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Generalized Fourier Frames in Terms of Balayage

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Abstract We develop a theory of non-uniform sampling in the context of the theory of frames for the settings of the short time fourier transform and pseudo-differential operators. Our theory is based on profound historical precedents including Beurling's theory of balayage, emanating from the nineteenth century work of Christoffel and Poincaré, the theory and results from spectral synthesis due to Wiener and Beurling and a host of the major harmonic analysts of the twentieth century, and the theory of sets of multiplicity, going back to Riemann and emerging fundamentally from the Russian school of harmonic analysis in the early twentieth century. Our results are meant to serve as the underpinnings for both theoretical and practical results in the realm of non-uniform sampling. They can also be compared with several other distinct forays into non-uniform sampling, including the settings of quasi-crystals and modulation spaces, where proofs for the latter setting require the analysis of convolution operators on the Heisenberg group. Our theory herein is the first step in which the ultimate goal is computational implementation for non-uniform sampling and its myriad applications, where balayage, spectral synthesis, and sets of multiplicity are computationally quantified. A critical component is to resurrect the formulation of balayage in terms of covering criteria.

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1 Introduction

1.1 Background and Theme

There has been a great deal of work during the past quarter century in analyzing, formulating, validating, and extending sampling formulas,

$$f(x) = \sum f(x_n)s_n, \tag{1}$$

for non-uniformly spaced sequences $\{x_n\}$, for specific sequences of sampling functions s_n depending on x_n, and for classes of functions f for which such formulas are true. For glimpses into the literature, see the *Journal of Sampling Theory in Signal and Image Processing*, the influential book by Young [82], edited volumes such as [9], and specific papers such as those by Jaffard [47] and Seip [74]. This surge of activity is intimately related to the emergence of wavelet and Gabor theories and more general frame theory. Further, it is firmly buttressed by the profound results of Paley–Wiener [70], Levinson [59], Duffin–Schaeffer [27], Beurling–Malliavin [18,19], Beurling (unpublished 1959–1960 lectures), and Landau [56], that themselves have explicit origins by Dini [26], as well as Birkhoff (1917), Walsh (1921), and Wiener (1927), see [70], p. 86, for explicit references. This is our *background*.

The setting will be in terms of classical spectral criteria to prove non-uniform sampling formulas such as (1). Our *theme* is to generalize non-uniform sampling in this setting to the Gabor theory [33,38,54], as well as to the setting of time-varying signals and pseudo-differential operators. The techniques are based on Beurling's methods from 1959–1960, [15,17], pp. 299–315, [17], pp. 341–350, which incorporate balayage, spectral synthesis, and strict multiplicity. Our formulation is in terms of the theory of frames.

1.2 Motivation

With an eye towards Eq. (1) and with a decidedly mathematical point of view, we note that there are extensions and analogues of the classical result that the set $\{e^{-2\pi i n\omega} : n \in \mathbb{Z}\}$ of exponentials forms an orthonormal basis for the space $L^2(\Lambda)$ of square-integrable functions on $\Lambda = [0, 1]$. As such, we ask if there is a unifying theory that ties together these analogues and extensions? Further, are there general theoretical justifications for the often intricate relations that occur between the sequences of sampling points and the support sets of the spectra of functions in equations such as (1)? Such questions are the basis for our *motivation*.

To be more precise with regard to these questions, and to illustrate specific cases of such intricate relations, we give the following example.

Example 1.1 a. This part of the example is a result of Olevskii and Ulanovskii [68] concerning universal sets of stable sampling for band-limited functions.

Consider an analogue of the aforementioned classical result, where the interval [0, 1] is now replaced by a set Λ (possibly unbounded), in which Λ has Lebesgue measure $|\Lambda|$ strictly less than 1 and, speaking intuitively, for which Λ is not too spread-out.

