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Moduli of Smoothness Related to the Laplace-Operator

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Abstract We introduce and study a series of new moduli of smoothness in the multivariate case in L_p-spaces of periodic functions. The main focus lies on the case $0 < p < 1$. We prove a direct Jackson-type estimate and provide necessary and sufficient conditions with respect to the dimension d and to integrability p for the equivalence of these moduli and polynomial K-functionals related to the Laplace-operator. As a consequence we obtain an inverse Bernstein-type estimate. Moreover, we are able to characterize the approximation error in case of approximation by families of linear polynomial operators which are generated by Bochner–Riesz kernels in terms of the introduced moduli.

Keywords Trigonometric approximation · Fourier multipliers · Moduli of smoothness · K-functionals · Jackson- and Bernstein-type theorems · Bochner–Riesz means and families

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1 Introduction

For a 2π-periodic function $f(x)$, $x = (x_1, \dots, x_d)$, of d variables in the space L_p, $0 < p \le +\infty$, equipped with the standard norm denoted by $\|\cdot\|_p$ and for a natural number m we introduce a new modulus of smoothness by ($\delta \ge 0$)

$$\omega_{m,d}(f,\delta)_p = \sup_{0 \le h \le \delta} \left\| \frac{\sigma_m}{d} \sum_{j=1}^{d} \sum_{\substack{\nu = -m \\ \nu \neq 0}}^{m} \frac{(-1)^\nu}{\nu^2} \binom{2m}{m - |\nu|} f(x + \nu h e_j) - f(x) \right\|_p, \tag{1.1}$$

where

$$\sigma_m = \left(2 \sum_{\nu=1}^{m} \frac{(-1)^\nu}{\nu^2} \binom{2m}{m-\nu} \right)^{-1} \tag{1.2}$$

and e_j, $j = 1, \dots, d$, are the unit vectors in direction of the coordinates of the d-dimensional torus $\mathbb{T}^d$. In analogy to the classical one-dimensional modulus of smoothness we call the operators given by (I is the identity operator)

$$T_h^{(m,d)} f(x) = \frac{\sigma_m}{d} \sum_{j=1}^{d} \sum_{\substack{\nu = -m \\ \nu \neq 0}}^{m} \frac{(-1)^\nu}{\nu^2} \binom{2m}{m - |\nu|} f(x + \nu h e_j), \tag{1.3}$$

$$\Delta_h^{(m,d)} = T_h^{(m,d)} - I, \tag{1.4}$$

translation operator and *difference operator*, respectively. It will be shown in Sect. 2 that at least on the set $\mathcal{T}$ of real-valued trigonometric polynomials the identities

$$\Delta_h^{(m,d)} g(x) = \sum_{\nu \in \mathbb{Z}^d} \theta^{\wedge}_{m,d}(\nu)\, g(x + h\nu) = \sum_{k \in \mathbb{Z}^d} \theta_{m,d}(hk)\, g^{\wedge}(k)\, e^{ikx} \tag{1.5}$$

hold true. Here $g^{\wedge}(k)$, $k \in \mathbb{Z}^d$, are the Fourier coefficients of g and the *generator* $\theta_{m,d}$ of modulus (1.1) is defined by

$$\theta_{m,d}(\xi) = \frac{1}{d} \sum_{j=1}^{d} \theta_m(\xi_j),\ \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d, \tag{1.6}$$

$$\theta_m(\xi) = \gamma_m \int_0^{\xi} \int_0^{t} \left(\sin^{2m}(\tau/2) - \alpha_m \right) d\tau\, dt, \quad \xi \in \mathbb{R}, \tag{1.7}$$

where α_m is the mean value of the function $\sin^{2m}(\tau/2)$ on $[0, 2\pi]$ and γ_m is chosen such that the condition $\theta_m^\wedge(0) = -1$ is satisfied, i. e.,

$$\alpha_m = 2^{-2m}\binom{2m}{m}, \; \gamma_m = -2^{2m}\,\sigma_m = 2^{2m-1}\left(\sum_{\nu=1}^{m} \frac{(-1)^{\nu-1}}{\nu^2}\binom{2m}{m-\nu}\right)^{-1}. \tag{1.8}$$

Relations (1.8) follow from the well-known formula

$$\sin^{2m}(\tau/2) = 2^{1-2m}\sum_{\nu=1}^{m}(-1)^\nu\binom{2m}{m-\nu}\cos\nu\tau + 2^{-2m}\binom{2m}{m} \tag{1.9}$$

in combination with (1.2).

By Taylor's formula applied to the function $\sin x$ we obtain from (1.6)–(1.7) for $\xi \to 0$

$$\theta_{m,d}(\xi) = \frac{-\alpha_m\gamma_m}{2d}\,|\,\xi\,|^2 + \frac{\gamma_m 2^{-2m}}{d(2m+1)(2m+2)}\sum_{j=1}^{d}\xi_j^{2(m+1)} + O\Big(\sum_{j=1}^{d}\xi_j^{2(m+2)}\Big), \tag{1.10}$$

where $|\,\xi\,|^2 = \xi_1^2 + \cdots + \xi_d^2$. Taking into account that

$$\Delta g(x) = \sum_{j=1}^{d}\frac{\partial^2 g}{\partial x_j^2}(x) = -\sum_{k\in\mathbb{Z}^d}|\,k\,|^2 g^\wedge(k)\,e^{ikx} \tag{1.11}$$

for sufficiently smooth functions g, in particular for g in $\mathcal{T}$, in view of (1.5) and (1.10) we get

$$\Delta = \frac{2d}{\alpha_m\gamma_m}\lim_{h\to+0}\frac{T_h^{(m,d)} - I}{h^2} \tag{1.12}$$

in L_p-sense at least on the set $\mathcal{T}$ of real-valued trigonometric polynomials. The operator relation (1.12) shows that all moduli $\omega_{m,d}(f,\delta)_p$ are related to the Laplace-operator independently on m.

Some special cases of construction (1.1) are well-known. For example, the modulus $2\omega_{1,1}(f,\delta)_p$ coincides with the classical modulus smoothness of second order $\omega_2(f,\delta)_p$. In the d-dimensional case ($d > 1$) one has

$$\omega_{1,d}(f,\delta)_p = (2d)^{-1}\,\widetilde{\omega}(f,\delta)_p, \quad f \in L_p,\; \delta \ge 0 \tag{1.13}$$

for each $0 < p \le +\infty$, where

$$\widetilde{\omega}(f,\delta)_p = \sup_{0\le h\le\delta}\left\|\sum_{j=1}^{d}\big(f(x+he_j) + f(x-he_j)\big) - 2d\,f(x)\right\|_p \tag{1.14}$$

is the modulus introduced and studied by Z. Ditzian for $1 \leq p \leq +\infty$ in [2]. In particular, it has been shown that for $1 \leq p \leq +\infty$ the modulus $\widetilde{\omega}(f,\delta)_p$ is equivalent to the K-functional related to the Laplace-operator which is defined by

$$K_{\Delta}(f,\delta)_p = \inf_{g \in C^2} \left\{ \| f - g \|_p + \delta^2 \| \Delta g \|_p \right\}, \quad f \in L_p,\ \delta \geq 0, \tag{1.15}$$

where C^2 is the space of twice continuously differentiable 2π-periodic functions. Clearly, this result is an extension of the well-known one-dimensional result of Johnen with respect to the equivalence of the classical modulus of smoothness and J. Peetre's K-functional (see e. g. [1], Ch. 6) to the multivariate case.

The above result is not true for $0 < p < 1$. It has been proved in [3,6] that in this case K-functionals with classical derivatives are identically equal to 0. For this reason the concept of a polynomial K-functional given by

$$K_{\Delta}^{(\mathcal{P})}(f,\delta)_p = \inf_{T \in \mathcal{T}_{1/\delta}} \left\{ \| f - g \|_p + \delta^2 \| \Delta g \|_p \right\}, \quad f \in L_p,\ \delta > 0, \tag{1.16}$$

where ($\overline{c}$ is a complex conjugate to c)

$$\mathcal{T}_\sigma = \left\{ T(x) = \sum_{|k| \leq \sigma} c_k e^{ikx} \, : \, c_{-k} = \overline{c_k} \right\}, \quad \sigma \geq 0, \tag{1.17}$$

has been introduced in [6]. Note that in (1.15) the infimum is taken over the infinitely dimensional space C^2, whereas in (1.16) C^2 is replaced by the finite dimensional space $\mathcal{T}_{1/\delta}$ of real-valued trigonometric polynomials of (spherical) order at most $1/\delta$. Functionals (1.15) and (1.16) are shown to be equivalent if $1 \leq p \leq +\infty$ in [4].

