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Lebesgue Points of Two-Dimensional Fourier Transforms and Strong Summability

Ferenc Weisz

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Abstract We introduce the concept of modified strong Lebesgue points and show that almost every point is a modified strong Lebesgue point of f from the Wiener amalgam space $W(L_1, \ell_\infty)(\mathbb{R}^2)$. A general summability method of two-dimensional Fourier transforms is given with the help of an integrable function θ. Under some conditions on θ we show that the Marcinkiewicz-θ-means of a function $f \in W(L_1, \ell_\infty)(\mathbb{R}^2)$ converge to f at each modified strong Lebesgue point. The same holds for a weaker version of Lebesgue points, for the so called modified Lebesgue points of $f \in W(L_p, \ell_\infty)(\mathbb{R}^2)$, whenever $1 < p < \infty$. As an application we generalize the classical one-dimensional strong summability results of Hardy and Littlewood, Marcinkiewicz, Zygmund and Gabisoniya for $f \in W(L_1, \ell_\infty)(\mathbb{R})$ and for strong θ-summability. Some special cases of the θ-summation are considered, such as the Weierstrass, Abel, Picar, Bessel, Fejér, de La Vallée-Poussin, Rogosinski and Riesz summations.

Keywords Fourier transforms · Fejér summability · θ-Summability · Marcinkiewicz summability · Lebesgue points · Strong summability

Mathematics Subject Classifications Primary 42B08 · Secondary 42A38, 42A24, 42B25

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F. Weisz (✉)
Department of Numerical Analysis, Eötvös L. University, Pázmány P. sétány 1/C., Budapest 1117, Hungary
e-mail: weisz@inf.elte.hu

1 Introduction

It was proved by Lebesgue [18] that the Fejér means [5] of the trigonometric Fourier series of an integrable function converge almost everywhere to the function, i.e.,

$$\frac{1}{n+1}\sum_{k=0}^{n}\Big(s_k f(x) - f(x)\Big) \to 0 \quad \text{as} \quad n \to \infty$$

for almost every $x \in \mathbb{T}$, where $\mathbb{T}$ denotes the torus and $s_k f$ the kth partial sum of the Fourier series of the one-dimensional function f. The set of convergence is characterized as the Lebesgue points of f.

Hardy and Littlewood [16] considered the so called strong summability and verified that the strong means

$$\frac{1}{n+1}\sum_{k=0}^{n}\left|s_k f(x) - f(x)\right|^q$$

tend to 0 at each Lebesgue-point of f, as $n \to \infty$, whenever $f \in L_p(\mathbb{T})(1 < p < \infty)$ (for Fourier transforms see Giang and Móricz [10]). This result does not hold for $p = 1$ (see Hardy and Littlewood [17]). However, the strong means tend to 0 almost everywhere for all $f \in L_1(\mathbb{T})$. This is due to Marcinkiewicz [19] for $q = 2$ and to Zygmund [33] for all $q > 0$ (see also Bary [1]). Later Gabisoniya [6,7] (see also Rodin [22]) characterized the set of convergence as the so called Gabisoniya points.

In the two-dimensional case Marcinkievicz [20] verified that

$$\sigma_n f(x, y) := \frac{1}{n+1}\sum_{k=0}^{n} s_{k,k} f(x, y) \to f(x, y) \quad \text{a.e., as} \quad n \to \infty$$

for all functions $f \in L\log L(\mathbb{T}^2)$. Here we take the Fejér means of the two-dimensional Fourier series over the diagonal. Later Zhizhiashvili [31,32] extended this convergence to all $f \in L_1(\mathbb{T}^2)$ and to Cesàro means. Recently the author [27,29] generalized this result for all $f \in L_1(\mathbb{R}^2)$. The set of the convergence is not yet known. In this direction the only result is due to Grünwald [15], he proved that if the integrable function f is continuous at (x, y), then the convergence holds at (x, y).

A general method of summation, the so called θ-summation method, which is generated by a single function θ and which includes the well known Fejér, Riesz, Weierstrass, Abel, etc. summability methods, is studied intensively in the literature (see e.g. Butzer and Nessel [2], Trigub and Belinsky [25], Gát [8,9], Goginava [11–13], Simon [23] and Weisz [28,30]). The Marcinkiewicz means generated by the θ-summation are defined by

$$\sigma_T^\theta f(x, y) = \frac{-1}{T}\int_0^\infty \theta'\left(\frac{t}{T}\right) s_{t,t} f(x, y)\, dt.$$

The choice $\theta(t) = \max(1-|t|, 0)$ yields the Fejér summation. We proved in [27,29] that $\sigma_T^\theta f \to f$ almost everywhere if $f \in L_1(\mathbb{R}^2)$.

In this paper we generalize this result for Wiener amalgam spaces and we characterize the set of convergence. We introduce the concept of modified Lebesgue points and modified strong Lebesgue points. We show that almost every point is a modified Lebesgue point and a modified strong Lebesgue point of $f \in L_1(\mathbb{R}^2)$ or $f \in W(L_1, \ell_\infty)(\mathbb{R}^2)$. Here $W(L_p, \ell_q)(\mathbb{R}^2)$ denotes the Wiener amalgam space. Under some conditions on θ we show that the Marcinkiewicz-θ-means of a function $f \in W(L_1, \ell_\infty)(\mathbb{R}^2)$ converge to f at each modified strong Lebesgue point. The same result holds for the modified Lebesgue points of $f \in W(L_p, \ell_\infty)(\mathbb{R}^2)$, whenever $1 < p < \infty$.

As an application we generalize the classical one-dimensional strong summability results mentioned above for $f \in W(L_1, \ell_\infty)(\mathbb{R})$ and for strong θ-summability. More exactly, we will show that

$$\lim_{T\to\infty} \frac{-1}{T} \int_0^\infty \theta'\Big(\frac{t}{T}\Big) \big|s_t f(x) - f(x)\big|^2 \, dt = 0$$

at each Lebesgue point x of $f \in W(L_1, \ell_q)(\mathbb{R}) \supset L_q(\mathbb{R}) (1 \le q < \infty)$ when f is locally bounded at x. The convergence holds at each Lebesgue point of f if $f \in W(L_p, \ell_q)(\mathbb{R}) \supset L_p(\mathbb{R})$ $(1 < p < \infty, 1 \le q < \infty)$. Moreover, it holds at each Gabisoniya point if $f \in W(L_1, \ell_q)(\mathbb{R})$ $(1 \le q < \infty)$. Finally, some special cases of the θ-summation are considered, such as the Weierstrass, Abel, Picar, Bessel, Fejér, de La Vallée-Poussin, Rogosinski and Riesz summations.

2 Wiener Amalgam Spaces

We briefly write $L_p(\mathbb{R}^2)$ instead of the $L_p(\mathbb{R}^2, \lambda)$ space equipped with the norm

$$\|f\|_p := \left(\int_{\mathbb{R}^2} |f(x)|^p \, d\lambda(x) \right)^{1/p} \qquad (1 \le p < \infty),$$

with the usual modification for $p = \infty$, where λ is the Lebesgue measure.

Now we generalize the L_p spaces. A measurable function f belongs to the *Wiener amalgam space* $W(L_p, \ell_q)(\mathbb{R}^2)$ $(1 \le p, q \le \infty)$ if

$$\|f\|_{W(L_p,\ell_q)} := \left(\sum_{k\in\mathbb{Z}^2} \|f(\cdot + k)\|_{L_p[0,1)^2}^q \right)^{1/q} < \infty,$$

with the obvious modification for $q = \infty$.

It is easy to see that $W(L_p, \ell_p)(\mathbb{R}^d) = L_p(\mathbb{R}^d)$ and the following continuous embeddings hold true:

$$W\big(L_{p_1}, \ell_q\big)\big(\mathbb{R}^2\big) \supset W\big(L_{p_2}, \ell_q\big)\big(\mathbb{R}^2\big) \qquad \big(p_1 \le p_2\big)$$

and

$$W\big(L_p, \ell_{q_1}\big)\big(\mathbb{R}^2\big) \subset W\big(L_p, \ell_{q_2}\big)\big(\mathbb{R}^2\big) \qquad \big(q_1 \le q_2\big),$$

$(1 \le p_1, p_2, q_1, q_2 \le \infty)$. Thus

$$W\big(L_\infty, \ell_1\big)\big(\mathbb{R}^2\big) \subset L_p\big(\mathbb{R}^2\big) \subset W\big(L_1, \ell_\infty\big)\big(\mathbb{R}^2\big) \qquad \big(1 \le p \le \infty\big).$$

In this paper the constants C and C_p may vary from line to line and the constants C_p are depending only on p.

3 The Kernel Functions

Let us recall some results for the inverse Fourier transforms. The *Fourier transform* of $f \in L_1(\mathbb{R}^2)$ is given by

$$\widehat{f}(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(u, v) e^{-\imath(xu+yv)}\, du\, dv \qquad (x, y \in \mathbb{R}),$$

where $\imath = \sqrt{-1}$. Suppose first that $f \in L_p(\mathbb{R}^2)$ for some $1 \le p \le 2$. The Fourier inversion formula

$$f(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \widehat{f}(u, v) e^{\imath(xu+yv)}\, du\, dv \qquad \big(x, y \in \mathbb{R}, \widehat{f} \in L_1\big(\mathbb{R}^2\big)\big)$$

motivates the definition of the Dirichlet integral $s_t f$ $(t > 0)$:

$$\begin{aligned} s_t f(x, y) &:= \frac{1}{2\pi} \int_{-t}^{t} \int_{-t}^{t} \widehat{f}(u, v) e^{\imath(xu+yv)}\, du\, dv \\ &= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} f(x-u, y-v) D_t(u, v)\, du\, dv, \end{aligned} \tag{1}$$

where the *Dirichlet kernel* is defined by

$$D_t(x, y) := \int_{-t}^{t} \int_{-t}^{t} e^{\imath(xu+yv)}\, dudv = 4 \frac{\sin tx}{x} \frac{\sin ty}{y}.$$

Obviously, $|D_t| \le Ct^2$.

It is easy to see that, with the help of the integral in (1), the definition of $s_t f$ can be extended to all $f \in W(L_1, \ell_q)(\mathbb{R}^2)$ with $1 \le q < \infty$. Note that $W(L_1, \ell_p)(\mathbb{R}^2) \supset L_p(\mathbb{R}^2)$, where $1 \le p < \infty$. It is known (see e.g. Grafakos [14] or [30]) that for $f \in L_p(\mathbb{R}^2)$, $1 < p < \infty$,

$$\lim_{T \to \infty} s_T f = f \qquad \text{in the } L_p(\mathbb{R}^2)\text{-norm and a.e.}$$

Note that $T \in \mathbb{R}_+$. This convergence does not hold for $p = 1$. However, using a summability method, we can generalize these results. We may take a general summability method, the so called Marcinkiewicz-θ-summation defined by a function $\theta : \mathbb{R}_+ \to \mathbb{R}$. This summation contains all well known summability methods, such as the Marcinkiewicz–Fejér, Riesz, Weierstrass, Abel, Picard, Bessel summations.

