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Cotlar's Ergodic Theorem Along the Prime Numbers

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Abstract The aim of this paper is to prove Cotlar's ergodic theorem modeled on the set of primes.

Keywords Maximal truncated Hilbert transform · Prime numbers · Pointwise convergence

1 Introduction

Let $(X, \mathcal{B}, \mu, S)$ be a dynamical system on a measure space X endowed with a σ-algebra $\mathcal{B}$, a σ-finite measure μ and an invertible measure preserving transformation $S : X \to X$. In 1955 Cotlar (see [4]) established the almost everywhere convergence of the ergodic truncated Hilbert transform

$$\lim_{N\to\infty} \sum_{1\leq|n|\leq N} \frac{f(S^n x)}{n}$$

for all $f \in L^r(\mu)$ with $1 \leq r < \infty$. The aim of the present paper is to obtain the corresponding result for the set of prime numbers $\mathbb{P}$. Let $\mathbb{P}_N = \mathbb{P} \cap (1, N]$. We prove

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Theorem 1 *For a given dynamical system $(X, \mathcal{B}, \mu, S)$ the almost everywhere convergence of the ergodic truncated Hilbert transform along $\mathbb{P}$*

$$\lim_{N\to\infty} \sum_{p\in\pm\mathbb{P}_N} \frac{f(S^p x)}{p} \log|p|$$

holds for all $f \in L^r(\mu)$ with $1 < r < \infty$.

In view of Calderón's transference principle, it is more convenient to work with the set of integers rather than an abstract measure space X. In these settings we consider discrete singular integrals with Calderón–Zygmund kernels. Given $K \in C^1\big(\mathbb{R} \setminus \{0\}\big)$ satisfying

$$|x||K(x)| + |x|^2\big|K'(x)\big| \le 1 \tag{1}$$

for $|x| \ge 1$, together with a cancellation property

$$\sup_{\lambda\ge 1}\bigg|\int_{1\le|x|\le\lambda} K(x)dx\bigg| \le 1 \tag{2}$$

a singular transform T along the set of prime numbers is defined for a finitely supported function $f : \mathbb{Z} \to \mathbb{C}$ as

$$Tf(n) = \sum_{p\in\pm\mathbb{P}} f(n-p)K(p)\log|p|.$$

Let T_N denote the truncation of T, i.e.

$$T_N f(n) = \sum_{p\in\pm\mathbb{P}_N} f(n-p)K(p)\log|p|.$$

We show

Theorem 2 *The maximal function*

$$T^* f(n) = \sup_{N\in\mathbb{N}} \big|T_N f(n)\big|$$

is bounded on $\ell^r(\mathbb{Z})$ for any $1 < r < \infty$. Moreover, the pointwise limit

$$\lim_{N\to\infty} T_N f(n)$$

exists and coincides with the Hilbert transform Tf which is also bounded on $\ell^r(\mathbb{Z})$ for any $1 < r < \infty$.

For $r = 2$, the proof of Theorem 2 is based on the Hardy and Littlewood circle method which allows us to construct appropriate approximating multipliers (see for instance (13)) and control the error terms as in Proposition 3.2. These ideas were pioneered by Bourgain (see [1–3]) in the context of pointwise ergodic theorems along integer valued polynomials. For $r \ne 2$, we shall compare the discrete norm $\|\cdot\|_{\ell^r}$ of our approximating multipliers with the continuous norm $\|\cdot\|_{L^r}$ of certain multipliers which are a priori bounded on L^r, we refer to the proof of Proposition 3.3 and Theorem 3. Initially we wanted to follow elegant arguments from [23] which used very specific features of the set of prime numbers. However, we identified an issue in [23] (see Appendix 1) which made the proof incomplete. Instead, we propose an approach (see Lemmas 1 and 2) which rectifies Wierdl's proof (see Appendix 1 for details) as well as simplifies Bourgain's arguments.

Bourgain's works have inspired many authors to investigate discrete analogues of classical operators with arithmetic features (see e.g. [5–7,12–14,17–19]). Nevertheless, not many have been proved for the operators and maximal functions modelled on the set of primes (see e.g. [9,10,23]). To the authors best knowledge, there are no other results dealing with maximal functions corresponding with truncated discrete singular integrals.

It is worth mentioning that Theorem 2 extends the result of Ionescu and Wainger [6] to the set of prime numbers. However, our approach is different and provides a stronger result since we study maximal functions corresponding with truncations of discrete singular integral rather than the whole singular integral. Furthermore, we are able to define the singular integral as a pointwise limit of its truncations. Theorem 2 encourages us to study maximal functions associated with truncations of the Radon transforms from [6]. For more details we refer the reader to the forthcoming article [8].

1.1 Notation

Throughout the paper, unless otherwise stated, $C > 0$ stands for a large positive constant whose value may vary from occurrence to occurrence. We will say that $A \lesssim B$ ($A \gtrsim B$) if there exists an absolute constant $C > 0$ such that $A \leq CB$ ($A \geq CB$). If $A \lesssim B$ and $A \gtrsim B$ hold simultaneously then we will shortly write that $A \simeq B$. We will write $A \lesssim_\delta B$ ($A \gtrsim_\delta B$) to indicate that the constant $C > 0$ depends on some $\delta > 0$. We always assume zero belongs to the set of natural numbers $\mathbb{N}$.

2 Preliminaries

We start by recalling some basic facts from number theory. A general reference is [11]. Given $q \in \mathbb{N}$ we define A_q to be the set of all $a \in \mathbb{Z} \cap [1, q]$ such that $(a, q) = 1$. By μ we denote Möbius function, i.e. for $q = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_n^{\alpha_n}$ where $p_1, \ldots, p_n \in \mathbb{P}$

$$\mu(q) = \begin{cases} (-1)^n & \text{if } \alpha_1 = \alpha_2 = \cdots = \alpha_n = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and $\mu(1) = 1$. In what follows, a significant role will be played by Ramanujan's identity

$$\mu(q) = \sum_{r \in A_q} e^{2\pi i r a/q} \quad \text{if} \quad (a, q) = 1,$$

and the Möbius inversion formula

$$\sum_{a \in A_q} F(a/q) = \sum_{d|q} \mu(q/d) \sum_{a=1}^{d} F(a/d) \tag{3}$$

satisfied by any function F. Let φ be the Euler's totient function, i.e. for $q \in \mathbb{N}$ the value $\varphi(q)$ is equal to the number of elements in A_q. Then for every $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that

$$\varphi(q) \geq C_\epsilon q^{1-\epsilon}. \tag{4}$$

If we denote by $d(q)$ the number of divisors of $q \in \mathbb{N}$, then for every $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that

$$d(q) \leq C_\epsilon q^\epsilon. \tag{5}$$

3 Maximal Function on $\mathbb{Z}$

The measure space $\mathbb{Z}$ with the counting measure and the bilateral shift operator will be our model dynamical system which permits us to prove Theorem 1.

From now on, all the maximal functions will be defined on non-negative finitely supported functions $f : \mathbb{Z} \to \mathbb{R}$ and unless otherwise stated f always has a finite support.

Let us fix $\tau \in (1, 2]$ and define a set $\Lambda = \{\tau^j : j \in \mathbb{N}\}$. Given a kernel $K \in C^1(\mathbb{R} \setminus \{0\})$ satisfying (1) and (2) we consider a sequence $\big(K_j : j \in \mathbb{N}\big)$ where

$$K_j(x) = \begin{cases} K(x) & \text{if } |x| \in (\tau^j, \tau^{j+1}], \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{F}$ denote the Fourier transform on $\mathbb{R}$ defined for any function $f \in L^1(\mathbb{R})$ as

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x) e^{2\pi i \xi x} dx.$$

If $f \in \ell^1(\mathbb{Z})$ we set

$$\hat{f}(\xi) = \sum_{n \in \mathbb{Z}} f(n) e^{2\pi i \xi n},$$

then for $\Phi_j = \mathcal{F}K_j$, integration by parts shows that

$$\big|\Phi_j(\xi)\big| \lesssim |\xi|^{-1} \tau^{-j}, \tag{6}$$

for $\xi \in \mathbb{R}$. We define a sequence $\big(m_j : j \in \mathbb{N}\big)$ of multipliers

$$m_j(\xi) = \sum_{p \in \pm\mathbb{P}} e^{2\pi i \xi p} K_j(p) \log |p|.$$

3.1 ℓ^2-Approximation

To approximate the multiplier m_j we adopt the argument introduced by Bourgain [3] (see also Wierdl [23]) which is based on the Hardy–Littlewood circle method (see e.g. [20]).

For any $\alpha > 0$ and $j \in \mathbb{N}$ major arcs are defined by

$$\mathfrak{M}_j^\alpha = \bigcup_{1 \le q \le j^\alpha} \bigcup_{a \in A_q} \mathfrak{M}_j^\alpha(a/q)$$

where

$$\mathfrak{M}_j^\alpha(a/q) = \left\{ \xi \in [0,1] : |\xi - a/q| \le \tau^{-j} j^\alpha \right\}.$$

Here and subsequently we will treat the interval $[0, 1]$ as the circle group $\Pi = \mathbb{R}/\mathbb{Z}$ identifying 0 and 1.

Proposition 3.1 *For $\xi \in \mathfrak{M}_j^\alpha(a/q) \cap \mathfrak{M}_j^\alpha$*

$$\left| m_j(\xi) - \frac{\mu(q)}{\varphi(q)} \Phi_j(\xi - a/q) \right| \le C_\alpha j^{-\alpha}.$$

The constant C_α depends only on α.