Let $E = \{n + 2^{-|n|} : n \in \mathbb{Z}\}$ and let $\mathcal{E}(E) = \{e_{-x} : x \in E\}$, where $e_x(\gamma) = e^{2\pi x\gamma}$. Then $\mathcal{E}(E)$ is complete in $L^2(\Lambda)$ for every measurable set $\Lambda \subseteq \mathbb{R}$ satisfying $|\Lambda| < 1$ and for which $|\Lambda \cap \{\gamma : k - 1 < |\gamma| < k\}| \leq C\,2^{-k}$, where C is independent of k. This means that for any $F \in L^2(\Lambda)$, that is orthogonal to each function in $\mathcal{E}(E)$, we can conclude that $F = 0\,a.e.$ This is equivalent to saying that for any $f \in L^2(\mathbb{R})$, for which $f(x) = \int_\Lambda F(\gamma)e^{2\pi i x\gamma}\,d\gamma$, for some $F \in L^2(\Lambda)$ (and so f is continuous on $\mathbb{R}$ since $|\Lambda| < \infty$), the condition that $f = 0$ on E implies that $f = 0\,a.e.$ The hypothesis, $|\Lambda \cap \{\gamma : k - 1 < |\gamma| < k\}| \leq C\,2^{-k}$, where C is independent of k, can be weakened but not eliminated. Thus, although Λ can be an unbounded set, there is a restriction that Λ cannot be too thin or too spread-out over $\mathbb{R}$. This illustrates that there is an intricate relation between the set E of sampling points and the support set Λ of the spectrum F of a function f.

b. This part is a result of Han and Wang [42].

Let $\mathcal{L} = A\mathbb{Z}^d$ and $\mathcal{K} = B\mathbb{Z}^d$, where A and B are real $d \times d$ nonsingular matrices. Let $g \in L^2(\mathbb{R}^d)$ and define the Gabor family,

$$G(\mathcal{L}, \mathcal{K}, g) = \{e^{2\pi i \ell\cdot x}\, g(x - k) : \ell \in \mathcal{L}, k \in \mathcal{K}\} \subseteq L^2(\mathbb{R}^d).$$

Then there exists $g \in L^2(\mathbb{R}^d)$ such that $G(\mathcal{L}, \mathcal{K}, g)$ is a frame for $L^2(\mathbb{R}^d)$ if and only if $|\det(AB)| \leq 1$. Frames are defined in Sect. 2.1, and they can be thought of as sequences of harmonics or sampling functions to provide decompositions of functions. Of course, bases have the same property and are a particular subset of frames giving unique decompositions. The value of a general frame is that it can be an overcomplete system so as to compensate for naturally occurring noises as well as erasures of information in applications.

1.3 Goal

Our *goal* in this paper is to establish a substantive, fundamental theory with which to understand and analyze a wide class of sampling phenomena in terms of basic, quantitative components of such phenomena. From our point of view, and following Beurling, three such components are the notions of balayage, spectral synthesis, and strict multiplicity. These notions will be defined and given context in Sect. 2.1. They are integrated in our theory in terms of the theory of frames. For now, and intuitively speaking, balayage is a means of spectrally identifying measures with their restrictions, spectral synthesis establishes spectral criteria to determine if a functional will or will not annihilate a given function, and strict multiplicity quantifies the required girth of the underlying spectral sets that arise. Our ultimate *goal* is the computational implementation of this theory for a variety of important applications.

1.4 Outline

Section 2 has three subsections. In Sect. 2.1 we give the definitions of frames, balayage, spectral synthesis, and strict multiplicity, that we have already described intuitively. Each of these notions is a major and deep topic in its own right, and so we have provided some context, history, and references. Beurling was the first to combine them in a profound and creative way, and an outline of some of his results in this area is the subject of Sect. 2.2. In Sect. 2.3, we extricate and reformulate one of these results, that we call *A fundamental identity of balayage*. This identity is a major technique that we use in establishing our theory.

In Sect. 3, we prove two theorems, that are the basis for our frame theoretic non-uniform sampling theory for the Short Time Fourier Transform (STFT). Both of these theorems are stated in terms of frame inequalities from which non-uniform sampling formulas can be deduced. The first of these theorems, Theorem 3.2, formally resembles an assertion in terms of Fourier frame inequalities (Definition 2.1), but in a significantly more general way. The generality is best understood in terms of so-called (X, μ) or continuous frames, and so we have also included a slight digression on such frames. The second of these theorems, Theorem 3.4, is compared with an earlier result of Gröchenig, that itself goes back to work of Feichtinger and Gröchenig. In the necessary give and take between various STFT non-uniform sampling formulas, we see that there is larger class of functions for which Gröchenig's theorem is valid than for the case of Theorem 3.4, but the sampling set E depends on the given window function in the case of Gröchenig's theorem but not so in the case of Theorem 3.4.