Moreover, it follows from [6] that in the case $0 < p < 1, d = 1$ the polynomial K-functional given by (1.16) is equivalent to the classical modulus of smoothness of second order. In the multivariate case ($d > 1$) and if $0 < p < 1$ modulus (1.14) has been systematically studied in [5]. In particular, it is proved that in this case modulus (1.14) and polynomial K-functional (1.16) are equivalent if and only if $d/(d+2) < p \leq +\infty$. The occurence of the critical value $d/(d+2)$ can be explained as follows. Analysing the proof given in [5] one observes that the equivalence problem can be reduced to the behavior of the Fourier transform of the second item of expansion (1.10) with $m = 1$ divided by the generator of the Laplace-operator. The Fourier transform of the function

$$|\xi|^{-2} \left(\sum_{j=1}^{d} \xi_j^4 \right) \eta(\xi),$$

where η is an infinitely differentiable function with compact support satisfying $\eta(0) \neq 0$ (test-function), belongs to $L_p(\mathbb{R}^d)$ if and only if $p > d/(d+2)$. This follows from (Theorem 4.1, [10]) where it has been proved that the Fourier transform of $\psi\eta$ for an infinitely differentiable (defined on $\mathbb{R}^d\backslash\{0\}$) homogeneous function ψ of order $\alpha > 0$, which is not polynomial, belongs to the space $L_p(\mathbb{R}^d)$ if and only if $p > d/(d+\alpha)$.

In the general case the order of homogeneity of the second item in (1.10) divided by the generator of the Laplace-operator becomes $2m$. Taking into account the above arguments one can expect that in the multivariate case ($d > 1$) the modulus $\omega_{m,d}(f,\delta)_p$ will be equivalent to $K_{\Delta}^{(\mathcal{P})}(f,\delta)_p$ at least for $p > d/(d+2m)$. It means that in contrast to the modulus of Z. Ditzian the collection of moduli (1.1) "covers" the range of all admissible parameters $0 < p \leq +\infty$ in the sense that for each p there exists a natural number m such that the moduli (1.14) and functionals (1.16) are equivalent in L_p. The confirmation of this hypothesis is one of our main goals and will be done in Theorem 4.3. Moreover, in the present paper we essentially improve and simplify the research scheme given in [5]. In future work it will enable us to introduce and study general moduli of smoothness generated by arbitrary periodic functions satisfying some natural conditions.

Let us mention that there exists an universal modulus of smoothness related to the Laplace-operator which is relevant for all $0 < p \leq +\infty$ in the sense of its equivalence to a correponding polynomial K-functional in L_p for all admissible p. As it follows from the results below, in order to construct such a modulus it is enough to choose the Fourier coefficients of a certain 2π-periodic infinitely differentiable function θ satisfying $\psi(\xi) = -|\xi|^2$ near the point $\xi = 0$ as coefficients of values $f(x+\nu h)$, $\nu \in \mathbb{Z}^d$. However, such a construction is of theoretical interst only, since in contrast to (1.1) the Fourier coefficients of such a function can not be presented in an explicit form.

The paper is organized as follows. Section 1 provides necessary definitions, notations and preliminaries. The basic properties of moduli (1.1) are studied in Sect. 2. Section 3 is devoted to the proof of a Jackson-type estimate. The equivalence of moduli (1.1) and polynomial K-functionals related to the Laplace-operator is studied in Sect. 4. Some applications, in particular, the description of the quality of approximation by families of linear polynomial operators generated by Bochner–Riesz kernels in terms of $\theta_{m,d}$-moduli are given in Sect. 5. In this sense our paper is a continuation of [12].

2 Notations, Preliminaries and Auxiliary Results

2.1 Notational Agreements

By the symbols $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{Z}^d, \mathbb{Z}_+^d, \mathbb{R}^d$ we denote the sets of natural, non-negative integer, integer, real, complex numbers and d-dimensional vectors with integer, non-negative integer and real components, respectively. The symbol $\mathbb{T}^d$ is reserved for the d-dimensional torus $[0,2\pi)^d$. We shall also use the notations $xy = x_1y_1+\cdots+x_dy_d$, $|x| = (x_1^2+\cdots+x_d^2)^{1/2}$, $|x|_1 = |x_1|+\cdots+|x_d|$ for the scalar product as well as for 2- and 1-norms of $x = (x_1,\ldots,x_d)$. We denote by

$$B_r = \{x \in \mathbb{R}^d : |x| < r\},\ \overline{B}_r = \{x \in \mathbb{R}^d : |x| \leq r\}$$

the open and closed ball of radius r, respectively. Unimportant positive constants denoted by c (with subscripts and superscripts) may have different values in different formulas (but not in the same formula). By $A \lesssim B$ we denote the relation $A \leq cB$, where c is a positive constant independent of f (function) and n or δ (approximation methods, K-functionals and moduli may depend on). The symbol $\asymp$ indicates equivalence which means that $A \lesssim B$ and $B \lesssim A$ simultaneously.

2.2 Spaces L_p

As usual, $L_p \equiv L_p(\mathbb{T}^d)$, where $0 \leq p < +\infty$, is the space of measurable real-valued 2π-periodic with respect variable functions $f(x)$, $x = (x_1, \ldots, x_d)$, such that

$$\|f\|_p = \Big(\int\limits_{\mathbb{T}^d} |f(x)|^p \, dx \Big)^{1/p} < +\infty.$$

Moreover $C \equiv C(\mathbb{T}^d)$ $(p = +\infty)$ is the space of real-valued 2π-periodic continuous functions equipped with the Chebyshev norm

$$\|f\|_\infty = \|f\|_C = \max_{x \in \mathbb{T}^d} |f(x)|.$$

Spaces L_p of non-periodic functions defined on $\mathbb{R}^d$ will be denoted $L_p(\mathbb{R}^d)$. The functional $\|\cdot\|_p$ is a norm if and only if $1 \leq p \leq +\infty$. For $0 < p < 1$ it is a quasi-norm and the "triangle" inequality is valid for its pth power. If we put $\widetilde{p} = \min(1, p)$, the inequality

$$\|f + g\|_p^{\widetilde{p}} \leq \|f\|_p^{\widetilde{p}} + \|g\|_p^{\widetilde{p}}, \quad f, g \in L_p, \tag{2.1}$$

holds for all $0 < p \leq +\infty$. Such a form of the "triangle" inequality is convenient because both cases can be treated uniformly. Moreover, for the sake of simplicity we shall use the notation "norm" also in the case $0 < p < 1$.

2.3 Best Approximation and Jackson Type Estimate

We define, as usual, the best approximation of f by trigonometric polynomials of order σ in L_p by

$$E_\sigma(f)_p = \inf_{T \in \mathcal{T}_\sigma} \|f - T\|_p, \quad \sigma \geq 0. \tag{2.2}$$

Here $\mathcal{T}_\sigma$ is given by (1.17). As it has been shown in [5] the Jackson type estimate

$$E_\sigma(f)_p \leq c \sum_{j=1}^{d} \omega_k^{(j)}\big(f, (\sigma + 1)^{-1}\big)_p, \quad f \in L_p, \sigma \geq 0, \tag{2.3}$$

where the positive constant c is independent of f and σ, holds for all $k \in \mathbb{N}$ and $0 < p \le +\infty$. In (2.3) we used the notations

$$\omega_k^{(j)}(f,\delta)_p = \sup_{0\le h\le\delta} \Big\| \sum_{\nu=1}^{k} (-1)^{\nu+1} \binom{k}{\nu} f(x+\nu h e_j) - f(x) \Big\|_p, \quad \delta \ge 0, \quad (2.4)$$

for the partial modulus of smoothness of order k in direction e_j.

2.4 Spaces $\boldsymbol{l_q}$

As usual, $l_q \equiv l_q(\mathbb{Z}^d)$, where $0 < q < +\infty$, is the space of complex-valued sequences $(a(\nu))_{\nu\in\mathbb{Z}^d}$ defined on $\mathbb{Z}^d$ and satisfying

$$\|a\|_{l_q} = \Big(\sum_{\nu\in\mathbb{Z}^d} |a(\nu)|^q\Big)^{1/q} < +\infty.$$

The convolution of elements $a,\ b$ in l_q is given by

$$a * b(\nu) = \sum_{j\in\mathbb{Z}^d} a(j)b(\nu-j), \quad \nu \in \mathbb{Z}^d. \quad (2.5)$$

If $0 < q \le 1$ and if $a, b \in l_q$ then we have $a * b \in l_q$ and, moreover,

$$\|a * b\|_{l_q} \le \|a\|_{l_q}\|b\|_{l_q}. \quad (2.6)$$

This follows from (2.5) and the elementary inequality

$$\Big|\sum_j c(j)\Big|^q \le \sum_j |c(j)|^q, \quad 0 < q \le 1.$$

2.5 Fourier Transform and Fourier Coefficients

The Fourier transform of $g \in L_1(\mathbb{R}^d)$ is defined pointwise by

$$\widehat{g}(x) = \int_{\mathbb{R}^d} g(\xi)\, e^{-ix\xi} d\xi, \quad x \in \mathbb{R}^d. \quad (2.7)$$

For convenience we shall sometimes use also the notation $\mathcal{F}g$ in place of $\widehat{f}$.