Proof Since for a prime number p, $p \mid q$ if and only if $(p \bmod q, q) > 1$, we have

$$\left| \sum_{\substack{1 \le r \le q \\ (r,q) > 1}} \sum_{\substack{p \in \mathbb{P} \\ q \mid (p-r)}} e^{2\pi i \xi p} K_j(p) \log p \right| \le \tau^{-j} \sum_{\substack{p \in \mathbb{P} \\ p \mid q}} \log p \lesssim \tau^{-j} \log j. \tag{7}$$

Let $\theta = \xi - a/q$. If $p \equiv r \pmod q$ then

$$\xi p \equiv \theta p + ra/q \pmod 1$$

and consequently

$$\sum_{r \in A_q} \sum_{\substack{p \in \mathbb{P} \\ q \mid (p-r)}} e^{2\pi i \xi p} K_j(p) \log p = \sum_{r \in A_q} e^{2\pi i r a/q} \sum_{\substack{p \in \mathbb{P} \\ q \mid (p-r)}} e^{2\pi i \theta p} K_j(p) \log p. \tag{8}$$

Using the summation by parts (see e.g. [11, p. 304]) for the inner sum on the right hand side in (8) we obtain

$$\begin{aligned}\sum_{\substack{n\in N_j\\ q|(n-r)}} e^{2\pi i\theta n}K(n)\mathbb{1}_{\mathbb{P}}(n)\log n &= \psi(\tau^{j+1};q,r)e^{2\pi i\theta\tau^{j+1}}K(\tau^{j+1})\\ &\quad-\psi(\tau^j;q,r)e^{2\pi i\theta\tau^j}K(\tau^j)\\ &\quad-\int_{\tau^j}^{\tau^{j+1}}\psi(t;q,r)\frac{d}{dt}\left(e^{2\pi i\theta t}K(t)\right)dt \end{aligned}\tag{9}$$

where $N_j=\mathbb{N}\cap(\tau^j,\tau^{j+1}]$ and for $x\ge 2$ we have set

$$\psi(x;q,r)=\sum_{\substack{p\in\mathbb{P}_x\\ q|(p-r)}}\log p.$$

Similar reasoning gives

$$\begin{aligned}\sum_{n\in N_j} e^{2\pi i\theta n}K(n) &= \tau^{j+1}e^{2\pi i\theta\tau^{j+1}}K\left(\tau^{j+1}\right)-\tau^j e^{2\pi i\theta\tau^j}K\left(\tau^j\right)\\ &\quad-\int_{\tau^j}^{\tau^{j+1}} t\frac{d}{dt}\left(e^{2\pi i\theta t}K(t)\right)dt.\end{aligned}\tag{10}$$

By Siegel–Walfisz theorem (see [16,22]) we know that for every $\alpha>0$ and $x\ge 2$

$$\left|\psi(x;q,r)-\frac{x}{\varphi(q)}\right|\lesssim x(\log x)^{-3\alpha}\tag{11}$$

where the implied constant depends only on α. Therefore (9) and (10) combined with the estimates (1) and (11) yield

$$\begin{aligned}&\left|\sum_{\substack{p\in\mathbb{P}\\ q|(p-r)}} e^{2\pi i\theta p}K_j(p)\log p-\frac{1}{\varphi(q)}\sum_{n\in\mathbb{N}}e^{2\pi i\theta n}K_j(n)\right|\\ &\lesssim\left|\psi(\tau^{j+1};q,r)-\frac{\tau^{j+1}}{\varphi(q)}\right|\left|K(\tau^{j+1})\right|+\left|\psi(\tau^j;q,r)-\frac{\tau^j}{\varphi(q)}\right||K(\tau^j)|\\ &\quad+\int_{\tau^j}^{\tau^{j+1}}\left|\psi(t;q,r)-\frac{t}{\varphi(q)}\right|\left(t^{-1}|\theta|+t^{-2}\right)dt\\ &\lesssim j^{-3\alpha}+\int_{\tau^j}^{\tau^{j+1}}(\log t)^{-3\alpha}\left(|\theta|+t^{-1}\right)dt\end{aligned}$$

what is bounded by $j^{-2\alpha}$. Finally, by (8),

$$\Big|\sum_{r\in A_q}\sum_{\substack{p\in\mathbb{P}\\ q|(p-r)}} e^{2\pi i\xi p}K_j(p)\log p-\frac{\mu(q)}{\varphi(q)}\sum_{n\in\mathbb{N}}e^{2\pi i\theta n}K_j(n)\Big|$$
$$=\Big|\sum_{r\in A_q}e^{2\pi ira/q}\Big(\sum_{\substack{p\in\mathbb{P}\\ q|(p-r)}}e^{2\pi i\theta p}K_j(p)\log p-\frac{1}{\varphi(q)}\sum_{n\in\mathbb{N}}e^{2\pi i\theta n}K_j(n)\Big)\Big|\lesssim qj^{-2\alpha}\le j^{-\alpha}. \tag{12}$$

Next, we can substitute an integral for the sum since for $n_0=\lceil\tau^j\rceil$ and $n_1=\lfloor\tau^{j+1}\rfloor$ we have

$$\int_{\tau^j}^{\tau^{j+1}}e^{2\pi i\theta t}K(t)dt=\int_{\tau^j}^{n_0}e^{2\pi i\theta t}K(t)dt+\sum_{n=n_0}^{n_1-1}\int_0^1 e^{2\pi i\theta(n+t)}K(n+t)dt$$
$$+\int_{n_1}^{\tau^{j+1}}e^{2\pi i\theta t}K(t)dt.$$

Since $|\theta|\le\tau^{-j}j^{\alpha}$ we get

$$\Big|\sum_{n=n_0}^{n_1-1}\Big(e^{2\pi i\theta n}K(n)-\int_0^1 e^{2\pi i\theta(n+t)}K(n+t)dt\Big)\Big|$$
$$\le\sum_{n=n_0}^{n_1-1}\int_0^1\Big|1-e^{-2\pi i\theta t}\Big||K(n)|dt+\sum_{n=n_0}^{n_1-1}\int_0^1|K(n)-K(n+t)|dt\lesssim\tau^{-j}j^{\alpha}.$$

Hence, by (7) and (12) we obtain

$$\Big|\sum_{p\in\mathbb{P}}e^{2\pi i\xi p}K_j(p)\log p-\frac{\mu(q)}{\varphi(q)}\int_0^\infty e^{2\pi i\theta t}K_j(t)dt\Big|\lesssim\tau^{-j}j^{\alpha}+j^{-\alpha}.$$

Repeating all the steps with p replaced by $-p$ we finish the proof. □

For $s\in\mathbb{N}$ we set

$$\mathscr{R}_s=\big\{a/q\in[0,1]\cap\mathbb{Q}:2^s\le q<2^{s+1}\text{ and }(a,q)=1\big\}.$$

Since we treat $[0,1]$ as the circle group identifying 0 and 1 we treat $\mathscr{R}_0=\{1\}$. Let us consider

$$\nu_j^s(\xi)=\sum_{a/q\in\mathscr{R}_s}\frac{\mu(q)}{\varphi(q)}\Phi_j(\xi-a/q)\eta_s(\xi-a/q) \tag{13}$$

where $\eta_s(\xi) = \eta(A^{s+1}\xi)$ and $\eta : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $0 \le \eta(x) \le 1$ and

$$\eta(x) = \begin{cases} 1 & \text{for } |x| \le 1/4, \\ 0 & \text{for } |x| \ge 1/2. \end{cases}$$

The value of A is chosen to satisfy (18). Additionally, we may assume (this will be important in Lemma 1) that η is a convolution of two smooth functions with compact supports contained in $[-1/2, 1/2]$. Let $\nu_j = \sum_{s\in\mathbb{N}} \nu_j^s$. For any $s \in \mathbb{N}$ the multiplier ν_j^s is meant to be 1-periodic.

Proposition 3.2 *For every* $\alpha > 16$

$$\left|m_j(\xi) - \nu_j(\xi)\right| \le C_\alpha j^{-\alpha/4}.$$

The constant C_α *depends only on* α.

Proof First of all notice that for a fixed $s \in \mathbb{N}$ and $\xi \in [0, 1]$ the sum (13) consists of the single term. Otherwise, there would be $a/q, a'/q' \in \mathscr{R}_s$ such that $\eta_s(\xi - a/q) \neq 0$ and $\eta_s(\xi - a'/q') \neq 0$. Therefore,

$$2^{-2s-2} \le \frac{1}{qq'} \le \left|\frac{a}{q} - \frac{a'}{q'}\right| \le \left|\xi - \frac{a}{q}\right| + \left|\xi - \frac{a'}{q'}\right| \le A^{-s-1}$$

which is not possible whenever $A > 4$, as it was assumed in (18).

