear in diffractive and Fourier optics.) The proof of this is constructive as follows.

Theorem 2.10 *Assume $D_1 + D_2 S$, where $S \in \mathbb{C}^{n \times n}$ is the cyclic shift and D_0 and D_1 are invertible diagonal matrices. Then there exist diagonal matrices $\hat{D}_1$ and $\hat{D}_2$ such that*

$$D_0 + D_1 S = \hat{D}_1 (I + \alpha S) \hat{D}_2 \qquad (2.10)$$

for a nonzero $\alpha \in \mathbb{C}$.

Proof Clearly, by using (2.1), we may conclude that the left-hand side is of more general type, including all the matrices of the type given on the right-hand side. Suppose therefore that $D_0 = \text{diag}(a_1, a_2, \ldots, a_n)$ and $D_1 = \text{diag}(b_1, b_2, \ldots, b_n)$ are given. Denote the variables by $\hat{D}_1 = \text{diag}(x_1, x_2, \ldots, x_n)$ and $\hat{D}_2 = \text{diag}(y_1, y_2, \ldots, y_n)$. Imposing the identity (2.10) yields us the equations

$$\begin{cases} x_1 y_1 &= a_1 \\ x_2 y_2 &= a_2 \\ &\vdots \\ x_{n-1} y_{n-1} &= a_{n-1} \\ x_n y_n &= a_n \end{cases} \quad \text{and} \quad \begin{cases} \alpha x_2 y_1 &= b_1 \\ \alpha x_3 y_2 &= b_2 \\ &\vdots \\ \alpha x_n y_{n-1} &= b_{n-1} \\ \alpha x_1 y_n &= b_n \end{cases}.$$

Solving y_j in terms of x_j from the first set of equations and inserting them into the second one yields the condition $\alpha^n = \frac{\prod_{j=1}^{n} b_j}{\prod_{j=1}^{n} a_j}$ for the parameter α to satisfy. This is necessary and sufficient for the existence of a solution, obtained now by a straightforward substitution process once, e.g., the value of x_1 has been assigned.

We may alternatively factor

$$\hat{D}_1 (I + \alpha S) \hat{D}_2 = \hat{D}_1 F^* D F \hat{D}_2, \qquad (2.11)$$

where F denotes the Fourier matrix and D is a diagonal matrix.

The existence of factoring (2.10) can hence be generically guaranteed in the following sense.

Corollary 2.11 $\mathcal{D}(I + \mathbb{C}S)\mathcal{D}$ *contains an open dense subset of $\mathcal{D} + \mathcal{D}S$.*

Consider the equivalence (2.5) In a generic case, using (2.10) with the blocks yields the simplest way to compute the inverse of the sum of two PD matrices.

3 Extending the Sum of Two PD Matrices: Polynomials in Permutation Matrices over Diagonal Matrices

By the fact that matrices representable as the sum of two PD matrices can be regarded as well-understood, consider extending this structure. The equivalence (2.5) provides an appropriate starting point to this end. There the canonical form consists of first degree polynomials in a permutation matrix P over diagonal matrices. More generally, define polynomials over the ring $\mathcal{D}$ with the indeterminate being an element of $\mathcal{P}$ as follows.

Definition 3.1 Let P be a permutation and $D_k \in \mathcal{D}$ for $k = 0, 1, \ldots, j$. Then

$$p(P) = \sum_{k=0}^{j} D_k P^k \tag{3.1}$$

is said to be a polynomial in P over $\mathcal{D}$.

In terms of this representation, due to (2.1), these matrices behave in essence like standard polynomials. To avoid redundancies, we are interested in polynomials p whose degree does not exceed $\deg(P)$. Then the degree of the matrix $p(P)$ is defined to be the degree of p. For algebraic operations, the sum of polynomials $p_1(P)$ and $p_2(P)$ is obvious. Whenever $\deg p_1 + \deg p_2 < \deg(P)$, the product behaves essentially classically, i.e., the degree of the product is the sum of the degrees of the factors.