Section 4 is devoted to examples that we formulated as avenues for further development integrating balayage with other theoretical notions.

In Sect. 5 we prove the frame inequalities necessary to provide a non-uniform sampling formula for pseudo-differential operators defined by a specific class of Kohn-Nirenberg symbols. We view this as the basis for a much broader theory.

Our last mathematical section, Sect. 6, is a brief recollection of Beurling's balayage results, but formulated in terms of covering criteria and due to a collaboration of one of the authors in 1990s with Dr. Hui-Chuan Wu. Such coverings in terms of polar sets of given band width are a natural vehicle for extending the theory developed herein. Finally, in the Epilogue, we note the important related contemporary research being conducted in terms of quasicrystals, as well as other applications.

2 Definitions and the Beurling Theory

2.1 Definitions

Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing smooth functions on d-dimensional Euclidean space $\mathbb{R}^d$. We define the Fourier transform and inverse Fourier transform of $f \in \mathcal{S}(\mathbb{R}^d)$ by the formulas,

$$\widehat{f}(\gamma) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\gamma}\, dx \quad \text{and} \quad (\widehat{f})^{\vee}(x) = f(x) = \int_{\widehat{\mathbb{R}}^d} \widehat{f}(\gamma)e^{2\pi i x\cdot\gamma}\, d\gamma,$$

respectively. $\widehat{\mathbb{R}}^d$ denotes $\mathbb{R}^d$ considered as the spectral domain. If $F \in \mathcal{S}(\widehat{\mathbb{R}}^d)$, then we write $F^\vee(x) = \int_{\widehat{\mathbb{R}}^d} F(\gamma)e^{2\pi i x\cdot\gamma}\, d\gamma$. The notation "$\int$" designates integration over $\mathbb{R}^d$ or $\widehat{\mathbb{R}}^d$. The Fourier transform extends to tempered distributions. If $X \subseteq \mathbb{R}^d$, where X is closed, then $M_b(X)$ is the space of bounded Radon measures μ with support, $\operatorname{supp}(\mu)$, contained in X. $C_b(\mathbb{R}^d)$ denotes the space of complex valued bounded continuous functions on $\mathbb{R}^d$.

Definition 2.1 (Frame) Let H be a separable Hilbert space. A sequence $\{x_n\}_{n\in\mathbb{Z}} \subseteq H$ is a *frame* for H if there are positive constants A and B such that

$$\forall\, f \in H, \quad A\|f\|^2 \le \sum_{n\in\mathbb{Z}} |\langle f, x_n\rangle|^2 \le B\|f\|^2.$$

The constants A and B are lower and upper frame bounds, respectively. We choose A to be the supremum over all lower frame bounds, and we choose B to be the infimum over all upper frame bounds. As such A and B are uniquely defined, and are called *the lower* and *upper frame bounds*, respectively, of the frame $\{x_n\}_{n\in\mathbb{Z}}$. If $A = B$, we say that the frame is a tight frame or an A-tight frame for H.

Definition 2.2 (Fourier frame) Let $E \subseteq \mathbb{R}^d$ be a sequence and let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a compact set. Notationally, let $e_x(\gamma) = e^{2\pi i x\cdot\gamma}$. The sequence $\mathcal{E}(E) = \{e_{-x} : x \in E\}$ is a *Fourier frame* for $L^2(\Lambda)$ if there are positive constants A and B such that

$$\forall\, F \in L^2(\Lambda), \quad A\|F\|^2_{L^2(\Lambda)} \le \sum_{x\in E} |\langle F, e_{-x}\rangle|^2 \le B\|F\|^2_{L^2(\Lambda)}.$$

Define the *Paley–Wiener space*,

$$PW_\Lambda = \{f \in L^2(\mathbb{R}^d) : \operatorname{supp}(\widehat{f}) \subseteq \Lambda\}.$$

Clearly, $\mathcal{E}(E)$ is a Fourier frame for $L^2(\Lambda)$ if and only if the sequence,

$$\{(e_{-x}\mathbb{1}_\Lambda)^\vee : x \in E\} \subseteq PW_\Lambda,$$

is a frame for PW_Λ, in which case it is called a *Fourier frame* for PW_Λ. Note that $\langle F, e_{-x}\rangle = f(x)$ for $f \in PW_\Lambda$, where $\widehat{f} = F \in L^2(\widehat{\mathbb{R}}^d)$ can be considered an element of $L^2(\Lambda)$.