The Fourier coefficients of $g \in L_1$ are defined by

$$g^\wedge(\nu) = (2\pi)^{-d} \int_{\mathbb{T}^d} g(\xi) e^{-i\nu\xi} d\xi, \quad \nu \in \mathbb{Z}^d. \quad (2.8)$$

To denote the sequence of Fourier coefficients of g we use shall the symbol $g^\wedge$, that is, $g^\wedge = \{g^\wedge(\nu)\}_{\nu\in\mathbb{Z}^d}$. It holds the equality

$$(g_1 \cdot g_2)^\wedge = g_1^\wedge * g_2^\wedge, \quad g_1, g_2 \in L_1. \tag{2.9}$$

Indeed, for trigonometric polynomials formula (2.9) can be proved by direct calculation applying (2.5). The extension to arbirary functions in L_1 is based on a density argument.

Henceforth, the symbol C^k, $k \in \mathbb{N}$, stands for the space of 2π-periodic k-times continuously differentiable functions of d variables.

Lemma 2.1 *Let* $0 < q < +\infty$ *and let* $g \in C^{d([1/q]+1)}$. *Then* $g^\wedge$ *belongs to* l_q.

Proof We put $k = [1/q] + 1$ and

$$M = \max_{1\le j_1<\cdots<j_n\le d} \left\| \frac{\partial^{nk} g}{\partial x_{j_1}^k, \ldots, \partial x_{j_n}^k} \right\|_C.$$

Since $nk \le d([1/q] + 1)$, the number M is finite. For any $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{Z}^d$ we choose the indices $1 \le j_1 < \cdots < j_n \le d$, for which $\nu_{j_r} \neq 0$, $r = 1, \ldots, n$. Integration by parts yields

$$\begin{aligned} |g^\wedge(\nu)| &= (2\pi)^{-d} \prod_{r=1}^{n} |\nu_{j_r}|^{-k} \left| \int_{\mathbb{T}^d} \frac{\partial^{nk} g(\xi)}{\partial\xi_{j_1}^k, \ldots, \partial\xi_{j_n}^k} e^{i\nu\xi} d\xi \right| \\ &\le M \prod_{r=1}^{n} |\nu_{j_r}|^{-k} \equiv M \prod_{r=1}^{n} \psi(\nu_j), \end{aligned} \tag{2.10}$$

where $\psi(\nu)$ is equal to $|\,\nu\,|^{-k}$ if $\nu \in \mathbb{Z}\backslash\{0\}$ and $\psi(0) = 1$. By means of (2.10) and taking into account that $kq > 1$ we obtain

$$\begin{aligned} \|g^\wedge\|_{l_q}^q &\le M \sum_{\nu\in\mathbb{Z}^d} \prod_{j=1}^{d} (\psi(\nu_j))^q = M \prod_{j=1}^{d} \sum_{\nu_j=-\infty}^{+\infty} (\psi(\nu_j))^q \\ &\le M \left(1 + \sum_{\nu\neq 0} |\nu|^{-kq}\right)^d < +\infty. \end{aligned}$$

Thus, the function $g^\wedge(\nu)$, $\nu \in \mathbb{Z}^d$, belongs to l_q. The proof of Lemma 1.1 is complete. □

2.6 Operators and Inequalities of Fourier Multiplier-Type

Let $\mathcal{X}(\xi)$, $\xi \in \mathbb{R}^d$, be real- or complex-valued satisfying $\mathcal{X}(-\xi) = \overline{\mathcal{X}(\xi)}$ for $\xi \in \mathbb{R}^d$. It generates the family of operators $\{A_\sigma(\mathcal{X})\}_{\sigma>0}$ putting

$$A_\infty(\mathcal{X}) \equiv \mathcal{X}(0)I; \quad A_\sigma(\mathcal{X})T(x) = \sum_{k\in\mathbb{Z}^d} \mathcal{X}\left(\frac{k}{\sigma}\right) T^\wedge(k) e^{ikx}, \quad T \in \mathcal{T}, \tag{2.11}$$

which is well-defined at least on the space $\mathcal{T}$ of real-valued trigonometric polynomials.

Let $0 < p \le +\infty$. We consider the inequality

$$\|A_\sigma(\mu)T\|_p \le c(p, \mu, \nu)\| A_\sigma(\nu)T\|_p, \quad T \in \mathcal{T}_\sigma, \ \sigma > 0. \tag{2.12}$$

Inequality (2.12) is said to be valid in L_p for some $0 < p \le +\infty$ if it holds in the L_p-norm for all $T \in \mathcal{T}_\sigma$ and for all $\sigma > 0$ with a certain positive constant independent of T and σ. Suppose that $\nu(\xi) \ne 0$ for $\xi \ne 0$. Then inequality

$$\|A_\sigma(\mathcal{X})T\|_p \le c'(p, \mu, \nu) \cdot \|T\|_p, \quad T \in \mathcal{T}_\sigma, \ \sigma > 0, \tag{2.13}$$

where

$$\mathcal{X}(\xi) = \frac{\mu(\xi)}{\nu(\xi)}, \quad \xi \in \mathbb{R}\backslash\{0\}, \tag{2.14}$$

is *associated with* (2.12). Clearly, (2.13) is of the same type, but the operator on the right-hand side is the identity I. Let (A) and (B) be inequalities of type (2.12). We say that inequality (A) implies inequality (B) for some p if the validity of (A) for p implies the validity of (B) for p. We also say that (A) implies (B) if this is the case for all $0 < p \le +\infty$.

Recall that $\widetilde{p} = \min(1, p)$. The following properties hold.

Proposition 2.2 *(i) If $\mu(0) = \nu(0) = 0$ then (2.13) implies (2.12) independently of the value $\mathcal{X}(0)$.*
(ii) Let $\mu(0) = \nu(0) = 0$. If $\mathcal{X}$ is continuous on $\mathbb{R}^d$ and if $\widehat{\mathcal{X}\eta} \in L_{\widetilde{p}}(\mathbb{R}^d)$ for a certain infinitely differentiable function η with compact support satisfying $\eta(\xi) = 1$ for $\xi \in B_1$, then inequality (2.12) is valid in L_p.
(iii) Let $\mathcal{X}$ be continuous on $\mathbb{R}^d$ and let η be an infinitely differentiable function with support contained the unit ball B_1. If (2.12) is valid for a certain parameter $0 < p \le +\infty$, then $\widehat{\mathcal{X}\eta} \in L_{p^*}(\mathbb{R}^d)$, where $p^* = p$ for $0 < p \le 2$ and $p^* = p/(p-1)$ for $2 < p \le +\infty$.

The continuity of $\mathcal{X}$ on $\mathbb{R}^d$ means that there exists $\lim_{\xi\to 0} \mathcal{X}(\xi)$. Proofs of (i)–(iii) can be found in [10] (Theorems 3.1 and 3.2) and [11]. For (ii) we also refer to [13], pp. 150–151.

2.7 Homogeneous Functions

Let $s > 0$. By H_s we denote the class of functions ψ satisfying the properties

(1) ψ is a complex-valued function defined on $\mathbb{R}^d$ and $\psi(-\xi) = \overline{\psi(\xi)}$ for $\xi \in \mathbb{R}^d$;
(2) ψ is continuous;
(3) ψ is infinitely differentiable on $\mathbb{R}^d\backslash\{0\}$;
(4) ψ is homogeneous of order s, i. e. $\psi(t\xi) = t^s\psi(\xi)$ for $t > 0$, $\xi \in \mathbb{R}^d\backslash\{0\}$;
(5) $\psi(\xi) \ne 0$ for $\xi \in \mathbb{R}^d\backslash\{0\}$.

Let η be an infinitely differentiable function defined on $\mathbb{R}^d$ satisfying $\eta(\xi) = 1$ for $|\xi| \le \rho_1$ and $\eta(\xi) = 0$ for $|\xi| \ge \rho_2$, where $0 < \rho_1 < \rho_2 < +\infty$. The following properties hold.