Suppose that θ is continuous on $\mathbb{R}_+$, the support of θ is $[0, c]$ for some $0 < c \leq \infty$ and θ is differentiable on $(0, c)$. Suppose further that

$$\theta(0) = 1, \qquad \int_0^\infty (t \vee 1)^2 |\theta'(t)|\, dt < \infty, \qquad \lim_{t\to\infty} t^2\theta(t) = 0, \tag{2}$$

where $\vee$ denotes the maximum and $\wedge$ the minimum.

For $T > 0$ the *Marcinkiewicz-θ-means* of a function $f \in L_p(\mathbb{R}^2)(1 \leq p \leq 2)$ are defined by

$$\sigma_T^\theta f(x, y) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \theta\left(\frac{|u| \vee |v|}{T}\right) \widehat{f}(u, v) e^{\imath(xu+yv)}\, du\, dv.$$

It is easy to see that

$$\sigma_T^\theta f(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(x-u, y-v) K_T^\theta(u, v)\, dudv, \tag{3}$$

where the *Marcinkiewicz-θ-kernel* is given by

$$\begin{aligned} K_T^\theta(x, y) &:= \frac{1}{2\pi} \int_{\mathbb{R}^2} \theta\left(\frac{|u| \vee |v|}{T}\right) e^{\imath(xu+yv)}\, dudv \\ &= \frac{-1}{2\pi T} \int_{\mathbb{R}^2} \int_{|u|\vee|v|}^\infty \theta'\left(\frac{t}{T}\right) dt\, e^{\imath(xu+yv)}\, du\, dv \\ &= \frac{-1}{2\pi T} \int_0^\infty \theta'\left(\frac{t}{T}\right) \int_{-t}^t \int_{-t}^t e^{\imath(xu+yv)}\, du\, dv\, dt \\ &= \frac{-1}{2\pi T} \int_0^\infty \theta'\left(\frac{t}{T}\right) D_t(x, y)\, dt. \end{aligned} \tag{4}$$

Observe that K_T^θ is well defined because

$$\begin{aligned} \frac{1}{4} \int_{\mathbb{R}^2} \theta(|u| \vee |v|)\, dudv &= \int_0^\infty \int_0^\infty 1_{\{u<v\}} \theta(v)\, du\, dv + \int_0^\infty \int_0^\infty 1_{\{u>v\}} \theta(u)\, du\, dv \\ &= 2\int_0^\infty u\theta(u)\, du = c^2\theta(c) - \int_0^c u^2\theta'(u)\, du, \end{aligned} \tag{5}$$

which is finite by (2). Hence

$$\sigma_T^\theta f(x, y) = \frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right) s_t f(x, y)\, dt.$$

Note that for the Marcinkiewicz–Fejér means (i.e. for $\theta(t) = \max((1 - |t|), 0))$ we get the usual definition

$$\sigma_T^\theta f(x, y) = \frac{1}{T} \int_0^T s_t f(x, y)\, dt.$$

We may suppose that $x > y > 0$. The first two inequalities of the next lemma follows from (4), the others were proved in Weisz [27].

Lemma 1 *If*

$$\left| \int_0^\infty \theta'(t) \cos(tu)\, dt \right| \le C u^{-\alpha}, \qquad \left| \int_0^\infty \theta'(t)\, t\, \sin(tu)\, dt \right| \le C u^{-\alpha} \tag{6}$$

for some $0 < \alpha < \infty$*, then*

$$\left| K_T^\theta(x, y) \right| \le C T^2, \tag{7}$$

$$\left| K_T^\theta(x, y) \right| \le C x^{-1} y^{-1}, \tag{8}$$

$$\left| K_T^\theta(x, y) \right| \le C T^{-\alpha} x^{-1} y^{-1} (x - y)^{-\alpha}, \tag{9}$$

$$\left| K_T^\theta(x, y) \right| \le C T^{1-\alpha} x^{-1} (x - y)^{-\alpha}. \tag{10}$$

We have proved the next lemma in Weisz [26].

Lemma 2 *If* (6) *is satisfied for some* $0 < \alpha < \infty$*, then* $\int_{\mathbb{R}^2} \left| K_T^\theta \right| d\lambda \le C$ $(T \in \mathbb{R}_+)$.

Now we can extend the definition of the Marcinkiewicz-θ-means $\sigma_T^\theta f$ with the formula (3) to all $f \in W(L_1, \ell_\infty)(\mathbb{R}^2)$.

4 Modified Lebesgue Points

$L_p^{loc}(\mathbb{R}^2)(1 \le p < \infty)$ denotes the space of measurable functions f for which $|f|^p$ is locally integrable. We say that f is locally bounded at (x, y) if there exists a neighborhood of (x, y) such that f is bounded on this neighborhood.

For $f \in L_p^{loc}(\mathbb{R}^2)$ the *Hardy-Littlewood maximal function* is defined by

$$M_p f(x, y) := \sup_{h>0} \left(\frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} |f(x - s, y - t)|^p \, ds\, dt \right)^{1/p}.$$

We are going to generalize the Hardy–Littlewood maximal function. Let $\mu(h)$ and $\nu(h)$ be two continuous functions of $h \ge 0$, strictly increasing to ∞ and 0 at $h = 0$. Let

$$M_p^{(1),\mu,\nu} f(x, y) := \sup_{h>0} \left(\frac{1}{4\mu(h)\nu(h)} \int_{-\mu(h)}^{\mu(h)} \int_{-\nu(h)}^{\nu(h)} |f(x - s, y - t)|^p \, ds\, dt \right)^{1/p},$$

where $f \in L_p^{loc}(\mathbb{R}^2)$. If $\mu(h) = \nu(h) = h$, then we get back the usual Hardy-Littlewood maximal function. For $p = 1$, we write simply Mf and $M^{(1),\mu,\nu} f$. It is known that the usual maximal function is of *weak type* (1, 1) and bounded on $L_p(\mathbb{R}^2)$ $(1 < p \leq \infty)$. We can prove in the same way that $M^{(1),\mu,\nu}$ has these properties as well, i.e.,

$$\sup_{\rho>0} \rho\lambda\big(M^{(1),\mu,\nu} f > \rho\big) \leq C\|f\|_1 \qquad \big(f \in L_1\big(\mathbb{R}^2\big)\big) \tag{11}$$

and

$$\big\|M^{(1),\mu,\nu} f\big\|_p \leq C_p\|f\|_p \qquad \big(f \in L_p\big(\mathbb{R}^2\big), 1 < p \leq \infty\big) \tag{12}$$

(see Zygmund [34], Stein [24] or Weisz [28]), where the constants C and C_p are independent of μ and ν.

For some $\tau > 0$ and $f \in L_p^{loc}(\mathbb{R}^2)$ let

$$\mathcal{M}_p^{(1)} f(x, y) := \sup_{i,j\in\mathbb{N},h>0} 2^{-\tau(i+j)} \left(\frac{1}{4 \cdot 2^{i+j}h^2} \int_{-2^i h}^{2^i h} \int_{-2^j h}^{2^j h} |f(x-s, y-t)|^p \, ds \, dt\right)^{1/p}.$$

Again $\mathcal{M}^{(1)} f := \mathcal{M}_1^{(1)} f$. Applying inequality (11) to $\mu(h) = 2^i h$ and $\nu(h) = 2^j h$, we obtain

$$\begin{aligned} \rho\lambda\big(\mathcal{M}^{(1)} f > \rho\big) &\leq \rho \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \lambda\Big(M^{(1),\mu,\nu} f > 2^{\tau(i+j)}\rho\Big) \\ &\leq C \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} 2^{-\tau(i+j)}\|f\|_1 \leq C\|f\|_1 \end{aligned} \tag{13}$$

for all $f \in L_1(\mathbb{R}^2)$ and $\rho > 0$. The inequality

$$\big\|\mathcal{M}^{(1)} f\big\|_p \leq C_p\|f\|_p \qquad \big(f \in L_p\big(\mathbb{R}^2\big), 1 < p \leq \infty\big) \tag{14}$$

can be shown similarly.

We modify slightly the definition of the maximal function. Let

$$M_p^{(2),\mu,\nu} f(x, y) := \sup_{h>0} \left(\frac{1}{4\mu(h)\nu(h)} \int_{-\mu(h)}^{\mu(h)} \int_{s-\nu(h)}^{s+\nu(h)} |f(x-s, y-t)|^p \, dt \, ds\right)^{1/p}$$

and

$$\mathcal{M}_p^{(2)} f(x, y) := \sup_{i,j\in\mathbb{N},h>0} 2^{-\tau(i+j)} \left(\frac{1}{4 \cdot 2^{i+j}h^2} \int_{-2^i h}^{2^i h} \int_{s-2^j h}^{s+2^j h} |f(x-s, y-t)|^p \, dt \, ds\right)^{1/p}.$$

With the same proof we can see that (11) and (12) holds also for $M^{(2),\mu,\nu} f := M_1^{(2),\mu,\nu} f$ and (13) and (14) for $\mathcal{M}^{(2)} f := \mathcal{M}_1^{(2)} f$ (see also Zhizhiashvili [32]). The next theorem can be proved with the method of Feichtinger and Weisz [4] for

$$\mathcal{M}_p f(x,y) := \mathcal{M}_p^{(1)} f(x,y) + \mathcal{M}_p^{(2)} f(x,y).$$

Theorem 1 *For* $1 \le p < \infty$,

$$\sup_{\rho>0} \rho\lambda(\mathcal{M}_p f > \rho)^{1/p} \le C\|f\|_p \qquad \big(f \in L_p(\mathbb{R}^2)\big),$$

$$\big\|\mathcal{M}_p f\big\|_r \le C_r \|f\|_r \qquad \big(f \in L_r(\mathbb{R}^2),\ p < r \le \infty\big)$$

and

$$\sup_{k\in\mathbb{Z}^d} \sup_{\rho>0} \rho\lambda(\mathcal{M}_p f > \rho, [k,k+1))^{1/p} \le C\|f\|_{W(L_p,\ell_\infty)} \qquad \big(f \in W(L_p,\ell_\infty)(\mathbb{R}^2)\big),$$

$$\big\|\mathcal{M}_p f\big\|_{W(L_r,\ell_\infty)} \le C_r \|f\|_{W(L_r,\ell_\infty)} \qquad \big(f \in W(L_r,\ell_\infty)(\mathbb{R}^2),\ p < r \le \infty\big).$$

A point $(x,y) \in \mathbb{R}^2$ is called a *p-Lebesgue point* (or a Lebesgue point of order p) of $f \in L_p^{loc}(\mathbb{R}^2)$ if

$$\lim_{h\to 0} \left(\frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h} |f(x-s,y-t) - f(x,y)|^p \, ds\, dt\right)^{1/p} = 0.$$

It was proved in Feichtinger and Weisz [3,4] that almost every point $(x,y) \in \mathbb{R}^2$ is a p-Lebesgue point of $f \in W(L_p,\ell_\infty)(\mathbb{R}^2)$ $(1 \le p < \infty)$.