Remark 2.3 Frames were first defined by Duffin and Schaeffer [27], but appeared explicitly earlier in Paley and Wiener's book [70], p. 115. See Christensen's book [21] and Kovačević and Chebira's articles [52], [53] for recent expositions of theory and applications. If $\{x_n\}_{n\in Z} \subseteq H$ is a frame, then there is a topological isomorphism $S : H \longrightarrow \ell^2(Z)$ such that

$$\forall x \in H, \quad x = \sum_{n\in\mathbb{Z}} \left\langle x, S^{-1}(x_n)\right\rangle x_n = \sum_{n\in\mathbb{Z}} \langle x, x_n\rangle\, S^{-1}(x_n). \tag{2}$$

Equation (2) illustrates the natural role that frames play in studying non-uniform sampling formulas (1), see Example 2.16.

Beurling introduced the following definition in his 1959-1960 lectures.

Definition 2.4 (Balayage) Let $E \subseteq \mathbb{R}^d$ and $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be closed sets. *Balayage* is possible for $(E, \Lambda) \subseteq \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ if

$$\forall \mu \in M_b(\mathbb{R}^d),\ \exists \nu \in M_b(E) \text{ such that } \widehat{\mu} = \widehat{\nu} \text{ on } \Lambda.$$

Remark 2.5 a. The set Λ is a collection of group characters in analogy to the Newtonian potential theoretic setting, e.g., [17], pp. 341–350, [56].

b. The notion of balayage in potential theory is due to Christoffel (1871), e.g., see the remarkable book [20], edited by Butzer and Fehér, and the article therein by Brelot. Then, Poincaré (1890 and 1899) used the idea of balayage as a method of solving the Dirichlet problem for the Laplace equation. Letting $D \subseteq \mathbb{R}^d$, $d \geq 3$, be a bounded domain, a balayage or sweeping of the measure $\mu = \delta_y$, $y \in D$, to ∂D is a measure $\nu_y \in M_b(\partial D)$ whose Newtonian potential coincides outside of D with the Newtonian potential of δ_y. In fact, ν_y is unique and is the harmonic measure on ∂D for $y \in D$, e.g., [24,51].

One then formulates a more general balayage problem: for a given mass distribution μ inside a closed bounded domain $\overline{D} \subseteq \mathbb{R}^d$, find a mass distribution ν on ∂D such that the potentials are equal outside $\overline{D}$ [58], cf. [1].

c. Given the general formulation of Definition 2.4, it is important to note that substantial families of pairs of sets can be constructed for which balayage is possible, see, e.g., [15].

Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a closed set. Define

$$\mathcal{C}(\Lambda) = \{f \in C_b(\mathbb{R}^d) : \text{supp}\,(\widehat{f}) \subseteq \Lambda\},$$

cf. the role of $\mathcal{C}(\Lambda)$ in [77].

Definition 2.6 (Spectral synthesis) A closed set $\Lambda \subseteq \widehat{\mathbb{R}}^d$ is a set of *spectral synthesis (S-set)* if

$$\forall f \in \mathcal{C}(\Lambda) \text{ and } \forall \mu \in M_b(\mathbb{R}^d), \quad \widehat{\mu} = 0 \text{ on } \Lambda \Rightarrow \int f\, d\mu = 0, \tag{3}$$

see [5].