Major arcs estimates: $\xi \in \mathfrak{M}_j^\alpha(a/q) \cap \mathfrak{M}_j^\alpha$. Let s_0 be such that

$$2^{s_0} \le q < 2^{s_0+1}. \tag{14}$$

We choose s_1 satisfying

$$2^{s_1+1} \le \tau^j j^{-2\alpha} < 2^{s_1+2}.$$

If $s < s_1$ then for any $a'/q' \in \mathscr{R}_s$, $a'/q' \neq a/q$ we have

$$\left|\xi - \frac{a'}{q'}\right| \ge \frac{1}{qq'} - \left|\xi - \frac{a}{q}\right| \ge 2^{-s-1} j^{-\alpha} - \tau^{-j} j^\alpha \ge \tau^{-j} j^\alpha.$$

Therefore, using (6)

$$\left|\Phi_j\big(\xi - a'/q'\big)\right| \lesssim \left(\left|\xi - a'/q'\right|\tau^j\right)^{-1} \lesssim j^{-\alpha}.$$

Combining the last estimate with (4), we obtain that for any $0 < \delta_1 < 1$

$$I_1 = \left|\sum_{s=0}^{s_1-1} \sum_{\substack{a'/q' \in \mathscr{R}_s \\ a'/q' \neq a/q}} \frac{\mu(q')}{\varphi(q')} \Phi_j\big(\xi - a'/q'\big)\eta_s\big(\xi - a'/q'\big)\right| \lesssim j^{-\alpha} \sum_{s=0}^{s_1-1} 2^{-\delta_1 s}.$$

Moreover, if $\eta_{s_0}(\xi - a/q) < 1$ then $|\xi - a/q| \geq 4^{-1}A^{-s_0-1}$. By (14) we have $2^{s_0} \leq j^{\alpha}$. Hence, (5) together with (6) implies

$$I_2 = \left|\frac{\mu(q)}{\varphi(q)}\Phi_j(\xi - a/q)\big(1 - \eta_{s_0}(\xi - a/q)\big)\right| \lesssim A^{s_0+1}\tau^{-j} \lesssim j^{-\alpha}.$$

In the last estimate it is important that the implied constant does not depend on s_0. Since Φ_j is bounded uniformly with respect to $j \in \mathbb{N}$, by (4) and the definition of s_1 we have

$$I_3 = \left|\sum_{s=s_1}^{\infty}\sum_{\substack{a'/q' \in \mathscr{R}_s \\ a'/q' \neq a/q}}\frac{\mu(q')}{\varphi(q')}\Phi_j(\xi - a'/q')\eta_s(\xi - a'/q')\right| \lesssim \sum_{s=s_1}^{\infty}2^{-\delta_2 s} \lesssim \big(\tau^{-j}j^{2\alpha}\big)^{\delta_2} \lesssim j^{-\alpha}$$

for appropriately chosen $\delta_2 > 0$. Finally, in view of Proposition 3.1 and definitions of s_0 and s_1 we conclude

$$\big|m_j(\xi) - \nu_j(\xi)\big| \leq C_\alpha j^{-\alpha} + I_1 + I_2 + I_3 \lesssim j^{-\alpha}.$$

Minor arcs estimates $\xi \notin \mathfrak{M}_j^{\alpha}$. Firstly, by the summation by parts, we get

$$\left|\sum_{p \in \mathbb{P}} e^{2\pi i \xi p} K_j(p) \log p\right| \leq \left|F_{\tau^{j+1}}(\xi)\right|\left|K\left(\tau^{j+1}\right)\right| + \left|F_{\tau^j}(\xi)\right|\left|K\left(\tau^j\right)\right| + \int_{\tau^j}^{\tau^{j+1}} |F_t(\xi)||K'(t)|dt \tag{15}$$

where

$$F_x(\xi) = \sum_{p \in \mathbb{P}_x} e^{2\pi i \xi p} \log p.$$

Using Dirichlet's principle there are $(a, q) = 1$, $j^{\alpha} \leq q \leq \tau^j j^{-\alpha}$ such that

$$|\xi - a/q| \leq q^{-1}\tau^{-j}j^{\alpha} \leq q^{-2}.$$

Thus, by Vinogradov's theorem (see [21, Theorem 1, Chapter IX] or [11, Theorem 8.5]) we get

$$|F_t(\xi)| \lesssim j^4\Big(\tau^j q^{-1/2} + \tau^{4j/5} + \tau^{j/2}q^{1/2}\Big) \lesssim \tau^j j^{4-\alpha/2}$$

for $t \in [\tau^j, \tau^{j+1}]$. Combining $\left|K'(t)\right| \lesssim \tau^{-2j}$ with the last bound and (15) we conclude

$$\big|m_j(\xi)\big| \lesssim j^{4-\alpha/2} \lesssim j^{-\alpha/4}$$

since $\alpha > 16$. In order to estimate the ν_j let us define s_2 by setting

$$2^{s_2} \le j^{\alpha/2} < 2^{s_2+1}.$$

If $a/q \in \mathscr{R}_s$ for $s < s_2$ then $q < j^\alpha$ and

$$\left|\xi - \frac{a}{q}\right| \ge 2^{-s-1}\tau^{-j} j^\alpha \gtrsim \tau^{-j} j^{\alpha/2}.$$

Again, by (6) we obtain

$$\left|\Phi_j(\xi - a/q)\right| \lesssim \left(|\xi - a/q|\tau^j\right)^{-1} \lesssim j^{-\alpha/2}.$$

Therefore, the first part of the sum may be majorized by

$$\left|\sum_{s=0}^{s_2-1} \nu_j^s(\xi)\right| \lesssim j^{-\alpha/2} \sum_{s=0}^{\infty} 2^{-\delta_1 s},$$

as for I_1. For the second part we proceed as for I_3 to get

$$\left|\sum_{s=s_2}^{\infty} \nu_j^s(\xi)\right| \lesssim \sum_{s=s_2}^{\infty} 2^{-\delta s} \lesssim j^{-\delta_2 \alpha/2} \lesssim j^{-\alpha/4}.$$

A suitable choice of $\delta_1, \delta_2 > 0$ in both estimates above was possible thanks to (4). □

3.2 ℓ^r-Theory

We start the section by proving two lemmas which will play a crucial role.

Lemma 1 *There is a constant $C > 0$ such that for all $s \in \mathbb{N}$ and $u \in \mathbb{R}$*

$$\left\| \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i \xi j} \eta_s(\xi) d\xi \right\|_{\ell^1(j)} \le C, \tag{16}$$

$$\left\| \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i \xi j} \left(1 - e^{2\pi i \xi u}\right) \eta_s(\xi) d\xi \right\|_{\ell^1(j)} \le C|u| A^{-s-1}. \tag{17}$$

Proof We only show (17) for $u \in \mathbb{R}$, since the proof of (16) is almost identical. Recall, $\eta = \phi * \psi$ for ψ, ϕ smooth functions with supports inside $[-1/2, 1/2]$. Hence, $\eta_s = A^{s+1}\phi_s * \psi_s$ and

$$A^{-s-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i \xi j} \left(1 - e^{2\pi i \xi u}\right) \eta_s(\xi) d\xi = \mathcal{F}^{-1}\phi_s(j) \mathcal{F}^{-1}\psi_s(j) - \mathcal{F}^{-1}\phi_s(j-u) \mathcal{F}^{-1}\psi_s(j-u).$$

By Cauchy–Schwarz's inequality and Plancherel's theorem

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \left|\mathcal{F}^{-1}\phi_s(j)\right|\left|\mathcal{F}^{-1}\psi_s(j)-\mathcal{F}^{-1}\psi_s(j-u)\right| \\ \leq \left\|\mathcal{F}^{-1}\phi_s\right\|_{\ell^2}\left\|\int_{\mathbb{R}} e^{-2\pi i\xi j}\big(1-e^{2\pi i\xi u}\big)\psi_s(\xi)d\xi\right\|_{\ell^2(j)} \\ = \|\phi_s\|_{L^2}\left\|\big(1-e^{2\pi i\xi u}\big)\psi_s(\xi)\right\|_{L^2(d\xi)}.\end{aligned}$$

Moreover, since

$$\int_{\mathbb{R}}\left|1-e^{-2\pi i\xi u}\right|^2|\psi_s(\xi)|^2d\xi \leq u^2\int_{\mathbb{R}}|\xi|^2|\psi_s(\xi)|^2d\xi \lesssim u^2A^{-3(s+1)}\|\psi\|_{L^2}^2$$

we obtain

$$\sum_{j\in\mathbb{Z}}\left|\mathcal{F}^{-1}\phi_s(j)\right|\left|\mathcal{F}^{-1}\psi_s(j)-\mathcal{F}^{-1}\psi_s(j-u)\right| \lesssim |u|A^{-2(s+1)}\|\phi\|_{L^2}\|\psi\|_{L^2}$$

which finishes the proof of (17). □

Lemma 2 *Let $r\geq 1$. For all $q\in[2^s,2^{s+1})$, $s\geq r$ and $l\in\{1,2,\ldots,q\}$*

$$\left\|\mathcal{F}^{-1}\big(\eta_s\hat{f}\big)(qj+l)\right\|_{\ell^r(j)} \simeq q^{-1/r}\left\|\mathcal{F}^{-1}\big(\eta_s\hat{f}\big)\right\|_{\ell^r}.$$

Proof We define a sequence $\big(J_1,J_2,\ldots,J_q\big)$ by

$$J_l=\left\|\mathcal{F}^{-1}\big(\eta_s\hat{f}\big)(qj+l)\right\|_{\ell^r(j)}.$$

