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Integral Representations for the Class of Generalized Metaplectic Operators

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Abstract This article gives explicit integral formulas for the so-called generalized metaplectic operators, i.e. Fourier integral operators of Schrödinger type, having a symplectic matrix as their canonical transformation. These integrals are over specific linear subspaces of $\mathbb{R}^d$, related to the $d \times d$ upper left-hand side submatrix of the underlying $2d \times 2d$ symplectic matrix. The arguments use the integral representations for the classical metaplectic operators obtained by Morsche and Oonincx in a previous paper, algebraic properties of symplectic matrices and time-frequency tools. As an application, we give a specific integral representation for solutions of the Cauchy problem of Schrödinger equations with bounded perturbations for every instant time $t \in \mathbb{R}$, even at the (so-called) caustic points.

Keywords Fourier integral operators · Metaplectic operators · Modulation spaces · Wigner distribution · Short-time Fourier transform · Schrödinger equation

Mathematics Subject Classification 42A38 · 47G30 · 42B10

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1 Introduction

The objective of this study is to find integral representations for generalized metaplectic operators. Starting from the original idea of extending the usual metaplectic representation of the symplectic group using a certain class of Fourier integral operators in Weinstein [29], these operators were introduced in [7] as examples of Wiener algebras of Fourier integral operators of Schrödinger type (cf. [4,9,10,12] and the extensive references therein) having symplectic matrices as canonical transformations. They appear for instance in quantum mechanics, as propagators for solutions to Cauchy problems for Schrödinger equations with bounded perturbations [5,8,11]. In the work [7] generalized metaplectic operators turns out to be the composition of classical metaplectic operators with pseudodifferential operators with symbols in suitable classes of modulation spaces. Classical metaplectic operators, which are unitary operators on $L^2(\mathbb{R}^d)$, arise as intertwining operators for the Schrödinger representation (see the next section for details).

Explicit integral representations for classical metaplectic operators, extending the results already contained in the literature [16,17,19,20,23], were given by Morsche and Oonincx in [25] and applied to energy localization problems and to fractional Fourier transforms in [24], see also [1,13,15,26] and the references therein. The novelty of [25], with respect to the classical works [13,16], is the explicit integral representation of metaplectic operators, covering all possible cases of symplectic matrices. Indeed, the integral representation of metaplectic operators in [13,16] covers only the cases of non-singular upper-left or upper-right component of the parameterizing matrix. This work can be considered as a completion of the study [7], since integral representations of generalized metaplectic operators are given for all possible cases of symplectic matrices parameterizing the phase function.

To make it easier to compare the results obtained in [25] and in this paper we use the same definition of Schrödinger representation and symplectic group given in [25]; these definitions are not the same as in [7,16]: to compare these results with the latter works, a symplectic matrix $\mathcal{A}$ must be replaced with its transpose $\mathcal{A}^T$.

The symplectic group $Sp(d,\mathbb{R})$ is the subgroup of $2d \times 2d$ invertible matrices $GL(2d,\mathbb{R})$, defined by

$$Sp(d,\mathbb{R}) = \left\{ \mathcal{A} \in GL(2d,\mathbb{R}) :\ \mathcal{A} J \mathcal{A}^T = J \right\}, \tag{1}$$

where J is the orthogonal matrix

$$J = \begin{pmatrix} 0_d & I_d \\ -I_d & 0_d \end{pmatrix},$$

(here $I_d, 0_d$ are the $d \times d$ identity matrix and null matrix, respectively). Observe that if $\mathcal{A}$ satisfies (1), then also the transpose $\mathcal{A}^T$ and the inverse $\mathcal{A}^{-1}$ fulfill (1) and so are symplectic matrices as well. Writing $\mathcal{A} \in Sp(d,\mathbb{R})$ in the following $d \times d$ block decomposition:

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2}$$

Morsche and Oonincx in [25, Theorem 1] represented a metaplectic operator by using r-dimensional integrals, were $r = \dim R(B) \in \mathbb{N}$, $0 \le r \le d$, is the range of the $d \times d$ block B. Their result is the starting point for our representation formula for generalized metaplectic operators.

For a phase-space point $z = (x, \xi) \in \mathbb{R}^{2d}$ and a function f defined on $\mathbb{R}^d$, we call a time-frequency shift (or phase-space shift) the operator

$$\pi(z) f(t) = M_\xi T_x f(t) = e^{2\pi i t\cdot\xi} f(t - x),$$

(that is, the composition of the modulation operator M_ξ with the translation T_x). The definition of a generalized metaplectic operator T is based on its kernel decay with respect to the set of phase-space shifts $\pi(z)g$, $z \in \mathbb{R}^{2d}$, for a given window function g in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$. The decay is measured using the smooth polynomial weight $\langle z\rangle = (1 + |z|^2)^{1/2}$, $z \in \mathbb{R}^{2d}$.

Definition 1.1 Consider $\mathcal{A} \in Sp(d, \mathbb{R})$, $g \in \mathcal{S}(\mathbb{R}^d)$ and $s \ge 0$. A linear operator $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is a generalized metaplectic operator (in short, $T \in FIO(\mathcal{A}, s)$) if its kernel satisfies the decay condition

$$|\langle T\pi(z)g, \pi(w)g\rangle| \le C\langle w - \mathcal{A}z\rangle^{-s}, \qquad w, z \in \mathbb{R}^{2d}. \tag{3}$$

The union $\bigcup_{\mathcal{A}\in Sp(d,\mathbb{R})} FIO(\mathcal{A}, s)$ is called the class of generalized metaplectic operators and denoted by $FIO(Sp, s)$. Simple examples of generalized metaplectic operators are provided by the classical metaplectic operators $\mu(\mathcal{A})$, $\mathcal{A} \in Sp(d, \mathbb{R})$, where μ is the metaplectic representation recalled below, which (according to our notation) satisfy $\mu(\mathcal{A}) \in \cap_{s\ge 0} FIO(\mathcal{A}^T, s)$ (cf. [7, Proposition 5.3]). More interesting examples are provided by composing classical metaplectic operators with pseudodifferential operators. A pseudodifferential operator (in the Weyl form) with a symbol σ is formally defined as

$$\sigma^w(x, D) f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi i (x-y)\cdot\xi} \sigma\left(\frac{x + y}{2}, \xi\right) f(y) dy\, d\xi. \tag{4}$$

We focus on symbols in sub-classes of the Sjöstrand class (or modulation space) $M^{\infty,1}(\mathbb{R}^{2d})$. This class is a special case of modulation spaces, introduced and studied by Feichtinger in [14] and later redefined and used to prove the Wiener property for pseudodifferential operators by Sjöstrand in [27,28]. The space $M^{\infty,1}(\mathbb{R}^{2d})$ consists of all continuous functions σ on $\mathbb{R}^{2d}$ whose norm, with respect to a fixed window $g \in \mathcal{S}(\mathbb{R}^{2d})$, satisfies

$$\|\sigma\|_{M^{\infty,1}} = \int_{\mathbb{R}^{2d}} \sup_{z\in\mathbb{R}^{2d}} |\langle \sigma, \pi(z, \zeta)g\rangle|\, d\zeta < \infty. \tag{5}$$

Note that in the space $M^{\infty,1}$ even the differentiability property can be lost. The scale of modulation spaces under our consideration are denoted by $M^{\infty}_{1\otimes v_s}(\mathbb{R}^{2d})$, $s \in \mathbb{R}$. They are Banach spaces of tempered distributions $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ such that their norm

$$\|\sigma\|_{M^\infty_{1\otimes v_s}} = \sup_{z,\zeta\in\mathbb{R}^{2d}} |\langle\sigma, \pi(z,\zeta)g\rangle| v_s(\zeta) < \infty, \tag{6}$$

where $v_s(\zeta) = \langle\zeta\rangle^s$ (it can be shown that their definition does not depend on the choice of the window $g \in \mathcal{S}(\mathbb{R}^{2d})$). For $s > 2d$, they turn out to be spaces of continuous functions contained in the Sjöstrand class $M^{\infty,1}(\mathbb{R}^{2d})$. The regularity of the class $M^\infty_{1\otimes v_s}(\mathbb{R}^{2d})$ increases with the parameter s. In particular, $\bigcap_{s>2d} M^\infty_{1\otimes v_s}(\mathbb{R}^{2d}) = S^0_{0,0}$, the Hörmander's class of smooth functions on $\mathbb{R}^{2d}$ satisfying, for every $\alpha \in \mathbb{N}^{2d}$,

$$|\partial^\alpha_z \sigma(z)| \le C_\alpha, \quad z \in \mathbb{R}^{2d},$$

for a suitable $C_\alpha > 0$.

