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The Non-Archimedean Stochastic Heat Equation Driven by Gaussian Noise

W. A. Zúñiga-Galindo

Abstract We introduce and study a new class of non-Archimedean stochastic pseudodifferential equations. These equations are the non-Archimedean counterparts of the classical stochastic heat equations. We show the existence and uniqueness of mild random field solutions for these equations.

1 Introduction

In this article we study a new class of stochastic pseudodifferential equations in $\mathbb{R}_+ \times \mathbb{Q}_p^N$, here $\mathbb{Q}_p$ denotes the field of p-adic numbers, driven by a spatially homogeneous Gaussian noise. More precisely, we consider pseudodifferential equations of the type

$$Lu\,(t,x) = \sigma\,(u\,(t,x))\,\dot{W}\,(t,x) + b\,(u(t,x) = 0, t \geq 0, x \in \mathbb{Q}_p^N,$$

where $L = \frac{\partial}{\partial t} + A\,(\partial, \beta)$, $\beta > 0$, with $A\,(\partial, \beta)$ a pseudodifferential operator of the form $\mathcal{F}_{x \to \xi}\,(A\,(\partial, \beta)\,\varphi) = |a\,(\xi)|_p^\beta\,\mathcal{F}_{x \to \xi}\,(A\varphi)$, and $a\,(\xi)$ an elliptic polynomial. The

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coefficients σ and b are real-valued functions and $\dot{W}(t, x)$ is the formal notation for a Gaussian random perturbation defined on some probability space. We assume that it is white in time and with a homogeneous spatial correlation given by a function f, see Sect. 5.2. Our main result, see Theorem 6.4, asserts the existence and uniqueness of mild random field solutions for these equations. The equations studied here are the non-Archimedean counterparts of the Archimedean stochastic heat equations studied for instance in [16,18,40].

The pseudodifferential equations of the form

$$\frac{\partial u(t, x)}{\partial t} + A(\partial, \beta) u(t, x) = 0$$

are the p-adic counterparts of the Archimedean heat equations. Indeed, the fundamental solutions of these equations (i.e. the heat kernels) are transition density functions of Markov processes on $\mathbb{Q}_p^N$, see Sect. 3.2. The one-dimensional p-adic heat equation was introduced in [39, Sect. XVI], since then the theory of such equations has been steadily developing, see e.g. [3,11,27,34,35,38,41] and the references therein. This type of equations appear in some new models of complex systems constructed by Avetisov et al. [5–7], see also [29], thus, the study of stochastic versions of these equations is a natural and relevant problem.

From a more general perspective, the stochastic processes over the p-adics, or more generally over ultrametric spaces, have attracted a lot of attention during the last thirty years, see e.g. [1,2,5,7–9,11,12,19,20,23–26,30,32–34,37–39,41], and the references therein. From the point of view of the mathematical physics, the interest on this type of stochastic processes comes from their connections with models of complex systems. It has been proposed that the space of states of certain complex systems, for instance proteins, have a hierarchical structure, see e.g. [21], which can be put in turn in connection with p-adic structures [5–7].

Stochastic equations over p-adics have been studied intensively by many authors, see e.g. [8,9,19,23,24,26,28,31,34]. The p-adic Gaussian noise and the corresponding stochastic integrals was studied in [8,19,23,26,28]. All these articles consider processes and stochastic integrals depending on p-adic variables. Here, we introduced a non-Archimedean, spatially homogeneous Gaussian noise parametrized by a nonnegative real variable, the time variable, and by a p-adic vector, the position variable. As far as we know such noises have been not studied before. On the other hand, in [33,34] Kochubei introduced stochastic integrals with respect to the 'p-adic Brownian motion' generated by the one-dimensional heat equation. This is a non-Gaussian process parametrized by non-negative real variable and by a p-adic variable.

The article is organized as follows. In Sect. 2, we review some basic facts about p-adic analysis. In Sect. 3, we review some aspects of the parabolic type pseudodifferential equations needed for other sections. In Sect. 4, we prove a p-adic version of the Bochner–Schwartz Theorem, see Theorem 4.1. In Sect. 5, we review the stochastic integration with respect to Hilbert-space-valued Wiener processes, and introduce the Gaussian noise W and its associated cylindrical process, see Proposition 5.7. We also give some results about the spectral measure of W, see Theorem 5.11. Finally, we give a result, Proposition 5.12, which gives us examples of random distributions that can

be integrated with respect to W. It is interesting to note that the proof of Proposition 5.12 is much more involved than the corresponding result in the Archimedean setting, see e.g. proof of Proposition 3.3 in [17]. This is due to the fact that in the p-adic setting, the smoothing of a process requires 'cutting' and convolution operations while, in the Archimedean setting, it requires only a convolution operation. In Sect. 6, we prove the main result, see Theorem 6.4. Like in [18] we prove Theorem 6.4 under the 'Hypotheses A and B,' here we give an explicit and sufficient conditions to fulfill these hypotheses in terms of the spectral measure of W, see Theorem 5.11 and Lemma 6.3.

2 p-Adic Analysis: Essential Ideas

In this section we fix the notation and collect some basic results on p-adic analysis that we will use through the article. For a detailed exposition the reader may consult [3,36,39].

2.1 The Field of p-Adic Numbers

Along this article p will denote a prime number. The field of p-adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-\gamma} & \text{if } x = p^{\gamma}\dfrac{a}{b}, \end{cases}$$

where a and b are integers coprime with p. The integer $\gamma := ord(x)$, with $ord(0) := +\infty$, is called the *p-adic order of* x. We extend the p-adic norm to $\mathbb{Q}_p^N$ by taking

$$||x||_p := \max_{1 \leq i \leq N} |x_i|_p, \qquad \text{for } x = (x_1, \ldots, x_N) \in \mathbb{Q}_p^N.$$

We define $ord(x) = \min_{1 \leq i \leq N}\{ord(x_i)\}$, then $||x||_p = p^{-ord(x)}$. The metric space $\left(\mathbb{Q}_p^N, ||\cdot||_p\right)$ is a complete ultrametric space. As a topological space $\mathbb{Q}_p$ is homeomorphic to a Cantor-like subset of the real line, see e.g. [3,39].

Any p-adic number $x \neq 0$ has a unique expansion $x = p^{ord(x)} \sum_{j=0}^{+\infty} x_j p^j$, where $x_j \in \{0, 1, 2, \ldots, p-1\}$ and $x_0 \neq 0$. By using this expansion, we define *the fractional part of* $x \in \mathbb{Q}_p$, denoted $\{x\}_p$, as the rational number

$$\{x\}_p = \begin{cases} 0 & \text{if } x = 0 \text{ or } ord(x) \geq 0 \\ p^{ord(x)} \sum_{j=0}^{-ord(x)-1} x_j p^j & \text{if } ord(x) < 0. \end{cases}$$

For $\gamma \in \mathbb{Z}$, denote by $B_{\gamma}^N(a) = \{x \in \mathbb{Q}_p^N : ||x - a||_p \leq p^{\gamma}\}$ *the ball of radius* p^{γ} *with center at* $a = (a_1, \ldots, a_N) \in \mathbb{Q}_p^N$, and take $B_{\gamma}^N(0) := B_{\gamma}^N$. Note that

$B_\gamma^N(a) = B_\gamma(a_1) \times \cdots \times B_\gamma(a_n)$, where $B_\gamma(a_i) := \{x \in \mathbb{Q}_p : |x - a_i|_p \leq p^\gamma\}$ is the one-dimensional ball of radius p^γ with center at $a_i \in \mathbb{Q}_p$. The ball B_0^N equals the product of N copies of $B_0 := \mathbb{Z}_p$, *the ring of p-adic integers*. We denote by $\Omega(\|x\|_p)$ the characteristic function of B_0^N. For more general sets, say Borel sets, we use $1_A(x)$ to denote the characteristic function of A.

2.2 The Bruhat–Schwartz Space

A complex-valued function φ defined on $\mathbb{Q}_p^N$ is *called locally constant* if for any $x \in \mathbb{Q}_p^N$ there exists an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x + x') = \varphi(x) \text{ for } x' \in B_{l(x)}^N. \tag{2.1}$$

A function $\varphi : \mathbb{Q}_p^N \to \mathbb{C}$ is called a *Bruhat–Schwartz function (or a test function)* if it is locally constant with compact support. The $\mathbb{C}$-vector space of Bruhat–Schwartz functions is denoted by $\mathcal{D}(\mathbb{Q}_p^N) := \mathcal{D}$. We will denote by $\mathcal{D}_\mathbb{R}(\mathbb{Q}_p^N)$ the $\mathbb{R}$-vector space of Bruhat–Schwartz functions.

For $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$, the largest of such number $l = l(\varphi)$ satisfying (2.1) is called *the exponent of local constancy of φ*.

Let $\mathcal{D}'(\mathbb{Q}_p^N) := \mathcal{D}'$ denote the set of all functionals (distributions) on $\mathcal{D}(\mathbb{Q}_p^N)$. All functionals on $\mathcal{D}(\mathbb{Q}_p^N)$ are continuous.

2.3 Fourier Transform

Set $\chi_p(y) = \exp(2\pi i \{y\}_p)$ for $y \in \mathbb{Q}_p$. The map $\chi_p(\cdot)$ is an additive character on $\mathbb{Q}_p$, i.e. a continuos map from $\mathbb{Q}_p$ into the unit circle satisfying $\chi_p(y_0 + y_1) = \chi_p(y_0)\chi_p(y_1)$, $y_0, y_1 \in \mathbb{Q}_p$.