Then $J_1^r+J_2^r+\cdots+J_q^r=I^r$ where $I=\big\|\mathcal{F}^{-1}\big(\eta_s\hat{f}\big)\big\|_{\ell^r(j)}$. Since $\eta_s=\eta_s\eta_{s-1}$, by Minkowski's inequality we obtain

$$\begin{aligned}\left\|\mathcal{F}^{-1}\big(\eta_s\hat{f}\big)(qj+l)-\mathcal{F}^{-1}\big(\eta_s\hat{f}\big)(qj+l')\right\|_{\ell^r(j)} \\ = \left\|\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i\xi(qj+l)}\big(1-e^{2\pi i\xi(l-l')}\big)\eta_s(\xi)\hat{f}(\xi)d\xi\right\|_{\ell^r(j)} \\ \leq \left\|\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i\xi j}\big(1-e^{2\pi i\xi(l-l')}\big)\eta_{s-1}(\xi)d\xi\right\|_{\ell^1(j)} I,\end{aligned}$$

what, by (17), is bounded by $CqA^{-s}I$. We notice, the constant $C>0$ depends only on η. Hence, for all $l,l'\in\{1,2,\ldots,q\}$

$$J_l\leq J_{l'}+CqA^{-s}I.$$

Since $q < 2^{s+1}$ taking

$$A > 32 \max\{1, C\} \tag{18}$$

we obtain $CqA^{-s} \leq 2^{-4s+1}$ thus

$$J_l^r \leq 2^{r-1} J_{l'}^r + 2^{r-1}\big(CqA^{-s}\big)^r I^r \leq 2^{r-1} J_{l'}^r + 2^{2r-4s-1} I^r. \tag{19}$$

Therefore,

$$I^r = J_1^r + J_2^r + \cdots + J_q^r \leq 2^{r-1} q J_l^r + q 2^{2r-4s-1} I^r \leq 2^{r-1} q J_l^r + 2^{3r-3s-1} I^r$$

and using $s \geq r$, we get $I^r \leq 2^r q J_l^r$. For the converse inequality, we use again (19) to conclude

$$q J_l^r \leq 2^{r-1}\big(J_1^r + J_2^r + \cdots + J_q^r\big) + q 2^{2r-4s-1} I^r \leq 2^r I^r.$$

□

Proposition 3.3 *For $r > 1$ and $s \in \mathbb{N}$*

$$\left\| \sup_{k\in\mathbb{N}} \left|\mathcal{F}^{-1}\big(\Psi_k \eta_s \hat{f}\big)\right| \right\|_{\ell^r} \leq C_r \big\|\mathcal{F}^{-1}\big(\eta_s \hat{f}\big)\big\|_{\ell^r}$$

where $\Psi_k = \sum_{j=0}^k \Phi_j$.

Proof Since $\eta_s = \eta_{s-1}\eta_s$ thus by Hölder's inequality we have

$$\begin{aligned}\sup_{k\in\mathbb{N}} \left|\mathcal{F}^{-1}\big(\Psi_k \eta_s \hat{f}\big)(m)\right|^r &\leq \left(\int_{\mathbb{R}} \sup_{k\in\mathbb{N}} \left|\mathcal{F}^{-1}\big(\Psi_k \eta_s \hat{f}\big)(t)\right| \left|\mathcal{F}^{-1}\eta_{s-1}(m-t)\right| dt\right)^r \\ &\leq \int_{\mathbb{R}} \sup_{k\in\mathbb{N}} \left|\mathcal{F}^{-1}\big(\Psi_k \eta_s \hat{f}\big)(t)\right|^r \left|\mathcal{F}^{-1}\eta_{s-1}(m-t)\right| dt \,\left\|\mathcal{F}^{-1}\eta_{s-1}\right\|_{L^1}^{r-1}.\end{aligned}$$

Now we note that $\left\|\mathcal{F}^{-1}\eta_{s-1}\right\|_{L^1} \lesssim 1$ and

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \left|\mathcal{F}^{-1}\eta_{s-1}(m-t)\right| &\lesssim A^{-s} \sum_{m\in\mathbb{Z}} \frac{1}{1 + (A^{-s}(m-t))^2} \\ &\lesssim A^{-s}\left(1 + \int_{\mathbb{R}} \frac{dx}{1 + (A^{-s}x)^2}\right) \lesssim A^{-s}(1 + A^s) \lesssim 1\end{aligned}$$

and the implied constants are independent of A. Thus we obtain

$$\left\| \sup_{k\in\mathbb{N}} \left|\mathcal{F}^{-1}\big(\Psi_k \eta_s \hat{f}\big)\right| \right\|_{\ell^r} \lesssim \left\| \sup_{k\in\mathbb{N}} \left|\mathcal{F}^{-1}\big(\Psi_k \eta_s \hat{f}\big)\right| \right\|_{L^r} \lesssim \big\|\mathcal{F}^{-1}\big(\eta_s \hat{f}\big)\big\|_{L^r}, \tag{20}$$

where the last inequality is a consequence of [15]. The proof will be completed if we show

$$\big\|\mathcal{F}^{-1}\big(\eta_s \hat{f}\big)\big\|_{L^r} \lesssim \big\|\mathcal{F}^{-1}\big(\eta_s \hat{f}\big)\big\|_{\ell^r}.$$

For this purpose we use (17) from Lemma 1. Indeed,

$$\begin{aligned}
&\|\mathcal{F}^{-1}(\eta_s \hat{f})\|_{L^r}^r \\
&= \sum_{j\in\mathbb{Z}} \int_0^1 |\mathcal{F}^{-1}(\eta_s \hat{f})(x+j)|^r dx \\
&\leq 2^{r-1}\|\mathcal{F}^{-1}(\eta_s \hat{f})\|_{\ell^r}^r + 2^{r-1}\sum_{j\in\mathbb{Z}}\int_0^1 |\mathcal{F}^{-1}(\eta_s \hat{f})(x+j) - \mathcal{F}^{-1}(\eta_s \hat{f})(j)|^r dx \\
&= 2^{r-1}\|\mathcal{F}^{-1}(\eta_s \hat{f})\|_{\ell^r}^r + 2^{r-1}\int_0^1 \Big\| \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i \xi j}\big(1-e^{-2\pi i \xi x}\big)\eta_s(\xi)\hat{f}(\xi)d\xi\Big\|_{\ell^r(j)}^r dx \\
&\leq 2^{r-1}\|\mathcal{F}^{-1}(\eta_s \hat{f})\|_{\ell^r}^r + 2^{r-1}\int_0^1 \Big\| \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i \xi j}\big(1-e^{-2\pi i \xi x}\big)\eta_{s-1}(\xi)d\xi\Big\|_{\ell^1(j)}^r \|\mathcal{F}^{-1}(\eta_s \hat{f})\|_{\ell^r}^r dx \\
&\lesssim \|\mathcal{F}^{-1}(\eta_s \hat{f})\|_{\ell^r}^r.
\end{aligned}$$

This finishes the proof of the proposition. □

Theorem 3 *For each $r > 1$ there are $\delta_r > 0$ and $C_r > 0$ such that*

$$\Big\| \sup_{k\in\mathbb{N}} \Big| \sum_{j=0}^{k} \mathcal{F}^{-1}(\nu_j^s \hat{f}) \Big| \Big\|_{\ell^r} \leq C_r 2^{-\delta_r s} \|f\|_{\ell^r}$$

for all $f \in \ell^r(\mathbb{Z})$.

There is an interesting question about the endpoint estimate for $r = 1$ in Theorem 3. Unfortunately, our method does not settle this issue. However, we hope to return to this problem at some point.

Proof Let us fix $r > 1$. For $s < r$, by Proposition 3.3 we have

$$\Big\| \sup_{k\in\mathbb{N}} \Big| \sum_{j=0}^{k} \mathcal{F}^{-1}(\nu_j^s \hat{f}) \Big| \Big\|_{\ell^r} \leq \sum_{a/q\in\mathscr{R}_s} \frac{1}{\varphi(q)} \Big\| \sup_{k\in\mathbb{N}} \Big| \mathcal{F}^{-1}\big(\Psi_k \hat{f}(\cdot + a/q)\big) \Big| \Big\|_{\ell^r} \leq C_r \|f\|_{\ell^r}.$$

Next, we consider $s \geq r$. Let $q \in [2^s, 2^{s+1})$ be fixed. We are going to show that for every $\epsilon > 0$ we have

$$\Big\| \sup_{k\in\mathbb{N}} \Big| \sum_{a\in A_q} \mathcal{F}^{-1}\big(\Psi_k(\cdot - a/q)\eta_s(\cdot - a/q)\hat{f}\big) \Big| \Big\|_{\ell^r} \leq C_\epsilon q^\epsilon \|f\|_{\ell^r}. \tag{21}$$

By Möbius inversion formula (3) we see that

$$\sum_{a\in A_q}\mathcal{F}^{-1}\big(\Psi_k(\cdot-a/q)\eta_s(\cdot-a/q)\hat{f}\big)(x)=\sum_{b|q}\mu(q/b)\sum_{a=1}^{b}e^{-2\pi iax/b}\mathcal{F}^{-1}\big(\Psi_k\eta_s\hat{f}(\cdot+a/b)\big)(x). \tag{22}$$

Moreover, for $x\equiv l\pmod q$ we may write

$$\sum_{a=1}^{b}e^{-2\pi iax/b}\mathcal{F}^{-1}\big(\Psi_k\eta_s\hat{f}(\cdot+a/b)\big)(x)=\mathcal{F}^{-1}\big(\Psi_k\eta_sF_b(\cdot\,;l)\big)(x) \tag{23}$$

where for $b\mid q$ we have set

$$F_b(\xi;l)=\sum_{a=1}^{b}\hat{f}(\xi+a/b)e^{-2\pi ila/b}.$$

Therefore, by formulas (22) and (23) we have

$$\Big\|\sup_{k\in\mathbb{N}}\Big|\sum_{a\in A_q}\mathcal{F}^{-1}\big(\Psi_k(\cdot-a/q)\eta_s(\cdot-a/q)\hat{f}\big)\Big|\Big\|_{\ell^r}$$
$$\le\sum_{b|q}\bigg(\sum_{l=1}^{q}\Big\|\sup_{k\in\mathbb{N}}\big|\mathcal{F}^{-1}\big(\Psi_k\eta_sF_b(\cdot\,;l)\big)(qj+l)\big|\Big\|_{\ell^r(j)}^r\bigg)^{1/r}.$$