In the works [5,7] is proved the following characterization for generalized metaplectic operators:

Theorem 1.2 (i) *An operator T is in $FIO(\mathcal{A}, s)$ if and only if there exist symbols σ_1 and $\sigma_2 \in M^\infty_{1\otimes v_s}(\mathbb{R}^{2d})$ such that*

$$T = \sigma_1^w(x,D)\mu(\mathcal{A}) = \mu(\mathcal{A})\sigma_2^w(x,D). \tag{7}$$

(ii) *Let $\mathcal{A} \in Sp(d,\mathbb{R})$ be a symplectic matrix with block decomposition (2) and such that* $\det A \neq 0$. *Define the phase function Φ as*

$$\Phi(x,\xi) = \frac{1}{2}CA^{-1}x\cdot x + A^{-1}x\cdot\xi - \frac{1}{2}A^{-1}B\xi\cdot\xi. \tag{8}$$

Then $T \in FIO(\mathcal{A}, s)$ if and only if T can be written as a type I Fourier integral operator (FIO), that is an operator in the form

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi i\Phi(x,\xi)}\sigma(x,\xi)\hat{f}(\xi)\,d\xi, \tag{9}$$

with symbol $\sigma \in M^\infty_{1\otimes v_s}(\mathbb{R}^{2d})$.

Integral formulas of the type (9) are also called Fresnel's formulas [19]. The main objective of this paper is to find an integral representation of the type (9) also when the block A is singular. The d-dimensional integral in (9) will be split up into two integrals: an r-dimensional integral on the range $R(A)$ of the block A, where $r = \dim R(A)$, the dimension of the linear space $R(A)$, and a $(d-r)$-dimensional integral on the kernel $N(A)$ of the block A (observe that $\dim N(A) = d - r$). Let us denote by $\mathcal{F}_{R(A)}$ the partial Fourier transform of a function $f \in L^1(\mathbb{R}^d)$ with respect to the linear space $R(A)$; that is, for $x = x_1 + x_2$, $\xi = \xi_1 + \xi_2 \in R(A) \oplus N(A^T)$,

$$\mathcal{F}_{R(A)}f(\xi) = \int_{R(A)} e^{-2\pi i x_1\cdot\xi_1} f(x_1 + x_2)\,dx_1 \quad \xi_1 \in R(A). \tag{10}$$

Since the $d \times d$ block $A : R(A^T) \to R(A)$ is an isomorphism, we denote by $A^{inv} : R(A) \to R(A^T)$ the pseudo-inverse of A. We first show this preliminary result for symplectic matrices.

Lemma 1.3 *Consider $\mathcal{A} \in Sp(d, \mathbb{R})$ with the 2×2 block decomposition in (2). Then the $d \times d$ block B is an isomorphism from $N(A)$ onto $N(A^T)$.*

We denote by $B^{inv} : N(A^T) \to N(A)$ the pseudo-inverse of B. Our main result reads as follows.

Theorem 1.4 (Integral Representations for generalized metaplectic operators) *With the notation introduced before, an operator T is in the class $FIO(\mathcal{A}^T, v_s)$ if and only if T admits the following integral representation: for $x = x_1 + x_2 \in R(A^T) \oplus N(A) = \mathbb{R}^d$, $\xi_2 \in N(A)$, $y \in R(A)$,*

$$Tf(x) = \int_{R(A)} \int_{N(A)} e^{\pi i \left(A^{inv} B x_1 \cdot x_1 - B^T D x_2 \cdot x_2 - C A^{inv} y \cdot y) + 2\pi i (x_1 \cdot A^{inv} y + x_2 \cdot \xi_2\right)} \\ \cdot \;\; \sigma\left(x, A^{inv} y + \xi_2\right) \mathcal{F}_{R(A)} f\left(y + (B^{inv})^T \xi_2\right) d\xi_2 \, dy, \tag{11}$$

where the symbol σ is in the class $M^{\infty}_{1 \otimes v_s}(\mathbb{R}^{2d})$.

Observe that, if $y \in R(A)$, then $A^{inv} y \in R(A^T)$ and for any $\xi_2 \in N(A)$, we obtain $\xi = A^{inv} y + \xi_2 \in R(A^T) \oplus N(A) = \mathbb{R}^d$.

When either the block A is the null matrix or A is nonsingular, the previous integral representation reduces to the following cases:

Corollary 1.5 *The integral representation (11) yields the following special cases:* (i) *If* $\dim\ R(A) = 0$ *(i.e. $A = 0_d$), then the operator $T \in FIO(\mathcal{A}^T, v_s)$ if and only if*

$$Tf(x) = \int_{\mathbb{R}^d} e^{-\pi i B^T D x \cdot x + 2\pi i B x \cdot t} \tilde{\sigma}_1(x, t) \, f(t) \, dt, \tag{12}$$

for a suitable symbol $\tilde{\sigma}_1 \in M^{\infty}_{1 \otimes v_s}(\mathbb{R}^{2d})$. (ii) *If* $\dim\ R(A) = d$, *then the operator $T \in FIO(\mathcal{A}^T, v_s)$ if and only if*

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi i \Phi_T(x,\xi)} \tilde{\sigma}_2(x, \xi) \hat{f}(\xi) \, d\xi \tag{13}$$

for a suitable symbol $\tilde{\sigma}_2 \in M^{\infty}_{1 \otimes v_s}(\mathbb{R}^{2d})$ and where the phase function

$$\Phi_T(x, \xi) = \frac{1}{2} A^{-1} B x \cdot x + A^{-T} x \cdot \xi - \frac{1}{2} C A^{-1} \xi \cdot \xi \tag{14}$$

is the generating function of the canonical transformation $\mathcal{A}^T$ (i.e., the integral representation of T in (9)).

Applications to the previous formulae can be found in quantum mechanics. The solutions to Cauchy problems for Schrödinger equations with bounded perturbations, provided by pseudodifferential operators $\sigma^w(x, D)$ having symbols σ in the classes $M^{\infty}_{1 \otimes v_s}(\mathbb{R}^{2d})$, are generalized metaplectic operators applied to the initial datum (cf. [5], see also [8]). So, formula (11) can be applied to find an integral representation of such operators. As simple example, one can consider the following Cauchy problem for the anisotropic perturbed harmonic oscillator in dimension $d = 2$ (see Sect. 4 below). For $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}$, $t \in \mathbb{R}$, we study

$$\begin{cases} i\partial_t u = Hu, \\ u(0, x) = u_0(x), \end{cases} \tag{15}$$

where

$$H = -\frac{1}{4\pi}\partial_{x_2}^2 + \pi x_2^2 + V(x_1, x_2), \tag{16}$$

with $V \in M^{\infty}_{1\otimes v_s}(\mathbb{R}^2)$, $s > 4$. The initial datum u_0 is in $\mathcal{S}(\mathbb{R}^2)$ or in a suitable rougher modulation space, cf. Sect. 4. The solution $u(t, x) = e^{-itH}u_0$, has the propagator e^{-itH} which turns out to be a one-parameter family of generalized metaplectic operators $FIO(\mathcal{A}_t, s)$, related to the symplectic matrices

$$\mathcal{A}_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos t & 0 & \sin t \\ 0 & 0 & 1 & 0 \\ 0 & -\sin t & 0 & \cos t \end{pmatrix} \quad t \in \mathbb{R}. \tag{17}$$

For $t \in \mathbb{R}$, the 2×2 block A_t is given by

$$A_t = \begin{pmatrix} 1 & 0 \\ 0 & \cos t \end{pmatrix}. \tag{18}$$

Observe that $\det A_t = \cos t$ so that A_t is a singular matrix whenever $t = \pi/2 + k\pi$, $k \in \mathbb{Z}$, the so-called *caustics* of the solution. In this case, using formula (11), we are able to give an integral representation as well.