Given $\xi = (\xi_1, \ldots, \xi_N)$ and $x = (x_1, \ldots, x_N) \in \mathbb{Q}_p^N$, we set $\xi \cdot x := \sum_{j=1}^N \xi_j x_j$. The Fourier transform of $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$ is defined as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^N} \chi_p(-\xi \cdot x)\varphi(\xi)d^N x \quad \text{for } \xi \in \mathbb{Q}_p^N,$$

where $d^N x$ is the Haar measure on $\mathbb{Q}_p^N$ normalized by the condition $vol(B_0^N) = 1$. The Fourier transform is a linear isomorphism from $\mathcal{D}(\mathbb{Q}_p^N)$ onto itself satisfying $(\mathcal{F}(\mathcal{F}\varphi))(\xi) = \varphi(-\xi)$. We will also use the notation $\mathcal{F}_{x \to \xi}\varphi$ and $\widehat{\varphi}$ for the Fourier transform of φ.

The Fourier transform $\mathcal{F}[T]$ of a distribution $T \in \mathcal{D}'\left(\mathbb{Q}_p^N\right)$ is defined by

$$(\mathcal{F}[T], \varphi) = (T, \mathcal{F}[\varphi]) \text{ for all } \varphi \in \mathcal{D}\left(\mathbb{Q}_p^N\right).$$

The Fourier transform $f \to \mathcal{F}[T]$ is a linear isomorphism from $\mathcal{D}'\left(\mathbb{Q}_p^N\right)$ onto $\mathcal{D}'\left(\mathbb{Q}_p^N\right)$. Furthermore, $T = \mathcal{F}[\mathcal{F}[T](-\xi)]$.

3 p-Adic Parabolic Type Pseudodifferential Equations

In this article we work exclusively with complex and real valued functions on $\mathbb{Q}_p^N$. Having complex and real valued functions defined on a locally compact topological group, we have the notion of continuous function and may use the functional spaces $L^\varrho(\mathbb{Q}_p^N, d^N x), \rho \geq 1$, defined in the standard way. We also use the following standard notation:

(i) $C(I, X)$ the space of continuous functions u on a time interval I with values in X;

(ii) $C^1(I, X)$ the space of continuously differentiable functions u on a time interval I such that $u' \in X$;

(iii) $L^1(I, X)$ the space of measurable functions u on I with values in X such that $\|u\|$ is integrable;

(iv) $W^{1,1}(I, X)$ the space of measurable functions u on I with values in X such that $u' \in L^1(I, X)$.

3.1 Elliptic Pseudodifferential Operators

Let $a(\xi) \in \mathbb{Q}_p[\xi_1, \ldots, \xi_n]$ be a non-constant polynomial. We say that $a(\xi)$ is *an elliptic polynomial of degree d*, if it satisfies: (i) $a(\xi)$ is a homogeneous polynomial of degree d, and (ii) $a(\xi) = 0 \Leftrightarrow \xi = 0$.

It is known that an elliptic polynomial satisfies

$$C_0 \|\xi\|_p^d \leq |a(\xi)|_p \leq C_1 \|\xi\|_p^d, \text{ for every } \xi \in \mathbb{Q}_p^n, \tag{3.1}$$

for some positive constants $C_0 = C_0(a)$, $C_1 = C_1(a)$, cf. [41, Lemma 1]. Without loss of generality we will assume that $a(\xi) \in \mathbb{Z}_p[\xi_1, \ldots, \xi_n]$.

Given a fixed $\beta > 0$, a pseudodifferential operator of the form $A(\partial, \beta)\phi(x) = \mathcal{F}_{\xi \to x}^{-1}\left(|a(\xi)|_p^\beta \mathcal{F}_{x \to \xi}\phi\right), \phi \in \mathcal{D}$, is called *an elliptic pseudodifferential operator of degree d with symbol $|a|_p^\beta$*.

Lemma 3.1 *With the above notation the following assertions hold:*

(i)

$$\mathcal{D} \to C\left(\mathbb{Q}_p^N, \mathbb{C}\right) \cap L^2\left(\mathbb{Q}_p^N, d^N x\right)$$

$$\phi \to \qquad A(\partial, \beta)\phi;$$

(ii) the closure of the operator $A(\partial, \beta)$, $\beta > 0$ (let us denote it by $A(\partial, \beta)$ again) with domain

$$Dom(A(\partial, \beta)) := Dom(A) = \left\{ f \in L^2 : |a|_p^\beta \, \widehat{f} \in L^2 \right\} \tag{3.2}$$

is a self-adjoint operator;

(iii) $-A(\partial, \beta)$ is the infinitesimal generator of a contraction C_0 semigroup $(\mathcal{T}(t))_{t \geq 0}$;

(iv) set

$$\Gamma(t, x) := \mathcal{F}_{\xi \to x}^{-1} \left(e^{-t|a(\xi)|_p^\beta} \right) \text{ for } t > 0, x \in \mathbb{Q}_p^N. \tag{3.3}$$

Then

$$(\mathcal{T}(t)f)(x) = \begin{cases} (\Gamma(t, \cdot) * f)(x) & \text{for } t > 0 \\[2mm] f(x) & \text{for } t = 0, \end{cases}$$

for $f \in L^2$.

Proof The results follow from the properties of the heat kernels given in [41] by using well-known techniques of semigroup theory, see e.g. [10]. Alternatively, the reader may consult [37, Lemma 3.21, Lemmas 3.23 and 7.4, Theorem 7.5] for same results in a more general setting. □

3.2 *p*-Adic Heat Equations

Consider the following Cauchy problem:

$$\begin{cases} \dfrac{\partial u(t, x)}{\partial t} + A(\partial, \beta) u(t, x) = f(t, x), \, x \in \mathbb{Q}_p^N, \, t \in [0, T] \\[3mm] u(0, x) = u_0(x) \in Dom(A). \end{cases} \tag{3.4}$$

We say that a function $u(x, t)$ is a *solution of* (3.4) if $u \in C([0, T], Dom(A)) \cap C^1([0, T], L^2)$ and u satisfies Eq. (3.4) for all $t \in [0, T]$.

Theorem 3.2 *Let $\beta > 0$ and let $f \in C([0, T], L^2)$. Assume that at least one of the following conditions is satisfied:*

(i) $f \in L^1((0, T), Dom(A))$;
(ii) $f \in W^{1,1}((0, T), L^2)$.

Then Cauchy problem (3.4) has a unique solution given by

$$u(t, x) = \int_{\mathbb{Q}_p^N} \Gamma(t, x - y) u_0(y) \, d^N y + \int_0^t \left\{ \int_{\mathbb{Q}_p^N} \Gamma(t - \tau, x - y) f(\tau, y) \, d^n y \right\} d\tau,$$

where Γ is defined in (3.3).

Proof The result follows from Lemma 3.1 by well-known results in semigroup theory, see e.g. [10]. Alternatively, the reader may consult [37, Theorem 7.9] for same result in a more general setting.   □

Theorem 3.3 *The heat kernel (or fundamental solution of (3.4))* $\Gamma(t, x)$, $t > 0$, *satisfies the following:*

(i) $\Gamma(t, x) \geq 0$ *for any* $t > 0$;

(ii) $\int_{\mathbb{Q}_p^N} \Gamma(t, x) \, d^N x = 1$ *for any* $t > 0$;

(iii) $\Gamma(t, \cdot) \in L^1(\mathbb{Q}_p^N)$ *for any* $t > 0$;

(iv) $\left(\Gamma(t, \cdot) * \Gamma(t', \cdot)\right)(x) = \Gamma(t + t', x)$ *for any* $t, t' > 0$;

(v) $\lim_{t \to 0+} \Gamma(t, x) = \delta(x)$ *in* $\mathcal{D}'$;

(vi) $\Gamma(t, x) \leq At\left(t^{\frac{1}{d\beta}} + \|x\|_p\right)^{-d\beta - N}$ *for any* $x \in \mathbb{Q}_p^N$ *and* $t > 0$;

(vii) $\Gamma(x, t)$ *is the transition density of a time- and space homogenous Markov process which is bounded, right-continuous and has no discontinuities other than jumps.*

Proof See Theorems 1, 2, 4, and Proposition 2 in [41].   □

The *p*-adic heat equation in dimension one was introduced in the book of Vladimirov, Volovich and Zelenov [39, Sect. XVI]. In [34, Chapters 4, 5] Kochubei presented a general theory for one-dimensional parabolic-type pseudodifferential equations with variable coefficients, whose fundamental solutions are transition density functions for Markov processes in the *p*-adic line. For a generalization of this theory see [11]. In [41] the author introduced the elliptic operators presented before and studied the corresponding *n*-dimensional heat equations and the associated Markov processes.

4 Positive-Definite Distributions and the Bochner–Schwartz Theorem

In this section, we establish a *p*-adic version of the Bochner–Schwartz Theorem on positive-definite distributions following to Gel'fand and Vilenkin [22, Chapter II].

4.1 The *p*-Adic Bochner–Schwartz Theorem

Along this section we work with complex-valued test functions. A distribution $F \in \mathcal{D}'\left(\mathbb{Q}_p^N\right)$ is called *positive*, if $(F, \varphi) \geq 0$ for every positive test function φ, i.e. if $\varphi(x) \geq 0$ for every x. In this case we will use the notation $F \geq 0$. We say that F is *multiplicatively positive*, if $(F, \varphi\overline{\varphi}) \geq 0$ for every test function φ, where $\overline{\varphi}$ denotes the complex conjugate of φ. A distribution F is positive-definite, if for every test function φ, the inequality $\left(F, \overline{\varphi} * \widetilde{\varphi}\right) \geq 0$ holds, where $\widetilde{\varphi}(x) = \overline{\varphi(-x)}$.

Theorem 4.1 (*p*-adic Bochner–Schwartz Theorem). *Every positive-definite distribution F on $\mathbb{Q}_p^N$ is the Fourier transform of a regular Borel measure μ on $\mathbb{Q}_p^N$, i.e.*

$$(F, \varphi) = \int_{\mathbb{Q}_p^N} \widehat{\varphi}(\xi) \, d\mu(\xi) \text{ for } \varphi \in \mathcal{D}\left(\mathbb{Q}_p^N\right).$$

Conversely, the Fourier transform of any regular Borel measure gives rise to a positive-definite distribution on $\mathbb{Q}_p^N$.