Proposition 2.3 *(i) If* $\psi_i \in H_{s_i}$, $i = 1, 2$, *then* $\psi_1\psi_2 \in H_{s_1+s_2}$.
(ii) If $\psi \in H_s$, $s > 1$, *and* $j = 1, \dots, d$, *then* $\partial\psi/\partial\xi_j \in H_{s-1}$.
(iii) If $\psi \in H_s$ *then there exists a positive constant* c_1 *such that*

$$|\widehat{\psi\eta}(x)| \le c_1 \left(|x| + 1\right)^{-(d+s)} \tag{2.15}$$

holds for all $x \in \mathbb{R}^d$.
(iv) If $\psi \in H_s$ *is not a polynomial, then there exist* $r_0 > 0$, $u_0 \in S^{d-1}$, *where* S^{d-1} *is the d-dimensional sphere, and* $0 < \theta_0 < \pi/2$ *such that*

$$|\widehat{\psi\eta}(x)| \ge c_2\left(|x| + 1\right)^{-(d+s)}, \quad x \in \Omega \equiv \Omega(r_0, u_0, \theta_0), \tag{2.16}$$

where

$$\Omega = \{x \in \mathbb{R}^d : x = ru, r \ge r_0, \ u \in S^{d-1}, (u, u_0) \ge 1 - \theta_0\} \tag{2.17}$$

and where the positive constant c_2 *is independent of* x.
(v) If $\psi \in H_s$ *is not a polynomial then the Fourier transform of* $\psi\eta$ *belongs to* $L_p(\mathbb{R}^d)$ *if and only if* $p > d/(d+s)$.

Statements (i)–(ii) are obvious. The proofs of (iii) and (iv) can be found in [10] (formulae (4.6) and (4.7)). Part (v) is a consequence of (iii) and (iv).

3 Basic Properties of the Moduli $\omega_{m,d}(f, \delta)_p$

Some elementary properties of modulus (1.1) are collected in the following.

Lemma 3.1 *Let* $m, d \in \mathbb{N}$, $0 < p \le +\infty$ *and let* $\widetilde{p} = \min(1, p)$.

(i) *The operators* $T_h^{(m,d)}$ *and* $\Delta_h^{(m,d)}$ *given by* (1.3) *and* (1.4), *respectively, are linear and uniformly bounded in* L_p.
(ii) *Modulus* (1.1) *is well-defined in* L_p *(convergence in* L_p*) and there exists a constant c such that*

$$\omega_{m,d}(f, \delta)_p \le c\|f\|_p < +\infty, \tag{3.1}$$

for each $f \in L_p$ *and* $\delta \ge 0$. *The function* $\omega_{m,d}(f, \cdot)$ *is increasing on* $[0, +\infty)$ *and it holds* $\omega_{m,d}(f, 0) = 0$.
(iii) *If* $f_1, f_2 \in L_p$ *and* $\delta \ge 0$ *then*

$$\omega_{m,d}(f_1 + f_2, \delta)_p^{\widetilde{p}} \le \omega_{m,d}(f_1, \delta)_p^{\widetilde{p}} + \omega_{m,d}(f_2, \delta)_p^{\widetilde{p}}. \tag{3.2}$$

(iv) *Let* A_σ *and* $\theta_{m,d}$ *be given by* (2.11) *and* (1.6)–(1.8)*, respectively. Then it holds*

$$\Delta_h^{(m,d)} = A_{h^{-1}}(\theta_{m,d}) \tag{3.3}$$

for each $h \geq 0$ *at least on the space* $\mathcal{T}$ *of real-valued trigonometric polynomials.*

Proof The linearity of translation and difference operator follows immediately from (1.3) and (1.4). Their uniform boundedness follows from the estimate

$$\|T_h^{(m,d)} f\|_p^{\widetilde{p}} \leq c \sum_{j=1}^{d} \sum_{\substack{\nu = -m \\ \nu \neq 0}}^{m} \|f(x + \nu h e_j)\|_p^{\widetilde{p}} \leq 2mdc\|f\|_p^{\widetilde{p}},$$

which can be derived from (1.3) and (2.1) for $f \in L_p$ and $h \geq 0$. Here the constant

$$c \equiv c(m, d) = \left(\frac{\sigma_m}{d}\right)^{\widetilde{p}} \max_{|\nu| \leq m} \binom{2m}{m - |\nu|}^{\widetilde{p}}$$

is independent of f and h. Part (i) is proved. Inequality (3.1) is a direct consequence of part (i) and the definition (1.1) of the modulus. The other statements of part (ii) immediately follow from (1.1). Inequality (3.2) follows from (1.1) in combination with (2.1).

It remains to prove part (iv). In view of (2.11) we have

$$\begin{aligned} A_{h^{-1}}(\theta_{m,d})T(x) &= \sum_{k \in \mathbb{Z}^d} \theta_{m,d}(hk) T^\wedge(k) e^{ikx} \\ &= \sum_{k \in \mathbb{Z}^d} T^\wedge(k) e^{ikx} \Big(\sum_{\nu \in \mathbb{Z}^d} \theta_{m,d}^\wedge(\nu)\, e^{i\nu kh} \Big) \\ &= \sum_{\nu \in \mathbb{Z}^d} \theta_{m,d}^\wedge(\nu) \Big(\sum_{k \in \mathbb{Z}^d} T^\wedge(k) e^{ik(x+\nu h)} \Big) \\ &= \sum_{\nu \in \mathbb{Z}^d} \theta_{m,d}^\wedge(\nu) T(x + \nu h) \end{aligned} \tag{3.4}$$

for each $T \in \mathcal{T}$ and $h \geq 0$. Applying formula (1.9) in combination with (1.2) and (1.6)–(1.8) we find the representation

$$\theta_{m,d}^\wedge(\nu) = \begin{cases} -1 & , \quad \nu = 0 \\ \dfrac{(-1)^{\nu_j}\sigma_m}{d\nu_j^2} \dbinom{2m}{m - |\nu_j|} & , \quad \nu = \nu_j e_j,\ 0 < |\nu_j| \leq m,\ j = 1, \ldots, d \\ 0 & , \quad \text{otherwise} \end{cases} \tag{3.5}$$

for the Fourier coefficients of the generator $\theta_{m,d}$. Now (3.3) follows from (3.4) and (3.5) by means of (1.3) and (1.4). This completes the proof. □

4 Jackson-Type Estimate

In this section we prove a Jackson-type estimate for modulus (1.1). Our approach is based on the comparison of $\omega_{m,g}(f,\delta)_p$ and the partial moduli $\omega_k^{(j)}(f,\delta)_p$ introduced in (2.4).

Lemma 4.1 *Let* $m, d \in \mathbb{N}$.

(i) *The generator* $\theta_{m,d}$ *given by* (1.6)–(1.8) *is analytic on* $\mathbb{R}^d$.
(ii) *We have* $\theta_{m,d}(\xi) < 0$ *for* $\xi \in \mathbb{R}^d \backslash 2\pi\mathbb{Z}^d$, *where* $2\pi\mathbb{Z}^d = \{2\pi\nu,\ \nu \in \mathbb{Z}^d\}$.
(iii) *The function* $1/\theta_{m,d}$ *is analytic on* $\mathbb{R}^d \backslash 2\pi\mathbb{Z}^d$.

Proof Part (i) follows immediately from (1.6)–(1.8). In view of (1.6) it is enough to prove part (ii) for $d = 1$. We consider the function

$$\varphi(t) = \int_0^t \big(\sin^{2m}(\tau/2) - \alpha_m\big)d\tau, \quad t \in \mathbb{R}. \tag{4.1}$$

Using (1.8) and (1.9) we obtain

$$\varphi(t) = 2^{1-2m}\sum_{\nu=1}^{m} \frac{(-1)^\nu}{\nu}\binom{2m}{m-\nu}\sin\nu t. \tag{4.2}$$

By (4.2) the function φ is a 2π-periodic, odd and satisfies $\varphi(\pi) = 0$. Using (4.1) and the properties of the function $\sin(\tau/2)$ it is easy to see that φ is decreasing on $[0, \xi_0]$ and increasing on $[\xi_0, \pi]$, where $\xi_0 \in (0, \pi)$ satisfies $\sin^{2m}(\xi_0/2) = \alpha_m$. According to these properties the function (see also 1.7, 1.8)

$$\gamma_m^{-1}\theta_m(\xi) = \int_0^\xi\int_0^t \big(\sin^{2m}(\tau/2) - \alpha_m\big)d\tau dt, \quad \xi \in \mathbb{R},$$

is 2π-periodic and even. It decreases on $[0, \pi]$ and it increases on $[\pi, 2\pi]$. Therefore,

$$\gamma_m^{-1}\,\theta_m(\xi) < 0, \quad \xi \in \mathbb{R}\backslash 2\pi\mathbb{Z}. \tag{4.3}$$

Combining (4.3) and (3.5) we get

$$\gamma_m^{-1} = -\gamma_m^{-1}\theta_m^{\wedge}(0) = -(2\pi)^{-1}\int_0^{2\pi}\gamma_m^{-1}\theta_m(\xi)d\xi > 0. \tag{4.4}$$

Now the statement of part (ii) immediately follows from (4.3) and (4.4). Part (iii) is a direct consequence of parts (i) and (ii). This completes the proof. □

Theorem 4.2 (Jackson-type estimate) *Let $m, d \in \mathbb{N}$, and let $0 < p \le \infty$. Then for any $\lambda > 0$*

$$E_\sigma(f)_p \;\le\; c_p(\lambda)\omega_{m,d}\big(f, \lambda(\sigma+1)^{-1}\big)_p, \quad f \in L_p,\ \sigma \ge 0, \tag{4.5}$$

where $c_p(\lambda)$ is a positive constant independent of f and σ.