Again, bearing in mind the equivalence (2.5), there is a need to relax Definition 3.1. For this purpose, take two permutations P_1 and P_2 and consider matrix subspaces of the form

$$P_1 \left\{ p(P) \,\middle|\, \deg(p) \le j \right\} P_2. \tag{3.2}$$

Since P_1 and P_2 can be chosen freely, by using (2.1) and (2.5) we may assume that $P = S_1 \oplus \cdots \oplus S_k$ with cyclic shifts $S_1, \ldots, S_k$. Consequently, the degrees of freedom lie in the choices of P_1 and P_2 and in the lengths of the cycles and j. Observe that (2.3) is covered by the case $j = 1$. For j even it many be worthwhile to make the sparsity structure symmetric by choosing $P_1 = P^{\frac{j}{2}T}$ and $P_2 = I$. (Then the sparsity structure obviously contains band matrices of bandwidth $j + 1$.) This gives rise to the respective notion of "bandwidth"; see Fig. 1.

Let us make some related graph theoretical remarks. It is natural to identify the sparsity structure of (3.2) with the $(0, 1)$-matrix having the same sparsity structure.[9] Namely, there are many decomposition results allowing one to express a $(0, 1)$-matrix as the sum of permutation matrices; see [2]. In this area of combinatorial matrix theory, we are not aware of any polynomial expressions of type (3.2).[10] In particular, it does not appear straightforward to see when a $(0, 1)$-matrix is a realization of such a polynomial structure. For example, by (2.8) we know that the matrix of all ones is. In

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[9] Since the study of matrix subspaces is (finite dimensional) operator space theory [16], this provides a link between analysis and discrete mathematics. Figure 1 corresponds to a circulant graph.

[10] It would be tempting to call such a $(0, 1)$-matrix a polynomial digraph. It has, however, another meaning [2, p.157].
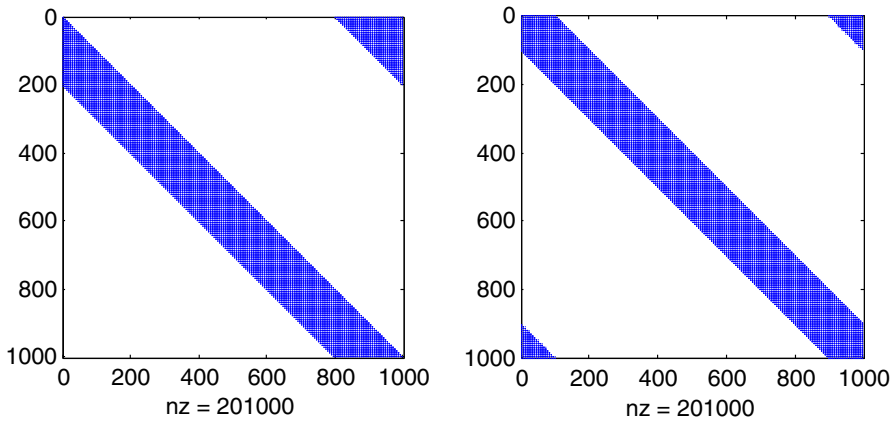
Fig. 1 On the *left* the sparsity pattern in (3.2) corresponding to $P = S$, $P_1 = P_2 = I$ for $n = 10^3$ and $j = 200$. On the *right* the corresponding symmetric sparsity pattern

particular, for any sparse standard matrix subspace this leads to the following notion of "graph bandwidth" in accordance with regular graphs.

Definition 3.2 Let $\mathcal{V}$ be a standard matrix subspace of $\mathbb{C}^{n \times n}$. The polynomial permutation degree of $\mathcal{V}$ is the smallest possible j allowing $\mathcal{V}$ to be included in (3.2) for some permutations P, P_1 and P_2.

Clearly, the polynomial degree is at most $n - 1$. When the degree is low, we have a sparse matrix structure. In particular, such a polynomial structure arises in connection with finite difference matrices with very small values of j.