Proof ($\Rightarrow$) By the Riesz-Markov-Kakutani Theorem every positive distribution F on $\mathbb{Q}_p^N$ has the form

$$(F, \phi) = \int_{\mathbb{Q}_p^N} \phi(\xi) \, d\mu(\xi) \text{ for } \phi \in \mathcal{D}\left(\mathbb{Q}_p^N\right),$$

where μ is a regular Borel measure. Conversely, every regular Borel measure μ defines a positive linear functional on $\mathcal{D}\left(\mathbb{Q}_p^N\right)$. On the other hand, since F is a multiplicatively positive distribution if and only if F is a positive distribution, we can replace positive by multiplicatively positive in the above assertion. We now note that the Fourier transform carries positive-definite distributions into multiplicatively positive distributions, and every multiplicatively positive distribution can be obtained in this manner. Indeed,

$$\left(\widehat{F}, \widehat{\varphi \overline{\varphi}}\right) = \left(\widehat{F}, \widehat{\varphi * \widetilde{\varphi}}\right)$$

$$= \left(F, \widehat{\widehat{\varphi * \widetilde{\varphi}}}\right) = (F(\xi), (\varphi * \widetilde{\varphi})(-\xi))$$

$$= \left(F, \overline{\varphi * \widetilde{\varphi}}\right),$$

since $\overline{(\varphi * \widetilde{\varphi})(-\xi)} = (\varphi * \widetilde{\varphi})(\xi)$. Now, let $F \in \mathcal{D}'\left(\mathbb{Q}_p^N\right)$ be a multiplicatively positive distribution, i.e. $\left(F, \psi \overline{\psi}\right) \geq 0$ for every $\psi \in \mathcal{D}\left(\mathbb{Q}_p^N\right)$. Then, there exist a distribution T and a test function ϕ satisfying $\widehat{T} = F$ and $\psi = \widehat{\phi}$, because the Fourier transform is an isomorphism on $\mathcal{D}'\left(\mathbb{Q}_p^N\right)$ and on $\mathcal{D}\left(\mathbb{Q}_p^N\right)$. From this observation we have $\left(F, \psi \overline{\psi}\right) = \left(T, \overline{\varphi * \widetilde{\varphi}}\right) \geq 0$.

($\Leftarrow$) It follows from this calculation:

$$\int_{\mathbb{Q}_p^N} \mathcal{F}\left(\overline{(\varphi * \widetilde{\varphi})}\right)(\xi) \, d\mu(\xi) = \int_{\mathbb{Q}_p^N} \left(\mathcal{F}^{-1}\varphi\right)(\xi) \overline{\left(\mathcal{F}^{-1}\varphi\right)}(\xi) \, d\mu(\xi)$$

$$= \int_{\mathbb{Q}_p^N} \left|\left(\mathcal{F}^{-1}\varphi\right)(\xi)\right|^2 \, d\mu(\xi) \geq 0.$$

$\square$

4.2 Positive-Definite Functions

We recall that a continuous function $g : \mathbb{Q}_p^N \to \mathbb{C}$ is positive-definite, if for any p-adic numbers $x_1, \ldots, x_m$ and any complex numbers $\sigma_1, \ldots, \sigma_m$, it verifies that $\sum_j \sum_i g(x_j - x_i) \sigma_j \overline{\sigma_i} \geq 0$. Such function g satisfies the following: $\overline{g}$ is positive-definite, $g(-x) = \overline{g}(x)$, $g(0) \geq 0$, and $|g(x)| \leq g(0)$. We associate to g the distribution $\int_{\mathbb{Q}_p^N} g(x)\varphi(x) \, d^N x$, while Gel'fand-Vilenkin attach to g the distribution

$\int_{\mathbb{Q}_p^N} \overline{g}(x)\varphi(x) \, d^N x$, for this reason our definition of positive-definite distribution is slightly different, but equivalent to the one given in [22, Chapter II]. Finally, we recall that g satisfies $\left(g, \overline{\varphi * \widetilde{\varphi}}\right) \geq 0$ for any test function φ, i.e. g generates a positive-definite distribution, see e.g. [13, Proposition 4.1].

5 Stochastic Integrals and Gaussian Noise

In this section we introduce the stochastic integration with respect to a spatially homogeneous Gaussian noise. Our exposition has been strongly influenced by [17]. There are two distinct approaches (or schools) of study for stochastic partial differential equations, based on different theories of stochastic integration: the Walsh theory [40], which uses integration with respect to worthy martingale measures, and a theory of integration with respect to Hilbert-space valued processes [16]. In [17] the authors discuss the connections between these theories. In this article we use Hilbert-space approach. In this section we present the non-Archimedean counterpart of this theory.

5.1 Stochastic Integrals with Respect to a Spatially Homogeneous Gaussian Noise

Let V be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_V$. Following [17] and the references therein, we define the general notion of cylindrical Wiener process in V as follows:

Definition 5.1 Let Q be a symmetric and non-negative definite bounded linear operator on V. A family of random variables $B = \{B_t(h), t \geq 0, h \in V\}$ is a cylindrical Wiener process if the following conditions hold:

(i) for any $h \in V$, $\{B_t(h), t \geq 0\}$ defines a Brownian motion with variance $t \langle Qh, h \rangle_V$;

(ii) for all $s, t \in \mathbb{R}_+$ and $h, g \in V$,

$$E(B_s(h) B_t(g)) = (s \wedge t) \langle Qh, g \rangle_V,$$

where $s \wedge t := \min\{s, t\}$. If $Q = I_V$ is the identity operator in V, then B will be called a standard cylindrical Wiener process. We will refer to Q as the covariance of B.

Let $\mathcal{F}_t$ be the σ-field generated by the random variables $\{B_s(h), h \in V, 0 \leq s \leq t\}$ and the P-null sets. We define the *predictable σ-field in* $[0, T] \times \Omega$ generated by the sets

$$\{(s, t] \times A, A \in \mathcal{F}_s, 0 \leq s < t \leq T\}.$$

We denote by V_Q the completion of the Hilbert space V endowed with the inner semi-product

$$\langle h, g \rangle_{V_Q} := \langle Qh, g \rangle_V, h, g \in V.$$

We define the stochastic integral of any predictable square-integrable process with values in V_Q as follows. Let $(v_j)_j$ be a complete orthonormal basis of V_Q. For any predictable process $g \in L^2(\Omega \times [0, T]; V_Q)$, the following series converges in $L^2(\Omega, \mathcal{F}, P)$ and the sum does not depend on the chosen orthonormal basis:

$$g \cdot B := \sum_{j=1}^{+\infty} \int_0^T \langle g_s, v_j \rangle_{V_Q} \, dB_s(v_j). \tag{5.1}$$

We note that each summand in the above series is a classical Itô integral with respect to a standard Brownian motion, and the resulting stochastic integral is a real-valued random variable. The stochastic integral $g \cdot B$ is also denoted by $\int_0^T g_s dB_s$. The independence of each of terms in series (5.1) leads to the isometry property

$$E\left((g \cdot B)^2\right) = E\left(\left(\int_0^T g_s dB_s\right)^2\right) = E\left(\left(\int_0^T \|g_s\|_{V_Q} \, ds\right)\right).$$

5.2 Spatially Homogeneous Gaussian Noise

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. We denote by $\mathcal{I}(\mathbb{R})$ the $\mathbb{R}$-vector space of functions of the form $\sum_{k=1}^m c_k 1_{I_k}(x)$ where $c_1, \dots, c_k$ are real numbers and each I_k is a bounded interval (open, closed, half-open). It is well-known that $\mathcal{I}(\mathbb{R})$ is dense in $L^\rho(\mathbb{R})$ for $1 \le \rho < \infty$.

We denote by $\mathcal{I}(\mathbb{R}) \otimes_{\text{alg}} \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N)$ the algebraic tensor product of the $\mathbb{R}$-vector spaces $\mathcal{I}(\mathbb{R})$ and $\mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N)$. Notice that $\mathcal{I}(\mathbb{R}) \otimes_{\text{alg}} \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N)$ is the $\mathbb{R}$-vector space spanned by $\sum_{j \in J} c_j(t) \Omega\left(p^m \|x - \tilde{x}_j\|_p\right)$, where $c_j(t) \in \mathcal{I}(\mathbb{R})$, $m \in \mathbb{Z}$, and J is a finite set.

On $(\Omega, \mathcal{F}, P)$, we consider a family of mean zero Gaussian random variables

$$\left\{ W(\varphi), \varphi \in \mathcal{I}(\mathbb{R}) \otimes_{\text{alg}} \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N) \right\} \tag{5.2}$$

with covariance

$$E(W(\varphi)W(\psi)) = \int_0^{+\infty} \int_{\mathbb{Q}_p^N} \int_{\mathbb{Q}_p^N} \varphi(t, x) f(x - y) \psi(t, y) d^N x \, d^N y \, dt$$

$$= \int_0^{+\infty} \int_{\mathbb{Q}_p^N} f(z) \left(\varphi(t) * \tilde{\psi}(t) \right)(z) \, d^N z \, dt, \tag{5.3}$$

where $\tilde{\psi}(t)(z) = \psi(t, -z)$ and f is non-negative continuous function on $\mathbb{Q}_p^N \smallsetminus \{0\}$. This function induces a positive distribution on $\mathbb{Q}_p^N$ and then f is the Fourier transform of a regular Borel measure μ on $\mathbb{Q}_p^N$, see Theorem 4.1. This measure is *called the spectral measure* of W. In this case

$$E\left(W\left(\varphi\right)W\left(\psi\right)\right) = \int_0^{+\infty}\int_{\mathbb{Q}_p^N}\mathcal{F}\varphi(t)\left(\xi\right)\overline{\mathcal{F}\psi(t)\left(\xi\right)}d\mu\left(\xi\right)dt.$$

5.2.1 Some Examples of Kernels

The basic example of kernel function is the white noise kernel: $f(x) = \delta(x)$, $d\mu(\xi) = d^N\xi$. Here are some typical examples:

Example 5.2 If $d\mu(\xi) = \|\xi\|_p^{-\alpha}d^N\xi$, $0 < \alpha < N$, then $f(x) = R_\alpha(x) = \frac{1-p^{-\alpha}}{1-p^{\alpha-N}}\|x\|_p^{\alpha-n}$, the Riesz kernel, see e.g. [36, Chapter III, Sect. 4].