We say that a point $(x,y) \in \mathbb{R}^2$ is a *modified p-Lebesgue point* (or a modified Lebesgue point of order p) of $f \in L_p^{loc}(\mathbb{R}^2)(1 \le p < \infty)$ if for all $\tau > 0$

$$\lim_{r\to 0} \sup_{\substack{i,j\in\mathbb{N},h>0\\ 2^i h<r,2^j h<r}} 2^{-\tau(i+j)} \left(\frac{1}{4\cdot 2^{i+j}h^2}\int_{-2^i h}^{2^i h}\int_{-2^j h}^{2^j h} |f(x-s,y-t) - f(x,y)|^p \, ds\, dt\right)^{1/p}$$
$$= 0. \tag{15}$$

If in addition

$$\lim_{r\to 0} \sup_{\substack{i,j\in\mathbb{N},h>0\\ 2^i h<r,2^j h<r}} 2^{-\tau(i+j)} \left(\frac{1}{4\cdot 2^{i+j}h^2}\int_{-2^i h}^{2^i h}\int_{s-2^j h}^{s+2^j h} |f(x-s,y-t) - f(x,y)|^p \, dt\, ds\right)^{1/p}$$
$$= 0, \tag{16}$$

then we say that $(x, y) \in \mathbb{R}^2$ is a *modified strong p-Lebesgue point* (or a modified strong Lebesgue point of order p). If $p = 1$, then we call the points *modified Lebesgue points* or *modified strong Lebesgue points*. Obviously, every modified (strong) p-Lebesgue point is a modified (strong) Lebesgue point.

Theorem 2 *Almost every point $(x, y) \in \mathbb{R}^2$ is a modified p-Lebesgue point and a modified strong p-Lebesgue point of $f \in W(L_p, \ell_\infty)(\mathbb{R}^2)$ $(1 \le p < \infty)$.*

Proof It is enough to prove the theorem for the modified strong Lebesgue points and for $f \in L_p(\mathbb{R}^2)$. Let $\tau > 0$ be arbitrary. If f is a continuous function, then (15) and (16) hold for all $(x, y \in \mathbb{R}^2)$. Let us denote

$$
\begin{aligned}
U_{r,p} f(x,y) &:= U_{r,p}^{(1)} f(x,y) + U_{r,p}^{(2)} f(x,y), \\
U_{r,p}^{(1)} f(x,y) &:= \sup_{\substack{i,j\in\mathbb{N}, h>0 \\ 2^i h<r, 2^j h<r}} 2^{-\tau(i+j)} \\
&\qquad \left(\frac{1}{4\cdot 2^{i+j} h^2} \int_{-2^i h}^{2^i h} \int_{-2^j h}^{2^j h} |f(x-s, y-t) - f(x,y)|^p \, ds \, dt \right)^{1/p}, \\
U_{r,p}^{(2)} f(x,y) &:= \sup_{\substack{i,j\in\mathbb{N}, h>0 \\ 2^i h<r, 2^j h<r}} 2^{-\tau(i+j)} \\
&\qquad \left(\frac{1}{4\cdot 2^{i+j} h^2} \int_{-2^i h}^{2^i h} \int_{s-2^j h}^{s+2^j h} |f(x-s, y-t) - f(x,y)|^p \, dt \, ds \right)^{1/p}.
\end{aligned}
$$

In case $p = 1$ we omit the notation p and write simply $U_r f$, $U_r^{(1)} f$ and $U_r^{(2)} f$. Then, by Theorem 1,

$$
\begin{aligned}
\rho^p \lambda\big(\sup_{r>0} U_{r,p} f > \rho\big) &\le \rho^p \lambda\big(\mathcal{M}_p^{(1)} f > \rho/4\big) + \rho^p \lambda\big(\mathcal{M}_p^{(2)} f > \rho/4\big) \\
&\quad + 2\rho^p \lambda\big(f > \rho/4\big) \\
&\le C \|f\|_p^p.
\end{aligned}
$$

Since the result holds for continuous functions and the continuous functions are dense in $L_p(\mathbb{R}^2)$, the theorem follows from the usual density argument due to Marcinkiewicz and Zygmund [21]. □

It is not sure that (x, x) is a modified (strong) p-Lebesgue point of a function $f \in W(L_p, \ell_\infty)(\mathbb{R}^2)$ for almost every $x \in \mathbb{R}$. However, under some conditions, we can prove this result.

Theorem 3 *Suppose that $f(x, y) = f_0(x) f_0(y)$. If x and y are p-Lebesgue points of $f_0 \in W(L_p, \ell_\infty)(\mathbb{R})$, then (x, y) is a modified p-Lebesgue point of $f \in W(L_p, \ell_\infty)(\mathbb{R}^2)$ $(1 \le p < \infty)$.*

Proof We have

$$\begin{aligned}
&\left(\frac{1}{4\cdot 2^{i+j}h^2}\int_{-2^i h}^{2^i h}\int_{-2^j h}^{2^j h}\left|f(x-s,y-t)-f(x,y)\right|^p\,dt\,ds\right)^{1/p}\\
&\le\left(\frac{1}{4\cdot 2^{i+j}h^2}\int_{-2^i h}^{2^i h}\int_{-2^j h}^{2^j h}\left|f_0(x-s)-f_0(x)\right|^p\left|f_0(y-t)\right|^p\,dt\,ds\right)^{1/p}\\
&\quad+\left(\frac{1}{4\cdot 2^{i+j}h^2}\int_{-2^i h}^{2^i h}\int_{-2^j h}^{2^j h}\left|f_0(x)\right|^p\left|f_0(y-t)-f_0(y)\right|^p\,dt\,ds\right)^{1/p}\\
&=A_1(x,y)+A_2(x,y).
\end{aligned}$$

It is easy to see that if y is a p-Lebesgue point of f_0, then $M_p f_0(y)$ is finite, where $M_p f_0$ denotes the maximal function of the one-dimensional function f_0. Since x is also a p-Lebesgue point of f_0,

$$A_1(x,y)\le M_p f_0(y)\left(\frac{1}{2\cdot 2^i h}\int_{-2^i h}^{2^i h}\left|f_0(x-s)-f_0(x)\right|^p\,ds\right)^{1/p}<\epsilon,$$

whenever $2^i h<r$ and r is small enough. The term A_2 can be handled similarly,

$$A_2(x,y)\le C\left(\frac{1}{2\cdot 2^j h}\int_{-2^j h}^{2^j h}\left|f_0(y-t)-f_0(y)\right|^p\,dt\right)^{1/p}<\epsilon,$$

whenever $2^j h<r$ and r is small enough. □

The following corollary can be seen in the same way.

Corollary 1 *Suppose that* $f(x,y)=f_0(x)f_0(y)$. *If* x *and* y *are p-Lebesgue points of* $f_0\in W(L_p,\ell_\infty)(\mathbb{R})$, *then* $\mathcal{M}_p^{(1)}f(x,y)$ *is finite* $(1\le p<\infty)$.

Proof It is easy to see that $\mathcal{M}_p^{(1)}f(x,y)\le M_p f_0(x)M_p f_0(y)$.

For the modified strong Lebesgue points we need in addition that f_0 is almost everywhere locally bounded.

Theorem 4 *Suppose that* $f(x,y)=f_0(x)f_0(y)$. *If* x *and* y *are p-Lebesgue points of* $f_0\in W(L_p,\ell_\infty)(\mathbb{R})$ *and* f_0 *is locally bounded at* x *and* y, *then* (x,y) *is a modified strong p-Lebesgue point of* $f\in W(L_p,\ell_\infty)(\mathbb{R}^2)$ $(1\le p<\infty)$.

Proof We will prove (16), only. Then

$$\begin{aligned}
&\left(\frac{1}{4\cdot 2^{i+j}h^2}\int_{-2^i h}^{2^i h}\int_{s-2^j h}^{s+2^j h}\left|f(x-s,y-t)-f(x,y)\right|^p\,dt\,ds\right)^{1/p}\\
&\le\left(\frac{1}{4\cdot 2^{i+j}h^2}\int_{-2^i h}^{2^i h}\int_{s-2^j h}^{s+2^j h}\left|f_0(x-s)-f_0(x)\right|^p\left|f_0(y-t)\right|^p\,dt\,ds\right)^{1/p}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{1}{4\cdot 2^{i+j}h^2}\int_{-2^ih}^{2^ih}\int_{s-2^jh}^{s+2^jh}\left|f_0(x)\right|^p\left|f_0(y-t)-f_0(y)\right|^p\,dt\,ds\right)^{1/p}\\
&= A_3(x,y)+A_4(x,y).
\end{aligned}
$$

Since x is a Lebesgue point of f_0 and f_0 is bounded in a neighborhood of y,

$$
A_3(x,y)\le C\left(\frac{1}{2\cdot 2^ih}\int_{-2^ih}^{2^ih}\left|f_0(x-s)-f_0(x)\right|^p\,ds\right)^{1/p}<\epsilon,
$$

whenever $2^ih<r,\ 2^jh<r$ and r is small enough.

On the other hand,

$$
\begin{aligned}
&A_4(x,y)\\
&=\left(\frac{1}{4\cdot 2^{i+j}h^2}\int_{-2^ih-2^jh}^{2^ih+2^jh}\int_{-2^ih\vee(t-2^jh)}^{2^ih\wedge(t+2^jh)}\left|f_0(x)\right|^p\left|f_0(y-t)-f_0(y)\right|^p\,ds\,dt\right)^{1/p}.
\end{aligned}
$$

If $i\ge j$, then

$$
\begin{aligned}
A_4f_0(x)&\le C\left(\frac{1}{4\cdot 2^{i+j}h^2}\int_{-2^{i+1}h}^{2^{i+1}h}\int_{t-2^jh}^{t+2^jh}\left|f_0(y-t)-f_0(y)\right|^p\,ds\,dt\right)^{1/p}\\
&=C\left(\frac{1}{2\cdot 2^ih}\int_{-2^{i+1}h}^{2^{i+1}h}\left|f_0(y-t)-f_0(y)\right|^p\,dt\right)^{1/p}<\epsilon
\end{aligned}
$$

and if $i<j$, then

$$
\begin{aligned}
A_4(x,y)&\le C\left(\frac{1}{4\cdot 2^{i+j}h^2}\int_{-2^{j+1}h}^{2^{j+1}h}\int_{-2^ih}^{2^ih}\left|f_0(y-t)-f_0(y)\right|^p\,ds\,dt\right)^{1/p}\\
&=C\left(\frac{1}{2\cdot 2^jh}\int_{-2^{j+1}h}^{2^{j+1}h}\left|f_0(y-t)-f_0(y)\right|^p\,dt\right)^{1/p}<\epsilon,
\end{aligned}
$$

whenever $2^ih<r,\ 2^jh<r$ and r is small enough. This proves the theorem. □

5 Pointwise Convergence of Marcinkiewicz Summation

Now we prove that the Marcinkiewicz means $\sigma_T^\theta f$ converge to f at each modified strong Lebesgue points.