Example 1 The set of tridiagonal matrices (and any of their permutation equivalences) is a matrix subspace of polynomial degree two. To see this, let P be the cyclic shift and set $j = 2$, $P_1 = P^T$ and $P_2 = I$. Then $\mathcal{V}$ includes tridiagonal matrices. In this manner, finite difference matrices including periodic problems [9, p. 159] are covered by the structure (3.2).

Aside from the polynomial permutation degree of Definition 3.2, there is another natural option to classify sparsity here. Recall that a polynomial is said to be sparse if most of its coefficients are zeros; see, e.g., [17]. Adapting this notion analogously, allow j to be large. Then a natural notion of sparseness arises when only a small number of coefficients are allowed to be nonzero diagonal matrices.

4 Factoring Polynomials in a Permutation Matrix over Diagonal Matrices

To demonstrate that the matrix structure (3.2) extending $\mathcal{PD} + \mathcal{PD}$ is genuinely polynomial, we want perform factoring. In forming products, we are concerned with the following algebraic structure.

Definition 4.1 Suppose $\mathcal{V}_1$ and $\mathcal{V}_2$ are matrix subspaces of $\mathbb{C}^{n \times n}$ over $\mathbb{C}$ (or $\mathbb{R}$). Then

$$\mathcal{V}_1 \mathcal{V}_2 = \{ V_1 V_2 \mid V_1 \in \mathcal{V}_1 \text{ and } V_2 \in \mathcal{V}_2 \}$$

is said to be the set of products of $\mathcal{V}_1$ and $\mathcal{V}_2$.

A matrix subspace $\mathcal{V}$ is said to be factorizable if, for some matrix subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$, there holds

$$\overline{\mathcal{V}_1 \mathcal{V}_2} = \mathcal{V}, \tag{4.1}$$

i.e., the closure of $\mathcal{V}_1 \mathcal{V}_2$ equals $\mathcal{V}$, assuming the dimensions satisfy $1 < \dim \mathcal{V}_j < \dim \mathcal{V}$ for $j = 1, 2$. As illustrated by the Gaussian elimination applied to band matrices, taking the closure may be necessary. For a wealth of information on computational issues related with band matrices, see [9, Chapter 4.3]. For the geometry of the set of products more generally, see [11].

Factoring of the matrix subspace (3.2) in the case $j = 2$ can be handled as follows.

Example 2 This is Example 1 continued. Let $\mathcal{V}_1 = \mathcal{D} + \mathcal{D}P$ and $\mathcal{V}_2 = \mathcal{D} + \mathcal{D}P^T$. Then (4.1) holds. Namely, to factor an element in a generic case, the problem reduces into solving a system of equations of the form

$$\begin{cases} x_1 + \frac{a_1}{x_n} = b_1 \\ x_2 + \frac{a_2}{x_1} = b_2 \\ x_3 + \frac{a_3}{x_2} = b_3 \\ \vdots \qquad\qquad \vdots \\ x_n + \frac{a_n}{x_{n-1}} = b_n \end{cases} \tag{4.2}$$

with $a_j \neq 0$ and $b_j \neq 0$ for $j = 1, \ldots, n$ given. From the first equation x_1 can be solved in terms of x_n and substituted into the second equation. Thereafter x_2 can be solved in terms of x_n and substituted into the third equation. Repeating this, the system eventually turns into a univariate polynomial in x_n. Solving this combined with back substitution yields a solution. Computationally a more practical approach is to execute Newton's method on (4.2). Solving linear systems at each step is inexpensive by implementing the method of Proposition 2.5. Consequently, under standard assumptions on the convergence of Newton's method, finding a factorization is an $O(n)$ computation.