Example 5.3 If $d\mu(\xi) = e^{-\|\xi\|_p^\beta}$, $\beta > 0$, then $f(x) = \mathcal{F}_{\xi\to x}\left(e^{-\|\xi\|_p^\beta}\right)$ is the p-adic heat kernel. Notice that we can replace $\|\xi\|_p^\beta$ by $|a(\xi)|_p^\beta$, where $a(\xi)$ is an elliptic polynomial.

Before presenting our next example, we recall the following result:

Lemma 5.4 ([36, Lemma 5.2]). *Suppose that* $\alpha > 0$. *Define*

$$K_\alpha(x) = \begin{cases} \frac{1-p^{-\alpha}}{1-p^{\alpha-N}}\left(\|x\|_p^{\alpha-N} - p^{\alpha-N}\right)\Omega\left(\|x\|_p\right) & \text{if } \alpha \neq N \\[2ex] \left(1-p^{-N}\right)\log_p\left(\frac{p}{\|x\|_p}\right)\Omega\left(\|x\|_p\right) & \text{if } \alpha = N. \end{cases}$$

Then $K_\alpha \in L^1$ *and* $\mathcal{F}K_\alpha(\xi) = \max\left(1, \|\xi\|_p\right)^{-\alpha}$.

The distribution K_α is called *the Bessel potential of order* α, see e.g. [36, Chapter III, Sect. 5].

Example 5.5 If $d\mu(\xi) = \max\left(1, \|\xi\|_p\right)^{-\alpha}$, $\alpha > 0$, then $f(x) = K_\alpha(x)$, the Bessel potential of order α.

5.2.2 A Cylindrical Wiener Process Associated with W

Let U be the completion of the Bruhat–Schwartz space $\mathcal{D}_\mathbb{R}(\mathbb{Q}_p^N)$ endowed with semi-inner product

$$\langle\varphi, \psi\rangle_U := \int_{\mathbb{Q}_p^N}\mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}d\mu(\xi), \quad \varphi, \psi \in \mathcal{D}_\mathbb{R}(\mathbb{Q}_p^N),$$

where μ is the spectral measure of W. We denote by $\|\cdot\|_U$ the corresponding norm. Then U is a separable Hilbert space (because $\mathcal{D}_\mathbb{R}(\mathbb{Q}_p^N)$ is separable) that may contain distributions.

We fix a time interval $[0, T]$ and set $U_T := L^2([0, T]; U)$. This set is equipped with the norm given by

$$\|g\|_{U_T}^2 := \int_0^T\|g(s)\|_U^2\,ds.$$

We now associate a cylindrical Wiener process to W as follows. A direct calculation using (5.3) shows that the generalized Gaussian random field $W(\varphi)$ is a random linear functional, in the sense that $W(a\varphi + b\psi) = aW(\varphi) + bW(\psi)$, almost surely, and $\varphi \to W(\varphi)$ is an isometry from $\left(\mathcal{I}([0,T]) \otimes_{\text{alg}} \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N), \|\cdot\|_{U_T}\right)$ into $L^2(\Omega, \mathcal{F}, P)$. The following lemma identifies the completion of $\mathcal{I}([0,T]) \otimes_{\text{alg}} \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N)$ with respect to $\|\cdot\|_{U_T}$.

Lemma 5.6 *The space $\mathcal{I}([0,T]) \otimes_{alg} \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N)$ is dense in $U_T = L^2([0,T]; U)$ for $\|\cdot\|_{U_T}$.*

Proof Let $\mathcal{C}$ denote the closure of $\mathcal{I}([0,T]) \otimes_{\text{alg}} \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N)$ in U_T for $\|\cdot\|_{U_T}$. Suppose that we are given $\varphi_1 \in L^2([0,T]; \mathbb{R})$ and $\varphi_2 \in \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N)$. We show that $\varphi_1\varphi_2 \in \mathcal{C}$. Indeed, let $\left(\varphi_1^{(n)}\right)_n \subset \mathcal{I}(\mathbb{R})$ such that, for all n, the support of $\varphi_1^{(n)}$ is contained in $[0,T]$ and $\varphi_1^{(n)} \to \varphi_1$ in $L^2([0,T]; \mathbb{R})$. Now $\varphi_1^{(n)}\varphi_2 \in \mathcal{I}([0,T]) \otimes_{\text{alg}} \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N) \subset \mathcal{C}$ and $\varphi_1^{(n)}\varphi_2 \xrightarrow{\|\cdot\|_{U_T}} \varphi_1\varphi_2$, therefore $\varphi_1\varphi_2 \in \mathcal{C}$.

Suppose that $\varphi \in U_T$. We show that $\varphi \in \mathcal{C}$. Indeed, let $(e_j)_j$ be a complete orthonormal basis of U with $e_j \in \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N)$ for all j. Then, since $\varphi(s) \in U$ for any $s \in [0,T]$,

$$\|\varphi\|_{U_T}^2 = \int_0^T \|\varphi(s)\|_U^2 \, ds = \sum_{j=1}^{+\infty} \int_0^T \langle \varphi(s), e_j \rangle_U^2 \, ds.$$

We now note that for any $j \geq 1$, the function $s \to \langle \varphi(s), e_j \rangle_U$ belongs to $L^2([0,T]; \mathbb{R})$. Thus, by the above considerations,

$$\varphi^{(n)}(\cdot) := \sum_{j=1}^n \langle \varphi(\cdot), e_j \rangle_U e_j \in \mathcal{C}.$$

Finally, since $\lim_{n\to+\infty} \left\|\varphi - \varphi^{(n)}\right\|_{U_T}^2 = 0$, we conclude that $\varphi \in \mathcal{C}$. $\qquad\square$

By using the above lemma, we can extend W to U_T following the standard methods for extending an isometry. This establishes the following result.

Proposition 5.7 *For $t \geq 0$ and $\varphi \in U$, set $W_t(\varphi) := W\left(1_{[0,t]}(\cdot)\,\varphi(\star)\right)$. Then the process $W = \{W_t(\varphi), t \geq 0, \varphi \in U\}$ is a cylindrical Wiener process as in Definition 5.1, with V there replaced by U and $Q = I_V$. In particular, for any $\varphi \in U$, $\{W_t(\varphi), t \geq 0\}$ is a Brownian motion with variance $t \|\varphi\|_U$ and for all $s, t \geq 0$ and $\varphi, \psi \in U$, $E(W_t(\varphi) W_t(\psi)) = (s \wedge t) \langle \varphi, \psi \rangle_U$.*

Remark 5.8 This proposition allow us to use the stochastic integration defined in Sect. 5.1. This defines the stochastic integral $g \cdot W$ for all $g \in L^2(\Omega \times [0,T]; U)$. In order to use the stochastic integral of Sect. 5.1, let $(e_j)_j \subset \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N)$ be a complete orthonormal basis of U, and consider the cylindrical Wiener process $\{W_t(\varphi)\}$ defined

in Proposition 5.7. For any predictable process g in $L^2\left(\Omega \times [0, T]; U\right)$, the stochastic integral with respect to W is

$$g \cdot W = \int_0^T g_s dW_s := \sum_{j=1}^{+\infty} \int_0^T \langle g_s, e_j\rangle_U \, dW_s\left(e_j\right),$$

and the isometry property is given by

$$E\left((g \cdot W)^2\right) = E\left(\left(\int_0^T g_s dW_s\right)^2\right) = E\left(\int_0^T \|g_s\|_U^2 \, ds\right). \qquad (5.4)$$

We also use the notation $\int_0^T \int_{\mathbb{Q}_p^N} g(s, y) W(ds, dy)$ instead of $\int_0^T g_s dW_s$. In later sections we will also use the notation $E\left(\int_0^T \|g(s)\|_U^2 \, ds\right)$ for $E\left((g \cdot W)^2\right)$.

5.3 The Spectral Measure

Recall that μ is the spectral measure of W. In the following we use a function Γ satisfying the following hypothesis:

Hypothesis A The function Γ is defined on $\mathbb{R}_+ := [0, +\infty)$ with values in $\mathcal{D}'_{\mathbb{R}}(\mathbb{Q}_p^N)$ such that, for all $t > 0$, $\Gamma(t)$ is a positive distribution satisfying

$$\int_0^T dt \int_{\mathbb{Q}_p^N} |\mathcal{F}\Gamma(t)(\xi)|^2 \, d\mu(\xi) < +\infty, \qquad (5.5)$$

and Γ is associated with a measure $\Gamma\left(t, d^N x\right)$ such that, for all $T > 0$,

$$\sup_{0 \le t \le T} \Gamma\left(t, \mathbb{Q}_p^N\right) < +\infty. \qquad (5.6)$$

We now set $\Gamma(t, x) = \mathcal{F}_{\xi \to x}^{-1}\left(e^{-t|a(\xi)|_p^\beta}\right)$, for $t > 0$, and $\Gamma(0, x) := \delta(x)$, i.e Γ is the fundamental solution of (3.4), and since $\Gamma(t, x) \in L^1\left(\mathbb{Q}_p^N, d^N x\right)$ for $t > 0$, it defines an element of $\mathcal{D}'_{\mathbb{R}}(\mathbb{Q}_p^N)$. In addition, $\Gamma\left(t, d^N x\right) := \Gamma(t, x) d^N x$, and by Theorem 3.3 (ii) and (v),

$$\sup_{0 \le t \le T} \Gamma\left(t, \mathbb{Q}_p^N\right) = \sup_{0 < t \le T} \int_{\mathbb{Q}_p^N} \Gamma(t, x) \, d^N x = 1.$$

Hence $\Gamma\left(t, d^N x\right)$ satisfies (5.6).