To compare with other results in the literature, we recall [20, Sects. 6-7, Chapter 7], which provides an overview of the classical results on caustics in the context of spectral asymptotics. The works [21,30] are relevant recent references on Fourier integral operators and their applications, from the point of view of the semiclassical limit, i.e. the limit with the Planck constant $\hbar$ tending to 0. The book by Zworski [30] (Chapters 10 and 11 are the most relevant in the context of the current manuscript) nicely complements the book by Folland [16]. It presents a current view of the topic with the orientation towards partial differential equations. The book [21] addresses directly many issues studied in the current manuscript, in the framework of semiclassical analysis. They study local representations of differential operators, even at caustics, and apply their representations to global asymptotic solutions of hyperbolic equations. We refer to [21, Chapters 4, 5, 8] for the most relevant results.

2 Preliminaries and Notation

Here and in the sequel, for $x, y \in \mathbb{R}^m$, $x \cdot y$ denotes the inner product in $\mathbb{R}^m$. As recalled above, given a matrix A, we call A^T the transpose of A and denote by $R(A)$ and $N(A)$ the range and the kernel of the matrix A, respectively.

Given $\mathcal{A} \in Sp(d, \mathbb{R})$ with the 2×2 block decomposition (2), from (1) it follows that the four blocks must satisfy the following properties:

$$D^T A - B^T C = I_d \tag{19}$$
$$A^T C - C^T A = 0_d \tag{20}$$
$$D^T B - B^T D = 0_d. \tag{21}$$

Moreover, since also

$$\mathcal{A}^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix},$$

is a symplectic matrix, relations (20) and (19) for $\mathcal{A}^{-1}$ give

$$CA^{-1} - A^{-T} C^T = 0_d \tag{22}$$
$$-AB^T + BA^T = 0_d. \tag{23}$$

The metaplectic representation μ of (the two-sheeted cover of) the symplectic group arises as intertwining operator between the Schrödinger representation ρ of the Heisenberg group $\mathbb{H}^d$ and the representation that is obtained from it by composing ρ with the action of $Sp(d, \mathbb{R})$ by automorphisms on $\mathbb{H}^d$. Namely, the Heisenberg group $\mathbb{H}^d$ is the group obtained by defining on $\mathbb{R}^{2d+1}$ the product law

$$(z, t) \cdot (z', t') = \left(z + z', t + t' + \frac{1}{2}\omega(z, z')\right), \quad z, z' \in \mathbb{R}^{2d},\ t,\ t' \in \mathbb{R},$$

where ω is the symplectic form

$$\omega(z, z') = z \cdot Jz', \qquad z, z' \in \mathbb{R}^{2d}.$$

The Schrödinger representation of the group $\mathbb{H}^d$ on $L^2(\mathbb{R}^d)$ is then defined by

$$\rho(p, q, t) f(x) = e^{2\pi it} e^{\pi i p \cdot q} e^{2\pi i p \cdot x} f(x + q), \quad x, q, p \in \mathbb{R}^d,\ t \in \mathbb{R}.$$

The symplectic group acts on $\mathbb{H}^d$ via automorphisms that leave the center $\{(0, t) : t \in \mathbb{R}\} \in \mathbb{H}^d \simeq \mathbb{R}$ of $\mathbb{H}^d$ pointwise fixed:

$$A \cdot (z, t) = (Az, t)\,.$$

Therefore, for any fixed $\mathcal{A} \in Sp(d, \mathbb{R})$ there is a representation

$$\rho_{\mathcal{A}^T} : \mathbb{H}^d \to \mathcal{U}(L^2(\mathbb{R}^d)), \qquad (z, t) \mapsto \rho\left(\mathcal{A}^T \cdot (z, t)\right)$$

whose restriction to the center is a multiple of the identity. By the Stone-von Neumann theorem, $\rho_{\mathcal{A}^T}$ is equivalent to ρ. So, there exists an intertwining unitary operator $\mu(\mathcal{A}) \in \mathcal{U}(L^2(\mathbb{R}^d))$ such that

$$\rho_{\mathcal{A}^T}(z,t) = \mu(\mathcal{A}) \circ \rho(z,t) \circ \mu(\mathcal{A})^{-1} \quad (z,t) \in \mathbb{H}^d. \tag{24}$$

By Schur's lemma, μ is determined up to a phase factor e^{is}, $s \in \mathbb{R}$. Actually, the phase ambiguity is only a sign, so that μ lifts to a representation of the (double cover of the) symplectic group.

An alternative definition of a metaplectic operator (cf. [16,23,25]), up to a constant c, with $|c| = 1$, involves a time-frequency representation, the so-called Wigner distribution W_f of a function $f \in L^2(\mathbb{R}^d)$, given by

$$W_f(x,\xi) = \int e^{-2\pi i y\cdot\xi} f\left(x + \frac{y}{2}\right) \overline{f\left(x - \frac{y}{2}\right)}\, dy. \tag{25}$$

The crucial property of the Wigner distribution W is that it intertwines $\mu(\mathcal{A})$ and the affine action on $\mathbb{R}^{2d}$:

$$W_{\mu(\mathcal{A})f} = W_f \circ \mathcal{A}, \quad \mathcal{A} \in Sp(d,\mathbb{R}). \tag{26}$$

Since $W_g = W_f$ if and only if there exists a constant $c \in \mathbb{C}$, with $|c| = 1$, such that $g = cf$, it is clear that, up to a constant c with $|c| = 1$, a metaplectic operator can be defined by the intertwining relation (26).

Morsche and Oonincx in [25] use the relation (26) to obtain an integral representation (up to a constant $c \in \mathbb{C}$, with $|c| = 1$) of every metaplectic operator $\mu(\mathcal{A})$, $\mathcal{A} \in Sp(d,\mathbb{R})$, extending the preceding results for special symplectic matrices contained in the pioneering work of Frederix [17], in Folland's book [16] and in Kaiblinger's thesis [23] (see also [13,18,22] and references therein).

To state the integral representation for metaplectic operators contained in [25], we need to introduce some preliminaries (cf. [2,3,25]). For a $d \times d$ matrix A and a linear subspace L of $\mathbb{R}^d$ with $\dim L = r$, $q_L(A)$ denotes the r-dimensional volume of the parallelepiped

$$X = \left\{x \in \mathbb{R}^d \,:\, x = \xi_1 A e_1 + \cdots + \xi_r A e_r,\ 0 \le \xi_i \le 1,\ i = 1,\ldots,r\right\}$$

spanned by the vectors $Ae_1, \ldots, Ae_r$, where $e_1, \ldots, e_r$ is any orthonormal basis of L. If $\dim A(L) = \dim L = r$, then the r-dimensional volume of X is positive, otherwise this volume is zero. The number $q_L(A)$ can be interpreted as a matrix volume as follows. We collect the vectors $e_1, \ldots, e_r$ as columns into the $d \times r$ matrix $E = [e_1, \ldots, e_r]$. Assuming $\dim A(L) = \dim L = r$, the matrix AE has full column rank and

$$q_L(A) = \operatorname{vol} AE = \sqrt{\det(E^T A^T A E)}.$$

If $L = \mathbb{R}^d$ and A is nonsingular, then $q_L(A) = |\det A|$.

The definition of $q_L(A)$ is extended to the following cases: we set $q_L(A) = 1$ either when L is the null space and A is nonsingular or A is the null matrix and $\dim L > 0$.

The number $q_L(A)$ appears in the change-of-variables formulas for more dimensional integral as follows.