Thus in view of (5) it will suffice to prove that

$$\bigg(\sum_{l=1}^{q}\Big\|\sup_{k\in\mathbb{N}}\big|\mathcal{F}^{-1}\big(\Psi_k\eta_sF_b(\cdot\,;l)\big)(qj+l)\big|\Big\|_{\ell^r(j)}^r\bigg)^{1/r}\le C_r\|f\|_{\ell^r} \tag{24}$$

where the constant does not depend on b. For the proof let us fix $f\in\ell^r(\mathbb{Z})$ and consider a sequence $(J_1,J_2,\ldots,J_q)$ defined by

$$J_l=\Big\|\sup_{k\in\mathbb{N}}\big|\mathcal{F}^{-1}\big(\Psi_k\eta_s\hat{f}\big)(qj+l)\big|\Big\|_{\ell^r(j)}.$$

By Proposition 3.3, we have

$$J_1^r+J_2^r+\cdots+J_q^r=I^r=\Big\|\sup_{k\in\mathbb{N}}\big|\mathcal{F}^{-1}\big(\Psi_k\eta_s\hat{f}\big)\big|\Big\|_{\ell^r}^r\lesssim\big\|\mathcal{F}^{-1}\big(\eta_s\hat{f}\big)\big\|_{\ell^r}^r.$$

Also for any $l, l' \in \{1, 2, \ldots, q\}$

$$\Big\| \sup_{k\in\mathbb{N}} \Big| \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i \xi (qj+l)} \big(1 - e^{2\pi i \xi (l-l')}\big) \Psi_k(\xi) \eta_s(\xi) \hat{f}(\xi) d\xi \Big| \Big\|_{\ell^r(j)} \lesssim \Big\| \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i \xi j} \big(1 - e^{2\pi i \xi (l-l')}\big) \eta_s(\xi) \hat{f}(\xi) d\xi \Big\|_{\ell^r(j)}.$$

Since $\eta_s = \eta_s \eta_{s-1}$, by Minkowski's inequality and Lemma 1 we obtain that the last expression can be dominated by

$$\Big\| \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i \xi j} \big(1 - e^{2\pi i \xi (l-l')}\big) \eta_{s-1}(\xi) d\xi \Big\|_{\ell^1(j)} \big\| \mathcal{F}^{-1}\big(\eta_s \hat{f}\big) \big\|_{\ell^r} \le C q A^{-s} \big\| \mathcal{F}^{-1}\big(\eta_s \hat{f}\big) \big\|_{\ell^r}.$$

Therefore, by (18)

$$J_l \le J_{l'} + q^{-1} \big\| \mathcal{F}^{-1}\big(\eta_s \hat{f}\big) \big\|_{\ell^r}.$$

Summing up over all $l' \in \{1, 2, \ldots, q\}$ we obtain

$$q J_l^r \le 2^{r-1} I^r + C 2^{r-1} q^{1-r} \big\| \mathcal{F}^{-1}\big(\eta_s \hat{f}\big) \big\|_{\ell^r}^r \lesssim \big\| \mathcal{F}^{-1}\big(\eta_s \hat{f}\big) \big\|_{\ell^r}^r.$$

Finally, by Lemma 2 we conclude

$$\Big\| \sup_{k\in\mathbb{N}} \big| \mathcal{F}^{-1}\big(\Psi_k \eta_s \hat{f}\big)(qj+l) \big| \Big\|_{\ell^r(j)} \lesssim \big\| \mathcal{F}^{-1}\big(\eta_s \hat{f}\big)(qj+l) \big\|_{\ell^r(j)}. \tag{25}$$

Next, we resume the analysis of (24). Using (25) we get

$$\Big(\sum_{l=1}^{q} \Big\| \sup_{k\in\mathbb{N}} \big| \mathcal{F}^{-1}\big(\Psi_k \eta_s F_b(\cdot\,; l)\big)(qj+l) \big| \Big\|_{\ell^r(j)}^r \Big)^{1/r} \lesssim \Big(\sum_{l=1}^{q} \Big\| \mathcal{F}^{-1}\big(\eta_s F_b(\cdot\,; l)\big)(qj+l) \Big\|_{\ell^r(j)}^r \Big)^{1/r}.$$

We observe that by the change of variables

$$\mathcal{F}^{-1}\big(\eta_s F_b(\cdot\,; l)\big)(qj+l) = \sum_{a=1}^{b} \mathcal{F}^{-1}\big(\eta_s(\cdot - a/b) \hat{f}\big)(qj+l).$$

Thus by Minkowski's inequality

$$\Big(\sum_{l=1}^{q} \Big\| \mathcal{F}^{-1}\big(\eta_s F_b(\cdot\,; l)\big)(qj+l) \Big\|_{\ell^r(j)}^r \Big)^{1/r} \le \Big\| \mathcal{F}^{-1}\Big(\sum_{a=1}^{b} \eta_s(\cdot - a/b) \Big) \Big\|_{\ell^1} \|f\|_{\ell^r}.$$

Since for $j \in \mathbb{Z}$

$$\sum_{a=1}^{b} e^{-2\pi i j a/b} = \begin{cases} b & \text{if } b \mid j, \\ 0 & \text{otherwise} \end{cases}$$

we conclude

$$\Big\| \mathcal{F}^{-1}\Big(\sum_{a=1}^{b} \eta_s(\cdot - a/b) \Big) \Big\|_{\ell^1} = \Big\| \mathcal{F}^{-1}\eta_s(j) \sum_{a=1}^{b} e^{-2\pi i j a/b} \Big\|_{\ell^1(j)} = b \Big\| \mathcal{F}^{-1}\eta_s(bj) \Big\|_{\ell^1(j)}.$$

Now Lemmas 1 and 2 imply

$$b \Big\| \mathcal{F}^{-1}\eta_s(bj) \Big\|_{\ell^1(j)} \lesssim \Big\| \mathcal{F}^{-1}\eta_s \Big\|_{\ell^1} \lesssim 1.$$

This completes the proof of (24). Finally, by (4) and (21) we obtain that

$$\Big\| \sup_{k \in \mathbb{N}} \Big| \sum_{j=0}^{k} \mathcal{F}^{-1}(\nu_j^s \hat{f}) \Big| \Big\|_{\ell^r} \lesssim 2^{\epsilon s} \| f \|_{\ell^r} \tag{26}$$

for any $\epsilon > 0$ and $s \in \mathbb{N}$. If $r = 2$ we may refine the estimate (26) (see also [1]). Let

$$G_q(\xi) = \sum_{a \in A_q} \eta_{s-1}(\xi - a/q) \hat{f}(\xi).$$

and note that

$$\sum_{a \in A_q} \mathcal{F}^{-1}\big(\Psi_k(\cdot - a/q)\eta_s(\cdot - a/q)\hat{f}\big) = \sum_{a \in A_q} \mathcal{F}^{-1}\big(\Psi_k(\cdot - a/q)\eta_s(\cdot - a/q) G_q\big)$$

since $\eta_s = \eta_s \eta_{s-1}$, and the supports of $\eta_s(\cdot - a/q)$'s are disjoint when a/q varies. By (21) we have

$$\Big\| \sup_{k \in \mathbb{N}} \Big| \sum_{a \in A_q} \mathcal{F}^{-1}(\Psi_k(\cdot - a/q)\eta_s(\cdot - a/q) G_q) \Big| \Big\|_{\ell^2} \lesssim q^{\epsilon} \Big\| \mathcal{F}^{-1} G_q \Big\|_{\ell^2}$$

whereas by (4), we have

$$\Big\| \sup_{k \in \mathbb{N}} \Big| \sum_{j=0}^{k} \mathcal{F}^{-1}\big(\nu_j^s \hat{f}\big) \Big| \Big\|_{\ell^2} \le \sum_{q=2^s}^{2^{s+1}-1} q^{-1+\epsilon} \Big\| \sup_{k \in \mathbb{N}} \Big| \sum_{a \in A_q} \mathcal{F}^{-1}(\Psi_k(\cdot - a/q)\eta_s(\cdot - a/q)\hat{f}) \Big| \Big\|_{\ell^2}.$$

These two bounds yield

$$\Big\| \sup_{k\in\mathbb{N}} \Big| \sum_{j=0}^{k} \mathcal{F}^{-1}\big(\nu_j^s \hat{f}\big)\Big|\Big\|_{\ell^2} \lesssim \sum_{q=2^s}^{2^{s+1}-1} q^{-1+2\epsilon} \big\|\mathcal{F}^{-1} G_q\big\|_{\ell^2}$$
$$\lesssim 2^{-s/2+2\epsilon s}\Big(\sum_{a/q\in\mathscr{R}_s} \big\|\mathcal{F}^{-1}\big(\eta_{s-1}(\cdot - a/q)\hat{f}\big)\big\|_{\ell^2}^2\Big)^{1/2},$$

where the last estimate follows from Cauchy–Schwarz inequality and the definition of G_q. Finally, by Plancherel's theorem we may write

$$\sum_{a/q\in\mathscr{R}_s} \big\|\mathcal{F}^{-1}\big(\eta_{s-1}(\cdot - a/q)\hat{f}\big)\big\|_{\ell^2}^2 = \sum_{a/q\in\mathscr{R}_s} \int_{\mathbb{R}} |\eta_{s-1}(\xi - a/q)|^2 \big|\hat{f}(\xi)\big|^2 d\xi$$

which is majorized by $\|f\|_{\ell^2}^2$. Thus for appropriately chosen $\epsilon > 0$ we obtain

$$\Big\| \sup_{k\in\mathbb{N}} \Big| \sum_{j=0}^{k} \mathcal{F}^{-1}\big(\nu_j^s \hat{f}\big)\Big|\Big\|_{\ell^2} \le 2^{-s/4}\|f\|_{\ell^2}. \tag{27}$$

Next, for $r \neq 2$ we can use Marcinkiewicz interpolation theorem and interpolate between (26) and (27) to conclude the proof. □

3.3 Maximal Function

We have gathered necessary tools to illustrate the proof of Theorem 2. First, we show the boundedness on $\ell^r(\mathbb{Z})$ of the maximal function T^*.