Proof We put

$$k = 2^{d([1/\widetilde{p}]+1)+1} + d([1/\widetilde{p}]+1) + 1\,. \tag{4.6}$$

Let $j \in \{1, \dots, d\}$. By Lemma 4.1 the function

$$\Theta_j(\xi) = -\frac{(1-e^{i\xi_j})^k}{\theta_{m,d}(\xi)} \tag{4.7}$$

is analytic on $\mathbb{R}^d \backslash 2\pi\mathbb{Z}^d$. Let $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{Z}_+^d$ such that $|\nu|_1 \le d([1/\widetilde{p}]+1)$. Applying standard differentiation formulas and taking into account (1.9) and (4.6) we find

$$\left|\frac{\partial^{|\nu|_1}\Theta_j(\xi)}{\partial\xi_1^{\nu_1}, \dots, \partial\xi_d^{\nu_d}}\right| \le c\frac{|\xi_j|^{k-\nu_j}}{|\xi|^{2^{|\nu|_1+1}}} \le c|\xi|^{k-|\nu|_1-2^{|\nu|_1+1}} \le c|\xi|,$$

for $|\,\xi\,| \le 1$. In particular, it follows

$$\lim_{\xi\to 0}\frac{\partial^{|\nu|_1}\Theta_j(\xi)}{\partial\xi_1^{\nu_1}, \dots, \partial\xi_d^{\nu_d}} = 0.$$

Thus, the function Θ_j belongs to the space $C^{d([1/\widetilde{p}]+1)}$. By Lemma 2.1 the sequence $(\Theta_j^\wedge(\nu))_{\nu\in\mathbb{Z}^d}$ of its Fourier coefficients belongs to the space $l_{\widetilde{p}}$. Taking into account formula (3.4) with Θ_j in place of $\theta_{m,d}$ we can extend the operator $A_{h^{-1}}(\Theta_j)$, $h \ge 0$, which is initially defined on $\mathcal{T}$, to the space L_p by the formula

$$A_{h^{-1}}(\Theta_j)f(x) = \sum_{\nu\in\mathbb{Z}^d}\Theta_j^\wedge(\nu)f(x+\nu h). \tag{4.8}$$

Using (4.8) we get

$$\big\|A_{h^{-1}}(\Theta_j)f(x)\,\big\|_p^{\widetilde{p}} \le \sum_{\nu\in\mathbb{Z}^d}\big|\Theta_j^\wedge(\nu)\big|^{\widetilde{p}}\|f(x+\nu h)\|_p^{\widetilde{p}} = \big\|\Theta_j^\wedge\big\|_{l_{\widetilde{p}}}^{\widetilde{p}}\|f\|_p^{\widetilde{p}}$$

for each $f \in L_p$. This implies that the series on the right-hand side of (4.8) converges in L_p and, moreover,

$$\big\|A_{h^{-1}}(\Theta_j)\big\|_{(p)} \equiv \sup_{\|f\|_p\le 1}\big\|A_{h^{-1}}(\Theta_j)f(x)\,\big\|_p \le \big\|\Theta_j^\wedge\big\|_{l_{\widetilde{p}}} < +\infty. \tag{4.9}$$

Taking into account that the coefficients in (2.4) are the Fourier coefficients of the function $\theta_j(\xi) = -(1 - e^{i\xi_j})^k$ and applying (3.4) with θ_j in place of $\theta_{m,d}$ we can rewrite the definition of the partial modulus of smoothness of order k defined in (2.4) as

$$\omega_k^{(j)}(f, \delta)_p = \sup_{0 \le h \le \delta} \left\| A_{h^{-1}}(\theta_j) f \right\|_p, \quad f \in L_p, \delta \ge 0. \tag{4.10}$$

In view of (2.11), (3.3) and (4.7) we have

$$A_{h^{-1}}(\theta_j) = A_{h^{-1}}(\Theta_j) \circ A_{h^{-1}}(\theta_{m,d}) = A_{h^{-1}}(\Theta_j) \circ \Delta_h^{(m,d)} \tag{4.11}$$

in L_p for each $h \ge 0$. Combining (4.9) and (4.11) we obtain

$$\begin{aligned} \left\| A_{h^{-1}}(\theta_j) f(x) \right\|_p &\le \left\| A_{h^{-1}}(\Theta_j) \right\|_{(p)} \left\| \Delta_h^{(m,d)} f(x) \right\|_p \\ &\le \left\| \Theta_j^{\wedge} \right\|_{l_p} \left\| \Delta_h^{(m,d)} f(x) \right\|_p \end{aligned} \tag{4.12}$$

for $f \in L_p$ and $h \ge 0$. Combining (1.1), (1.3), (1.4), (4.10), and (4.12) we get the estimate

$$\omega_k^{(j)}(f, \delta)_p \le c\omega_{m,d}(f, \delta)_p, \quad f \in L_p, \delta \ge 0, \tag{4.13}$$

where the positive constant c is independent of f and δ.

Recall that the inequality

$$\omega_k^{(j)}(f, t\delta)_p \le \ (1 + t)^{k/\widetilde{p}}\, \omega_k^{(j)}(f, \delta)_p, \quad f \in L_p, t, \delta \ge 0, \tag{4.14}$$

holds for the classical moduli of smoothness (see e. g. [1]). Combining (2.3), (4.13), and (4.14) we find the estimates

$$\begin{aligned} E_\sigma(f)_p &\le c \sum_{j=1}^{d} \omega_k^{(j)}\big(f, (\sigma + 1)^{-1}\big)_p \le c_1 \sum_{j=1}^{d} \omega_k^{(j)}\big(f, \lambda\,(\sigma + 1)^{-1}\big)_p \\ &\le c_2\, \omega_{m,d}\big(f, \lambda\,(\sigma + 1)^{-1}\big)_p \end{aligned}$$

for $f \in L_p, \sigma \ge 0$ and $\lambda > 0$, where the positive constants c, c_1 and c_2 are independent of f and σ. The proof is complete. □

5 Equivalence of $\omega_{m,d}(f, \delta)_p$ and $K_{\Delta}^{(\mathcal{P})}(f, \delta)_p$

In order prove the main result of this paper on the equivalence of moduli (1.1) and functionals (1.16) we need some auxiliary results.

Lemma 5.1 *Let $s, d \in \mathbb{N}$ and assume $s, d > 1$. The polynomial $P_s(\xi) = \xi_1^{2s} + \cdots + \xi_d^{2s}$ is divisible by $|\,\xi\,|^2 = \xi_1^2 + \cdots + \xi_d^2$ if and only if $d = 2$ and s is an odd number.*

Proof Sufficiency Let $d = 2$ and let $s = 2k + 1, k \in \mathbb{N}$. Then

$$|\xi|^2 \sum_{\nu=0}^{s-1} (-1)^j \, \xi_1^{2(s-j-1)} \, \xi_2^{2j} = \xi_1^{2s} + (-1)^{s+1} \xi_2^{2s} = P_s(\xi).$$

Necessity Suppose that $P_s(\xi)$ is divisible by $|\, \xi \,|^2$. Then the function

$$Q_s(x) = \frac{x_1^s + \cdots + x_d^s}{x_1 + \cdots + x_d}, \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d,$$

is a polynomial as well. In particular, $\lim_{x \to x^0} Q_s(x)$ exists for any $x^0 \in \mathbb{R}^d$. Let first $d = 2$. We put $x^0 = (1, -1)$. Since $x_1 + x_2$ tends to 0 for $x \to x^0$ the sum $x_1^s + x_2^s$ should also tend to 0. It yields that $1 + (-1)^s = 0$ and that s is an odd number. Let now $d \geq 3$. If $x^0 = (1, 1, -2, 0, \ldots, 0)$ then $x_1 + \cdots + x_d$ tends to 0. Therefore, $x_1^s + \cdots + x_d^s$ also tends to 0. It implies $1 + 1 + (-2)^s = 0$. Hence, s should be equal to 1. The proof of Lemma 5.1 is complete. □

Let v and w be continuous functions defined on $\mathbb{R}^d$ and let $0 < p \leq +\infty$. In the following we write $v(\cdot) \overset{(p)}{\prec} w(\cdot)$, if there exists a function η infinitely differentiable on $\mathbb{R}^d$, satisfying $\eta(\xi) = 1$ for $|\, \xi \,| \leq \rho_1$ and $\eta(\xi) = 0$ for $|\, \xi \,| \geq \rho_2$, where $0 < \rho_1 < \rho_2 < +\infty$, such that $\mathcal{F}(\,(\eta v)/w\,)$ belongs to $L_p(\mathbb{R}^d)$. The notation $v(\cdot) \overset{(p)}{\asymp} w(\cdot)$ indicates equivalence. It means that $v(\cdot) \overset{(p)}{\prec} w(\cdot)$ and $w(\cdot) \overset{(p)}{\prec} v(\cdot)$ hold simultaneously.