Lemma 2.1 *Under the assumptions above, if* $\dim\ A(L) = \dim\ L$ *we have*

$$\int_L \varphi(Ax)\,dx = \frac{1}{q_L(A)} \int_{A(L)} \varphi(x)\,dx, \tag{27}$$

for every function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ *or, more generally, any function* φ *for which the above integrals exist.*

Corollary 2.2 *Under the assumptions of Lemma 2.1, for any* $y \in A(L)$*, we have*

$$\int_L \varphi(Ax + y)\,dx = \frac{1}{q_L(A)} \int_{A(L)} \varphi(x)\,dx. \tag{28}$$

Proof It is an immediate consequence of Lemma 2.1, since by assumption $\dim\ A(L) = \dim\ L$ so that A is a an isomorphism from L onto $A(L)$. □

We associate to a symplectic matrix $\mathcal{A}$ with block decomposition (2) a constant

$$c(\mathcal{A}) = \sqrt{\frac{s(A)}{q_{N(A)}(C)}}, \tag{29}$$

where $s(A)$ denotes the product of the nonzero singular values of the $d \times d$ block A, or equivalently

$$s(A) = q_{R(A^T)}(A). \tag{30}$$

The integral representation of a metaplectic operator proved in [25, Theorem 1] and applied to the matrix

$$\mathcal{B} = \mathcal{A}J = \begin{pmatrix} -B & A \\ -D & C \end{pmatrix}$$

gives the following integral representation.

Theorem 2.3 *Consider* $\mathcal{A} \in Sp(d, \mathbb{R})$ *with the* 2×2 *block decomposition in (2) and set* $r = \dim\ R(A)$*. Then, for* $f \in \mathcal{S}(\mathbb{R}^d)$*, the metaplectic operator* $\mu(\mathcal{A})$*, up to a constant* $c \in \mathbb{C}$*, with* $|c| = 1$*, can be represented as follows:*

(i) If $r > 0$ *then*

$$\mu(\mathcal{A})f(x) = c(\mathcal{A}) \int_{R(A^T)} e^{-\pi i B^T Dx\cdot x - \pi i A^T Ct\cdot t + 2\pi i A^T Dx\cdot t} \hat{f}(At - Bx)\,dt. \tag{31}$$

(ii) If $r = 0$ *then*

$$\mu(\mathcal{A})f(x) = \sqrt{|\det B|} \int_{\mathbb{R}^d} e^{-\pi i B^T Dx\cdot x + 2\pi i Bx\cdot t} f(t)\,dt. \tag{32}$$

In the sequel the integral representations of metaplectic operators will be always meant "up to a constant" $c \in \mathbb{C}$, with $|c| = 1$.

Corollary 2.4 *Under the assumptions of Proposition 2.3, if $R(A) = d$, that is the block A is nonsingular, then*

$$\mu(\mathcal{A}) f(x) = |\det A|^{-1/2} \int_{\mathbb{R}^d} e^{2\pi i \Phi_T(x,\xi)} \hat{f}(\xi)\, d\xi, \tag{33}$$

where the phase function Φ_T is defined in (14). (Observe that $A^{-1}B$ and CA^{-1} are symmetric matrices by (23) and (22) respectively).

Proof Since A is nonsingular, $N(A) = 0$ and $R(A^T) = \mathbb{R}^d$ so that $c(\mathcal{A}) = \sqrt{|\det A|}$. We make the change of variables $At - Bx = \xi$ in the integrals in (31) so that $dx = |\det A|^{-1} d\xi$. Making straightforward computations and using the following properties: the matrix CA^{-1} is symmetric by relation (22) and $D - CA^{-1}B = A^{-T}$ by (19), the result immediately follows. □

Remark 2.5 (i) If $\Phi_T(x, \xi)$ is as in (14), we have

$$\nabla_x \Phi_T(x,\xi) = A^{-1}Bx + A^{-1}\xi, \quad \nabla_\xi \Phi_T(x,\xi) = A^{-T}x - CA^{-1}\eta$$

and using $D^T = B^T A^{-T} C^T + A^{-1}$ (by relation (19)) and $A^{-1}B = B^T A^{-T}$ (by relation (23)), we obtain

$$\begin{pmatrix} x \\ \nabla_x \Phi_T \end{pmatrix} = \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \begin{pmatrix} \nabla_\xi \Phi_T \\ \xi \end{pmatrix} = \mathcal{A}^T \begin{pmatrix} \nabla_\xi \Phi_T \\ \xi \end{pmatrix},$$

that is the function Φ_T is the generating phase function of the canonical transformation $\mathcal{A}^T$. Indeed, the phase function Φ_T in (14) coincides with the generating phase function Φ in (8) when $\mathcal{A}$ is replaced by $\mathcal{A}^T$. The fact we obtain $\mathcal{A}^T$ instead of $\mathcal{A}$ depends on our definition of the Schrödinger representation, with follows the one in [25]. Hence, under our notations, $\mu(\mathcal{A}) \in FIO(\mathcal{A}^T, v_s)$, for every $s \geq 0$. Observe that, up to a constant, this is also the integral representation of Theorem (4.51) in [16].

(ii) If $0 < \dim R(A) = r < d$, then the integral representation in (31) can be interpreted as a degenerate form of a type I generalized metaplectic operator in $FIO(\mathcal{A}^T, v_s)$, with constant symbol $\sigma = |\det A|^{-1/2}$.

(iii) If $\dim R(A) = 0$, then

$$\mathcal{A} = \begin{pmatrix} 0_d & B \\ B^{-T} & D \end{pmatrix} \tag{34}$$

and the integral representation in (32) is, up to a constant factor, the one of Theorem (4.53) in [16] (with $\mathcal{A}$ replaced by $\mathcal{A}^T$, so that the block B is replaced by B^{-1} in formula (4.54) of [16]).

We recall the integral representation of Theorem 2.3 for elements of $Sp(d, \mathbb{R})$ in special form, which we shall use in the sequel. For $f \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\mu\left(\begin{pmatrix} A & 0_d \\ 0_d & A^{-T} \end{pmatrix}\right) f(x) = \sqrt{|\det A|} f(Ax) \tag{35}$$

$$\mu\left(\begin{pmatrix} I_d & 0_d \\ C & I_d \end{pmatrix}\right) f(x) = e^{-\pi i Cx\cdot x} f(x) \tag{36}$$

$$\mu(J) = \mathcal{F}^{-1}, \tag{37}$$

where $\mathcal{F}$ denotes the Fourier transform

$$\mathcal{F} f(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi}\, dx, \qquad f \in L^1\left(\mathbb{R}^d\right).$$

2.1 Time-Frequency Methods

We recall here the time-frequency tools we shall use to prove the integral representation for generalized metaplectic operators. The polarized version of the Wigner distribution in (25), is the so called cross-Wigner distribution $W_{f,g}$, given by

$$W_{f,g}(x,\xi) = \int e^{-2\pi i y\cdot\xi} f\left(x + \frac{y}{2}\right) \overline{g\left(x - \frac{y}{2}\right)}\, dy, \quad f, g \in L^2(\mathbb{R}^d). \tag{38}$$

A pseudodifferential operator in the Weyl form (4) with symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ can be also defined by

$$\langle \sigma^w(x, D) f, g\rangle = \langle \sigma, W(g, f)\rangle \quad f, g \in \mathcal{S}(\mathbb{R}^d), \tag{39}$$

where the brackets $\langle\cdot,\cdot\rangle$ denote the extension to $\mathcal{S}' \times \mathcal{S}$ of the inner product $\langle f, g\rangle = \int f(t)\overline{g(t)}dt$ on L^2. Observe that by the intertwining relation (26) and the definition of Weyl operator (39), it follows the property

$$\sigma^w(x, D)\mu(\mathcal{A}) = \mu(\mathcal{A})(\sigma \circ \mathcal{A}^{-1})^w(x, D). \tag{40}$$

2.1.1 Weighted Modulation Spaces

We shall recall the definition of modulation spaces related to the weight functions

$$v_s(z) = \langle z\rangle^s = \left(1 + |z|^2\right)^{\frac{s}{2}}, \quad s \in \mathbb{R}. \tag{41}$$

Observe that for $\mathcal{A} \in Sp(d, \mathbb{R})$, $|\mathcal{A}z|$ defines an equivalent norm on $\mathbb{R}^{2d}$, hence for every $s \in \mathbb{R}$, there exist $C_1, C_2 > 0$ such that

$$C_1 v_s(z) \le v_s(\mathcal{A}z) \le C_2 v_s(z), \quad \forall z \in \mathbb{R}^{2d}. \tag{42}$$

The time-frequency representation which occurs in the definition of modulation spaces is the short-time Fourier Transform (STFT) of a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to a function $g \in \mathcal{S}(\mathbb{R}^d)$ (so-called window), given by

$$V_g f(z) = \langle f, \pi(z)g \rangle, \quad z = (x, \xi) \in \mathbb{R}^{2d}.$$

The short-time Fourier transform is well-defined whenever the bracket $\langle \cdot, \cdot \rangle$ makes sense for dual pairs of function or distribution spaces, in particular for $f \in \mathcal{S}'(\mathbb{R}^d)$, $g \in \mathcal{S}(\mathbb{R}^d)$, or for $f, g \in L^2(\mathbb{R}^d)$.