Theorem 4 *The maximal function T^* is bounded on $\ell^r(\mathbb{Z})$ for each $1 < r < \infty$.*

Proof Let us observe that for a non-negative function f

$$T^* f(n) \lesssim \sup_{k\in\mathbb{N}} \Big| \sum_{j=0}^{k} \mathcal{F}^{-1}\big(m_j \hat{f}\big)(n)\Big| + \mathcal{M} f(n)$$

where $\mathcal{M} f = \sup_{N\in\mathbb{N}} |A_N f|$ is a maximal function corresponding with Bourgain–Wierdl's averages

$$A_N f(n) = N^{-1} \sum_{p\in\pm\mathbb{P}_N} f(n-p)\log|p|.$$

Indeed, suppose $\tau^k \leq N < \tau^{k+1}$ for $k \in \mathbb{N}$. Then

$$T_N f(n) = \sum_{j=0}^{k} \sum_{p \in \pm\mathbb{P}} f(n-p) K_j(p) \log|p| - \sum_{p \in \pm R_N} f(n-p) K(p) \log|p|.$$

where $R_N = \mathbb{P} \cap (N, \tau^{k+1})$. Therefore, by (1), we see

$$\Big| \sum_{p \in R_N} f(n-p) K(p) \log|p| \Big| \lesssim \tau^{-k} \sum_{p \in \pm\mathbb{P}_{\tau^{k+1}}} f(n-p) \log|p| \lesssim A_{\tau^{k+1}} f(n).$$

Since the maximal function $\mathcal{M}$ is bounded on $\ell^r(\mathbb{Z})$ for any $r > 1$ (see [3] or Appendix 1) thus we have reduced the boundedness of T^* to proving

$$\Big\| \sup_{k \in \mathbb{N}} \Big| \sum_{j=0}^{k} \mathcal{F}^{-1}\big(m_j \hat{f}\big) \Big| \Big\|_{\ell^r} \lesssim \|f\|_{\ell^r}.$$

Let us consider $f \in \ell^r(\mathbb{Z})$ for $r > 1$. By Theorem 3 we know that for $j \in \mathbb{N}$

$$\begin{aligned}
\Big\| \mathcal{F}^{-1}\big(\nu_j \hat{f}\big) \|_{\ell^r} &\leq \sum_{s \in \mathbb{N}} \Big\| \mathcal{F}^{-1}\big(\nu_j^s \hat{f}\big) \|_{\ell^r} \leq \sum_{s \in \mathbb{N}} \Big\| \sup_{k \in \mathbb{N}} \Big| \sum_{j=0}^{k} \mathcal{F}^{-1}\big(\nu_j^s \hat{f}\big) - \sum_{j=0}^{k-1} \mathcal{F}^{-1}\big(\nu_j^s \hat{f}\big) \Big| \Big\|_{\ell^r} \\
&\lesssim \sum_{s \in \mathbb{N}} \Big\| \sup_{k \in \mathbb{N}} \Big| \sum_{j=0}^{k} \mathcal{F}^{-1}\big(\nu_j^s \hat{f}\big) \Big| \Big\|_{\ell^r} \lesssim \sum_{s \in \mathbb{N}} 2^{-\delta s} \|f\|_{\ell^r} \lesssim \|f\|_{\ell^r}.
\end{aligned}$$

If f is non-negative then

$$\Big| \sum_{p \in \pm\mathbb{P}} f(x-p) K_j(p) \log|p| \Big| \leq \tau^{-j} \sum_{p \in \pm\mathbb{P}_{\tau^{j+1}}} f(x-p) \log|p|$$

thus by Prime Number Theorem,

$$\|\mathcal{F}^{-1}\big(m_j \hat{f}\big)\|_{\ell^r} \leq \tau^{-j} \Bigg(\sum_{p \in \mathbb{P}_{\tau^{j+1}}} \log p \Bigg) \|f\|_{\ell^r} \lesssim \|f\|_{\ell^r}.$$

Hence,

$$\Big\| \mathcal{F}^{-1}\big((m_j - \nu_j)\hat{f}\big) \Big\|_{\ell^r} \lesssim \|f\|_{\ell^r}. \tag{28}$$

For $r = 2$ we use Proposition 3.2 to get

$$\Big\| \mathcal{F}^{-1}\big((m_j - \nu_j)\hat{f}\big) \Big\|_{\ell^2} \leq \big\| m_j - \nu_j \big\|_{L^\infty} \|f\|_{\ell^2} \lesssim j^{-\alpha} \|f\|_{\ell^2} \tag{29}$$

for any $\alpha > 0$ big enough. If $r \neq 2$ we apply Marcinkiewicz interpolation theorem to interpolate between (28) and (29) and obtain

$$\left\| \mathcal{F}^{-1}\big((m_j - \nu_j)\hat{f}\big) \right\|_{\ell^r} \lesssim j^{-2} \|f\|_{\ell^r}. \tag{30}$$

Since

$$\left\| \sup_{k \in \mathbb{N}} \left| \sum_{j=0}^{k} \mathcal{F}^{-1}\big((m_j - \nu_j)\hat{f}\big) \right| \right\|_{\ell^r} \leq \sum_{j \in \mathbb{N}} \left\| \mathcal{F}^{-1}\big((m_j - \nu_j)\hat{f}\big) \right\|_{\ell^r}$$

by (30) and Theorem 3 we finish the proof. □

Next, we demonstrate the pointwise convergence of $(T_N : N \in \mathbb{N})$.

Proposition 3.4 *If $f \in \ell^r(\mathbb{Z})$, $1 < r < \infty$ then for every $n \in \mathbb{Z}$*

$$\lim_{N \to \infty} T_N f(n) = T f(n) \tag{31}$$

and T is bounded on $\ell^r(\mathbb{Z})$.

Proof If $N \in \mathbb{N}$ we define an operator T^N by setting

$$T^N f(n) = \sum_{\substack{p \in \pm\mathbb{P} \\ |p| > N}} f(n-p) K(p) \log |p|$$

for any $f \in \ell^r(\mathbb{Z})$. By Hölder's inequality we see that for every $n \in \mathbb{Z}$

$$\left| T^N f(n) \right| \leq 2 \Big(\sum_{\substack{p \in \mathbb{P} \\ p > N}} \big(p^{-1} \log p\big)^{r'} \Big)^{1/r'} \|f\|_{\ell^r}$$

where r' stands for the conjugate exponent to r, i.e. $1/r + 1/r' = 1$. The last inequality shows that, on the one hand, T is well defined for any $f \in \ell^r(\mathbb{Z})$, on the other proves (31). Next, Fatou's lemma with boundedness of T^* yield

$$\|Tf\|_{\ell^r} = \left\| \liminf_{N \to \infty} T_N f \right\|_{\ell^r} \leq \liminf_{N \to \infty} \left\| T_N f \right\|_{\ell^r} \leq \left\| T^* f \right\|_{\ell^r} \lesssim_r \|f\|_{\ell^r}$$

which completes the proof. □

3.4 Oscillatory Norm for H_N

Let $\big(N_j : j \in \mathbb{N}\big)$ be a strictly increasing sequence of Λ elements. We set $N_j = \tau^{k_j}$ and $\Lambda_j = \Lambda \cap (N_j, N_{j+1}]$. In this Section we consider the kernel $K(x) = x^{-1}$. Since each K_j for $j \in \mathbb{N}$ has mean zero we have

$$\left|\Phi_j(\xi)\right| \leq \int_{\mathbb{R}} \left|1 - e^{2\pi i \xi x}\right| \left|K_j(x)\right| dx \lesssim |\xi| \tau^j. \tag{32}$$

Let H_N denote the truncated Hilbert transform

$$H_N f(n) = \sum_{p \in \pm \mathbb{P}_N} \frac{f(n-p)}{p} \log|p|.$$

The following argument is based on [1, Section 7].