For $d, m \in \mathbb{N}$ we introduce the number as

$$p_{m,d} = \begin{cases} 0 & , \quad d = 1 \\ \dfrac{d}{d + 2(m+1)} & , \quad d = 2, m = 2k, k \in \mathbb{N} \\ \dfrac{d}{d + 2m} & , \quad \text{otherwise} \end{cases}. \tag{5.1}$$

Lemma 5.2 *Let $m, d \in \mathbb{N}$ and let $0 < p < +\infty$.*

(i) *It holds $|\cdot|^2 \overset{(p)}{\asymp} \theta_{m,d}(\cdot)$ for $p > p_{m,d}$.*

(ii) *If $d > 1$ and $0 < p \leq p_{m,d}$, then both relations $|\cdot|^2 \overset{(p)}{\prec} \theta_{m,d}(\cdot)$ and $\theta_{m,d}(\cdot) \overset{(p)}{\prec} |\cdot|^2$ are false.*

Proof First we consider the most general case $d \geq 3$ or $d = 2, m = 2k - 1, k \in \mathbb{N}$. Combining (1.6), (1.7), (1.9) and using the power series representation of $\cos x$ we see that

$$|\xi|^{-2}\, \theta_{m,d}(\xi) = \alpha^{(m,d)} + \sum_{\nu=0}^{+\infty} \alpha_\nu^{(m,d)} \psi_{2(m+\nu)}(\xi), \quad \xi \in \mathbb{R}^d, \tag{5.2}$$

where

$$\psi_{2(s-1)}(\xi) = |\,\xi\,|^{-2} \sum_{j=1}^{d} \xi_j^{2s}, \quad s \in \mathbb{N}. \tag{5.3}$$

Clearly, the series on the right-hand side of (5.2) converges absolutely and uniformly on each compact set $K \subset \mathbb{R}^d \backslash \{0\}$ and it holds $\psi_{2(s-1)} \in H_{2(s-1)}$ for $s > 1$.

By means of (5.2) and (5.3) we obtain

$$\begin{aligned} |\,\xi\,|^{-2}\,\theta_{m,d}(\xi)\,\eta(\xi) &= \alpha^{(m,d)}\,\eta(\xi) + \alpha_0^{(m,d)}\,\psi_{2m}(\xi)\,\eta(\xi) \\ &\quad + \Big(\sum_{\nu=1}^{N-1} \alpha_\nu^{(m,d)}\,\psi_{2(m+\nu)}(\xi)\Big)\,\eta(\xi) \\ &\quad + \Big(\sum_{\nu=N}^{+\infty} \alpha_\nu^{(m,d)}\,\psi_{2(m+\nu)}(\xi)\Big)\,\eta(\xi) \\ &\equiv \Psi_1(\xi) + \Psi_2(\xi) + \Psi_3(\xi) + \Psi_4(\xi), \quad \xi \in \mathbb{R}^d, \end{aligned} \tag{5.4}$$

where $N = [d/2] + 3$ is chosen. Since the function Ψ_1 is infinitely differentiable on $\mathbb{R}^d$ we get

$$|\,\widehat{\Psi}_1(x)\,| \;\le\; c_1\,(|\,x\,| + 1)^{-(d+2(m+1))}, \quad x \in \mathbb{R}^d. \tag{5.5}$$

In view of (2.15) (Proposition 2.3, part (iii)) we have

$$|\,\widehat{\Psi}_2(x)\,| \;\le\; c_2\,(|\,x\,| + 1)^{-(d+2m)} \tag{5.6}$$

$$|\,\widehat{\Psi}_3(x)\,| \;\le\; c \sum_{\nu=1}^{[d/2]+1} (|\,x\,| + 1)^{-(d+2(m+\nu))} \;\le\; c_3\,(|\,x\,| + 1)^{-(d+2(m+1))}. \tag{5.7}$$

for $x \in \mathbb{R}^d$. Because of Proposition 2.3, part (ii), and taking into account that

$$2(m+\nu) - (d + 2m + 2) \;\ge\; 2(m + [d/2] + 3) - (d + 2m + 2) \;>\; 1$$

for $\nu \ge N$ we conclude that

$$\lim_{\xi \to 0} \frac{\partial^{|\,j\,|_1}\,\Psi_3(\xi)}{\partial \xi_1^{j_1}, \ldots, \partial \xi_d^{j_d}} = 0$$

for each $j \in \mathbb{Z}_+^d$ satisfying $|\,j\,|_1 \le d + 2(m+1)$. It means that the function Ψ_3 has continuous derivatives on $\mathbb{R}^d$ up to the order $d + 2(m+1)$. In view of elementary properties of the Fourier transform this observation implies that

$$|\,\widehat{\Psi}_4(x)\,| \;\le\; c_4\,(|\,x\,| + 1)^{-(d+2(m+1))}, \quad x \in \mathbb{R}^d. \tag{5.8}$$

Applying (5.4)–(5.8) we obtain

$$\|\, \mathcal{F}\big(|\cdot|^{-2}\,\theta_{m,d}(\cdot)\,\eta(\cdot)\big)\,\|_p^p \le c\left(1+\int\limits_{|x|>1}\frac{dx}{|x|^{p\,(d+2m)}}\right) \le c'\left(1+\int\limits_{1}^{+\infty}\frac{dr}{r^{p\,(d+2m)-d+1}}\right) < +\infty$$

for $p\,(d+2m)-d+1>1$. Thus, in the case $d\ge 3$ or $d=2, m=2k-1$ $(k\in\mathbb{N})$ the estimate

$$\theta_{m,d}(\cdot) \overset{(p)}{\prec} |\cdot|^2 \tag{5.9}$$

follows for $p>p_{m,d}$.

Now, assume $0<p\le p_{m,d}$. Note that $\alpha_0^{(m,d)}\neq 0$. Moreover, the function ψ_{2m} is not a polynomial by Lemma 5.1. Hence, using Proposition 2.3, part (iv), as well as formulae (1.7), (1.8), (1.10) we get

$$|\,\widehat{\Psi}_2(x)\,| \ \ge\ c_0\,(|\,x\,|+1)^{-(d+2m)}, \quad x\in\Omega, \tag{5.10}$$

where the positive constant c_0 is independent of x and where $\Omega\equiv\Omega(r_0,u_0,\theta_0)$ is given by (2.17). We put

$$r_1 \ =\ \max\big\{r_0,\ (2(c_1+c_3+c_4)/c_0)^{1/2}\big\}. \tag{5.11}$$

Combining (5.4), (5.5), (5.7), (5.8) with (5.10) and (5.11) we obtain

$$\begin{aligned}\big|\,\mathcal{F}\big(|\cdot|^{-2}\,\theta_{m,d}(\cdot)\,\eta(\cdot)\big)(x)\,\big| &\ge |\,\widehat{\Psi}_2(x)\,| - \sum_{j=1,3,4}|\,\widehat{\Psi}_j(x)\,| \\ &\ge c_0\,(|\,x\,|+1)^{-(d+2m)} \\ &\quad -(c_1+c_3+c_4)\,(|\,x\,|+1)^{-(d+2(m+1))} \\ &> (c_0/2)\,(|\,x\,|+1)^{-(d+2m)}\end{aligned} \tag{5.12}$$

for $x\in\Omega_1\equiv\Omega(r_1,u_0,\theta_0)$. By means of (5.12) we get

$$\|\,\mathcal{F}\big(|\cdot|^{-2}\,\theta_{m,d}(\cdot)\,\eta(\cdot)\big)\,\|_p^p \ \ge\ c\int\limits_{\Omega_1}\frac{dx}{|\,x\,|^{p\,(d+2m)}} = c'\int\limits_{r_1}^{+\infty}\frac{dr}{r^{p\,(d+2m)-d+1}} = +\infty$$

for $0<p\le p_{m,d}$. Thus, for such p relation (5.9) is false.