Definition 2.6 Given $g \in \mathcal{S}(\mathbb{R}^d)$, $s \geq 0$, and $1 \leq p, q \leq \infty$, the *modulation space* $M^{p,q}_{1\otimes v_s}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_g f \in L^{p,q}_{1\otimes v_s}(\mathbb{R}^{2d})$ (weighted mixed-norm spaces). The norm on $M^{p,q}_{1\otimes v_s}(\mathbb{R}^d)$ is

$$\|f\|_{M^{p,q}_{1\otimes v_s}} = \|V_g f\|_{L^{p,q}_{1\otimes v_s}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, \xi)|^p dx \right)^{q/p} v_s(\xi)^q d\xi \right)^{1/q} \tag{43}$$

(with obvious modifications for $p = \infty$ or $q = \infty$).

When $p = q$, we write $M^p_{1\otimes v_s}(\mathbb{R}^d)$ instead of $M^{p,p}_{1\otimes v_s}(\mathbb{R}^d)$; when $s = 0$ (unweighted case) we simply write $M^{p,q}(\mathbb{R}^d)$ instead of $M^{p,q}_{1\otimes 1}(\mathbb{R}^d)$. The spaces $M^{p,q}_{1\otimes v_s}(\mathbb{R}^d)$ are Banach spaces, and every nonzero $g \in M^1_{1\otimes v_s}(\mathbb{R}^d)$ yields an equivalent norm in (43), so that their definition is independent of the choice of $g \in M^1_{1\otimes v_s}(\mathbb{R}^d)$. We shall use modulation spaces as symbol spaces, so the dimension of the spaces will be $\mathbb{R}^{2d}$ instead of $\mathbb{R}^d$. Moreover, in our setting $p = q = \infty$ (similar results occur for symbols in the weighted Sjöstrand classes $M^{\infty,1}_{1\otimes v_s}(\mathbb{R}^{2d})$, $s \geq 0$).

The modulation spaces $M^\infty_{1\otimes v_s}(\mathbb{R}^d)$ are invariant under linear and, in particular, symplectic transformations. This property is crucial to infer our main result and is proved in [5, Lemma 2.2] (see also [8, Lemma 2.2]) for the case of symplectic transformations. The proof for linear transformations goes exactly in the same way, just by adding $|\det \mathcal{A}|$ in formula (44), which is a consequence of a change of variables (observe that $|\det \mathcal{A}| = 1$ if $\mathcal{A}$ is a symplectic matrix). We denote by $GL(2d, \mathbb{R})$ the class of $2d \times 2d$ invertible matrices. Then we can state:

Lemma 2.7 *If $\sigma \in M^\infty_{1\otimes v_s}(\mathbb{R}^{2d})$ and $\mathcal{A} \in GL(2d, \mathbb{R})$, then $\sigma \circ \mathcal{A} \in M^\infty_{1\otimes v_s}(\mathbb{R}^{2d})$ and*

$$\|\sigma \circ \mathcal{A}^{-1}\|_{M^\infty_{1\otimes v_s}} \leq |\det \mathcal{A}| \, \|\left(\mathcal{A}^T\right)^{-1}\|^s \, \|V_{\Phi\circ\mathcal{A}}\Phi\|_{L^1_{v_s}} \|\sigma\|_{M^\infty_{1\otimes v_s}}, \tag{44}$$

where $\Phi \in \mathcal{S}(\mathbb{R}^{2d})$ is the window used to compute the norms of σ and $\sigma \circ \mathcal{A}^{-1}$.

In the sequel it will be useful to pass from the Weyl to the Kohn–Nirenberg form of a pseudodifferential operator. The latter form can be formally defined by

$$\sigma(x, D) f(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \sigma(x, \xi) \hat{f}(\xi) \, d\xi,$$

for a suitable symbol σ on $\mathbb{R}^{2d}$. The previous correspondences are related by $\sigma^w(x, D) = (\mathcal{U}\sigma)(x, D)$, where

$$\widehat{\mathcal{U}\sigma}(\eta_1, \eta_2) = e^{\pi i \eta_1\cdot\eta_2} \widehat{\sigma}(\eta_1, \eta_2) \tag{45}$$

(see, e.g., [18, formula 14.17]). The classes $M^{\infty}_{1\otimes v_s}(\mathbb{R}^{2d})$ are invariant under the action of the unitary operator $\mathcal{U}$, as shown below.

Lemma 2.8 *If* $\sigma \in M^{\infty}_{1\otimes v_s}(\mathbb{R}^{2d})$ *then* $\mathcal{U}\sigma \in M^{\infty}_{1\otimes v_s}(\mathbb{R}^{2d})$ *with*

$$\|\mathcal{U}\sigma\|_{M^{\infty}_{1\otimes v_s}} \le C\|\sigma\|_{M^{\infty}_{1\otimes v_s}}.$$

Proof Observe that, up to a constant c with

$$\mathcal{U}\sigma(z) = \mathcal{F}^{-1}e^{\pi i z\cdot Cz}\mathcal{F}\sigma = \mu\big(J\left(I_{2d}\ 0_{2d} - C\ I_{2d}\right)J^T\big)\sigma = \mu(\mathcal{D})\sigma$$

where $C = \begin{pmatrix} 0_d & 1/2\, I_d \\ 1/2\, I_d & 0_d \end{pmatrix}$ and $\mathcal{D} = \begin{pmatrix} I_{2d} & C \\ 0_{2d} & I_{2d} \end{pmatrix} \in Sp(2d,\mathbb{R})$. Consider now a window function $\Phi \in \mathcal{S}(\mathbb{R}^{2d})$. A straightforward computation shows

$$V_{\mu(\mathcal{D})\Phi}(\mu(\mathcal{D}))\sigma(z,\zeta) = V_\Phi f(\mathcal{D}^{-T}(z,\zeta)) = V_\Phi f(z - C\zeta, \zeta).$$

Since $\mu(\mathcal{D})\Phi \in \mathcal{S}(\mathbb{R}^{2d})$ and different window functions yield equivalent norms, we obtain

$$\begin{aligned}\|\mathcal{U}\sigma\|_{M^{\infty}_{1\otimes v_s}} &\le C\|V_{\mu(\mathcal{D})\Phi}\mu(\mathcal{D})\sigma\|_{L^{\infty}_{1\otimes v_s}} \\ &= \sup_{z,\zeta\in\mathbb{R}^{2d}} |V_\Phi f(z - C\zeta,\zeta)|v_s(\zeta) \\ &= \|V_\Phi\sigma\|_{L^{\infty}_{1\otimes v_s}} = \|\sigma\|_{M^{\infty}_{1\otimes v_s}}\end{aligned}$$

as desired. □

3 Integral Representations of Generalized Metaplectic Operators

The aim of this section is to give integral representations for generalized metaplectic operators $T \in FIO(\mathcal{A}, v_s)$, extending the integral representations (9) in Theorem 1.2, valid only in the special case $\det A \neq 0$. To obtain integral representations for generalized metaplectic operators $T \in FIO(\mathcal{A}^T, v_s)$, we use the characterization of generalized metaplectic operators of Theorem 1.2 and we write $T = \sigma^w(x,D)\mu(\mathcal{A})$ where $\sigma^w(x,D)$ is a Weyl operator with symbol $\sigma \in M^{\infty}_{1\otimes v_s}(\mathbb{R}^{2d})$. Then we study the composition of a pseudodifferential operator in the Weyl form with a metaplectic operator whose integral representation is given by Theorem 2.3. Define for a $d \times d$ matrix A the pre-image of a linear subspace L of $\mathbb{R}^d$:

$$\overleftarrow{A}(L) = \big\{x \in \mathbb{R}^d\ :\ Ax \in L\big\}. \tag{46}$$

The following property will be useful to study the previous composition.