With these preparations, for $j > 2$, consider the problem of factoring a matrix subspace (3.2) into the product of lower degree factors of the same type. As described, it suffices to consider factoring a given polynomial p of degree $j \leq n - 1$ in a cyclic shift $S \in \mathbb{C}^{n \times n}$ into linear factors. That is, assume having

$$A = p(S) = \sum_{k=0}^{j} F_k S^k \tag{4.3}$$

with given diagonal matrices F_k, for $k = 0, \ldots, j$. Then the task is to find diagonal matrices D_0 and D_1 and $E_0, \ldots, E_{j-1}$ such that

$$(D_0 + D_1 S) \sum_{k=0}^{j-1} E_k S^k = \sum_{k=0}^{j} F_k S^k \tag{4.4}$$

holds. Once solved, this can then be repeated with $\sum_{k=0}^{j-1} E_k S^k$. For a solution, there are several ways to proceed. To begin with, by using the identity (2.1), the problem separates into the equations $D_0 E_0 = F_0$ and $D_1 E_{j-1}^{S^T} = F_j$ and

$$D_0 E_{k+1} + D_1 E_k^{S^T} = F_{k+1} \tag{4.5}$$

for $k = 0, \ldots, j - 2$.

There are, however, redundancies. These can be removed so as to attain maximal simplicity in terms of a univariate polynomial-like factorization result. To formulate a precise statement for performing this, let us invoke the following lemma.

Lemma 4.2 *Let $f : \mathbb{C}^n \to \mathbb{C}^k$ be a polynomial function. If there exists a point $x \in \mathbb{C}^n$ such that the derivative $Df(x)$ has full rank, then $f(\mathbb{C}^n)$ contains an open set whose complement is of zero measure. In particular, the open set is dense and $f(\mathbb{C}^n)$ contains almost all points of $\mathbb{C}^k$ (in the sense of Lebesgue-measure.)*

Proof This follows from [14, Theorem10.2]. ∎

This is of use in proving the following theorem underscoring how the matrix structure (3.2) is in every sense univariate polynomial.

Theorem 4.3 *There exists an open dense set $G \subset \mathbb{C}^{n \times n}$ containing almost all matrices of $\mathbb{C}^{n \times n}$ (in the sense of Lebesgue-measure) such that if $A \in G$, then*

$$A = (S - D_1)(S - D_2) \cdots (S - D_{n-1}) D_n \tag{4.6}$$

for diagonal matrices D_i, $i = 1, \ldots, n$.

Proof For $1 \le j \le n$, define the following nj-dimensional subspaces of $\mathbb{C}^{n \times n}$

$$\mathcal{A}_j = \left\{ A \in \mathbb{C}^{n \times n} \,\middle|\, A = \sum_{k=0}^{j-1} E_k S^k \text{ for some diagonal } E_k \in \mathbb{C}^{n \times n} \right\}.$$

Consider the polynomial functions $f_j : \mathcal{A}_1 \times \mathcal{A}_{j-1} \to \mathcal{A}_j$ defined by

$$f_j(D, E) = (S - D)E.$$

After differentiating, we have

$$Df_j(D, E)(\Delta D, \Delta E) = (S - D)(\Delta E) + (-\Delta D)E.$$

Now choose $D = 0$, $E = I$ to obtain

$$Df_j(0, I)(\Delta D, \Delta E) = S(\Delta E) - \Delta D.$$

Hence $Df_j(0, I)$ is of full rank. By Lemma 4.2 it follows that the equation

$$f_j(D, E) = F$$

is solvable for D and E for almost all matrices $F \in \mathcal{A}_j$. Denote the subset of those matrices F by $\mathcal{B}_j = f_j(\mathcal{A}_1 \times \mathcal{A}_{j-1})$. Define $\widetilde{\mathcal{B}}_2 = \mathcal{B}_2$ and, furthermore, define

$$\widetilde{\mathcal{B}}_j = \mathcal{B}_j \cap f_j(\mathcal{A}_1 \times \widetilde{\mathcal{B}}_{j-1}), \quad j = 3, \ldots, n.$$

Then $\mathcal{A}_j \setminus \widetilde{\mathcal{B}}_j$ is of measure zero (in $\mathcal{A}_j$) and it follows that when $A \in \widetilde{\mathcal{B}}_n$ we can solve for $D_1, \ldots, D_n$ in (4.6) by successively solving the equations (where $E_1 = A$)

$$f_j(D_j, E_{j+1}) = E_j, \quad j = 1, 2, \ldots, n-1$$

and finally setting $D_n = E_n$. Hence almost all matrices $A \in \mathbb{C}^{n \times n}$ have a factorization (4.6). That the set of these matrices contains an open set with complement of zero measure follows by applying [14, Theorem 10.2].