Next we prove that in the case under consideration ($d\ge 3$ or $d=2, m=2k-1$, $k\in\mathbb{N}$) the inverse relation, i. e.

$$|\cdot|^2 \overset{(p)}{\prec} \theta_{m,d}(\cdot) \tag{5.13}$$

also holds if and only if $p > p_{m,d}$. We put

$$\Psi_{m,d}(\xi) = \sum_{\nu=0}^{+\infty} \frac{\alpha_\nu^{(m,d)}}{\alpha^{(m,d)}} \psi_{2(m+\nu)}(\xi), \ \xi \in \mathbb{R}^d. \tag{5.14}$$

Since $\lim_{\xi\to 0} \Psi_{m,d}(\xi) = 0$ there exists $\rho_0 > 0$ such that

$$|\Psi_{m,d}(\xi)| \le 1/2, \quad \xi \in \overline{B}_{\rho_0}. \tag{5.15}$$

Using the Taylor expansion of the function $(x+1)^{-1}$ at the point 0 we get

$$\alpha_{m,d} |\xi|^2 \left(\theta_{m,d}(\xi)\right)^{-1} \eta(\xi) = \frac{\eta(\xi)}{1+\Psi_{m,d}(\xi)} = \sum_{j=0}^{+\infty} (-1)^j \left(\Psi_{m,d}(\xi)\right)^j \eta(\xi) \tag{5.16}$$

by (5.2), (5.14) and (5.15) for each $\xi \in \mathbb{R}^d$. Here η is an infinitely differentiable function satisfying $\eta(\xi) = 1$ for $\xi \in \overline{B}_{\rho_1}$, where $0 < \rho_1 < \rho_0$, and $\eta(\xi) = 0$ for $\xi \notin B_{\rho_0}$. In view of (5.5) the series on the right-hand side of (5.16) converges absolutely and uniformly on each compact set $K \subset \mathbb{R}^d$. Note that

$$\left(\Psi_{m,d}(\xi)\right)^j = \sum_{\nu_1=0}^{+\infty} \cdots \sum_{\nu_d=0}^{+\infty} \prod_{i=1}^{j} \frac{\alpha_{\nu_i}^{(m,d)}}{\alpha^{(m,d)}} \psi_{2(m+\nu_i)}(\xi), \xi \in \mathbb{R}^d,$$

for $j \in \mathbb{N}$. Combining (5.14) and (5.16) and applying Propsition 2.3, part (i), we obtain

$$\begin{aligned} \alpha_{m,d} |\xi|^2 \left(\theta_{m,d}(\xi)\right)^{-1} \eta(\xi) = \eta(\xi) + \frac{\alpha_0^{(m,d)}}{\alpha^{(m,d)}} \psi_{2m}(\xi)\, \eta(\xi) \\ + \left(\sum_{\nu=1}^{+\infty} \beta_\nu^{(m,d)} \zeta_{2(m+\nu)}(\xi)\right) \eta(\xi) \end{aligned} \tag{5.17}$$

for $\xi \in \mathbb{R}^d$, where $\zeta_{2(m+\nu)} \in H_{2(m+\nu)}$, $\nu \in \mathbb{N}$. Formula (5.17) is similar to representation (5.4). Now the further proof of (5.13) follows the arguments above to prove (5.9).

Now let us consider the other remaining cases. For $d = 1$ the functions $|\cdot|^{-2}\theta_m(\cdot)\,\eta(\cdot)$ and $|\cdot|^2 \left(\theta_m(\cdot)\right)^{-1}\eta(\cdot)$ are infinitely differentiable by (5.2), (5.3), (5.17). Therefore, relations (5.9) and (5.13) are valid for all $0 < p \le +\infty$. If $d = 2$, $m = 2k$ ($k \in \mathbb{N}$) then the function ψ_{2m} is a polynomial by Lemma 5.1. To study relation (5.9) in this case we modify representation (5.4) as follows

$$
\begin{aligned}
|\,\xi\,|^{-2}\,\theta_{m,d}(\xi)\,\eta(\xi) &= \big(\alpha^{(m,d)}\,\eta(\xi) + \alpha_0^{(m,d)}\,\psi_{2m}(\xi)\,\eta(\xi)\big)\\
&\quad +\alpha_1^{(m,d)}\psi_{2(m+1)}(\xi)\,\eta(\xi)\\
&\quad +\Big(\sum_{\nu=2}^{N_1-1}\alpha_\nu^{(m,d)}\psi_{2(m+\nu)}(\xi)\Big)\,\eta(\xi)\\
&\quad +\Big(\sum_{\nu=N_1}^{+\infty}\alpha_\nu^{(m,d)}\,\psi_{2(m+\nu)}(\xi)\Big)\,\eta(\xi)\\
&\equiv \Theta_1(\xi)+\Theta_2(\xi)+\Theta_3(\xi)+\Theta_4(\xi), \quad \xi\in\mathbb{R}^d,
\end{aligned}
\tag{5.18}
$$

where $N_1 = [d/2] + 4$ is chosen. Similarly to (5.5)–(5.8) and (5.10) we obtain

$$
|\,\widehat{\Theta}_1(x)\,| \le c_1\,(|\,x\,|+1)^{-(d+2(m+2))}, \quad x\in\mathbb{R}^d, \tag{5.19}
$$
$$
|\,\widehat{\Theta}_2(x)\,| \le c_2\,(|\,x\,|+1)^{-(d+2(m+1))}, \quad x\in\mathbb{R}^d, \tag{5.20}
$$
$$
|\,\widehat{\Psi}_3(x)\,| \le c_3\,(|\,x\,|+1)^{-(d+2(m+2))}, \quad x\in\mathbb{R}^d, \tag{5.21}
$$
$$
|\,\widehat{\Theta}_4(x)\,| \le c_4\,(|\,x\,|+1)^{-(d+2(m+2))}, \quad x\in\mathbb{R}^d, \tag{5.22}
$$
$$
|\,\widehat{\Psi}_2(x)\,| \ge c_0\,(|\,x\,|+1)^{-(d+2(m+1))}, \quad x\in\Omega(r_0,u_0,\theta_0). \tag{5.23}
$$

The further proofs of the statements connected with (5.9) coincide with the proofs given above for the first case with obvious modifications. In order to study relation (5.13) for $d=2$, $m=2k$, $k\in\mathbb{N}$ we modify representation (5.17) similarly to (5.18) and apply the arguments given for (5.9).

The proof of Lemma 5.2 is complete. □

Theorem 5.3 (Equivalence Theorem) *Let* $m,\ d\in\mathbb{N}$. *Then it holds*

$$
\omega_{m,d}(f,\delta)_p \asymp K_\Delta^{(\mathcal{P})}(f,\delta)_p, \quad f\in L_p,\ \delta\ge 0, \tag{5.24}
$$

if and only if $p > p_{m,d}$.

Proof It follows from Proposition 2.2, Lemma 4.1 and Lemma 5.2 that the inequalities

$$
\|\,A_{h^{-1}}(\theta_{m,d})T\,\|_p \le c\,\|\,A_{h^{-1}}(|\cdot|^2)T\,\|_p, \quad T\in\mathcal{T}_{h^{-1}}, \quad h\ge 0, \tag{5.25}
$$
$$
\|\,A_{h^{-1}}(|\cdot|^2)T\,\|_p \le c\,\|\,A_{h^{-1}}(\theta_{m,d})T\,\|_p, \quad T\in\mathcal{T}_{h^{-1}}, \quad h\ge 0, \tag{5.26}
$$

are valid if and only if $p > p_{m,d}$.

Sufficiency Let $p > p_{m,d}$. We have $A_{h^{-1}}(\theta_{m,d}) = \Delta_h^{(m,d)}$ by part (iv) of Lemma 3.1 and $A_{h^{-1}}(|\cdot|^2) = -h^2\Delta$. Hence, the equivalence

$$
\|\,\Delta_h^{(m,d)}T\,\|_p \asymp h^2\,\|\Delta T\,\|_p, \quad T\in\mathcal{T}_{h^{-1}},\ h\ge 0, \tag{5.27}
$$

follows from (5.25) and (5.26). Applying (1.1)–(1.4), (3.1), (3.2) (parts (ii) and (iii) of Lemma 3.1), and (5.27) we obtain

$$\begin{aligned}\omega_{m,d}(f,\delta)_p^{\widetilde{p}} &\le \omega_{m,d}(f-T,\delta)_p^{\widetilde{p}} + \omega_{m,d}(T,\delta)_p^{\widetilde{p}} \le c \,\|\, f-T\,\|_p^{\widetilde{p}} \\ &\quad + \sup_{0\le h\le\delta} \|\,\Delta_h^{(m,d)} T\,\|_p^{\widetilde{p}} \le c_1 \left(\|\, f-T\,\|_p^{\widetilde{p}} + \delta^{2\widetilde{p}} \,\|\,\Delta T\,\|_p^{\widetilde{p}} \right) \\ &\le c_2 \,(\,\|\, f-T\,\|_p + \delta^2 \,\|\,\Delta T\,\|_p\,)^{\widetilde{p}}\end{aligned}$$

for each $T \in \mathcal{T}_{\delta^{-1}}$. This implies the upper estimate in (5.24). To prove the lower estimate we consider the polynomial $T^* \in \mathcal{T}_{\delta^{-1}}$ of best approximation of f in L_p by trigonometric polynomials of order δ^{-1}. Using the Jackson-type inequality from Theorem 4.2 with $\lambda = 1$ and taking into account that $(\delta^{-1}+1)^{-1} \le \delta$ we get

$$\|\, f - T^*\,\|_p = E_{\delta^{-1}}(f)_p \le c\,\omega_{m,d}\big(f, (\delta^{-1}+1)^{-1}\big)_p \le c\,\omega_{m,d}(f,\delta)_p. \tag{5.28}$$