Proposition 3.1 *Assume $\mathcal{A} \in Sp(d, \mathbb{R})$ admits the block decomposition (2). Then*

$$\overleftarrow{C^T}\left(R(A^T)\right) = R(A) \tag{47}$$

$$\dim\ C(N(A)) = \dim\ N(A) \tag{48}$$

$$\overleftarrow{B}\,(R(A)) = R\left(A^T\right) \tag{49}$$

$$B^T(N(A^T)) = N(A). \tag{50}$$

Proof Since the matrix $\mathcal{B} = \mathcal{A}J \in Sp(d, \mathbb{R})$, its block decomposition satisfies [25, Property 1] which gives relations (47) and (48). Analogously, the matrix

$$\mathcal{B}^{-1} = \begin{pmatrix} C^T & -A^T \\ -D^T & -B^T \end{pmatrix} \in Sp(d, \mathbb{R})$$

satisfies [25, Property 1], so that the other relations are fulfilled. □

We are now in position to prove Lemma 1.3.

Proof of Lemma 1.3 Observe that by relation (49), $B : N(A) \to N(A^T)$. By (19), for every $x \in N(A)$ it follows $-B^t C x = x$, hence $N(A) \subset R(B^T) = N(B)^\perp$. This gives $N(A) \cap N(B) = \{0\}$, so B is an injective mapping and $\dim\ N(A) \leq \dim\ N(A^T)$. Repeating the same argument for the symplectic matrix

$$\mathcal{A}^T = \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}, \tag{51}$$

we obtain $\dim\ N(A^T) \leq \dim\ N(A)$, hence $\dim\ N(A) = \dim\ N(A^T)$, i.e., B is onto and its pseudo-inverse $B^{inv} : N(A^T) \to N(A)$ is well-defined. □

Assume that the matrix $\mathcal{A} \in Sp(d, \mathbb{R})$ admits the block decomposition (2) with $\dim\ R(A) > 0$. We first work on the integral representation of $\mu(\mathcal{A})$ in (31).

Theorem 3.2 *Consider $\mathcal{A} \in Sp(d, \mathbb{R})$ with the 2×2 block decomposition in (2) and assume $\dim\ R(A) = r > 0$. For $f \in \mathcal{S}(\mathbb{R}^d)$ and*

$$x = x_1 + x_2 \in R(A^T) \oplus N(A) = \mathbb{R}^d$$

we have the following integral representation

$$\mu(\mathcal{A}) f(x) = c_1(\mathcal{A}) \int_{R(A)} e^{\pi i\left[(A^{inv})^T B x_1 \cdot x_1 - D^T B x_2 \cdot x_2\right] - C A^{inv} y \cdot y + 2\pi i A^{inv} y \cdot x_1} \times \hat{f}(y - Bx_2)\, dy \tag{52}$$

where

$$c_1(\mathcal{A}) = \frac{1}{\sqrt{s(A) q_{N(A)}(C)}}. \tag{53}$$

Proof Since dim $R(A) = r > 0$, the integral representation of $\mu(\mathcal{A})$ is given by (31). We set

$$Q(x) := \int_{R(A^T)} e^{-\pi i A^T Ct\cdot t + 2\pi i A^T Dx\cdot t} \hat{f}(At - Bx)\, dt.$$

We write $x = x_1 + x_2$, with $x_1 \in R(A^T)$ and $x_2 \in N(A)$. By relation (49) we obtain $Bx_1 \in R(A)$. Making the change of variables $y = At - Bx_1$ and applying Corollary 2.2 the integral $Q(x)$ becomes

$$Q(x) = \frac{1}{q_{R(A^T)}(A)} \int_{R(A)} e^{-\pi i C\left(A^{inv}y + A^{inv}Bx_1\right)\cdot(y+Bx_1) + 2\pi i D(x_1+x_2)\cdot(y+Bx_1)} \cdot \hat{f}(y - Bx_2)\, dy,$$

where $q_{R(A^T)}(A) = s(A)$. By the equality (19) we obtain $CA^{inv}Bx_1 - Dx_1 = -(A^{inv})^T x_1$ and relation (22) yields $(CA^{inv})^T = CA^{inv}$, so that we can write

$$\mu(\mathcal{A}) f(x_1 + x_2) = \frac{c(\mathcal{A})}{s(A)} e^{\pi i \left(A^{inv}\right)^T x_1\cdot Bx_1 + \pi i \left(Dx_2\cdot Bx_1 - Dx_1\cdot Bx_2 - Dx_2\cdot Bx_2\right)} \cdot \int_{R(A)} e^{-\pi i CA^{inv}y\cdot y + 2\pi i \left(A^{inv}\right)^T x_1\cdot y + Dx_2\cdot y)} \hat{f}(y - Bx_2)\, dy. \tag{54}$$

Observe that

$$\frac{c(\mathcal{A})}{s(A)} = \sqrt{\frac{s(A)}{q_{N(A)}(C)}} \frac{1}{s(A)} = \frac{1}{\sqrt{s(A) q_{N(A)}(C)}},$$

which is (53). Now, we shall prove that the $d \times d$ block D satisfies

$$D : N(A) \to N\left(A^T\right). \tag{55}$$

First, by Lemma 1.3, $B : N(A) \to N(A^T)$ whereas by (47) it follows $C^T : N(A^T) \to N(A)$. Hence, using (19), for $x_2 \in N(A)$, we obtain

$$A^T Dx_2 = C^T Bx_2 + x_2 \in N(A).$$

Now A^T maps $R(A)$ onto $R(A^T)$ bijectively, this implies $Dx_2 \in R(A)^\perp = N(A^T)$ and (55) is proved. Relation (55) yields $Dx_2 \cdot y = 0$ for $y \in R(A)$ and $Dx_2 \cdot Bx_1 = 0$ since $Bx_1 \in R(A)$ whenever $x_1 \in R(A^T)$. Moreover $C^T Bx_1 \in R(A^T)$, by relation (47), so that

$$A^T D = C^T B + I_d : R\left(A^T\right) \to R\left(A^T\right).$$

This gives $Dx_1 \in R(A)$ whenever $x_1 \in R(A^T)$, and $Dx_1 \cdot Bx_2 = 0$, for $Bx_2 \in N(A^T) = R(A)^\perp$ by relation (49). These observations allow to simplify the expression of $\mu(\mathcal{A}) f(x_1 + x_2)$ in (54) and give the representation (52), as desired. □

Remark 3.3 If dim $R(A) = d$, that is A is nonsingular, then $N(A) = \{0\}$, $R(A) = \mathbb{R}^d$, $x_2 = 0$, $x = x_1$, $s(A) = |\det A|$, $q_{N(A)}(C) = 1$ so that $c_1(\mathcal{A}) = |\det A|^{-1/2}$. Hence the integral representation (52) coincides with (33), as expected.

We now possess all the instruments to prove our main result.