Theorem 5 *Suppose that* (6) *is satisfied for some* $0<\alpha<\infty$ *and* $f\in W(L_1,\ell_\infty)(\mathbb{R}^2)$. *If* (x,y) *is a modified strong Lebesgue point of* f *and* $\mathcal{M}f(x,y)$ *is finite, then*

$$
\lim_{T\to\infty}\sigma_T^\theta f(x,y)=f(x,y).
$$

Proof Let $\theta_0(s,t) := \theta(|s| \vee |t|)$. The first equation of (4) implies that

$$K_T^\theta(s,t) := T^2 \widehat{\theta_0}(Ts, Tt).$$

Since $\theta_0 \in L_1(\mathbb{R}^2)$ by (5) and $\widehat{\theta_0} \in L_1(\mathbb{R}^2)$ by Lemma 2, the Fourier inversion formula yields that

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} K_T^\theta(s,t)\, ds\, dt = \frac{1}{2\pi} \int_{\mathbb{R}^2} \widehat{\theta_0}(s,t)\, ds\, dt = \theta(0) = 1.$$

Thus

$$\left|\sigma_T^\theta f(x,y) - f(x,y)\right| \le \frac{1}{2\pi} \int_{\mathbb{R}^2} \left|f(x-s, y-t) - f(x,y)\right| \left|K_T^\theta(s,t)\right| ds\, dt. \quad (17)$$

It is enough to integrate over the set $\{(s,t) \in \mathbb{R}^2 : s > t > 0\}$. Let us decompose this set into the union $\cup_{i=1}^5 A_i$, where

$$\begin{aligned}
A_1 &:= \{(s,t) : 0 < s \le 2/T, 0 < t < s\},\\
A_2 &:= \{(s,t) : s > 2/T, 0 < t \le 1/T\},\\
A_3 &:= \{(s,t) : s > 2/T, 1/T < t \le s/2\},\\
A_4 &:= \{(s,t) : s > 2/T, s/2 < t \le s - 1/T\},\\
A_5 &:= \{(s,t) : s > 2/T, s - 1/T < t \le s\}.
\end{aligned}$$

The sets A_i can be seen on Fig. 1. Let $\tau < \alpha/2 \wedge 1$. Since (x,y) is a modified strong Lebesgue point of f, we can fix a number $r < 1$ such that

$$U_r f(x,y) < \epsilon.$$

Let us denote the square $[0, r/2] \times [0, r/2]$ by $S_{r/2}$ and let $2/T < r/2$. We will integrate the right hand side of (17) over the sets

$$\bigcup_{i=1}^5 (A_i \cap S_{r/2}) \quad \text{and} \quad \bigcup_{i=1}^5 \left(A_i \cap S_{r/2}^c\right),$$

where S^c denotes the complement of the set S. Of course, $A_1 \subset S_{r/2}$. By (7),

$$\begin{aligned}
&\int_{A_1} \left|f(x-s, y-t) - f(x,y)\right| \left|K_T^\theta(s,t)\right| ds\, dt\\
&\quad \le CT^2 \int_0^{2/T} \int_0^{2/T} \left|f(x-s, y-t) - f(x,y)\right| ds\, dt \le C U_r^{(1)} f(x,y) < C\epsilon.
\end{aligned}$$

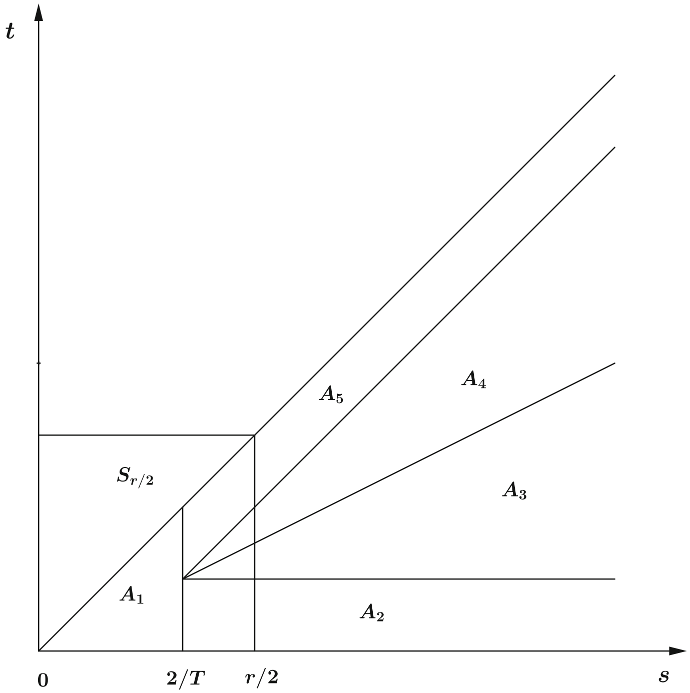

Fig. 1 The sets A_i

Let us denote by r_0 the largest number i, for which $r/2 \le 2^{i+1}/T < r$. By (10),

$$\begin{aligned}
&\int_{A_2 \cap S_{r/2}} \left| f(x-s, y-t) - f(x,y) \right| \left| K_T^\theta(s,t) \right| ds\, dt \\
&\le C \sum_{i=1}^{r_0} T^{1-\alpha} \left(\frac{2^i}{T} \right)^{-1} \left(\frac{2^i}{T} - \frac{1}{T} \right)^{-\alpha} \int_{2^i/T}^{2^{i+1}/T} \int_0^{1/T} \left| f(x-s, y-t) - f(x,y) \right| ds\, dt \\
&\le C \sum_{i=1}^{r_0} 2^{(\tau-\alpha)i} 2^{-\tau i} \left(\frac{T^2}{2^i} \right) \int_{2^i/T}^{2^{i+1}/T} \int_0^{1/T} \left| f(x-s, y-t) - f(x,y) \right| ds\, dt \\
&\le C \sum_{i=1}^{r_0} 2^{(\tau-\alpha)i} U_r^{(1)} f(x,y) < C\epsilon,
\end{aligned}$$

because $\tau < \alpha$.

Since $s - t > s/2$ and $s - t > t$ on A_3, we obtain by (9) that

$$\left| K_T^\theta(s,t) \right| \le C T^{-\alpha} s^{-1-\alpha/2} t^{-1-\alpha/2}. \tag{18}$$

Hence

$$\begin{aligned}
&\int_{A_3\cap S_{r/2}} \left|f(x-s,y-t)-f(x,y)\right|\left|K_T^\theta(s,t)\right| ds\,dt\\
&\leq C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} T^{-\alpha}\left(\frac{2^i}{T}\right)^{-1-\alpha/2}\left(\frac{2^j}{T}\right)^{-1-\alpha/2}\\
&\quad\int_{2^i/T}^{2^{i+1}/T}\int_{2^j/T}^{2^{j+1}/T}\left|f(x-s,y-t)-f(x,y)\right| ds\,dt\\
&\leq C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} 2^{(\tau-\alpha/2)(i+j)}\\
&\quad 2^{-\tau(i+j)}\left(\frac{T^2}{2^{i+j}}\right)\int_{2^i/T}^{2^{i+1}/T}\int_{2^j/T}^{2^{j+1}/T}\left|f(x-s,y-t)-f(x,y)\right| ds\,dt\\
&\leq C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} 2^{(\tau-\alpha/2)(i+j)} U_r^{(1)} f(x,y) < C\epsilon.
\end{aligned}\tag{19}$$

Since $t > s/2$ on A_4, (9) implies

$$\left|K_T^\theta(s,t)\right| \leq CT^{-\alpha}s^{-2}\big(s-t\big)^{-\alpha},\tag{20}$$

and so

$$\begin{aligned}
&\int_{A_4\cap S_{r/2}} \left|f(x-s,y-t)-f(x,y)\right|\left|K_T^\theta(s,t)\right| ds\,dt\\
&\leq C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} T^{-\alpha}\left(\frac{2^i}{T}\right)^{-2}\left(\frac{2^j}{T}\right)^{-\alpha}\int_{2^i/T}^{2^{i+1}/T}\int_{s-2^{j+1}/T}^{s-2^j/T}\left|f(x-s,y-t)-f(x,y)\right| dt\,ds\\
&\leq C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} 2^{(\tau-1)i}2^{(\tau+1-\alpha)j}\\
&\quad 2^{-\tau(i+j)}\left(\frac{T^2}{2^{i+j}}\right)\int_{2^i/T}^{2^{i+1}/T}\int_{s-2^{j+1}/T}^{s-2^j/T}\left|f(x-s,y-t)-f(x,y)\right| dt\,ds\\
&\leq C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} 2^{(\tau-1)i}2^{(\tau+1-\alpha)j} U_r^{(2)} f(x,y) < C\epsilon
\end{aligned}$$

if $2\leq\alpha<\infty$ and $\tau<1$. If $0<\alpha<2$, then

$$2^{(\tau-1)i}2^{(\tau+1-\alpha)j} = 2^{(\tau-\alpha/2)i}2^{(\alpha/2-1)i}2^{(\tau+1-\alpha)j} \leq 2^{(\tau-\alpha/2)i}2^{(\tau-\alpha/2)j}$$

and so

$$\int_{A_4\cap S_{r/2}} \left|f(x-s,y-t)-f(x,y)\right|\left|K_T^\theta(s,t)\right| ds\, dt$$
$$\le C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} 2^{(\tau-\alpha/2)(i+j)}U_r^{(2)} f(x,y) < C\epsilon$$

because $\tau < \alpha/2$.