With the help of Lemma 3.1 (part (i)), (5.27), and (5.28) we obtain

$$\begin{aligned}\delta^{2\widetilde{p}} \,\|\,\Delta T^*\|_p^{\widetilde{p}} &\le c\,\|\,\Delta_\delta^{(m,d)} T^*\|_p^{\widetilde{p}} \le c \left(\|\,\Delta_\delta^{(m,d)}(f-T^*)\,\|_p^{\widetilde{p}} + \|\,\Delta_\delta^{(m,d)} f\,\|_p^{\widetilde{p}}\right) \\ &\le c_1 \left(\left(\|\, f - T^*\,\|_p^{\widetilde{p}} + \omega_{m,d}(f,\delta)_p^{\widetilde{p}}\right) \le c_2\,\omega_{m,d}(f,\delta)_p^{\widetilde{p}}.\right.\end{aligned} \tag{5.29}$$

As a consequence of (5.28) and (5.29) we finally get

$$K_\Delta^{(\mathcal{P})}(f,\delta)_p \le c\left(\|\, f - T^*\|_p + \delta^2 \,\|\,\Delta T^*\|_p\right) \le c_1\,\omega_{m,d}(f,\delta)_p.$$

Necessity Suppose that (5.24) holds. Then one has

$$\|\,\Delta_h^{(m,d)} T\,\|_p \le \omega_{m,d}(T,h)_p \le c\,K_\Delta^{(\mathcal{P})}(T,h)_p \le c\,h^{-2}\,\|\,\Delta T\,\|_p$$

for each $T \in \mathcal{T}_{h^{-1}}$. This means that (5.25) holds and therefore $p > p_{m,d}$ follows.

The proof of Theorem 5.3 is complete. □

As it was already mentioned in the Introduction, Theorem 5.3 contains some known results as special cases. If $d = 1$, $m = 1$ and $1 \le p \le +\infty$ then the equivalence (5.24) is the well-known result of Johnen (see e. g. [1], Ch. 6, §2, Theorem 2.4) for the classical modulus of smoothness of second order $\omega_2(f,\delta)_p$ and Peetre's K-functional related to the derivative of the second order. The equivalence of $\omega_2(f,\delta)_p$ and the corresponding polynomial K-functional related to the second derivative in the case $0 < p < 1$ is proved in [6]. The multivariate case for $m = 1$ is studied in [5].

6 Applications

Combining Theorem 5.3 and the properties of polynomial K-functionals described in [9] we immediately obtain corresponding results for moduli of smoothness defined in (1.1). Recall that $\widetilde{p} = \min(1, p)$.

Theorem 6.1 *Let $m,\ d \in \mathbb{N}$ and let $p_{m,d} < p \le +\infty$. Then there exists a positive constant c such that*

$$\omega_{m,d}(f, t\delta)_p \ \le\ c\ \max\left(1, t^{\,d(1/\widetilde{p}-1)+2}\right) \omega_{m,d}(f, \delta)_p \tag{6.1}$$

holds for all $f \in L_p$ and $\delta, \quad t \ge 0$.

Theorem 6.2 (Bernstein-type estimate) *Let $m,\ d \in \mathbb{N}$ and let $p_{m,d} < p \le +\infty$. Then there exists a positive constant c such that*

$$\omega_{m,d}(f, \delta)_p \le c \min(\delta^2, 1)\left(\sum_{0\le\nu<1/\delta} (\nu+1)^{2\widetilde{p}-1}\, E_\nu(f)_p^{\widetilde{p}}\right)^{1/\widetilde{p}} \tag{6.2}$$

holds for all $f \in L_p$ and $\delta \ge 0$.

Obviously the definition in (5.1) implies that $p_{m,d} \le p_{m_0,d}$ for $m > m_0$. Hence, the following equivalence result follows from Theorem 5.3.

Theorem 6.3 (Equivalence of moduli (1) for different m) *Let $m_0,\ d \in \mathbb{N}$ and let $p_{m_0,d} < p \le +\infty$. Then we have*

$$\omega_{m,d}(f, \delta)_p \asymp \omega_{m_0,d}(f, \delta)_p, \quad f \in L_p,\ \delta \ge 0 \tag{6.3}$$

for $m > m_0$.

Finally we describe the quality of approximation by families of linear polynomial operators generated by Bochner–Riesz kernels in terms of moduli of smoothness $\omega_{m,d}(f, \delta)_p$. Let $\lambda,\ x \in \mathbb{R}^d$ and let $n \in \mathbb{N}_0$. We put (see also [12])

$$\mathcal{B}^{(\alpha)}_{n;\lambda}(f; x) = (2n+1)^{-d} \sum_{\nu=0}^{2n} f\left(t_n^\nu + \lambda\right) B_n^{(\alpha)}\left(x - t_n^\nu - \lambda\right). \tag{6.4}$$

Here we used the notations

$$t_n^\nu = \frac{2\pi\nu}{2n+1},\ \nu \in \mathbb{Z}^d;\ \sum_{\nu=0}^{2n} \equiv \sum_{\nu_1=0}^{2n} \cdots \sum_{\nu_d=0}^{2n},$$

$$B_0^{(\alpha)}(h) = 1,\ B_n^{(\alpha)}(h) = \sum_{|k|\le n} \left(1 - \frac{|k|^2}{n^2}\right)^\alpha e^{ikh}, \quad n \in \mathbb{N},\ h \in \mathbb{R}^d. \tag{6.5}$$

The functions $B_n^{(\alpha)}$ are the well-known Bochner–Riesz kernels with parameter $\alpha > 0$. It has been proved in ([7] Theorem 4.1 and Section 5) that in the super-critical case

$\alpha > (d-1)/2$ this family converges in L_p if and only if $p > 2d/(d+2\alpha+1)$. More precisely this means that

$$\| f - \mathcal{B}^{(\alpha)}_{n;\lambda}(f) \|_{\overline{p}} := (2\pi)^{-d/p} \left(\int_{\mathbb{T}^d} \| f - \mathcal{B}^{(\alpha)}_{n;\lambda}(f) \|_p^p d\lambda \right)^{1/p} \to 0 (n \to \infty) \quad (6.6)$$

if and only if $2d/(d+2\alpha+1) < p < +\infty$ and that

$$\| f - \mathcal{B}^{(\alpha)}_{n;\lambda}(f) \|_{\overline{C}} := \max_{\lambda \in \mathbb{R}^d} \| f - \mathcal{B}^{(\alpha)}_{n;\lambda}(f) \|_C \to 0 (n \to \infty) \quad (6.7)$$

in the case $p = +\infty$. In this sense the family of operators $\mathcal{B}^{(\alpha)}_n$ acting from $L_p(\mathbb{T}^d)$ into $L_p(\mathbb{T}^d \times T^d)$ can be considered as a constructive approximation method, in particular in the case $0 < p < 1$. More information can be found in [7] and [8]. We have proved in [12], Theorems 2 and 3, that

$$\| f - \mathcal{B}^{(\alpha)}_{n;\lambda}(f) \|_{\overline{p}} \asymp K^{(\mathcal{P})}_{\Delta}(f, 1/n)_p, \quad f \in L_p,\ n \in \mathbb{N} \quad (6.8)$$

if $\alpha > (d-1)/2$ and $p > 2d/(d+2\alpha+1)$ (see also [8] Theorems 6.1 and 7.3 for a more general approach). Combining Theorem 5.3 and (6.8) we obtain the following equivalence theorem.

Theorem 6.4 (Quality of approximation by Bochner–Riesz families) *Suppose that $m_0,\ d \in \mathbb{N}$ and $\alpha > (d-1)/2$. It holds*

$$\| f - \mathcal{B}^{(\alpha)}_{n;\lambda}(f) \|_{\overline{p}} \asymp \omega_{m,d}\big(f, (n)^{-1}\big)_p, \quad f \in L_p, n \in \mathbb{N}, \quad (6.9)$$

for $p > \max(p_{m,d},\ 2d/(d+2\alpha+1))$, where $p_{m,d}$ is given by (5.1).

Of peculiar interest is the case $0 < p < 1$ and $d \geq 2$. Theorem 6.4 extends the result of Theorem 5 in [12] which corresponds to the case $m = 1$ and which is restricted to $p > d/(d+2)$. Note that $p_{m,d} \to 0$ if $m \to \infty$. Hence, we are now able to characterize the approximation error $\| f - \mathcal{B}^{(\alpha)}_{n;\lambda}(f) \|_{\overline{p}}$ by an appropriate modulus of smoothness $\omega_{m,d}\big(f, (n)^{-1}\big)_p$ for a given $p > 0$ by choosing m large enough.

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