Proof of Theorem 1.4 By Theorem 1.2, a linear operator T belongs to the class $FIO(\mathcal{A}^T, v_s)$ if and only if there exists a symbol $\sigma_1 \in M^\infty_{1\otimes v_s}$ such that $T = \sigma_1^w(x, D)\mu(\mathcal{A})$. Consider $\mathcal{A}$ with the block decomposition (2). Observe that the symbols involved in the sequel are the results of compositions of symbols in $M^\infty_{1\otimes v_s}(\mathbb{R}^{2d})$ with suitable symplectic transformations, so that by Lemma 2.7 they all belong to the same class $M^\infty_{1\otimes v_s}(\mathbb{R}^{2d})$. First, assume $0 < r = \dim R(A)$. We shall prove that the composition $T = \sigma_1^w(x, D)\mu(\mathcal{A})$ admits the integral representation in (11). We use the integral representation of the metaplectic operator $\mu(\mathcal{A})$ in (52). Setting

$$Pf(x_1 + x_2) := \int_{R(A)} e^{-\pi i C A^{inv} y\cdot y + 2\pi i A^{inv} y \cdot x_1} \hat{f}(y - Bx_2)\, dy. \tag{56}$$

we will show that

$$\sigma_1^w(x, D)\mu(\mathcal{A}) f(x_1 + x_2) = e^{\pi i \left[A^{inv} Bx_1 x_1\cdot x_1 - B^T Dx_2\cdot x_2\right]} \sigma_2^w(x, D) Pf(x_1 + x_2), \tag{57}$$

where, for $x = x_1 + x_2$, $\xi = \xi_1 + \xi_2 \in R(A^T) \oplus N(A)$, we define

$$\sigma_2(x_1 + x_2, \xi_1 + \xi_2) = c_1(\mathcal{A})\sigma_1\big(x_1 + x_2, A^{inv} Bx_1 + \xi_1 - D^T Bx_2 + \xi_2\big). \tag{58}$$

Indeed, define on $\mathbb{R}^d = R(A^T) \oplus N(A)$ the symplectic matrix $\mathcal{C} \in Sp(d, \mathbb{R})$ as follows

$$\mathcal{C} = \begin{pmatrix} I_r & 0_{d-r} & 0_r & 0_{d-r} \\ 0_r & I_{d-r} & 0_r & 0_{d-r} \\ -A^{inv} B & 0_{d-r} & I_r & 0_{d-r} \\ 0_r & D^T B & 0_r & I_{d-r} \end{pmatrix}$$

(observe that the $d \times d$ block $\begin{pmatrix} -A^{inv} B & 0_{d-r} \\ 0_r, & D^T B \end{pmatrix}$ is a symmetric matrix, by relations (23) and (19)). The inverse of $\mathcal{C}$ is

$$\mathcal{C}^{-1} = \begin{pmatrix} I_r & 0_{d-r} & 0_r & 0_{d-r} \\ 0_r & I_{d-r} & 0_r & 0_{d-r} \\ A^{inv} B & 0_{d-r} & I_r & 0_{d-r} \\ 0_r & -D^T B & 0_r & I_{d-r} \end{pmatrix}.$$

We have that $\mu(\mathcal{C})f(x_1+x_2) = e^{\pi i[A^{inv}Bx_1x_1\cdot x_1 - B^TDx_2\cdot x_2]}f(x_1+x_2)$, by relation (36), so that $\sigma_1^w(x,D)\mu(\mathcal{C}) = \mu(\mathcal{C})(\sigma_1\circ\mathcal{C}^{-1})^w$ by means of (40). The equality (57) immediately follows.

Next, we pass from the Weyl to the Kohn–Nirenberg form of a pseudodifferential operator: $\sigma_2^w(x,D) = \sigma_3(x,D)$, for the new symbol $\sigma_3 = \mathcal{U}\sigma_2$ where $\mathcal{U}$ is defined in (45). Hence $\sigma_3 \in M^\infty_{1\otimes v_s}(\mathbb{R}^{2d})$ by Lemma 2.8. Using $x = x_1 + x_2$, $\xi = \xi_1 + \xi_2 \in R(A^T)\oplus N(A) = \mathbb{R}^d$, we can express the operator $\sigma_3(x,D)$ by means of integrals over the subspaces $R(A^T)$ and $N(A)$: for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$\sigma_3(x,D)\varphi(x_1+x_2) = \int_{R(A^T)}\int_{N(A)} e^{2\pi i x_2\cdot\xi_2}e^{2\pi i x_1\cdot\xi_1} \times \sigma_3(x_1+x_2,\xi_1+\xi_2)\hat{\varphi}(\xi_1+\xi_2)\,d\xi_2\,d\xi_1.$$

The previous decomposition helps to compute $\sigma_3(x,D)Pf(x)$, where the operator P is defined in (56). Indeed, computing first the integral over $R(A^T)$, we obtain

$$\begin{aligned}&\int_{R(A^T)} e^{2\pi i x_1\cdot\xi_1}\sigma_3\big(x_1+x_2,\xi_1+\xi_2\big)\mathcal{F}_{R(A^T)}\big(e^{2\pi i A^{inv}y\cdot x_1}\big)(\xi_1)\,d\xi_1\\ &\quad= e^{2\pi i A^{inv}y\cdot x_1}\sigma_3\big(x_1+x_2, A^{inv}y+\xi_2\big),\end{aligned}$$

and the expression of $\sigma_3(x,D)Pf(x)$ reduces to

$$\begin{aligned}\sigma_3(x,D)Pf(x_1+x_2) &= \int_{R(A)} e^{2\pi i x_1\cdot A^{inv}y - \pi i CA^{inv}y\cdot y}\\ &\quad\times\left(\int_{N(A)} e^{2\pi i x_2\cdot\xi_2}\sigma_3(x_1+x_2, A^{inv}y+\xi_2)\right.\\ &\quad\left.\cdot\left(\int_{N(A)} e^{-2\pi i\xi_2\cdot t}\hat{f}(y-Bt)\,dt\right)d\xi_2\right)dy\\ &= \int_{R(A)} e^{2\pi i x_1\cdot A^{inv}y - \pi i CA^{inv}y\cdot y}\\ &\quad\times\left(\int_{N(A)} e^{2\pi i x_2\cdot\xi_2}\sigma_3(x_1+x_2, A^{inv}y+\xi_2)\right.\\ &\quad\left.\cdot\left(\frac{1}{q_{N(A)}(B)}\int_{N(A^T)} e^{2\pi i (B^{inv})^T\xi_2\cdot z}\hat{f}(y+z)\,dz\right)d\xi_2\right)dy,\end{aligned}$$

where the last equality is the consequence of Lemma 2.1, with the change of variables $z = -Bt$ and using Lemma 1.3. Observe that the transpose of B^{inv}, denoted by $(B^{inv})^T$, maps $N(A)$ to $N(A^T)$. Finally, the Fourier inversion formula on the subspace $R(A^T)$ gives the desired result in (11), with symbol $\sigma = \frac{1}{q_{N(A)}(B)}\sigma_3$. Consider now the case dim $R(A) = 0$. Then the block B is nonsingular and the matrix $\mathcal{A}$ is the one in (34), whereas the integral representation of $\mu(\mathcal{A})$ is given by (32). Using similar arguments as in the previous case, we compute $Tf(x) = \sigma_1^w(x,D)\mu(\mathcal{A})f(x)$. We observe that $\sigma_1^w(x,D)(e^{-\pi i B^TDx\cdot x}) = e^{-\pi i B^TDx\cdot x}\sigma_4^w(x,D)$ where $\sigma_4(x,\xi) = \sigma_1(x,\xi -$

$B^T Dx$); this follows by the relation $\sigma_1^w(x,D)\mu(\mathcal{E}) = \mu(\mathcal{E})(\sigma\circ\mathcal{E}^{-1})^w(x,D)$, with $\mathcal{E} = \begin{pmatrix} I_d & 0_d \\ B^T D & I_d \end{pmatrix} \in Sp(d,\mathbb{R})$ and $\mathcal{E}^{-1} = \begin{pmatrix} I_d & 0_d \\ -B^T D & I_d \end{pmatrix}$. Next we rewrite $\sigma_4^w(x,D)$ in the Kohn–Nirenberg form $\sigma_5(x,D)$, with $\sigma_5 = \mathcal{U}\sigma_4$, and the operator $\mathcal{U}$ defined in (45). Finally, since $\sigma_5(x,D)(e^{2\pi i x\cdot B^T t}) = \sigma_5(x, B^T t)$, we obtain the representation (12). This formula can be recaptured from (11) when $R(A) = \{0\}$, $y = 0$ so that $N(A) = \mathbb{R}^d$. The block B is invertible on $\mathbb{R}^d$, hence $B^{inv} = B^{-1}$, and making the change of variables $B^{-T}\xi_2 = \eta$ we obtain the claim. This completes the proof. □

Proof of Corollary 1.5 Item (i) is already proved in Theorem 1.4.