We get from (8) that

$$\left|K_T^\theta(s,t)\right| \le Cs^{-2}$$

on the set A_5. This implies

$$\int_{A_5\cap S_{r/2}} \left|f(x-s,y-t)-f(x,y)\right|\left|K_T^\theta(s,t)\right| ds\, dt$$
$$\le C\sum_{i=1}^{r_0}\left(\frac{2^i}{T}\right)^{-2}\int_{2^i/T}^{2^{i+1}/T}\int_{s-1/T}^{s}\left|f(x-s,y-t)-f(x,y)\right| dt\, ds$$
$$\le C\sum_{i=1}^{r_0} 2^{(\tau-1)i}2^{-\tau i}\left(\frac{T^2}{2^i}\right)\int_{2^i/T}^{2^{i+1}/T}\int_{s-1/T}^{s}\left|f(x-s,y-t)-f(x,y)\right| dt\, ds$$
$$\le C\sum_{i=1}^{r_0} 2^{(\tau-1)i}U_r^{(2)} f(x,y) < C\epsilon.$$

Similarly, we can show that

$$\int_{A_2\cap S_{r/2}^c} \left|f(x-s,y-t)-f(x,y)\right|\left|K_T^\theta(s,t)\right| ds\, dt$$
$$\le C\sum_{i=r_0}^{\infty} 2^{(\tau-\alpha)i}\mathcal{M}^{(1)} f(x,y) + C\sum_{i=r_0}^{\infty} 2^{-\alpha i} f(x,y)$$
$$\le C2^{(\tau-\alpha)r_0}\mathcal{M}^{(1)} f(x,y) + C2^{-\alpha r_0} f(x,y)$$
$$\le C(Tr)^{\tau-\alpha}\mathcal{M}^{(1)} f(x,y) + C(Tr)^{-\alpha} f(x,y) \to 0$$

and

$$\int_{A_3\cap S_{r/2}^c} \left|f(x-s,y-t)-f(x,y)\right|\left|K_T^\theta(s,t)\right| ds\, dt$$
$$\le C\sum_{i=r_0}^{\infty}\sum_{j=0}^{i-1} 2^{(\tau-\alpha/2)(i+j)}\mathcal{M}^{(1)} f(x,y) + C\sum_{i=r_0}^{\infty}\sum_{j=0}^{i-1} 2^{-\alpha/2(i+j)} f(x,y)$$
$$\le C2^{(\tau-\alpha/2)r_0}\mathcal{M}^{(1)} f(x,y) + C2^{-\alpha/2r_0} f(x,y) \to 0,$$

as $T \to \infty$. If $0 < \alpha \le 2$, then

$$\begin{aligned}\int_{A_4\cap S_{r/2}^c} &\left|f(x-s,y-t)-f(x,y)\right|\left|K_T^\theta(s,t)\right| ds\,dt\\ &\le C\sum_{i=r_0}^{\infty}\sum_{j=0}^{i-1}2^{(\tau-\alpha/2)(i+j)}\mathcal{M}^{(2)}f(x,y)+C\sum_{i=r_0}^{\infty}\sum_{j=0}^{i-1}2^{-\alpha/2(i+j)}f(x,y)\\ &\le C2^{(\tau-\alpha/2)r_0}\mathcal{M}^{(2)}f(x,y)+C2^{-\alpha/2r_0}f(x,y)\to 0\end{aligned}$$

and if $2 < \alpha < \infty$, then

$$\begin{aligned}\int_{A_4\cap S_{r/2}^c} &\left|f(x-s,y-t)-f(x,y)\right|\left|K_T^\theta(s,t)\right| ds\,dt\\ &\le C\sum_{i=r_0}^{\infty}\sum_{j=0}^{i-1}2^{(\tau-1)i}2^{(\tau+1-\alpha)j}\mathcal{M}^{(2)}f(x,y)+C\sum_{i=r_0}^{\infty}\sum_{j=0}^{i-1}2^{-i}2^{(1-\alpha)j}f(x,y)\\ &\le C2^{(\tau-1)r_0}\mathcal{M}^{(2)}f(x,y)+C2^{-r_0}f(x,y)\to 0.\end{aligned}$$

Finally,

$$\begin{aligned}\int_{A_5\cap S_{r/2}^c} &\left|f(x-s,y-t)-f(x,y)\right|\left|K_T^\theta(s,t)\right| ds\,dt\\ &\le C\sum_{i=r_0}^{\infty}2^{(\tau-1)i}\mathcal{M}^{(2)}f(x,y)+C\sum_{i=r_0}^{\infty}2^{-i}f(x,y)\\ &\le C2^{(\tau-1)r_0}\mathcal{M}^{(2)}f(x,y)+C2^{-r_0}f(x,y)\to 0,\end{aligned}$$

as $T \to \infty$. Note that $A_1 \cap S_{r/2}^c = \emptyset$. This completes the proof of the theorem. □

Since by Theorems 1 and 2 almost every point is a modified strong Lebesgue point and the maximal operator $\mathcal{M}f$ is almost everywhere finite for $f \in W(L_1, \ell_\infty)(\mathbb{R}^2)$, Theorem 5 imply

Corollary 2 *Suppose that (6) is satisfied for some $0 < \alpha < \infty$ and $f \in W(L_1, \ell_\infty)(\mathbb{R}^2)$. Then*

$$\lim_{T\to\infty}\sigma_T^\theta f(x,y)=f(x,y) \qquad a.e.$$

If $1 \le \alpha < \infty$, then in Theorem 5 we can omit the condition that $\mathcal{M}f(x,y)$ is finite. We will use this result later in the theory of strong summability.

Theorem 6 *Suppose that (6) is satisfied for some $1 \le \alpha < \infty$ and $f \in W(L_1, \ell_\infty)(\mathbb{R}^2)$. If (x, y) is a modified strong Lebesgue point of f, then*

$$\lim_{T\to\infty}\sigma_T^\theta f(x,y)=f(x,y).$$

Proof The estimation of the integral (17) over the square $S_{r/2}$ can be found in Theorem 5. Similarly,

$$\int_{S_{r/2}^c} \left|f(x,y)\right|\left|K_T^\theta(s,t)\right| ds\, dt \to 0,$$

as $T \to \infty$. Hence we have to estimate the integral

$$\int_{\bigcup_{i=1}^5 (A_i \cap S_{r/2}^c)} \left|f(x-s,y-t)\right|\left|K_T^\theta(s,t)\right| ds\, dt.$$

For small $\delta > 0$ let us introduce the sets

$$\begin{aligned} B_1 &:= \{(s,t) : s > r/2, 0 < t \le \delta\}, \\ B_2 &:= \{(s,t) : s > r/2, \delta < t \le s-\delta\}, \\ B_3 &:= \{(s,t) : s > r/2, s-\delta < t \le s\}. \end{aligned}$$

Then we have to integrate over these three sets. On B_1 we use estimation (10) to obtain

$$\begin{aligned} &\int_{B_1} \left|f(x-s,y-t)\right|\left|K_T^\theta(s,t)\right| ds\, dt \qquad (21) \\ &\le CT^{1-\alpha} \sum_{i=0}^{N_0-1} (i \vee 1)^{-1-\alpha} \int_i^{i+1} \int_0^\delta \left|f(x-s,y-t)\right| ds\, dt \\ &\quad + CT^{1-\alpha} \sum_{i=N_0}^{\infty} i^{-1-\alpha} \int_i^{i+1} \int_0^\delta \left|f(x-s,y-t)\right| ds\, dt \\ &\le CT^{1-\alpha} \left\| f\, 1_{[r/2,N_0]\times[0,\delta]} \right\|_{W(L_1,\ell_\infty)} + CT^{1-\alpha} N_0^{-\alpha} \left\| f \right\|_{W(L_1,\ell_\infty)}. \end{aligned}$$

The second term is less than ϵ if N_0 is large enough and the first term is less than ϵ if δ is small enough. The rest of the proof works for all $0 < \alpha < \infty$. Indeed, by (8),

$$\begin{aligned} &\int_{B_3} \left|f(x-s,y-t)\right|\left|K_T^\theta(s,t)\right| ds\, dt \\ &\le C \sum_{i=0}^{N_0-1} (i \vee 1)^{-2} \int_i^{i+1} \int_{s-\delta}^{s} \left|f(x-s,y-t)\right| ds\, dt \\ &+ C \sum_{i=N_0}^{\infty} i^{-2} \int_i^{i+1} \int_{s-\delta}^{s} \left|f(x-s,y-t)\right| ds\, dt \\ &\le C \left\| f\, 1_{\{(s,t): r/2 < s < N_0, s-\delta < t \le s\}} \right\|_{W(L_1,\ell_\infty)} + CN_0^{-1} \left\| f \right\|_{W(L_1,\ell_\infty)} < \epsilon \end{aligned}$$

if N_0 is large enough and δ is small enough. Moreover, by (18),

$$\begin{aligned}
&\int_{B_2\cap A_3} \left|f(x-s,y-t)\right|\left|K_T^\theta(s,t)\right| ds\, dt \\
&\leq CT^{-\alpha}\sum_{i=0}^{\infty}(i\vee 1)^{-1-\alpha/2}\delta^{-1-\alpha/2}\int_i^{i+1}\int_\delta^1 \left|f(x-s,y-t)\right| ds\, dt \\
&+CT^{-\alpha}\sum_{i=1}^{\infty}\sum_{j=1}^{i} i^{-1-\alpha/2}j^{-1-\alpha/2}\int_i^{i+1}\int_j^{j+1}\left|f(x-s,y-t)\right| ds\, dt \\
&\leq CT^{-\alpha}\|f\|_{W(L_1,\ell_\infty)} \to 0,
\end{aligned}$$

as $T\to\infty$. If $2\leq\alpha<\infty$, then by (20),

$$\begin{aligned}
&\int_{B_2\cap A_4} \left|f(x-s,y-t)\right|\left|K_T^\theta(s,t)\right| ds\, dt \\
&\leq CT^{-\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{i}(i\vee 1)^{-2}(j\vee 1)^{-\alpha}\int_i^{i+1}\int_{s-j-1}^{s-j}\left|f(x-s,y-t)\right| ds\, dt \\
&\leq CT^{-\alpha}\|f\|_{W(L_1,\ell_\infty)} \to 0.
\end{aligned}$$

If $0<\alpha<2$, then

$$\begin{aligned}
&\int_{B_2\cap A_4} \left|f(x-s,y-t)\right|\left|K_T^\theta(s,t)\right| ds\, dt \\
&\leq CT^{-\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{i}(i\vee 1)^{-1-\alpha/2}(j\vee 1)^{-1-\alpha/2}\int_i^{i+1}\int_{s-j-1}^{s-j}\left|f(x-s,y-t)\right| ds\, dt \\
&\leq CT^{-\alpha}\|f\|_{W(L_1,\ell_\infty)} \to 0,
\end{aligned}$$

which finishes the proof. □

The preceding result holds also for $0<\alpha<1$ when $f(x,y)=f_0(x)f_0(y)$.