The identity (4.6) allows regarding matrices as polynomials which have been factored. The indeterminate is a permutation (now S) while the role of $\mathbb{C}$ is taken by $\mathcal{D}$. The representation is optimal in the sense that the number of factors (and diagonal matrices) cannot be reduced further in general. Of course, if $D_k = \alpha_k I$ with $\alpha_k \in \mathbb{C}$ for $k = 1, \ldots, n$, then we are dealing with circulant matrices, a classical polynomial structure among matrices [6].

Like with polynomials, this gives rise to a notion of degree.

Definition 4.4 The polynomial permutation degree of $A \in \mathbb{C}^{n \times n}$ is the smallest possible j admitting a representation $A = P_1 \sum_{k=0}^{j} D_k P^k P_2$ for some permutations P, P_1 and P_2 and diagonal matrices D_k for $k = 0, \ldots, j$.

To compute the diagonal matrices D_i in (4.6) for a matrix $A \in \mathbb{C}^{n \times n}$, the equations (4.4) hence simplify as follows. Let $j = n - 1$ and $A = \sum_{k=0}^{j} F_k S^k$ with given diagonal matrices F_k. For an integer i, define $[i] = 1 + ((i - 1) \mod n)$. Denote $D_{n-j} = \text{diag}(x_1, x_2, \ldots, x_n)$. Then eliminating the diagonal matrices E_k by imposing

$$(S - D_{n-j}) \sum_{k=0}^{j-1} E_k S^k = A \tag{4.7}$$

we obtain the following system of polynomial equations

$$
\begin{aligned}
a_{[1],1} + a_{[2],1}x_{[1]} + a_{[3],1}x_{[1]}x_{[2]} + \cdots + a_{[j+1],1}x_{[1]}x_{[2]}\cdots x_{[j]} &= 0 \\
a_{[2],2} + a_{[3],2}x_{[2]} + a_{[4],2}x_{[2]}x_{[3]} + \cdots + a_{[j+2],2}x_{[2]}x_{[3]}\cdots x_{[j+1]} &= 0 \\
&\vdots \\
a_{[n],n} + a_{[n+1],n}x_{[n]} + a_{[n+2],n}x_{[n]}x_{[n+1]} + \cdots + a_{[j+n],n}x_{[n]}x_{[n+1]}\cdots x_{[n+j-1]} &= 0.
\end{aligned}
$$

This system of polynomial equations obviously possesses a very particular structure. (At this point we are not sure how it should be exploited.) After being solved, the diagonal matrices E_k are determined by performing the substitutions

$$
\begin{aligned}
E_{j-1} &= F_j^S, \\
E_k &= (F_{k+1} + D_{n-j}E_{k+1})^S, \quad k = j-2, j-3, \ldots, 0.
\end{aligned}
$$

Then let $A = \sum_{k=0}^{j-1} E_k S^k$, decrease j by one and repeat the solving of (4.7) accordingly.

Equipped with this recurrence, consider now the problem of factoring a matrix $A = p(S)$ into the product of circulant and diagonal matrices. First apply Theorem 4.3 to have a factorization (4.6) after completing the prescribed recurrence. Thereafter apply Theorem 2.10 to transform each of the factors according to (2.10). This yields (1.2).

For another approach to factor a matrix $A = p(S)$ into the product of circulant and diagonal matrices, consider imposing (4.4). Apply then Theorem 2.10 to invert the first factor by assuming D_0 and D_1 to be invertible. We obtain

$$
\sum_{k=0}^{j-1} \hat{E}_k S^k = (I + \alpha S)^{-1} \tilde{D}_1 p(S) \tag{4.8}
$$

with $\hat{E}_k = \hat{D}_2 E_k$, $\alpha \in \mathbb{C}$ and $\tilde{D}_1 = \hat{D}_1^{-1}$. We may hence conclude that $\hat{D}_2$ is redundant. Thereby the task reduces to finding α and $\tilde{D}_1 = \mathrm{diag}(x_1, x_2, \ldots, x_n)$ in such a way that the right-hand side of the identity attains the zero structure imposed by the left-hand side. Any solution is homogeneous in $\tilde{D}_1$. Therefore we can further set $x_1 = 1$ to reduce the problem to n free complex parameters. Once the equations are solved, $\hat{E}_k$'s are determined by α and $\tilde{D}_1$ according to (4.8).