Remark 5.9 If $H(t, x)$ is a function on $\mathbb{R} \times \mathbb{Q}_p^N$, we use $H(t)$ instead of $H(t, \cdot)$. If $G(t, x, \omega)$ is a function on $\mathbb{R} \times \mathbb{Q}_p^N \times \Omega$, we use $G(t, x)$ instead of $G(t, x, \omega)$, as it is customary in probability, in certain special cases we will use $G(t, x)(\omega)$.

On the other hand, by using Fubini's Theorem and inequality (3.1), it is easy to check that condition (5.5) is equivalent to

$$\int_{\mathbb{Q}_p^N} \frac{d\mu(\xi)}{\max\left(1, \|\xi\|_p\right)^{d\beta}} < +\infty. \tag{5.7}$$

Lemma 5.10 *With the notation of Lemma 5.4, assuming* $\int_{\mathbb{Q}_p^N} f(x) K_{d\beta}(x) d^N x < +\infty$ *and (5.7), we have*

$$\int_{\mathbb{Q}_p^N} \frac{d\mu(\xi)}{\max\left(1, \|\xi\|_p\right)^{d\beta}} = \int_{\mathbb{Q}_p^N} f(x) K_{d\beta}(x) d^N x.$$

Proof Set

$$\delta_n(x) := p^{Nn} \Omega\left(p^n \|x\|_p\right), \text{ for } n \in \mathbb{N}. \tag{5.8}$$

Then $\int_{\mathbb{Q}_p^N} \delta_n(x) d^N x = 1$ for any n, $\delta_n \xrightarrow{\mathcal{D}'} \delta$ and $\mathcal{F}\delta_n(\xi) = \Omega\left(p^{-n} \|\xi\|_p\right)$ pointwise 1. Notice that $\left(K_{d\beta} * \delta_n\right)(x) = K_{d\beta}(x)$, for $x \in \mathbb{Q}_p^N \smallsetminus \{0\}$ and for any $n > N(x)$, since $K_{d\beta}$ is radial, then

$$f(x)\left(K_{d\beta} * \delta_n\right)(x) = f(x) K_{d\beta}(x), \text{ for } x \in \mathbb{Q}_p^N \smallsetminus \{0\} \text{ and } n \text{ big enough.} \tag{5.9}$$

Now, by the Riesz–Markov–Kakutani Theorem, μ is an element of $\mathcal{D}'\left(\mathbb{Q}_p^N\right)$ and since $K_{d\beta} * \delta_n \in \mathcal{D}\left(\mathbb{Q}_p^N\right)$, we have

$$\left(\mu, \mathcal{F}\left(K_{d\beta} * \delta_n\right)\right) = \left(\mathcal{F}\mu, K_{d\beta} * \delta_n\right) = \left(f, K_{d\beta} * \delta_n\right)$$
$$= \int_{\mathbb{Q}_p^N} f(x) \left(K_{d\beta} * \delta_n\right)(x) d^N x.$$

Then by applying the Dominated Convergence Theorem and using (5.9) and the hypothesis $\int_{\mathbb{Q}_p^N} f(x) K_{d\beta}(x) d^N x < +\infty$, we get

$$\lim_{n \to +\infty} \int_{\mathbb{Q}_p^N} f(x) \left(K_{d\beta} * \delta_n\right)(x) d^N x = \int_{\mathbb{Q}_p^N} f(x) K_{d\beta}(x) d^N x.$$

On the other hand, by the Riesz-Markov-Kakutani Theorem,

$$\left(\mu, \mathcal{F}\left(K_{d\beta} * \delta_n \right) \right) = \left(\mu, \frac{(\mathcal{F}\delta_n)(\xi)}{\max\left(1, \|\xi\|_p \right)^{d\beta}} \right)$$

$$= \int_{\mathbb{Q}_p^N} \frac{(\mathcal{F}\delta_n)(\xi)}{\max\left(1, \|\xi\|_p \right)^{d\beta}} d\mu(\xi),$$

now, by the Dominated Convergence Theorem and Hypothesis (5.7),

$$\lim_{n \to +\infty} \int_{\mathbb{Q}_p^N} \frac{(\mathcal{F}\delta_n)(\xi)}{\max\left(1, \|\xi\|_p \right)^{d\beta}} d\mu(\xi) = \int_{\mathbb{Q}_p^N} \frac{d\mu(\xi)}{\max\left(1, \|\xi\|_p \right)^{d\beta}}.$$

$\square$

From Lemmas 5.4–5.10, we obtain the following result:

Theorem 5.11

$$\int_{\mathbb{Q}_p^N} \frac{d\mu(\xi)}{\max\left(1, \|\xi\|_p \right)^{d\beta}} < +\infty$$

$$\Leftrightarrow \begin{cases} \frac{1-p^{-d\beta}}{1-p^{d\beta-N}} \int_{\|x\|_p \leq 1} \left(\|x\|_p^{d\beta-N} - p^{d\beta-N} \right) f(x)\, d^N x < +\infty \text{ if } d\beta \neq N \\ \\ \left(1 - p^{-N} \right) \int_{\|x\|_p \leq 1} \log_p \left(\frac{p}{\|x\|_p} \right) f(x)\, d^N x < +\infty \qquad \text{if } d\beta = N. \end{cases}$$

5.4 Examples of Integrands

The main examples of integrands are provided by the following result:

Proposition 5.12 *Assume that Γ satisfies Hypothesis A. Let*

$$Y = \left\{ Y(t, x), (t, x) \in [0, T] \times \mathbb{Q}_p^N \right\}$$

be a predictable process such that

$$C_Y := \sup_{(t,x) \in [0,T] \times \mathbb{Q}_p^N} E\left(|Y(t, x)|^2 \right) < +\infty.$$

Then, the random element $G = G(t, x) = Y(t, x) \Gamma(t, x)$ is a predictable process with values in $L^2(\Omega \times [0, T]; U)$. Moreover,

$$E\left(\|G\|^2_{U_T}\right) = E\left[\int_0^T \int_{\mathbb{Q}_p^N} |\mathcal{F}(\Gamma(t) Y(t)(\xi))|^2 d\mu(\xi) dt\right]$$

$$\leq C_Y \int_0^T \int_{\mathbb{Q}_p^N} |\mathcal{F}(\Gamma(t) Y(t)(\xi))|^2 d\mu(\xi) dt$$

and

$$E\left(|G \cdot W|^2\right) \leq \int_0^T \left(\sup_{x \in \mathbb{Q}_p^N} E\left(|Y(s, x)|^2\right)\right) \int_{\mathbb{Q}_p^N} |\mathcal{F}(\Gamma(s)(\xi))|^2 d\mu(\xi) ds.$$

$$(5.10)$$

Remark 5.13 The integral of G with respect to W will be also denoted by

$$G \cdot W = \int_0^T \int_{\mathbb{Q}_p^N} \Gamma(s, y) Y(s, y) W\left(ds, d^N y\right).$$

Proof The proof will be accomplished through several steps.

§1. **Assertion A**: $G(t) \in L^1\left(\mathbb{Q}_p^N\right)$, for $t \in (0, T]$ a.s.

Indeed, by the Hölder inequality,

$$\int_{\mathbb{Q}_p^N} E\left(|Y(t, x)|^2\right) \Gamma(t, x) d^N x \leq C_Y \|\Gamma(t)\|_{L^1\left(\mathbb{Q}_p^N\right)}, \text{ for } t \in (0, T],$$

cf. Theorem 3.3 (iii). Hence,

$$\int_\Omega \int_{\mathbb{Q}_p^N} |Y(t, x)(\omega)|^2 \Gamma(t, x) d^N x dP(\omega) < +\infty,$$

and by Fubini's Theorem, $|Y(t, x)|^2 \Gamma(t, x) \in L^1\left(\mathbb{Q}_p^N\right)$, for $t \in (0, T]$ a.s. Now,

$$\int_{\mathbb{Q}_p^N} |Y(t, x)| \Gamma(t, x) d^N x = \int_{|Y(t,x)|>1} |Y(t, x)| \Gamma(t, x) d^N x$$

$$+ \int_{|Y(t,x)|\leq 1} |Y(t, x)| \Gamma(t, x) d^N x$$

$$\leq \int_{\mathbb{Q}_p^N} |Y(t, x)|^2 \Gamma(t, x) d^N x$$

$$+ \int_{\mathbb{Q}_p^N} \Gamma(t, x) d^N x < +\infty \text{ for } t \in (0, T] \text{ a.s.}$$

By using the above reasoning, one verifies that $\int_0^T \int_{\mathbb{Q}_p^N} |Y(t,x)| \, h(t,x) \, d^N x \, dt <$ $+\infty$, for every $h \in L^1\left([0,T] \times \mathbb{Q}_p^N\right)$ satisfying $h \geq 0$, therefore

$$Y(t,x) \in L^\infty\left([0,T] \times \mathbb{Q}_p^N\right) \text{ a.s.} \tag{5.11}$$

As a consequence of Assertion A, we have $G(t) \in \mathcal{D}_{\mathbb{R}}'\left(\mathbb{Q}_p^N\right)$, for $t \in (0,T]$ a.s. We now proceed to regularize this distribution. We set $\Delta_l(x) := \Omega\left(p^{-l} \|x\|_p\right)$, $\delta_k(x) = p^{kN} \Omega\left(p^k \|x\|_p\right)$ for $k, l \in \mathbb{N}$. Then $\int \delta_k(x) \, d^N x = 1$, $\mathcal{F}(\Delta_k) = \delta_k$, $\delta_k \xrightarrow{\mathcal{D}_{\mathbb{R}}'} \delta$ (Dirac distribution), and Δ_l pointwise 1, as before.