Theorem 7 *Suppose that (6) is satisfied for some $0<\alpha<\infty$ and $f(x,y)=f_0(x)f_0(y)$ with $f_0\in W(L_1,\ell_\infty)(\mathbb{R})$. If x and y are Lebesgue points of f_0 and f_0 is locally bounded at x and y, then*

$$\lim_{T\to\infty}\sigma_T^\theta f(x,y)=f(x,y).$$

Proof Taking into account Theorems 4 and 6, we have to estimate the integral (21) for $0<\alpha<1$, only. Let δ_0 the largest number j, for which $\delta/2\leq 2^j/T<\delta$. If $T\geq t^{-1}$ then (9) implies

$$\left|K_T^\theta(s,t)\right|\leq Cs^{-1}t^{\alpha-1}(s-t)^{-\alpha}\leq Ct^{\alpha-1}(s-t)^{-1-\alpha}.$$

If $T < t^{-1}$ then we get the same inequality from (10). Using this we get similarly to (19) that

$$\begin{aligned}
&\int_{B_1} \left|f(x-s, y-t)\right|\left|K_T^\theta(s,t)\right| ds\, dt \\
&\leq C \sum_{i=r_0}^{\infty} \sum_{j=-\infty}^{\delta_0} \left(\frac{2^i}{T}\right)^{-1-\alpha} \left(\frac{2^j}{T}\right)^{\alpha-1} \int_{2^i/T}^{2^{i+1}/T} \int_{2^j/T}^{2^{j+1}/T} \left|f(x-s, y-t)\right| ds\, dt \\
&\leq C \sum_{i=r_0}^{\infty} \sum_{j=-\infty}^{\delta_0} 2^{-\alpha i+\alpha j} \left(\frac{T^2}{2^{i+j}}\right) \int_{2^i/T}^{2^{i+1}/T} \int_{2^j/T}^{2^{j+1}/T} \left|f_0(x-s)\right|\left|f_0(y-t)\right| ds\, dt \\
&\leq C \sum_{i=r_0}^{\infty} \sum_{j=-\infty}^{\delta_0} 2^{-\alpha i+\alpha j} M f_0(x) M f_0(y) \\
&\leq C(Tr)^{-\alpha}(T\delta)^{\alpha} M f_0(x) M f_0(y) < \epsilon,
\end{aligned}$$

if δ is small enough, because $Mf_0(x)$ is finite for a Lebesgue point x of $f_0 \in W(L_1, \ell_\infty)(\mathbb{R})$. This completes the proof of the theorem. □

In the next theorem we do not need the maximal operator $\mathcal{M}^{(2)} f$.

Theorem 8 *Suppose that (6) is satisfied for some $0 < \alpha < \infty$ and $f \in W(L_p, \ell_\infty)(\mathbb{R}^2)$ $(1 < p < \infty)$. If (x, y) is a modified p-Lebesgue point of f and $\mathcal{M}_p^{(1)} f(x, y)$ is finite, then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x, y) = f(x, y).$$

Proof We have to integrate the integral in (17) again on the sets $\cup_{i=1}^5 A_i$. Now let $\tau < \alpha/2 \wedge 1/4 \wedge 1/(2q)$, where $1/p + 1/q = 1$. Since (x, y) is a modified p-Lebesgue point of f, we can fix a number r such that

$$U_{r,p}^{(1)} f(x, y) < \epsilon.$$

Since

$$U_{r,1}^{(1)} f \leq U_{r,p}^{(1)} f \qquad \text{and} \qquad \mathcal{M}^{(1)} f \leq \mathcal{M}_p^{(1)} f,$$

we can prove in the same way as in Theorem 5 that

$$\int_{A_i \cap S_{r/2}} \left|f(x-s, y-t) - f(x, y)\right|\left|K_T^\theta(s,t)\right| ds\, dt < C\epsilon$$

and

$$\begin{aligned}&\int_{A_i\cap S^c_{r/2}} \left|f(x-s,y-t)-f(x,y)\right|\left|K^\theta_T(s,t)\right| ds\,dt\\&\quad\le C2^{(\tau-\alpha/2)r_0}\mathcal{M}^{(1)}f(x,y)+C2^{-\alpha/2r_0}f(x,y)\\&\quad\le C(Tr)^{\tau-\alpha/2}\mathcal{M}^{(1)}f(x,y)+C(Tr)^{-\alpha/2}f(x,y)\to 0,\end{aligned}$$

for $i=1,2,3$, as $T\to\infty$.

So we have to consider the sets A_4 and A_5, only. It is easy to see that

$$\begin{aligned}&\int_{A_4\cap S_{r/2}} \left|f(x-s,y-t)-f(x,y)\right|\left|K^\theta_T(s,t)\right| ds\,dt\\&\le\sum_{i=1}^{r_0}\sum_{j=i-1}^{i}\int_{2^i/T}^{2^{i+1}/T}\int_{2^j/T}^{2^{j+1}/T}\left|f(x-s,y-t)-f(x,y)\right|\left|K^\theta_T(s,t)\right|1_{A_4}\,dt\,ds.\end{aligned}$$

By (20) and Hölder's inequality,

$$\begin{aligned}&\int_{A_4\cap S_{r/2}} \left|f(x-s,y-t)-f(x,y)\right|\left|K^\theta_T(s,t)\right| ds\,dt\\&\quad\le\sum_{i=1}^{r_0}\sum_{j=i-1}^{i}\left(\int_{2^i/T}^{2^{i+1}/T}\int_{2^j/T}^{2^{j+1}/T}\left|f(x-s,y-t)-f(x,y)\right|^p dt\,ds\right)^{1/p}\\&\left(\int_{2^i/T}^{2^{i+1}/T}\int_{2^{i-1}/T}^{s-1/T}T^{-\alpha q}s^{-2q}(s-t)^{-\alpha q}1_{A_4}\,dt\,ds\right)^{1/q}.\end{aligned}$$

If $q<1/\alpha$, then

$$\begin{aligned}&\int_{2^i/T}^{2^{i+1}/T}\int_{2^{i-1}/T}^{s-1/T}T^{-\alpha q}s^{-2q}(s-t)^{-\alpha q}\,dt\,ds\\&\quad\le CT^{-\alpha q}\int_{2^i/T}^{2^{i+1}/T}s^{-2q}\left(s-\frac{2^{i-1}}{T}\right)^{-\alpha q+1}ds\\&\quad\le CT^{-\alpha q}\left(\frac{2^i}{T}\right)^{-\alpha q+1}\left(\frac{2^i}{T}\right)^{-2q+1}\\&\quad\le C\left(\frac{T}{2^i}\right)^{2q-2}2^{-i\alpha q}.\end{aligned}$$

Thus

$$\begin{aligned}&\int_{A_4\cap S_{r/2}} \left|f(x-s,y-t)-f(x,y)\right|\left|K^\theta_T(s,t)\right| ds\,dt\\&\quad\le C_p\sum_{i=1}^{r_0}\sum_{j=i-1}^{i}2^{(\tau-\alpha/2)(i+j)}\end{aligned}$$

$$2^{-\tau(i+j)}\left(\frac{T^2}{2^{i+j}}\int_{2^i/T}^{2^{i+1}/T}\int_{2^j/T}^{2^{j+1}/T}\left|f(x-s,y-t)-f(x,y)\right|^p\,dt\,ds\right)^{1/p}$$
$$\leq C_p\sum_{i=1}^{r_0}\sum_{j=i-1}^{i}2^{(\tau-\alpha/2)(i+j)}U_{r,p}^{(1)}f(x,y)<C\epsilon.$$

For $q>1/2\alpha$ we have

$$\int_{2^i/T}^{2^{i+1}/T}\int_{2^{i-1}/T}^{s-1/T}T^{-\alpha q}s^{-2q}(s-t)^{-\alpha q}\,dt\,ds$$
$$\leq\int_{2^i/T}^{2^{i+1}/T}\int_{2^{i-1}/T}^{s-1/T}T^{-\alpha q}s^{-2q+1/2}(s-t)^{-\alpha q-1/2}\,dt\,ds$$
$$\leq CT^{-\alpha q}\left(\frac{1}{T}\right)^{-\alpha q+1/2}\left(\frac{2^i}{T}\right)^{-2q+3/2}$$
$$\leq C\left(\frac{T}{2^i}\right)^{2q-2}2^{-i/2}$$

and

$$\int_{A_4\cap S_{r/2}}\left|f(x-s,y-t)-f(x,y)\right|\left|K_T^{\theta}(s,t)\right|ds\,dt$$
$$\leq C_p\sum_{i=1}^{r_0}\sum_{j=i-1}^{i}2^{(\tau-1/4)(i+j)}$$
$$2^{-\tau(i+j)}\left(\frac{T^2}{2^{i+j}}\int_{2^i/T}^{2^{i+1}/T}\int_{2^j/T}^{2^{j+1}/T}\left|f(x-s,y-t)-f(x,y)\right|^p\,dt\,ds\right)^{1/p}$$
$$\leq C_p\sum_{i=1}^{r_0}\sum_{j=i-1}^{i}2^{(\tau-1/4)(i+j)}U_{r,p}^{(1)}f(x,y)<C\epsilon.$$

Similarly, for $q<1/\alpha$,

$$\int_{A_4\cap S_{r/2}^c}\left|f(x-s,y-t)-f(x,y)\right|\left|K_T^{\theta}(s,t)\right|ds\,dt$$
$$\leq C_p\sum_{i=r_0}^{\infty}\sum_{j=i-1}^{i}2^{(\tau-\alpha/2)(i+j)}\mathcal{M}_p^{(1)}f(x,y)+C_p\sum_{i=r_0}^{\infty}\sum_{j=i-1}^{i}2^{-\alpha(i+j)/2}f(x,y)$$
$$\leq C_p2^{(2\tau-\alpha)r_0}\mathcal{M}_p^{(1)}f(x,y)+C_p2^{-\alpha r_0}f(x,y)\to 0,$$

and for $q > 1/2\alpha$,

$$\int_{A_4 \cap S_{r/2}^c} \left| f(x-s, y-t) - f(x,y) \right| \left| K_T^\theta(s,t) \right| ds\, dt \\ \le C_p 2^{(2\tau-1/2)r_0} \mathcal{M}_p^{(1)} f(x,y) + C_p 2^{-r_0/2} f(x,y) \to 0.$$

For the set A_5 we obtain

$$\int_{A_5 \cap S_{r/2}} \left| f(x-s, y-t) - f(x,y) \right| \left| K_T^\theta(s,t) \right| ds\, dt \\ \le \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} \left(\int_{2^i/T}^{2^{i+1}/T} \int_{2^j/T}^{2^{j+1}/T} \left| f(x-s, y-t) - f(x,y) \right|^p dt\, ds \right)^{1/p} \\ \left(\int_{2^i/T}^{2^{i+1}/T} \int_{s-1/T}^{s} s^{-2q}\, dt\, ds \right)^{1/q} \\ \le \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} \left(\int_{2^i/T}^{2^{i+1}/T} \int_{2^j/T}^{2^{j+1}/T} \left| f(x-s, y-t) - f(x,y) \right|^p dt\, ds \right)^{1/p} \\ T^{-1/q} \left(\frac{2^i}{T} \right)^{-2+1/q} \\ \le C_p \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} 2^{(\tau-1/(2q))(i+j)} \\ 2^{-\tau(i+j)} \left(\frac{T^2}{2^{i+j}} \int_{2^i/T}^{2^{i+1}/T} \int_{2^j/T}^{2^{j+1}/T} \left| f(x-s, y-t) - f(x,y) \right|^p dt\, ds \right)^{1/p} \\ \le C_p \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} 2^{(\tau-1/(2q))(i+j)} U_{r,p}^{(1)} f(x,y) < C\epsilon.$$

Finally,

$$\int_{A_5 \cap S_{r/2}^c} \left| f(x-s, y-t) - f(x,y) \right| \left| K_T^\theta(s,t) \right| ds\, dt \\ \le C_p \sum_{i=r_0}^{\infty} \sum_{j=i-1}^{i} 2^{(\tau-1/(2q))(i+j)} \mathcal{M}_p^{(1)} f(x,y) + C_p \sum_{i=r_0}^{\infty} \sum_{j=i-1}^{i} 2^{-(i+j)/(2q)} f(x,y) \\ \le C_p 2^{(\tau-1/(2q))r_0} \mathcal{M}_p^{(1)} f(x,y) + C_p 2^{-r_0/(2q)} f(x,y) \to 0,$$

as $T \to \infty$. This finishes the proof of the theorem. □

With the same proof as in Theorem 6 we can see that the finiteness of $\mathcal{M}_p^{(1)} f(x,y)$ can be omitted.