Proposition 3.5 *There is $C > 0$ such that for every $J \in \mathbb{N}$ and $s \in \mathbb{N}$ we have*

$$\sum_{j=0}^{J} \Big\| \sup_{\tau^k \in \Lambda_j} \left| \mathcal{F}^{-1}\big((\Psi_k - \Psi_{k_j}) \eta_s \hat{f}\big) \right| \Big\|_{\ell^2}^2 \leq C \big\| \mathcal{F}^{-1}\big(\eta_s \hat{f}\big) \big\|_{\ell^2}^2.$$

Proof Let $B_j = \{x \in (-1/2, 1/2) : |x| \leq N_j^{-1}\}$. By Plancherel's theorem we have

$$\begin{aligned}
&\sum_{j=0}^{J} \Big\| \sup_{\tau^k \in \Lambda_j} \left| \mathcal{F}^{-1}\big((\Psi_k - \Psi_{k_j}) \mathbb{1}_{B_{j+1}} \eta_s \hat{f}\big) \right| \Big\|_{\ell^2}^2 \\
&\leq \sum_{j=0}^{J} \sum_{k=k_j}^{k_{j+1}} \Big\| \mathcal{F}^{-1}\big((\Psi_k - \Psi_{k_j}) \mathbb{1}_{B_{j+1}} \eta_s \hat{f}\big) \Big\|_{\ell^2}^2 \\
&\leq \Big\| \sum_{j=0}^{J} \mathbb{1}_{B_{j+1}} \sum_{k=k_j}^{k_{j+1}} \left|\Psi_k - \Psi_{k_j}\right|^2 \Big\|_{L^\infty} \big\| \mathcal{F}^{-1}\big(\eta_s \hat{f}\big) \big\|_{\ell^2}^2.
\end{aligned}$$

By (32) we have

$$\left|\Psi_k(\xi) - \Psi_{k_j}(\xi)\right| = \left| \sum_{l=k_j+1}^{k} \Phi_l(\xi) \right| \lesssim |\xi| \tau^k.$$

Hence,

$$\begin{aligned}
&\sum_{j=0}^{J} \mathbb{1}_{B_{j+1}}(\xi) \sum_{k=k_j}^{k_{j+1}} \left|\Psi_k(\xi) - \Psi_{k_j}(\xi)\right|^2 \\
&\lesssim |\xi|^2 \sum_{j=0}^{J} \mathbb{1}_{B_{j+1}}(\xi) \sum_{k=k_j}^{k_{j+1}} \tau^{2k} \lesssim |\xi|^2 \sum_{j : N_{j+1} \leq |\xi|^{-1}} N_{j+1}^2 \lesssim 1.
\end{aligned}$$

Therefore, we obtain

$$\sum_{j=0}^{J}\Big\|\sup_{\tau^k\in\Lambda_j}\big|\mathcal{F}^{-1}\big((\Psi_k-\Psi_{k_j})\mathbb{1}_{B_{j+1}}\eta_s\hat{f}\big)\big|\Big\|_{\ell^2}^2\lesssim\Big\|\mathcal{F}^{-1}\big(\eta_s\hat{f}\big)\Big\|_{\ell^2}^2.$$

Similar for B_j^c, replacing Ψ_{k_j} by $\Psi_{k_{j+1}}$ under the supremum, we can estimate

$$\begin{aligned}\sum_{j=0}^{J}&\Big\|\sup_{\tau^k\in\Lambda_j}\big|\mathcal{F}^{-1}\big((\Psi_k-\Psi_{k_j})\mathbb{1}_{B_j^c}\eta_s\hat{f}\big)\big|\Big\|_{\ell^2}^2\\&\le\sum_{j=0}^{J}\sum_{k=k_j}^{k_{j+1}}\Big\|\mathcal{F}^{-1}\big((\Psi_{k_{j+1}}-\Psi_k)\mathbb{1}_{B_j^c}\eta_s\hat{f}\big)\Big\|_{\ell^2}^2\\&\le\Big\|\sum_{j=0}^{J}\mathbb{1}_{B_j^c}\sum_{k=k_j}^{k_{j+1}}\big|\Psi_{k_{j+1}}-\Psi_k\big|^2\Big\|_{L^\infty}\Big\|\mathcal{F}^{-1}\big(\eta_s\hat{f}\big)\Big\|_{\ell^2}^2.\end{aligned}$$

Now, using (6) we get

$$\big|\Psi_{k_{j+1}}(\xi)-\Psi_k(\xi)\big|\lesssim|\xi|^{-1}\tau^{-k}$$

thus

$$\begin{aligned}\sum_{j=0}^{J}\mathbb{1}_{B_j^c}(\xi)\sum_{k=k_j}^{k_{j+1}}\big|\Psi_{k_{j+1}}(\xi)-\Psi_k(\xi)\big|^2&\lesssim|\xi|^{-2}\sum_{j=0}^{J}\mathbb{1}_{B_j^c}(\xi)\sum_{k=k_j}^{k_{j+1}}\tau^{-2k}\\&\lesssim|\xi|^{-2}\sum_{j:N_j\ge|\xi|^{-1}}N_j^{-2}\lesssim 1.\end{aligned}$$

Therefore, we conclude

$$\sum_{j=0}^{J}\Big\|\sup_{\tau^k\in\Lambda_j}\Big|\mathcal{F}^{-1}\big((\Psi_k-\Psi_{k_j})\mathbb{1}_{B_j^c}\eta_s\hat{f}\big)\Big|\Big\|_{\ell^2}^2\lesssim\big\|\mathcal{F}^{-1}\big(\eta_s\hat{f}\big)\big\|_{\ell^2}^2.$$

Finally, by Proposition 3.3

$$\sum_{j=0}^{J}\Big\|\sup_{\tau^k\in\Lambda_j}\big|\mathcal{F}^{-1}\big((\Psi_k-\Psi_{k_j})\mathbb{1}_{B_j}\mathbb{1}_{B_{j+1}^c}\eta_s\hat{f}\big)\big|\Big\|_{\ell^2}^2\lesssim\sum_{j=0}^{J}\Big\|\mathcal{F}^{-1}\big(\mathbb{1}_{B_j}\mathbb{1}_{B_{j+1}^c}\eta_s\hat{f}\big)\Big\|_{\ell^2}^2$$

which is bounded by $\big\|\mathcal{F}^{-1}\big(\eta_s\hat{f}\big)\big\|_{\ell^2}^2$. □

Theorem 5 *For every $J \in \mathbb{N}$ there is C_J such that*

$$\sum_{j=0}^{J} \Big\| \sup_{\tau^k \in \Lambda_j} \big| H_{\tau^k} f - H_{N_j} f \big| \Big\|_{\ell^2}^2 \le C_J \|f\|_{\ell^2}^2$$

and $\lim_{J\to\infty} C_J/J = 0$.

Proof By Proposition 3.2, we have

$$\sum_{j=0}^{J} \Big\| \sup_{\tau^k \in \Lambda_j} \Big| \sum_{l=k_j+1}^{k} \mathcal{F}^{-1}\big((m_l - \nu_l)\hat{f}\big) \Big| \Big\|_{\ell^2}^2 \lesssim \Big(\sum_{j=0}^{J} \sum_{l=k_j+1}^{k_{j+1}} l^{-2} \Big) \|f\|_{\ell^2}^2 \lesssim \|f\|_{\ell^2}^2.$$

Consequently, it is enough to demonstrate

$$\sum_{j=0}^{J} \Big\| \sup_{\tau^k \in \Lambda_j} \Big| \sum_{l=k_j+1}^{k} \mathcal{F}^{-1}\big(\nu_l \hat{f}\big) \Big| \Big\|_{\ell^2}^2 \le C_J \|f\|_{\ell^2}^2$$

where $\lim_{J\to\infty} C_J/J = 0$.

Let $s_0 \in \mathbb{N}$ be defined as $2^{s_0} \le J^{1/3} < 2^{s_0+1}$. By Theorem 3 we have

$$\Big\| \sup_{\tau^k \in \Lambda_j} \Big| \sum_{s=s_0}^{\infty} \sum_{l=k_j+1}^{k} \mathcal{F}^{-1}\big(\nu_l^s \hat{f}\big) \Big| \Big\|_{\ell^2} \lesssim \sum_{s=s_0}^{\infty} \Big\| \sup_{k\in\mathbb{N}} \Big| \sum_{l=0}^{k} \mathcal{F}^{-1}\big(\nu_l^s \hat{f}\big) \Big| \Big\|_{\ell^2} \lesssim J^{-\delta/3} \|f\|_{\ell^2}.$$

We set

$$D_J = \sum_{s=0}^{s_0-1} \sum_{a/q \in \mathscr{R}_s} \frac{1}{\varphi(q)^2}.$$

By the change of variables, Cauchy–Schwarz inequality and by Proposition 3.5 we get

$$\begin{aligned}
&\sum_{j=0}^{J} \Big\| \sup_{\tau^k \in \Lambda_j} \Big| \sum_{s=0}^{s_0-1} \sum_{l=k_j+1}^{k} \mathcal{F}^{-1}\big(\nu_l^s \hat{f}\big) \Big| \Big\|_{\ell^2}^2 \\
&\le \sum_{j=0}^{J} \Big(\sum_{s=0}^{s_0-1} \sum_{a/q \in \mathscr{R}_s} \frac{1}{\varphi(q)} \Big\| \sup_{\tau^k \in \Lambda_j} \Big| \sum_{l=k_j+1}^{k} \mathcal{F}^{-1}\big(\Phi_l \eta_s \hat{f}(\cdot + a/q)\big) \Big| \Big\|_{\ell^2} \Big)^2 \\
&\le D_J \sum_{s=0}^{s_0-1} \sum_{a/q \in \mathscr{R}_s} \sum_{j=0}^{J} \Big\| \sup_{\tau^k \in \Lambda_j} \Big| \mathcal{F}^{-1}\big((\Psi_k - \Psi_{k_j})\eta_s \hat{f}(\cdot + a/q)\big) \Big| \Big\|_{\ell^2}^2
\end{aligned}$$

$$\lesssim D_J \sum_{s=0}^{s_0-1} \sum_{a/q\in\mathscr{R}_s} \left\| \mathcal{F}^{-1}\big(\eta_s(\cdot - a/q)\hat{f}\big) \right\|_{\ell^2}^2 \lesssim D_J s_0 \|f\|_{\ell^2}^2.$$

By the definition of $\mathscr{R}_s$ we see that $D_J \lesssim 2^{s_0} \le J^{1/3}$ thus we achieve

$$\sum_{j=0}^{J} \left\| \sup_{\tau^k\in\Lambda_j} \left| \sum_{l=k_j+1}^{k} \mathcal{F}^{-1}\big(\nu_l \hat{f}\big) \right| \right\|_{\ell^2}^2 \lesssim J\Big(J^{-\delta/3} + J^{-1/3}\log J\Big)\|f\|_{\ell^2}^2$$

which finishes the proof. □

4 Dynamical Systems

Let $(X, \mathcal{B}, \mu, S)$ be a dynamical system on a measure space X. Let $S : X \to X$ be an invertible measure preserving transformation. For $N > 0$ we set

$$\mathcal{H}_N f(x) = \sum_{p\in\pm\mathbb{P}_N} \frac{f(S^{-p}x)}{p} \log|p|.$$

We are going to show Theorem 1. We start from oscillatory norm.