We also set $G_{k,l}(t) := (\Delta_l Y(t) \overrightarrow{\Gamma(t)}) * \delta_k$, $k, l \in \mathbb{N}$, $t \in (0,T]$, and $\Gamma_k(t) := \Gamma(t) * \delta_k$. Then, $G_{k,l}(t) \in \mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)$ for $t \in (0,T]$, more precisely,

$$G_{k,l}(t) = \sum_j c_j(t; k, l) \, \Omega\left(p^{m(k)} \|x - x_j\|_p\right).$$

Now, since

$$|(\Delta_l Y(t) \Gamma(t)) * \delta_k| \leq |Y(t)| \, \Gamma(t) * \delta_k \leq \|Y\|_{L^\infty\left([0,T] \times \mathbb{Q}_p^N\right)} \|\Gamma(t) * \delta_k\|_{L^1\left(\mathbb{Q}_p^N\right)}$$

$$\leq p^{Nk} \|Y\|_{L^\infty\left([0,T] \times \mathbb{Q}_p^N\right)} \text{ a.s.,}$$

cf. (5.11), we have $c_j(t; k, l) \in L^\infty([0,T])$ a.s. Therefore

$$G_{k,l}(t) \in L^2([0,T]) \otimes_{\text{alg}} \mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right) \text{ a.s.} \tag{5.12}$$

§2. A bound for $\sup_{k,l \geq 1} E \left\|G_{k,l}\right\|_{U_T}^2$.

By using the definition of the convolution and the uniform bound for the square moments of Y, we get:

$$\sup_{k,l \geq 1} E \left\|G_{k,l}\right\|_{U_T}^2 = \sup_{k,l \geq 1} E \int_0^T \int_{\mathbb{Q}_p^N} \int_{\mathbb{Q}_p^N} G_{k,l}(t,x) \, f(x-y) \, G_{k,l}(t,y) \, d^N x \, d^N y \, dt$$

$$= \sup_{k,l \geq 1} E \int_0^T \int_{\mathbb{Q}_p^N} \int_{\mathbb{Q}_p^N} \left[\int_{\mathbb{Q}_p^N} \Delta_l(z) \, Y(t,z) \, \Gamma(t,z) \, \delta_k(x-z) \, d^N z\right]$$

$$\times f(x-y) \left[\int_{\mathbb{Q}_p^N} \Delta_l(z') \, Y(t,z') \, \Gamma(t,z') \, \delta_k(y-z') \, d^N z'\right] d^N x \, d^N y \, dt$$

$$\leq C_Y \sup_{k \geq 1} \int_0^T \int_{\mathbb{Q}_p^N} \int_{\mathbb{Q}_p^N} \Gamma_k(t,x) \, f(x-y) \, \Gamma_k(t,y) \, d^N x \, d^N y \, dt$$

$$= C_Y \sup_{k \geq 1} \int_0^T |\mathcal{F}\Gamma_k(t)(\xi)|^2 \, d\mu(\xi) \, dt \leq C_Y \int_0^T |\mathcal{F}\Gamma(t)(\xi)|^2 \, d\mu(\xi) \, dt,$$

because $\mathcal{F}\Gamma_k(t)(\xi) = \mathcal{F}\Gamma(t)(\xi) \cdot \Omega\left(p^{-k}\|\xi\|_p\right) \leq \mathcal{F}\Gamma(t)(\xi)$. Therefore

$$\sup_{k,l \geq 1} E\left(\|G_{k,l}\|_{U_T}^2\right) \leq C_Y \int_0^T |\mathcal{F}\Gamma(t)(\xi)|^2 \, d\mu(\xi) \, dt < +\infty \qquad (5.13)$$

for $t \in (0, T]$ a.s.

As a consequence, we get that $G_{k,l} \in L^2\left(\Omega \times [0, T]; U\right)$ for $k, l \geq 1$, since by (5.12), $G_{k,l}(t) \in L^2\left([0, T]\right) \otimes_{\mathrm{alg}} \mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)$ a.s.

§3. $\lim_{k \to +\infty} G_{k,l} \in L^2\left(\Omega \times [0, T]; U\right)$.

We set $G_l(t) := \Delta_l Y(t)\Gamma(t)$, $l \in \mathbb{N}$, $t \in (0, T]$. By using the reasoning given in Paragraph 2, we get

$$E\int_0^T \int_{\mathbb{Q}_p^N} |\mathcal{F}G_l(t)(\xi)|^2 \, d\mu(\xi) \, dt \leq C_Y \int_0^T \int_{\mathbb{Q}_p^N} |\mathcal{F}\Gamma(t)(\xi)|^2 \, d\mu(\xi) \, dt < +\infty,$$
$$\qquad (5.14)$$

for any $l \in \mathbb{N}$.

We now assert that $G_{k,l} \xrightarrow{U_T} G_l$ as $k \to +\infty$. Indeed,

$$E\int_0^T \int_{\mathbb{Q}_p^N} \left|\mathcal{F}G_{k,l}(t)(\xi) - \mathcal{F}G_l(t)(\xi)\right|^2 \, d\mu(\xi) \, dt$$

$$= E\int_0^T \int_{\mathbb{Q}_p^N} |\mathcal{F}G_l(t)(\xi)|^2 \, |\Delta_k(\xi) - 1|^2 \, d\mu(\xi) \, dt \to 0 \text{ as } k \to +\infty,$$

by the Dominated Convergence Theorem and (5.14). Hence $G_l \in L^2\left(\Omega \times [0, T]; U\right)$ and by (5.14),

$$\sup_{l \geq 1} E\left(\|G_l\|_{U_T}^2\right) \leq C_Y \int_0^T |\mathcal{F}\Gamma(t)(\xi)|^2 \, d\mu(\xi) \, dt. \qquad (5.15)$$

§4. $G_l \xrightarrow{U_T} G$, i.e. $\lim_{l \to +\infty} E\left(\|G - G_l\|_{U_T}^2\right) = 0$.
Indeed,

$$E\left(\|G - G_l\|_{U_T}^2\right) = E\int_0^T \int_{\mathbb{Q}_p^N} |\mathcal{F}G(t)(\xi) - \mathcal{F}G_l(t)(\xi)|^2 \, d\mu(\xi) \, dt$$

$$\leq 2E\int_0^T \int_{\mathbb{Q}_p^N} |\mathcal{F}G(t)(\xi)|^2 \, d\mu(\xi) \, dt$$

$$+ 2E\int_0^T \int_{\mathbb{Q}_p^N} |\mathcal{F}G_l(t)(\xi)|^2 \, d\mu(\xi) \, dt$$

$$\leq 4E\int_0^T \int_{\mathbb{Q}_p^N} |\mathcal{F}G(t)(\xi)|^2 \, d\mu(\xi) \, dt$$

$$\leq 4C_Y \int_0^T \int_{\mathbb{Q}_p^N} |\mathcal{F}\Gamma(t)(\xi)|^2 \, d\mu(\xi) \, dt < +\infty,$$

the last inequality was obtained by using the reasoning given in Paragraph 2. On the other hand,

$$E\left(\|G - G_l\|_{U_T}^2\right) = E\int_0^T \int_{\mathbb{Q}_p^N} |\mathcal{F}G(t)(\xi) - \mathcal{F}G(t)(\xi) * \delta_l(\xi)|^2 \, d\mu(\xi) \, dt.$$

Now by using the Dominated Convergence Theorem and the fact that

$$\lim_{l \to +\infty} \mathcal{F}G(t)(\xi) * \delta_l(\xi) = \mathcal{F}G(t)(\xi) \text{ almost everywhere,}$$

cf. [36, Theorem 1.14], we get that $\lim_{l \to +\infty} E\left(\|G - G_l\|_{U_T}^2\right) = 0$, which implies $G \in L^2(\Omega \times [0, T]; U)$.

Moreover, we deduce that

$$E\left(\|G\|_{U_T}^2\right) = E\left(\int_0^T \int_{\mathbb{Q}_p^N} |\mathcal{F}G(t)(\xi)|^2 \, d\mu(\xi) \, dt\right) = \lim_{l \to +\infty} E\left(\|G_l\|_{U_T}^2\right)$$

$$\leq C_Y \int_0^T \int_{\mathbb{Q}_p^N} |\mathcal{F}\Gamma(t)(\xi)|^2 \, d\mu(\xi) \, dt,$$

cf. (5.15).

§5. **A bound for** $E\left(|G \cdot W|^2\right)$.

The announced bound for $E\left(|G \cdot W|^2\right)$ is obtained from (5.4) by using a reasoning similar to the one used in Paragraph 2. □

Remark 5.14 Let Y be a process as in Proposition 5.12. Consider the processes of the form

$$\left\{Y(t, x), (t, x) \in [T_0, T] \times \mathbb{Q}_p^N\right\}$$

where $0 \leq T_0 < T$, then

$$E\left(|G \cdot W|^2\right) \leq \int_{T_0}^T \left(\sup_{x \in \mathbb{Q}_p^N} E\left(|Y(s, x)|^2\right)\right) \int_{\mathbb{Q}_p^N} |\mathcal{F}(\Gamma(s)(\xi))|^2 \, d\mu(\xi) \, ds.$$

$$(5.16)$$

6 Stochastic Pseudodifferential Equations Driven by a Spatially Homogeneous Noise

In this section we introduce a new class of stochastic pseudodifferential equations in $\mathbb{Q}_p^N$ driven by a spatially homogeneous noise, more precisely, we study the following class of stochastic equations:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) + A(\partial, \beta) u(t, x) = \sigma(u(t, x)) \dot{W}(t, x) + b(u(t, x)) \\ u(0, x) = u_0(x), \ t \geq 0, \ x \in \mathbb{Q}_p^N, \end{cases} \tag{6.1}$$

where the coefficients σ and b are real-valued functions and $\dot{W}(t, x)$ is the formal notation for the Gaussian random perturbation described in Sect. 5.2.

Recall that we are working with a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, where $(\mathcal{F}_t)_t$ is a filtration generated by the standard cylindrical Wiener process of Proposition 5.7. We fix a time horizon $T > 0$.