Remark 2.7 a. The problem of characterizing S-sets emanated from Wiener's Tauberian theorem ideas, and was developed by Beurling in the 1940s. It is "synthesis" in that one wishes to approximate $f \in L^\infty(\mathbb{R}^d)$ in the $\sigma(L^\infty(\mathbb{R}^d), L^1(\mathbb{R}^d))$ (weak-$*$) topology by finite sums of characters $\gamma : L^\infty(\mathbb{R}^d) \to \mathbb{C}$, where γ can be considered an element of $\widehat{\mathbb{R}}^d$ and where supp $(\delta_\gamma) \subseteq$ supp $(\widehat{f})$, which is the so-called spectrum of f. Such an approximation is elementary to achieve by convolutions of the measures δ_γ, but in this case we lose the essential property that the spectra of the approximants be contained in the spectrum of f. It is a fascinating problem whose complete resolution is equivalent to the characterization of the ideal structure of $L^1(\mathbb{R}^d)$, a veritable Nullstellensatz of harmonic analysis.

b. We obtain the annihilation property of (3) in the case that f and μ have balancing smoothness and irregularity. For example, if $\widehat{f} \in D'(\widehat{\mathbb{R}}^d)$, $\widehat{\mu} = \phi \in C_c^\infty(\widehat{\mathbb{R}}^d)$, and $\phi = 0$ on supp $(\widehat{f})$, then $\widehat{f}(\phi) = 0$, where $\widehat{f}(\phi)$ is sometimes written $\langle \widehat{f}, \phi \rangle$. The sphere $S^2 \subseteq \widehat{\mathbb{R}}^3$ is not an S-set (Laurent Schwartz, 1947), and every non-discrete locally compact abelian group $\widehat{G}$, e.g., $\widehat{\mathbb{R}}^d$, contains non-S-sets (Paul Malliavin 1959). On the other hand, polyhedra are S-sets, whereas the 1/3-Cantor set is an S-set with non-S-subsets. We refer to [5] for an exposition of the theory.

Definition 2.8 (Strict multiplicity) A closed set $\Gamma \subseteq \widehat{\mathbb{R}}^d$ is a set of *strict multiplicity* if

$$\exists \mu \in M_b(\Gamma) \setminus \{0\} \text{ such that } \lim_{\|x\| \to \infty} |\mu^\vee(x)| = 0.$$

Remark 2.9 The study of sets of strict multiplicity has its origins in Riemann's theory of sets of uniqueness for trigonometric series, see [4,83]. An early, important, and difficult result is due to Menchov (1916):

$$\exists \Gamma \subseteq \widehat{\mathbb{R}}/\mathbb{Z} \text{ and } \exists \mu \in M_b(\Gamma) \setminus \{0\} \text{ such that } |\Gamma| = 0 \text{ and } \mu^\vee(n) \\ = O((\log |n|)^{-1/2}),\ |n| \to \infty.$$

($|\Gamma|$ is the Lebesgue measure of Γ.) There are refinements of Menchov's result, aimed at increasing the rate of decrease, due to Bary (1927), Littlewood (1936), Salem (1942, 1950), and Ivašev-Mucatov (1952, 1956).

2.2 Results of Beurling

The results in this subsection stem from 1959 to 1960, and the proofs are sometimes sophisticated, see [17], pp. 341–350. Throughout, $E \subseteq \mathbb{R}^d$ is closed and $\Lambda \subseteq \widehat{\mathbb{R}}^d$ is compact. The following is a consequence of the open mapping theorem.

Proposition 2.10 *Assume balayage is possible for* (E, Λ). *Then*

$$\exists K > 0 \text{ such that } \forall \mu \in M_b(\mathbb{R}^d),\ \inf\{\|\nu\|_1 : \nu \in M_b(E) \text{ and } \widehat{\nu} = \widehat{\mu} \text{ on } \Lambda\} \le K\, \|\mu\|_1 .$$

($\|\ldots\|_1$ designates the total variation norm.)

The smallest such K is denoted by $K(E, \Lambda)$, and we say that balayage is not possible if $K(E, \Lambda) = \infty$. In fact, *if* Λ *is a set of strict multiplicity, then balayage is possible for* (E, Λ) *if and only if* $K(E, \Lambda) < \infty$, e.g., see Lemma 1 of [17], pp. 341–350. Let $J(E, \Lambda)$ be the smallest $J \ge 0$ such that

$$\forall f \in \mathcal{C}(\Lambda),\ \sup_{x \in \mathbb{R}^d} |f(x)| \le J \sup_{x \in E} |f(x)|.$$

$J(E, \Lambda)$ could be ∞.