Theorem 9 *Suppose that* (6) *is satisfied for some* $1 \le \alpha < \infty$ *and* $f \in W(L_p, \ell_\infty)(\mathbb{R}^2)$ $(1 < p < \infty)$. *If* (x, y) *is a modified p-Lebesgue point of* f, *then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x, y) = f(x, y).$$

The next corollary follows from Theorem 3, Corollary 1 and Theorem 8.

Corollary 3 *Suppose that* (6) *is satisfied for some* $0 < \alpha < \infty$ *and* $f(x, y) = f_0(x) f_0(y)$ *with* $f_0 \in W(L_p, \ell_\infty)(\mathbb{R})$ $(1 < p < \infty)$. *If* x *and* y *are p-Lebesgue points of* f_0, *then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x, y) = f(x, y).$$

6 Strong Summability

In this section we generalize the classical one-dimensional strong summability results and prove some new ones.

Theorem 10 *Suppose that* (6) *is satisfied for some* $0 < \alpha < \infty$ *and* $f_0 \in W(L_1, \ell_q)(\mathbb{R})$ *for some* $1 \le q < \infty$. *If* x *and* y *are Lebesgue points of* f_0 *and* f_0 *is locally bounded at* x *and* y, *then*

$$\lim_{T\to\infty} \frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right) \big(s_t f_0(x) - f_0(x)\big)\big(s_t f_0(y) - f_0(y)\big)\, dt = 0.$$

Proof Note that $s_t f_0$ is well defined when $f_0 \in W(L_1, \ell_q)(\mathbb{R})$ for some $1 \le q < \infty$. One can show that

$$\begin{aligned} &\frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right) \big(s_t f_0(x) - f_0(x)\big)\big(s_t f_0(y) - f_0(y)\big)\, dt \\ &\quad = \frac{1}{2\pi} \int_{\mathbb{R}^2} \big(f_0(x-s) - f_0(x)\big)\big(f_0(y-t) - f_0(y)\big) K_T^\theta(s, t) ds\, dt. \end{aligned} \tag{22}$$

The result can be proved as Theorems 5 and 7. □

Writing $x = y$, we obtain

Corollary 4 *Suppose that* (6) *is satisfied for some* $0 < \alpha < \infty$ *and* $f_0 \in W(L_1, \ell_q)(\mathbb{R})$ *for some* $1 \le q < \infty$. *If* x *is a Lebesgue point of* f_0 *and* f_0 *is locally bounded at* x, *then*

$$\lim_{T\to\infty} \frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right) \left|s_t f_0(x) - f_0(x)\right|^2 dt = 0.$$

If f_0 is almost everywhere locally bounded, then the corollary holds almost everywhere. It is not true that an integrable function is almost everywhere locally bounded. Let us denote the Cantor set of Lebesgue measure $1/2$ by $H \subset [0, 1]$. We obtain H in the following way. In the first step we omit the interval I_1^1 of measure $1/4$ from the middle of $[0, 1]$. In the second step we omit the intervals I_2^1 and I_2^2 of length $1/16$ from the middle of the remaining two intervals, in the kth step we omit the intervals $I_k^1, \dots, I_k^{2^{k-1}}$ of length $1/4^k$. We define the function f_0 by $f_0 = 0$ on H and $f_0(x) = (x - a_k^j)^{-1/2}/k^2$ if $x \in I_k^j = (a_k^j, b_k^j)$. Then f_0 is integrable and

$$\int_0^1 f_0\, d\lambda = 2 \sum_{k=1}^{\infty} \sum_{j=1}^{2^{k-1}} \frac{\left(b_k^j - a_k^j\right)^{1/2}}{k^2} = 2 \sum_{k=1}^{\infty} 2^{k-1} \frac{1}{k^2 2^k} = \frac{\pi^2}{6}.$$

On the other hand f_0 is not almost everywhere locally bounded, because for every $x \in H$ and every neighborhood of x there are a_k^j's contained in this neighborhood, and so f_0 is not locally bounded at x.

We will extend Corollary 4 to each $f_0 \in W(L_1, \ell_q)(\mathbb{R})$ $(1 \le q < \infty)$ later. For the convergence of $f_0 \in W(L_p, \ell_q)(\mathbb{R})$ $(1 < p < \infty, 1 \le q < \infty)$ at p-Lebesgue points we get the following result. Note that $W(L_p, \ell_p)(\mathbb{R}) = L_p(\mathbb{R})$. With the help of Theorem 9, the next result can be proved as Theorem 10.

Theorem 11 *Suppose that* (6) *is satisfied for some* $0 < \alpha < \infty$ *and* $f_0 \in W(L_p, \ell_q)(\mathbb{R})$ *for some* $1 < p < \infty$ *and* $1 \le q < \infty$. *If* x *and* y *are p-Lebesgue points of* f_0, *then*

$$\lim_{T\to\infty} \frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right) \big(s_t f_0(x) - f_0(x)\big)\big(s_t f_0(y) - f_0(y)\big)\, dt = 0.$$

Corollary 5 *Suppose that* (6) *is satisfied for some* $0 < \alpha < \infty$ *and* $f_0 \in W(L_p, \ell_q)(\mathbb{R})$ *for some* $1 < p < \infty$ *and* $1 \le q < \infty$. *If* x *is a p-Lebesgue point of* f_0, *then*

$$\lim_{T\to\infty} \frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right) \left|s_t f_0(x) - f_0(x)\right|^2 dt = 0.$$

Obviously, the convergence holds almost everywhere. We will extend this result to $p = 1$. Omitting the Lebesgue point property, we can show that almost everywhere convergence holds for $W(L_1, \ell_q)(\mathbb{R})$ functions $(1 \le q < \infty)$. More exactly, if in Theorem 10 we suppose that x and y are so called Gabisoniya points of f_0 instead of x and y are Lebesgue points of f_0 and f_0 is locally bounded, then a similar result holds.

A point x is called a *Gabisoniya point* of $f_0 \in W(L_1, \ell_\infty)(\mathbb{R})$ if

$$\lim_{T\to\infty} \sum_{i=1}^{\lfloor T \rfloor} \left(\frac{T}{i} \int_{(i-1)/T}^{i/T} \left| f_0(x-u) - f_0(x) \right| du \right)^2 = 0.$$

Here $\lfloor T \rfloor$ denotes the integer part of T. Note that the exponent 2 can be changed to any $1 < \gamma < \infty$. The next theorem is due to Gabisoniya [6] for $f \in L_1(\mathbb{T})$. However, Theorem 12 can be proved similarly.

Theorem 12 *Almost every point $x \in \mathbb{R}$ is a Gabisoniya point of $f \in W(L_1, \ell_\infty)(\mathbb{R})$.*

Theorem 13 *Suppose that (6) is satisfied for some $1 < \alpha < \infty$. If $f_0 \in W(L_1, \ell_q)(\mathbb{R})$ for some $1 \leq q < \infty$ and x and y are Gabisoniya points of f_0, then*

$$\lim_{T\to\infty} \frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right)\big(s_t f_0(x) - f_0(x)\big)\big(s_t f_0(y) - f_0(y)\big)\, dt = 0.$$

Proof By (22) we have to prove that

$$\lim_{T\to\infty} \int_{\mathbb{R}^2} \big|(f_0(x-s) - f_0(x))(f_0(y-t) - f_0(y))\big|\big|K_T^\theta(s,t)\big|\, ds\, dt = 0.$$

Since every Gabisoniya point is a Lebesgue point, we can prove as in Theorem 5 that for $i = 1, 2, 3$,

$$\int_{A_i \cap S_{r/2}} \big|(f_0(x-s) - f_0(x))(f_0(y-t) - f_0(y))\big|\big|K_T^\theta(s,t)\big| ds dt < C\epsilon$$

if T is large enough.

On the sets A_4 and A_5 we decompose the integrals in another way. Let us denote by r_1 the largest number i, for which $r/2 \leq (i+1)/T < r$. By (20),

$$\begin{aligned}
&\int_{A_4 \cap S_{r/2}} \Big|\big(f_0(x-s) - f_0(x)\big)\big(f_0(y-t) - f_0(y)\big)\Big|\big|K_T^\theta(s,t)\big|\, ds\, dt \\
&\leq C \sum_{i=2}^{r_1} \sum_{1 \leq j < (i+1)/2} T^{-\alpha} \left(\frac{i}{T}\right)^{-2} \left(\frac{j}{T}\right)^{-\alpha} \\
&\quad \int_{i/T}^{(i+1)/T} \int_{s-(j+1)/T}^{s-j/T} \Big|\big(f_0(x-s) - f_0(x)\big)\big(f_0(y-t) - f_0(y)\big)\Big|\, dt\, ds \\
&\leq C \sum_{i=2}^{r_1} \sum_{1 \leq j < (i+1)/2} T^{-\alpha} \left(\frac{i}{T}\right)^{-2} \left(\frac{j}{T}\right)^{-\alpha} \\
&\quad \int_{i/T}^{(i+1)/T} \big|f_0(x-s) - f_0(x)\big|\, ds \int_{i/T-(j+1)/T}^{(i+1)/T-j/T} \big|f_0(y-t) - f_0(y)\big| dt.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\int_{A_4\cap S_{r/2}} \Big|\big(f_0(x-s)-f_0(x)\big)\big(f_0(y-t)-f_0(y)\big)\Big|\big|K_T^\theta(s,t)\big|\,ds\,dt\\
&\le C\sum_{i=2}^{r_1}\left(\frac{T}{i}\int_{i/T}^{(i+1)/T}\big|f_0(x-s)-f_0(x)\big|\,ds\right)^2\\
&\quad + C\sum_{j=1}^{(r_1+1)/2}\sum_{i=2j}^{r_1} j^{-\alpha}\left(\frac{T}{i}\int_{i/T-(j+1)/T}^{(i+1)/T-j/T}\big|f_0(y-t)-f_0(y)\big|\,dt\right)^2\\
&\le C\sum_{i=2}^{r_1}\left(\frac{T}{i}\int_{i/T}^{(i+1)/T}\big|f_0(x-s)-f_0(x)\big|\,ds\right)^2\\
&\quad + C\sum_{j=1}^{(r_1+1)/2} j^{-\alpha}\sum_{k=0}^{r_1}\left(\frac{T}{k+1}\int_{k/T}^{(k+1)/T}\big|f_0(y-t)-f_0(y)\big|\,dt\right)^2\\
&\le C\sum_{i=0}^{\lfloor T\rfloor}\left(\frac{T}{i}\int_{i/T}^{(i+1)/T}\big|f_0(x-s)-f_0(x)\big|\,ds\right)^2\\
&\quad + C\sum_{i=0}^{\lfloor T\rfloor}\left(\frac{T}{i+1}\int_{i/T}^{(i+1)/T}\big|f_0(y-t)-f_0(y)\big|\,dt\right)^2 \to 0,
\end{aligned}$$