Definition 6.1 A real-valued adapted stochastic process

$$\left\{ u(t, x), (t, x) \in [0, T] \times \mathbb{Q}_p^N \right\}$$

is *a mild random field solution of* (6.1), if the following stochastic integral equation is satisfied:

$$u(t, x) = (\Gamma(t) * u_0)(x) + \int_0^t \int_{\mathbb{Q}_p^N} \Gamma(t - s, x - y) \sigma(u(s, y)) W\left(ds, d^N y\right)$$

$$+ \int_0^t ds \int_{\mathbb{Q}_p^N} \Gamma(s, y) b(u(t - s, x - y)) d^N y, \ \text{a.s.,} \tag{6.2}$$

for all $(t, x) \in [0, T] \times \mathbb{Q}_p^N$.

The stochastic integral on the right-hand side of (6.2) is as defined in Remark 5.8. In particular, we need to assume that for any (t, x) the fundamental solution $\Gamma(t - \cdot, x - \star)$ satisfies Hypothesis A, and to require that

$$s \to \Gamma(t - s, x - \star) \sigma(u(s, \star)), \text{ for } s \in [0, t],$$

defines a predictable process taking values in the space U such that

$$E\left(\int_0^t \|\Gamma(t - s, x - \star) \sigma(u(s, \star))\|_U^2 ds \right) < +\infty,$$

see Sect. 5.4. These assumptions will be satisfied by imposing that b and σ are Lipschitz continuous functions (see Theorem 6.4). The last integral on the right-hand side of (6.2) is considered in the pathwise sense.

The aim of this section is to prove the existence and uniqueness of a mild random field solution for stochastic integral equation (6.2). We are interested in solutions that are $L^2(\Omega)$-bounded and $L^2(\Omega)$-continuous.

Lemma 6.2 *Assume that $u_0 : \mathbb{Q}_p^N \to \mathbb{R}$ is measurable and bounded. Then*

$$(t, x) \to I_0(t, x) := (\Gamma(t) * u_0)(x)$$

is continuous and $\sup_{(t,x)\in[0,T]\times\mathbb{Q}_p^N} |I_0(t,x)| < +\infty.$

Proof Notice that

$$|I_0(t,x)| \leq \begin{cases} \|u_0\|_{L^\infty} \|\Gamma(t)\|_{L^1} & \text{for } t > 0 \\ \|u_0\|_{L^\infty} & \text{for } t = 0, \end{cases} \tag{6.3}$$

and

$$\sup_{(t,x)\in(0,T]\times\mathbb{Q}_p^N} |I_0(t,x)| \leq \|u_0\|_{L^\infty} \sup_{(t,x)\in(0,T]\times\mathbb{Q}_p^N} \|\Gamma(t)\|_{L^1} = \|u_0\|_{L^\infty}. \tag{6.4}$$

By combining (6.3)–(6.4), we get $\sup_{(t,x)\in[0,T]\times\mathbb{Q}_p^N} |I_0(t,x)| \leq \|u_0\|_{L^\infty}$.

The continuity of $I_0(t,x)$ at a point of the form (t_0, x_0), with $t_0 > 0$, follows by the Dominated Convergence Theorem and Theorem 3.3 (vi). The continuity of $I_0(t,x)$ at $(0, x_0)$ is a consequence of the fact that $\lim_{(t,x)\to(0,x_0)} I_0(t,x) = u_0(x_0) = I_0(0, x_0)$. □

Hypothesis B Let Γ be the fundamental solution of (3.4) as before. We assume that

$$\lim_{h\to 0^+} \int_0^T \int_{\mathbb{Q}_p^N} \sup_{|r-t|<h} |\mathcal{F}\Gamma(r)(\xi) - \mathcal{F}\Gamma(t)(\xi)|^2 \, d\mu(\xi) \, dt = 0. \tag{6.5}$$

Lemma 6.3 *If* $\int_{\mathbb{Q}_p^N} \|\xi\|_p^{d\beta} \, d\mu(\xi) < +\infty$, *then Hypothesis B holds. Furthermore, the condition* $\int_{\mathbb{Q}_p^N} \|\xi\|_p^{d\beta} \, d\mu(\xi) < +\infty$ *also implies* (5.5).

Proof By applying the Mean Value Theorem to the function $e^{-t|a(\xi)|_p^\beta}$, we have

$$\sup_{|r-t|<h} |\mathcal{F}\Gamma(r)(\xi) - \mathcal{F}\Gamma(t)(\xi)|^2 \leq h^2 |a(\xi)|_p^{2\beta} e^{-2(t-h)|a(\xi)|_p^\beta}$$

$$\leq C_1^{2\beta} h^2 \|\xi\|_p^{2d\beta} e^{-2C_0^\beta(t-h)\|\xi\|_p^{d\beta}},$$

cf. (3.1), and thus

$$\int_0^T \int_{\mathbb{Q}_p^N} \sup_{|r-t|<h} |\mathcal{F}\Gamma(r)(\xi) - \mathcal{F}\Gamma(t)(\xi)|^2 \, d\mu(\xi) \, dt$$

$$\leq h^2 \int_0^T \int_{\mathbb{Q}_p^N} \|\xi\|_p^{2d\beta} e^{-2C_0^\beta(t-h)\|\xi\|_p^{d\beta}} \, d\mu(\xi) \, dt.$$

In order to prove the result, it is sufficient to show that

$$\lim_{h\to 0^+} \int_0^T \int_{\mathbb{Q}_p^N} \|\xi\|_p^{2d\beta} e^{-2C_0^\beta(t-h)\|\xi\|_p^{d\beta}} \, d\mu(\xi) \, dt < +\infty. \tag{6.6}$$

Now, if

$$\int_0^T \int_{\mathbb{Q}_p^N} \|\xi\|_p^{2d\beta} \, e^{-2C_0^\beta t \|\xi\|_p^{d\beta}} \, d\mu\,(\xi)\, dt \, < \, +\infty, \tag{6.7}$$

then (6.6) follows by applying the Dominated Convergence Theorem. On the other hand, it is easy to check that $\int_{\mathbb{Q}_p^N} \|\xi\|_p^{d\beta} \, d\mu\,(\xi) < +\infty$ implies

$$\int_{\mathbb{Q}_p^N} \|\xi\|_p^{2d\beta} \int_0^T e^{-2C_0^\beta t \|\xi\|_p^{d\beta}} \, dt d\mu\,(\xi) \, < \, +\infty, \tag{6.8}$$

now (6.7) follows from (6.8) by using the Fubini Theorem. The last assertion in the statement follows from the fact that (5.5) is equivalent to (5.7).   □

Theorem 6.4 *Assume that b, σ are Lipschitz continuous functions, u_0 is measurable and bounded function, and that*

$$\int_{\mathbb{Q}_p^N} \|\xi\|_p^{d\beta} \, d\mu\,(\xi) \, < \, +\infty.$$

Then, there exists a unique mild random field solution $\left\{ u\,(t, x),\ (t, x) \in [0, T] \times \mathbb{Q}_p^N \right\}$ of (6.2). Moreover, u is $L^2\,(\Omega)$-continuous and

$$\sup_{(t,x)\in[0,T]\times\mathbb{Q}_p^N} E\left(|u\,(t, x)|^2\right) \, < \, +\infty. \tag{6.9}$$

Proof The proof involves similar techniques and ideas to those of [4,17,18]. We use the following Picard iteration scheme:

$$u^0\,(t, x) = I_0\,(t, x), \tag{6.10}$$

$$u^{n+1}(t, x) = u^0\,(t, x) + \int_0^t \int_{\mathbb{Q}_p^N} \Gamma\,(t - s, x - y)\, \sigma\left(u^n\,(s, y)\right) W\left(ds, d^N y\right)$$

$$+ \int_0^t \int_{\mathbb{Q}_p^N} b\left(u^n\,(t - s, x - y)\right) \Gamma\,(s, y)\, d^N y ds$$

$$=: u^0\,(t, x) + \mathcal{I}^n\,(t, x) + \mathcal{J}^n\,(t, x), \tag{6.11}$$

for $n \in \mathbb{N}$.

The proof will be accomplished through several steps.

§1. $u^n\,(t, x)$ is a well-defined measurable process.

We prove by induction on n that $\left\{ u^n\left(t, x\right), \left(t, x\right) \in [0, T] \times \mathbb{Q}_p^N \right\}$ is a well-defined measurable process satisfying

$$\sup_{(t,x)\in[0,T]\times\mathbb{Q}_p^N} E\left(\left| u^n\left(t, x\right) \right|^2 \right) < +\infty, \tag{6.12}$$

for $n \in \mathbb{N}$. By Lemma 6.2, $u^0\left(t, x\right)$ satisfies (6.12), and the Lipschitz property of σ implies that

$$\sup_{(t,x)\in[0,T]\times\mathbb{Q}_p^N} \sigma\left(u^0\left(t, x\right) \right) < +\infty.$$

By Proposition 5.12, the stochastic integral

$$\mathcal{I}^0\left(t, x\right) = \int_0^t \int_{\mathbb{Q}_p^N} \Gamma\left(t - s, x - y\right) \sigma\left(u^0\left(s, y\right) \right) W\left(ds, d^N y\right)$$

is well-defined and

$$E\left(\left| \mathcal{I}^0\left(t, x\right) \right|^2 \right) \leq C \int_0^t \sup_{z\in\mathbb{Q}_p^N} \left(1 + \left| u^0\left(s, z\right) \right|^2 \right) \int_{\mathbb{Q}_p^N} \left| \mathcal{F}\Gamma\left(t - s\right)\left(\xi\right) \right|^2 d\mu\left(\xi\right) ds$$

$$\leq C \sup_{(s,z)\in[0,T]\times\mathbb{Q}_p^N} \left(1 + \left| u^0\left(s, z\right) \right|^2 \right) \int_0^T J\left(s\right) ds, \tag{6.13}$$

where

$$J\left(s\right) = \int_{\mathbb{Q}_p^N} \left| \mathcal{F}\Gamma\left(s\right)\left(\xi\right) \right|^2 d\mu\left(\xi\right).$$

We now consider the pathwise integral

$$\mathcal{J}^0\left(t, x\right) = \int_0^t \int_{\mathbb{Q}_p^N} b\left(u^0\left(t - s, x - y\right) \right) \Gamma\left(s, y\right) d^N y ds.$$

By applying the Cauchy–Schwartz inequality with respect to the finite measure $\Gamma\left(s, y\right) d^N y ds$ on $[0, T] \times \mathbb{Q}_p^N$ and by using the Lipschitz property of b, one gets

$$\left| \mathcal{J}^0\left(t, x\right) \right|^2 \leq C \int_0^t \int_{\mathbb{Q}_p^N} \left(1 + \left| u^0\left(t - s, x - y\right) \right|^2 \right) \Gamma\left(s, y\right) d^N y ds, \tag{6.14}$$

which is uniformly bounded with respect to t and x. This fact together with (6.13) imply that $\left\{ u^1\left(t, x\right), \left(t, x\right) \in [0, T] \times \mathbb{Q}_p^N \right\}$ is a well-defined measurable process, cf.