The Riesz representation theorem is used to prove the following result. Part c is a consequence of parts a and b.

Proposition 2.11 *a. If Λ is a set of strict multiplicity, then $K(E, \Lambda) \leq J(E, \Lambda)$.*

b. If Λ is an S-set, then $J(E, \Lambda) \leq K(E, \Lambda)$.

c. Assume that Λ is an S-set of strict multiplicity and that balayage is possible for (E, Λ). If $f \in \mathcal{C}(\Lambda)$ and $f = 0$ on E, then f is identically 0.

Proposition 2.12 *Assume that Λ is an S-set of strict multiplicity. Then, balayage is possible for $(E, \Lambda) \Leftrightarrow$*

$$\exists K(E, \Lambda) > 0 \text{ such that } \forall f \in \mathcal{C}(\Lambda), \quad \|f\|_\infty \leq K(E, \Lambda) \sup_{x \in E} |f(x)|.$$

The previous results are used in the intricate proof of Theorem 2.13.

Theorem 2.13 *Assume that Λ is an S-set of strict multiplicity, and that balayage is possible for (E, Λ) and therefore $K(E, \Lambda) < \infty$. Let $\Lambda_\epsilon = \{\gamma \in \widehat{\mathbb{R}}^d : dist(\gamma, \Lambda) \leq \epsilon\}$. Then,*

$$\exists \epsilon_0 > 0 \text{ such that } \forall\, 0 < \epsilon < \epsilon_0,\ K(E, \Lambda_\epsilon) < \infty,$$

i.e., balayage is possible for (E, Λ_ϵ).

The following result for $\mathbb{R}^d$ is not explicitly stated in [17], pp. 341–350, but it goes back to his 1959–1960 lectures, see [81], Theorem E in [56], Landau's comment on its origins [57], and Example 2.20. In fact, using Theorem 2.13 and Ingham's theorem (Theorem 2.18), Beurling obtained Theorem 2.15. We have chosen to state Ingham's theorem (Theorem 2.18) in Sect. 2.3 as a basic step in the proof of Theorem 2.19, which supposes Theorem 2.13 and which we chose to highlight as *A fundamental identity of balayage* and in terms of its quantitative conclusion, (6) and (7). In fact, Theorem 2.19 essentially yields Theorem 2.15, see Example 2.20.

Definition 2.14 A sequence $E \subseteq \mathbb{R}^d$ is *separated* if

$$\exists\, r > 0 \text{ such that } \inf\{\|x - y\| : x, y \in E \text{ and } x \neq y\} \geq r.$$

Theorem 2.15 *Assume that $\Lambda \subseteq \widehat{\mathbb{R}}^d$ is an S-set of strict multiplicity and that $E \subseteq \mathbb{R}^d$ is a separated sequence. If balayage is possible for (E, Λ), then $\mathcal{E}(E)$ is a Fourier frame for $L^2(\Lambda)$, i.e., $\{(e_{-x}\mathbb{1}_\Lambda)^\vee : x \in E\}$ is a Fourier frame for PW_Λ.*

Example 2.16 The conclusion of Theorem 2.15 is the assertion

$$\forall f \in PW_\Lambda, \quad f = \sum_{x \in E} f(x) S^{-1}(f_x) = \sum_{x \in E} \left\langle f, S^{-1}(f_x) \right\rangle f_x,$$

where

$$f_x(y) = (e_{-x}\mathbb{1}_\Lambda)^\vee(\gamma)$$

and

$$S(f) = \sum_{x \in E} f(x)(e_{-x}\mathbb{1}_\Lambda)^\vee,$$

cf. (1) and (2). Clearly, f_x is a type of sinc function. Smooth sampling functions can be introduced into this setup, e.g., Theorem 7.45 of [10], Chapter 7.

Remark 2.17 Theorem 2.15 and results in [15] led to the Beurling covering theorem, see Sect. 6.

2.3 A Fundamental Identity of Balayage

By construction, and slightly paraphrased, Ingham [46] proved the following result for the case $d = 1$, see [15], p. 115 for a modification which gives the $d > 1$ case. In fact, Beurling gave a version for $d > 1$ in 1953; it is unpublished. In 1962, Kahane [49] went into depth about the $d > 1$ case.