as $T\to\infty$. Similarly,

$$\begin{aligned}
&\int_{A_5\cap S_{r/2}} \Big|\big(f_0(x-s)-f_0(x)\big)\big(f_0(y-t)-f_0(y)\big)\Big|\big|K_T^\theta(s,t)\big|\,ds\,dt\\
&\le C\sum_{i=2}^{r_1}\left(\frac{i}{T}\right)^{-2}\int_{i/T}^{(i+1)/T}\int_{s-1/T}^{s}\Big|\big(f_0(x-s)-f_0(x)\big)\big(f_0(y-t)-f_0(y)\big)\Big|\,dt\,ds\\
&\le C\sum_{i=2}^{r_1}\left(\frac{i}{T}\right)^{-2}\int_{i/T}^{(i+1)/T}\big|f_0(x-s)-f_0(x)\big|\,ds\int_{(i-1)/T}^{(i+1)/T}\big|f_0(y-t)-f_0(y)\big|\,dt\\
&\le C\sum_{i=2}^{r_1}\left(\frac{T}{i}\int_{i/T}^{(i+1)/T}\big|f_0(x-s)-f_0(x)\big|\,ds\right)^2\\
&\quad + C\sum_{i=2}^{r_1}\left(\frac{T}{i}\int_{(i-1)/T}^{(i+1)/T}\big|f_0(y-t)-f_0(y)\big|\,dt\right)^2 \to 0,
\end{aligned}$$

as $T\to\infty$. Finally,

$$\lim_{T\to\infty}\int_{A_i\cap S_{r/2}^c}\big|f(x-s,y-t)-f(x,y)\big|\big|K_T^\theta(s,t)\big|\,ds\,dt=0$$

for $i = 1, \ldots, 5$ can be proved in the same way as in Theorem 6. The proof of the theorem is complete. □

Corollary 6 *Suppose that* (6) *is satisfied for some* $1 < \alpha < \infty$. *If* $f_0 \in W(L_1, \ell_q)(\mathbb{R})$ *for some* $1 \le q < \infty$ *and* x *is a Gabisoniya point of* f_0, *then*

$$\lim_{T\to\infty} \frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right) \left|s_t f_0(x) - f_0(x)\right|^2 dt = 0.$$

Since almost every point is a Gabisoniya point of $f_0 \in W(L_1, \ell_q)(\mathbb{R})$ $(1 \le q < \infty)$ (see Theorem 12), the convergence holds almost everywhere. Remark that $W(L_1, \ell_p)(\mathbb{R}) \supset L_p(\mathbb{R})$ for all $1 \le p < \infty$. The following two corollaries follow easily from Corollaries 4 and 5.

Corollary 7 *Suppose that* $f_0 \in W(L_1, \ell_q)(\mathbb{R})$ *for some* $1 \le q < \infty$. *If* x *is a Lebesgue point of* f_0 *and* f_0 *is locally bounded at* x, *then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left|s_t f_0(x) - f_0(x)\right|^2 dt = 0.$$

Corollary 8 *Suppose that* $f_0 \in W(L_p, \ell_q)(\mathbb{R})$ *for some* $1 < p < \infty$ *and* $1 \le q < \infty$. *If* x *is a p-Lebesgue point of* f_0, *then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left|s_t f_0(x) - f_0(x)\right|^2 dt = 0.$$

The last corollary was proved by Giang and Móricz [10] for $f_0 \in L_p(\mathbb{R})$ $(1 < p < \infty)$. The next result is an easy consequence of Corollary 6. Note that Corollary 9 is due to Gabisoniya [6] for $f_0 \in L_1(\mathbb{T})$.

Corollary 9 *If* $f_0 \in W(L_1, \ell_q)(\mathbb{R})$ *for some* $1 \le q < \infty$ *and* x *is a Gabisoniya point of* f_0, *then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left|s_t f_0(x) - f_0(x)\right|^2 dt = 0.$$

Proof It is easy to see that $\theta(t) := e^{-t}$ satisfies the condition of Corollary 6 with $\alpha = 2$ (see also Example 6). Then the proof follows from the inequality $1/e \le e^{-t/T}$ on the interval $[0, T]$. □

Of course, the corollary holds almost everywhere. Note that this is the strong summability with respect to the Fejér summation. The Fejér summation does not satisfy the condition of Corollary 6, because $\alpha = 1$ in this case. Marcinkiewicz [19] and Zygmund [33] proved that the convergence holds almost everywhere for all $f_0 \in L_1(\mathbb{T})$, but it does not hold at each Lebesgue point of f_0 (see Hardy and Littlewood [17]). However, if f_0 is almost everywhere locally bounded, resp. if $f_0 \in L_p(\mathbb{R})$ or $W(L_p, \ell_q)(\mathbb{R})$ $(1 < p < \infty, 1 \le q < \infty)$, then it holds at each Lebesgue point, resp. p-Lebesgue point (see Corollary 5). The strong summability also holds for smaller exponents than 2.

Corollary 10 *Suppose that θ is non-increasing and $0 < r \le 2$. Under the same conditions as in Corollaries* 4, 5 or 6, *respectively, we get that*

$$\lim_{T\to\infty} \frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right) \left|s_t f_0(x) - f_0(x)\right|^r dt = 0.$$

Proof Since $\theta' \le 0$, by Hölder's inequality

$$\begin{aligned}
&\frac{1}{T} \int_0^\infty \left|\theta'\left(\frac{t}{T}\right)\right| \left|s_t f_0(x) - f_0(x)\right|^r dt \\
&= \frac{1}{T} \int_0^\infty \left|\theta'\left(\frac{t}{T}\right)\right|^{r/2} \left|s_t f_0(x) - f_0(x)\right|^r \left|\theta'\left(\frac{t}{T}\right)\right|^{1-r/2} dt \\
&\le \frac{1}{T} \left(\int_0^\infty \left|\theta'\left(\frac{t}{T}\right)\right| \left|s_t f_0(x) - f_0(x)\right|^2 dt\right)^{r/2} \left(\int_0^\infty \left|\theta'\left(\frac{t}{T}\right)\right| dt\right)^{1-r/2} \\
&\le C \left(\frac{1}{T} \int_0^\infty \left|\theta'\left(\frac{t}{T}\right)\right| \left|s_t f_0(x) - f_0(x)\right|^2 dt\right)^{r/2},
\end{aligned}$$

which shows the corollary. □

Similarly, for the strong Fejér summation we have

Corollary 11 *Suppose that $0 < r \le 2$. Under the same conditions as in Corollaries 7, 8 or 9, respectively, we get that*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left|s_t f_0(x) - f_0(x)\right|^r dt = 0.$$

7 Applications to Various Summability Methods

In this section we consider some summability methods as special cases of the Marcinkiewicz-θ-summation. Of course, there are a lot of other summability methods which could be considered as special cases. The elementary computations in the examples below are left to the reader (see also Weisz [27]).

Example 1 (Fejér summation) Let

$$\theta(t) := \begin{cases} 1 - |t| & \text{if } |t| \le 1 \\ 0 & \text{if } |t| > 1. \end{cases}$$

Example 2 (de La Vallée-Poussin summation) Let

$$\theta(t) = \begin{cases} 1 & \text{if } |t| \le 1/2 \\ -2|t| + 2 & \text{if } 1/2 < |t| \le 1 \\ 0 & \text{if } |t| > 1. \end{cases}$$

Example 3 (Jackson-de La Vallée-Poussin summation) Let

$$\theta(t) = \begin{cases} 1 - 3t^2/2 + 3|t|^3/4 & \text{if } |t| \le 1 \\ (2 - |t|)^3/4 & \text{if } 1 < |t| \le 2 \\ 0 & \text{if } |t| > 2. \end{cases}$$

The next example generalizes Examples 1–3.

Example 4 Let $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_m$ and $\beta_0, \ldots, \beta_m$ ($m \in \mathbb{N}$) be real numbers, $\beta_0 = 1$, $\beta_m = 0$. Suppose that θ is even, $\theta(\alpha_j) = \beta_j$ ($j = 0, 1, \ldots, m$), $\theta(t) = 0$ for $t \ge \alpha_m$, θ is a polynomial on the interval $[\alpha_{j-1}, \alpha_j]$ ($j = 1, \ldots, m$).

Example 5 (Rogosinski summation) Let

$$\theta(t) = \begin{cases} \cos \pi t/2 & \text{if } |t| \le 1 + 2j \\ 0 & \text{if } |t| > 1 + 2j \end{cases} \qquad (j \in \mathbb{N}).$$

Example 6 (Weierstrass summation) Let $\theta(t) = e^{-|t|^\gamma}$ for some $1 \le \gamma < \infty$. Note that if $\gamma = 1$, then we obtain the *Abel summation*.

Example 7 $\theta(t) = e^{-(1+|t|^q)^\gamma}$ ($t \in \mathbb{R}, 1 \le q < \infty, 0 < \gamma < \infty$).

Example 8 (Picard and Bessel summations) $\theta(t) = (1 + |t|^\gamma)^{-\delta}$ ($0 < \delta < \infty, 1 \le \gamma < \infty, \gamma\delta > 2$).

Example 9 (Riesz summation) Let

$$\theta(t) = \begin{cases} (1 - |t|^\gamma)^\delta & \text{if } |t| \le 1 \\ 0 & \text{if } |t| > 1 \end{cases}$$

for some $0 < \delta < \infty, 1 \le \gamma < \infty$.

By an easy computation we get that the conditions (2) and (6) are satisfied for Examples 1–5 and for Example 9 if $1 \le \delta, \gamma < \infty$ with $\alpha = 1$. Moreover, Examples 6–8 satisfy (2) and (6) with $\alpha = 2$ and Example 9 with $\alpha = \delta$ if $0 < \delta \le 1 \le \gamma < \infty$.

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