Proposition 5.12. In addition , by (6.13)–(6.14) and Hypothesis A,

$$\sup_{(t,x)\in[0,T]\times\mathbb{Q}_p^N} E\left(\left|u^1(t,x)\right|^2\right) < +\infty.$$

Consider now the case $n > 1$ and assume that $\left\{u^n(t,x),(t,x)\in[0,T]\times\mathbb{Q}_p^N\right\}$ is a well-defined measurable process satisfying (6.12). By the same arguments as above, one proves that

$$E\left(\left|\mathcal{I}^{n+1}(t,x)\right|^2\right)\leq C\int_0^t \sup_{z\in\mathbb{Q}_p^N} E\left(1+\left|u^n(s,z)\right|^2\right)\int_{\mathbb{Q}_p^N}|\mathcal{F}\Gamma(t-s)(\xi)|^2\,d\mu(\xi)\,ds,\tag{6.15}$$

and that

$$E\left(\left|\mathcal{J}^{n+1}(t,x)\right|^2\right)\leq C\int_0^t \sup_{y\in\mathbb{Q}_p^N} E\left(1+\left|u^n(t-s,x-y)\right|^2\right)\int_{\mathbb{Q}_p^N}\Gamma(s,y)\,d^Ny\,ds.\tag{6.16}$$

Hence the integrals $\mathcal{I}^{n+1}(t,x)$ and $\mathcal{J}^{n+1}(t,x)$ exist, so that u^{n+1} is a well-defined measurable process satisfying (6.12).

§2. We now show that

$$\sup_{n\geq0}\sup_{(t,x)\in[0,T]\times\mathbb{Q}_p^N} E\left(\left|u^n(t,x)\right|^2\right) < +\infty.\tag{6.17}$$

Indeed, by using the estimates (6.15)–(6.16), we have

$$E\left(\left|u^{n+1}(t,x)\right|^2\right)\leq C\left(1+\int_0^t\left(1+\sup_{z\in\mathbb{Q}_p^N} E\left(\left|u^n(s,z)\right|^2\right)\right)(J(t-s)+1)\right)ds.$$

Now (6.17) follows from the version of Gronwall's Lemma presented in [18, Lemma 15].

§3. $u^n(t,x)\xrightarrow{L^2(\Omega)}u(t,x)$ uniformly in $x\in\mathbb{Q}_p^N$, $t\in[0,T]$.

Following the same ideas as in the proof of [18, Theorem 13], we take

$$M_n(t):=\sup_{(s,x)\in[0,t]\times\mathbb{Q}_p^N} E\left(\left\{u^{n+1}(s,x)-u^n(s,x)\right\}^2\right).$$

By using Proposition 5.12, the Lipschitz property of b and σ, and by applying the same arguments as above, one gets

$$M_n(t)\leq C\int_0^t M_{n-1}(s)(J(t-s)+1)\,ds.$$

Now by applying Gronwall's Lemma presented in [18, Lemma 15], we get

$$\lim_{n \to +\infty} \left(\sup_{(s,x) \in [0,t] \times \mathbb{Q}_p^N} E \left(\left| u^{n+1}(s,x) - u^n(s,x) \right|^2 \right) \right) = 0.$$

Hence $\{u^n(t,x)\}_{n \in \mathbb{N}}$ converges uniformly in $L^2(\Omega)$ to a limit $u(t,x)$. From this fact, we get

$$\lim_{n \to +\infty} \sup_{(s,x) \in [0,T] \times \mathbb{Q}_p^N} E \left(\left| u^n(t,x) - u(t,x) \right|^2 \right) = 0. \tag{6.18}$$

Finally, by (6.18)–(6.17),

$$E \left(\left| u(t,x) \right|^2 \right) = \lim_{n \to +\infty} E \left(\left| u^n(t,x) \right|^2 \right) \leq \sup_{n \geq 0} \sup_{(t,x) \in [0,T] \times \mathbb{Q}_p^N} E \left(\left| u^n(t,x) \right|^2 \right) < +\infty.$$

§4. The process $\left\{ u(t,x), (t,x) \in [0,T] \times \mathbb{Q}_p^N \right\}$ is $L^2(\Omega)$-continuous and has a jointly measurable version.

The proof of this fact is based on the following result. Let $\mathcal{L}$ be a complete separable metric space, and $\mathcal{B}(\mathcal{L})$ the σ-algebra of Borel sets of $\mathcal{L}$, and let X_s, $s \in \mathcal{L}$ be a real stochastic process on $(\Omega, \mathcal{F}, P)$, where real means $[-\infty, +\infty]$-valued. The process $X_s, s \in \mathcal{L}$, is jointly measurable if the map $(s, \omega) \to X_s(\omega)$ is $\mathcal{B}(\mathcal{L}) \times \mathcal{F}$-measurable. Let $\mathcal{M}$ be the space of all real random variables on $(\Omega, \mathcal{F}, P)$ with the topology of convergence in probability. Then $X_s, s \in \mathcal{L}$, has a jointly measurable modification if and only if the map from $\mathcal{L}$ to $\mathcal{M}$ taking s to $[X_s]$, the class of X_s in $\mathcal{M}$, is measurable, see [14,15, Theorem 3].

In our case, $\mathcal{L} = \left([0,T] \times \mathbb{Q}_p^N, d \right)$ with

$$d \left((t,x), (t',x') \right) := \max \left\{ |t - t'|, \| x - x' \|_p \right\}. \tag{6.19}$$

Then $\mathcal{B}([0,T] \times \mathbb{Q}_p^N) = \mathcal{B}([0,T]) \times \mathcal{B}(\mathbb{Q}_p^N)$. It is sufficient to show that the map from $[0,T] \times \mathbb{Q}_p^N$ to $\mathcal{M}$ taking (t,x) to $u(t,x)$ is continuos in $L^2(\Omega)$. And since the convergence of $u^{n+1}(t,x)$ to $u(t,x)$ is uniform in $L^2(\Omega)$, it is sufficient to show that $u^{n+1}(t,x)$ is L^2-continuous. In order to do this, we have to verify that

$$\lim_{h \to 0} E \left(\left| u^{n+1}(t,x) - u^{n+1}(t+h,x) \right|^2 \right) = 0 \tag{6.20}$$

and

$$\lim_{x \to y} E \left(\left| u^{n+1}(t,x) - u^{n+1}(t,y) \right|^2 \right) = 0. \tag{6.21}$$

Indeed, (6.20) implies that $x \to u^{n+1}(t,x)$ is uniformly continuous in $L^2(\Omega)$ and (6.21) implies that $t \to u^{n+1}(t,x)$ is continuous in $L^2(\Omega)$, therefore $(t,x) \to$

$u^{n+1}(t, x)$ is continuous in $L^2(\Omega)$. The proof of this fact follows from Hypotheses A and B by using the technique given in [18] to prove Lemma 19.

§5. $u(t, x)$ is a solution of (6.2).

We set

$$\mathcal{I}(t, x) := \int_0^t \int_{\mathbb{Q}_p^N} \Gamma(t - s, x - y) \sigma(u(s, y)) W(ds, d^N y)$$

and

$$\mathcal{J}(t, x) := \int_0^t \int_{\mathbb{Q}_p^N} b(u(t - s, x - y)) \Gamma(s) d^N y ds.$$

In order to to establish Step 5, it is sufficient to show that

$$\lim_{n \to +\infty} \sup_{(t,x) \in [0,T] \times \mathbb{Q}_p^N} E\left(\left|\mathcal{I}^n(t, x) - \mathcal{I}(t, x)\right|^2\right) = 0, \tag{6.22}$$

and that

$$\lim_{n \to +\infty} \sup_{(t,x) \in [0,T] \times \mathbb{Q}_p^N} E\left(\left|\mathcal{J}^n(t, x) - \mathcal{J}(t, x)\right|^2\right) = 0. \tag{6.23}$$

To show (6.22) we proceed as follows. By The Lipschitz property of σ, Proposition 5.12 and Hypothesis A,

$$E\left(\left|\mathcal{I}^n(t, x) - \mathcal{I}(t, x)\right|^2\right)$$

$$\leq E\left(\left|\int_0^t \int_{\mathbb{Q}_p^N} \Gamma(t - s, x - y)\left[\sigma\left(u^{n-1}(s, y)\right) - \sigma(u(s, y))\right] W(ds, d^N y)\right|^2\right)$$

$$\leq C \int_0^t \sup_{z \in \mathbb{Q}_p^N} \left(\left|u^{n-1}(s, z) - u(s, z)\right|^2\right) \int_{\mathbb{Q}_p^N} |\mathcal{F}\Gamma(t - s)(\xi)|^2 d\mu(\xi) ds$$

$$\leq C \sup_{z \in \mathbb{Q}_p^N} \left(\left|u^{n-1}(s, z) - u(s, z)\right|^2\right),$$

this last term tends to zero as n tend to infinity. The case (6.23) can be treated in a similar form. By the results of Paragraph 4, the process $\left\{u(t, x), (t, x) \in [0, T] \times \mathbb{Q}_p^N\right\}$ has a measurable version that satisfies (6.2).

§6. $u(t, x)$ is the unique solution of (6.2) satisfying (6.9).

This fact can be checked by using standards arguments.   □

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