Proposition 4.1 *For each $J \in \mathbb{N}$ there is C_J such that*

$$\sum_{j=0}^{J} \left\| \sup_{N\in\Lambda_j} \left|\mathcal{H}_N f - \mathcal{H}_{N_j} f\right| \right\|_{L^2(\mu)}^2 \le C_J \|f\|_{L^2(\mu)}^2$$

and $\lim_{J\to\infty} C_J/J = 0$.

Proof Let $R \ge N_J$. For a fixed $x \in X$ we define a function on $\mathbb{Z}$ by

$$\phi(n) = \begin{cases} f(S^n x) & |n| \le R, \\ 0 & \text{otherwise.} \end{cases}$$

Then for $|n| \le R - N$

$$\mathcal{H}_N f(S^n x) = \sum_{p\in\pm\mathbb{P}_N} \frac{f(S^{n-p}x)}{p} \log|p| = \sum_{p\in\pm\mathbb{P}_N} \frac{\phi(n-p)}{p} \log|p| = H_N\phi(n).$$

Hence,

$$\sum_{|n|=0}^{R-N_J} \sup_{N\in\Lambda_j} \left|\mathcal{H}_N f(S^n x) - \mathcal{H}_{N_j} f(S^n x)\right|^2 \le \Big\| \sup_{N\in\Lambda_j} \left|H_N\phi - H_{N_j}\phi\right| \Big\|_{\ell^2}^2.$$

Therefore, by Theorem 5 we can estimate

$$\sum_{|n|=0}^{R-N_J} \sum_{j=0}^{J} \sup_{N\in\Lambda_j} \left|\mathcal{H}_N f(S^n x) - \mathcal{H}_{N_j} f(S^n x)\right|^2 \le C_J \|\phi\|_{\ell^2}^2 = C_J \sum_{|n|=0}^{R} \left|f(S^n x)\right|^2.$$

Since S is a measure preserving transformation integration with respect to $x \in X$ implies

$$(R - N_J) \sum_{j=0}^{J} \left\| \sup_{N\in\Lambda_j} \left|\mathcal{H}_N f - \mathcal{H}_{N_j} f\right| \right\|_{L^2(\mu)}^2 \le C_J R \|f\|_{L^2(\mu)}^2.$$

Finally, if we divide both sides by R and take $R \to \infty$ we conclude the proof. □

Corollary 1 *The maximal function*

$$\mathcal{H}^* f(x) = \sup_{N\in\mathbb{N}} \left|\mathcal{H}_N f(x)\right|$$

is bounded on $L^r(\mu)$ for each $1 < r < \infty$.

Next, we show the pointwise convergence of $(\mathcal{H}_N : N \in \mathbb{N})$.

Theorem 6 *Let $f \in L^r(\mu)$, $1 < r < \infty$. For μ-almost every $x \in X$*

$$\lim_{N\to\infty} \mathcal{H}_N f(x) = \mathcal{H} f(x)$$

and $\mathcal{H}$ is bounded on $L^r(\mu)$.

Proof Let $f \in L^2(\mu)$, since the maximal function $\mathcal{H}^*$ is bounded on $L^2(\mu)$ we may assume f is bounded by 1. Suppose $(\mathcal{H}_N f : N \in \mathbb{N})$ does not converge μ-almost everywhere. Then there is $\epsilon > 0$ such that

$$\mu\left\{x \in X : \limsup_{M,N\to\infty} \left|\mathcal{H}_N f(x) - \mathcal{H}_M f(x)\right| > 4\epsilon\right\} > 4\epsilon.$$

Now one can find a strictly increasing sequence of integers $\left(k_j : j \in \mathbb{N}\right)$ such that for each $j \in \mathbb{N}$

$$\mu\left\{x \in X : \sup_{N_j\le N\le N_{j+1}} \left|\mathcal{H}_N f(x) - \mathcal{H}_{N_j} f(x)\right| > \epsilon\right\} > \epsilon$$

where $N_j = \tau^{k_j}$ and $\tau = 1+\epsilon/4$. If $\tau^k \le N < \tau^{k+1}$ then setting $P_k = \mathbb{P} \cap (\tau^k, \tau^{k+1}]$ we get

$$\left|\mathcal{H}_N f(x) - \mathcal{H}_{\tau^k} f(x)\right| \le \tau^{-k} \sum_{p\in P_k} \log p.$$

By Siegel–Walfisz theorem we get

$$\sum_{p\in\mathbb{P}_N} \log p = N + \mathcal{O}\left(N(\log N)^{-1}\right)$$

thus there is $C > 0$ such that

$$\left|\tau^{-k}\sum_{p\in P_k} \log p - \tau + 1\right| \le Ck^{-1}(\log\tau)^{-1}.$$

Hence, whenever $k \ge 4C\epsilon^{-1}(\log\tau)^{-1}$ we have

$$\left|\mathcal{H}_N f(x) - \mathcal{H}_{\tau^k} f(x)\right| \le \epsilon/2.$$

In particular, we conclude

$$\mu\left\{x\in X : \sup_{\tau^k\in\Lambda_j}\left|\mathcal{H}_{\tau^k} f(x) - \mathcal{H}_{N_j} f(x)\right| > \epsilon/2\right\} > \epsilon$$

for each $k_j \ge 4C\epsilon^{-1}(\log\tau)^{-1}$ which contradicts to Proposition 4.1. Indeed,

$$\epsilon^3 \lesssim \frac{1}{J-J_0}\sum_{j=0}^{J}\left\|\sup_{\tau^k\in\Lambda_j}\left|\mathcal{H}_{\tau^k} f - \mathcal{H}_{N_j} f\right|\right\|_{L^2(\mu)}^2 \le \frac{C_J}{J-J_0}\|f\|_{L^2(\mu)}^2$$

where $J_0 = \min\{j\in\mathbb{N} : k_j \ge 4C\epsilon^{-1}(\log\tau)^{-1}\}$. Now, the standard density argument implies pointwise convergence for each $f\in L^r(\mu)$ where $r > 1$, and the proof of the theorem is completed. □

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Appendix: Boundedness of $\mathcal{M}$

In the Appendix we discuss why the maximal function

$$\mathcal{M} f(n) = \sup_{N\in\mathbb{N}}\left|N^{-1}\sum_{p\in\pm\mathbb{P}_N} f(n-p)\log|p|\right|$$

is bounded on $\ell^r(\mathbb{Z})$. This fact was published by Wierdl in [23], however, on p. 331 in the last equality for ** the factor q has the power 1 in place of p. Therefore, it is not sufficient to show an estimate (24) from [23] to conclude the proof. In fact, one has to prove the estimate corresponding to (25) from the present paper.

For the completeness we provide the sketch of the proof based on the method used in Sect. 3. First, we may restrict supremum to dyadic N. We modify the definition of the multiplier m_j by setting

$$m_j(\xi) = 2^{-j} \sum_{p \in \pm \mathbb{P}_N} e^{2\pi i \xi p} \log |p|.$$

Hence, it suffices to show that for $r > 1$

$$\Big\| \sup_{k \in \mathbb{N}} \big| \mathcal{F}^{-1}\big(m_k \hat{f}\big) \big| \Big\|_{\ell^r} \lesssim \|f\|_{\ell^r}.$$

Keeping the definition of the major arcs and setting

$$\Psi_j(\xi) = 2^{-j} \int_{1 \le |x| \le 2^j} e^{2\pi i \xi x} dx$$

Proposition 3.1 holds true. For proof we use the well-known result that for $\xi \in \mathfrak{M}_j^\alpha(a/q) \cap \mathfrak{M}_j^\alpha$ (see e.g. [11, Lemma 8.3])

$$\left| m_{2^j}(\xi) - 2^{-j} \frac{\mu(q)}{\varphi(q)} \sum_{1 \le |n| \le 2^j} e^{2\pi i \theta n} \right| \lesssim j^{-\alpha}$$

and then, as in the proof of Proposition 3.1, we replace the sum by Ψ_j. Also the demonstration of Proposition 3.2 has to be modified. There, the estimate for $\xi \notin \mathfrak{M}_j^\alpha$ is a direct application of Vinogradov's theorem. In the proof of Proposition 3.3 in the place of (20) we use L^r-boundedness of Hardy–Littlewood maximal function. Finally, in the proof of Theorem 4 we replace the sum $\sum_{j=0}^k m_j$ with a single term m_k.

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