(ii) If dim $R(A) = d$, that is the block A is nonsingular, then $N(A)$ is the null space, $A^{inv} = A^{-1}$, the inverse of A on $\mathbb{R}^d$, $x_2 = \xi_2 = 0$ so that $x_1 = x \in \mathbb{R}^d$. In this case the operator T reduces to the following representation

$$
\begin{aligned}
Tf(x) &= \int_{\mathbb{R}^d} e^{\pi i A^{-1}Bx\cdot x + 2\pi i x\cdot A^{-1}y - \pi i C A^{-1} y\cdot y}\sigma\left(x, A^{-1}y\right)\hat{f}(y)\,dy \\
&= \int_{\mathbb{R}^d} e^{2\pi i \Phi_T(x,y)}\tilde{\sigma}(x,y)\hat{f}(y)\,dy
\end{aligned}
$$

where the phase function Φ_T is the one defined in (14). Observe that the phase Φ_T is the generating function of the canonical transformation $\mathcal{A}^T$ (see Remark 2.5). Moreover, by Lemma 2.7, the symbol $\tilde{\sigma}(x,y) = \sigma(x, A^{-1}y)$ is in $M^\infty_{1\otimes v_s}(\mathbb{R}^{2d})$, whenever $\sigma \in M^\infty_{1\otimes v_s}(\mathbb{R}^{2d})$. We then recapture the integral representation of T in (9), as expected.

□

4 Applications to Schrödinger Equations

We now focus on the Cauchy problem for the anisotropic harmonic oscillator stated in (15). The main result of [5] says that the propagator is a generalized metaplectic operator. Let us first recall this issue. Consider the Cauchy problem

$$
\begin{cases} i\dfrac{\partial u}{\partial t} = (a^w(x,D) + \sigma^w(x,D))u \\ u(0,x) = u_0(x), \end{cases} \tag{59}
$$

where the hamiltonian $a^w(x,D)$ is the Weyl quantization of a real-valued homogeneous quadratic polynomial and $\sigma^w(x,D)$ is a pseudodifferential operator with a symbol $\sigma \in M^\infty_{1\otimes v_s}(\mathbb{R}^{2d})$, $s > 2d$. Then, a simplified version of [5, Theorem5.1] reads as follows

Theorem 4.1 *Consider the Cauchy problem (59) above and set* $\tilde{H} = a^w(x,D) + \sigma^w(x,D)$. *Then the evolution operator* $e^{-it\tilde{H}}$ *is a generalized metaplectic operator for every* $t \in \mathbb{R}$. *Specifically, we have*

$$
e^{-itH} = \mu(\mathcal{A}_t) b^w_{1,t}(x,D) = b^w_{2,t}(x,D)\mu(\mathcal{A}_t), \quad t \in \mathbb{R} \tag{60}
$$

for some symbols $b_{1,t}, b_{2,t} \in M^{\infty}_{1\otimes v_s}(\mathbb{R}^{2d})$ *and where* $\mu(\mathcal{A}_t) = e^{-ita^w(x,D)}$ *is the solution to the unperturbed problem* ($\sigma^w(x, D) = 0$)*. In particular, for* $1 \le p \le \infty$*, if the initial datum* $u_0 \in M^p$*, then* $u(t, \cdot) = e^{-it\tilde{H}} u_0 \in M^p$*, for all* $t \in \mathbb{R}$*.*

The example (17) falls in this setting. Indeed, consider first the unperturbed problem

$$\begin{cases} i\partial_t u = H_0 u, \\ u(0, x) = u_0(x), \end{cases} \tag{61}$$

where $u_0 \in \mathcal{S}(\mathbb{R}^2)$ or in $M^p(\mathbb{R}^2)$, and $H_0 = -\frac{1}{4\pi}\partial^2_{x_2} + \pi x_2^2$. In this case the propagator is a classical metaplectic operator and the solution is provided by

$$u(t, x_1, x_2) = e^{-itH_0} u_0(x_1, x_2) = \mu(\mathcal{A}_t) u_0(x_1, x_2),$$

where the simplectic matrices $\mathcal{A}_t$ are defined in (17). For details, we refer for instance to [5, Sect. 4] or [16, Chapter 4]. Observe that the 2×2 block in (18) is singular when $t = \pi/2 + k\pi$, $k \in \mathbb{Z}$, (the so-called caustic points).

We now consider the perturbed problem (16), where the potential $V(x_1, x_2)$ is a multiplication operator and so a particular example of a pseudodifferential operator with symbol $\sigma(x_1, x_2, \xi_1, \xi_2) = V(x_1, x_2) \in M^{\infty}_{1\otimes v_s}(\mathbb{R}^4)$, $s > 4$ (observe that $d = 2$), which satisfies the assumptions of Theorem 4.1. Indeed, we choose a window function $\Phi(x, \xi) = g_1(x) g_2(\xi)$, where $g_1, g_2 \in \mathcal{S}(\mathbb{R}^2)$. The STFT of the symbol then splits as follows:

$$V_\Phi \sigma(z_1, z_2, \zeta_1, \zeta_2) = V_{g_1}(V)(z_1, \zeta_1) V_{g_2}(1)(z_2, \zeta_2), \quad z_1, z_2, \zeta_1, \zeta_2 \in \mathbb{R}^2.$$

Using $\langle(\zeta_1, \zeta_2)\rangle \le \langle\zeta_1\rangle\langle\zeta_2\rangle$ and the fact that $1 \in S^0_{0,0} \subset M^{\infty}_{1\otimes v_s}(\mathbb{R}^2)$, for every $s \ge 0$, the claim follows.

Hence, the representation of the solution $u(t, x)$ of (15) is a generalized metaplectic operator applied to the initial datum u_0. For $t \ne \pi/2 + k\pi$, $k \in \mathbb{Z}$, the representation of $u(t, x)$ is provided by the type I FIO stated in (9), which in this case reads

$$\begin{aligned} u(t, x_1, x_2) = \int_{\mathbb{R}^2} & e^{2\pi i (x_1 \cdot \xi_1 + (\sec t) x_2 \cdot \xi_2) - \pi i (\tan t)\left(x_2^2 + \xi_2^2\right)} \\ & \times b_t(x_1, x_2, \xi_1, \xi_2) \hat{u_0}(\xi_1, \xi_2)\, d\xi_1 d\xi_2, \end{aligned}$$

for suitable symbols $b_t \in M^{\infty}_{1\otimes v_{\mu+1}}(\mathbb{R}^4)$. We are interested in the caustic points $t = \pi/2 + k\pi$, $k \in \mathbb{Z}$. The corresponding matrix in (17) is

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \text{with transpose} \quad \mathcal{A}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Applying Theorem 1.4 for the transpose matrix $\mathcal{A}^T$, we observe that in this case $A = A^T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The range and the kernel of A are given by $R(A) = R(A^T) = \{(\lambda, 0),\ \lambda \in \mathbb{R}\}$ and $N(A) = \{(0, \nu),\ \nu \in \mathbb{R}\}$. In this case $A^{inv} : R(A) \to R(A^T)$ is the identity mapping. Observe that $D^T B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; if $y \in R(A)$, then $Cy = 0$ and, for $x \in R(A) \oplus N(A)$, we have $x = (x_1, x_2)$, $x_1, x_2 \in \mathbb{R}$.

Setting $T = u(\pi/2+k\pi, \cdot)$, the integral representation in (11), for a suitable symbol $b \in M^{\infty}_{1\otimes v_{\mu+1}}(\mathbb{R}^4)$, reduces in this case to

$$Tf(x_1, x_2) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{2\pi i\left(x_1 y + x_2\xi_2\right)} b\big((x_1, x_2), (y, \xi_2)\big)(\mathcal{F}_1 u_0)(y, -\xi_2)d\xi_2 dy$$

where $\mathcal{F}_1 u_0(\xi_1, \xi_2) = \int_{\mathbb{R}} e^{-2\pi i \xi_1 t} u_0(t, \xi_2)\, dt$ is the one-dimensional Fourier transform of the initial datum u_0 restricted to the first variable x_1.

Finally, we observe that, if the symbol $b \equiv 1 \in M^{\infty}_{1\otimes v_s}(\mathbb{R}^4)$, for every $s \geq 0$, then the operator T reduces to

$$Tu_0(x_1, x_2) = (\mathcal{F}_2 u_0)(x_1, x_2)$$

the one-dimensional Fourier transform of u_0 restricted to the second variable x_2. This example of fractional Fourier transform was already studied in [25, Sect. 6.2].

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