Consider the first factorization step in (4.8) by letting $j = n - 1$. Then zeros on the left-hand side in (4.8) appear at the positions where $S^{n-1} = S^T$ has ones, i.e., at $(j, j+1)$, for $j = 1, \ldots, n-1$, and at $(n, 1)$. To recover the entries at these positions on right-hand side, by Theorem 2.9 the inverse of $I + \alpha S$ is the circulant matrix $\frac{1}{1+(-1)^{n-1}\alpha^n}C$ with C having the first row

$$
(1, (-1)^{n-1}\alpha^{n-1}, (-1)^{n-2}\alpha^{n-2}, \ldots, \alpha^2, -\alpha). \tag{4.9}
$$

Because on the left-hand side of the equations there are zeros, the factor $\frac{1}{1+(-1)^{n-1}\alpha^n}$ can be ignored and we are left with $C\tilde{D}_1 p(S)$. Forcing its entries to be zeros at $(j, j+1)$, for $j = 1, \ldots, n-1$, and at $(n, 1)$ yields n polynomial equations in which the highest power of α is $n-1$ while d_j's appear linearly. Solve these, then let $A = \sum_{k=0}^{j-1} \hat{E}_k S^k$, decrease j by one and repeat the solving of accordingly.

Once the factorization is completed, we obtain (1.2). By the fact that now the circulant matrices C_k are of the particular form $I + \alpha_k S$ with $\alpha_k \in \mathbb{C}$, the number of free parameters in our factorization is only $n^2 + n - 1$. Thereby we have only $n-1$ "excess" free parameters.

Since the circulant matrices were of particular form, let us end the paper with a speculative deliberation on the optimal number of factors and related compressions. After all, the subspace of circulant matrices in $\mathbb{C}^{n \times n}$ is of dimension n. Thereby, to factor a generic matrix into the minimal number of circulant and diagonal factors, we make the following conjecture.

Conjecture 1 *There exists an open dense set $G \subset \mathbb{C}^{n \times n}$ containing almost all matrices of $\mathbb{C}^{n \times n}$ (in the sense of Lebesgue-measure) such that if $A \in G$, then*

$$A = B_1 B_2 \cdots B_{n+1},$$

where $B_i \in \mathbb{C}^{n \times n}$ is circulant for odd i and diagonal for even i.

This is supported by calculations. That is, we have verified the conjecture for the dimensions n satisfying $2 \leq n \leq 20$ by computer calculations utilizing Lemma 4.2 (with randomly chosen integer coordinates for the point x resulting in an integer matrix for the derivative). Observe that, by a simple count of free parameters, no lower number of factors can suffice.

In reality, approximate factorizations and expansions are typically of major practical interest. In this connection it is natural to formulate the problem more Fourier analytically. Denote by $F \in \mathbb{C}^{n \times n}$ the Fourier matrix. For a given $A \in \mathbb{C}^{n \times n}$, the respective multiplicative Fourier compression problem then reads

$$\inf_{D_1,\ldots,D_j \in \mathcal{D}} \|A - D_1 F^* D_2 F D_3 F^* D_4 \cdots F^* D_{j-1} F D_j\|, \tag{4.10}$$

for $j = 1, 2, \ldots$, with respect to a unitarily invariant norm $\| \cdot \|$. This is a nonincreasing sequence of numbers while j grows. Attaining zero with $j = 1$ means that A is a diagonal matrix while attaining zero with $j = 2$ means that A is a product of a diagonal and a circulant matrix. This paper is concerned with a constructive demonstration showing that $j = 2n - 1$ yields zero. From the outset, solving (4.10) appears challenging.

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