Theorem 2.18 *Let $\epsilon > 0$ and let $\Omega : [0, \infty) \to (0, \infty)$ be a continuous function, increasing to infinity. Assume the following conditions:*

$$\int_1^\infty \Omega(r)\,\frac{dr}{r^2} < \infty, \tag{4}$$

$$\int exp(-\Omega(\|x\|))\,dx < \infty, \tag{5}$$

and $\Omega(r) > r^a$ on some interval $[r_0, \infty)$ and for some $a < 1$. Then, there is $h \in L^1(\mathbb{R}^d)$ for which $h(0) = 1$, supp $(\widehat{h}) \subseteq \overline{B(0, \epsilon)}$, and

$$|h(x)| = O(e^{-\Omega\|x\|}), \quad \|x\| \to \infty.$$

Ingham also proved the converse, which, in fact, requries the Denjoy–Carleman theorem for quasi-analytic functions.

If balayage is possible for (E, Λ) and $E \subseteq \mathbb{R}^d$ is a closed sequence, e.g., if E is separated, then Proposition 2.10 allows us to write $\widehat{\mu} = \sum_{x\in E} a_x(\mu)\widehat{\delta_x}$ on Λ, where $\sum_{x\in E} |a_x(\mu)| \le K(E, \Lambda)\,\|\mu\|_1$. In the case $\mu = \delta_y$, we write $a_x(\mu) = a_x(y)$.

We refer to the following result as *A fundamental identity of balayage*.

Theorem 2.19 *Let Ω satisfy the conditions of Ingham's Theorem 2.18. Assume that Λ is a compact S-set of strict multiplicity, that E is a separated sequence, and that balayage is possible for (E, Λ). Choose $\epsilon > 0$ from Beurling's Theorem 2.13 so that $K(E, \Lambda_\epsilon) < \infty$. For this $\epsilon > 0$, take h from Ingham's Theorem 2.18. Then, we have*

$$\forall y \in \mathbb{R}^d \text{and } \forall f \in \mathcal{C}(\Lambda), \quad f(y) = \sum_{x \in E} f(x)a_x(y)h(x-y), \tag{6}$$

where

$$\sup_{y\in\mathbb{R}^d}\sum_{x\in E}|a_x(y)|\le K(E,\Lambda_\epsilon)<\infty. \tag{7}$$

In particular, we have

$$\forall y\in\mathbb{R}^d \textit{and } \forall\gamma\in\Lambda,\quad e^{2\pi i y\cdot\gamma}=\sum_{x\in E}a_x(y)h(x-y)e^{2\pi i x\cdot\gamma}.$$

Proof Since balayage is possible for (E,Λ_ϵ), we have that $(\delta_y)^\wedge=(\sum_{x\in E}a_x(y)\delta_x)^\wedge$ on Λ_ϵ and that

$$\sum_{x\in E}|a_x(y)|\le K(E,\Lambda_\epsilon)\left\|\delta_y\right\|_1$$

for each $y\in\mathbb{R}^d$. Thus, (7) is obtained. Next, for each fixed $y\in\mathbb{R}^d$, define the measure,

$$\eta_y(w)=h_y(w)\left(\delta_y-\sum_{x\in E}a_x(y)\delta_x\right)(w)\in M_b(\mathbb{R}^d),$$

where $h_y(w)=h(w-y)$. Then, we have

$$\begin{aligned}(\eta_y)^\wedge(\gamma)&=\left[(h_y)^\wedge*\left(\delta_y-\sum_{x\in E}a_x(y)\delta_x\right)^\wedge\right](\gamma)\\&=\int\widehat{h}(\gamma-\lambda)e^{-2\pi i y\cdot(\gamma-\lambda)}\left(\delta_y-\sum_{x\in E}a_x(y)\delta_x\right)^\wedge(\lambda)\,d\lambda\\&=\int_{(\Lambda_\epsilon)^c}\widehat{h}(\gamma-\lambda)e^{-2\pi i y\cdot(\gamma-\lambda)}\left(\delta_y-\sum_{x\in E}a_x(y)\delta_x\right)^\wedge(\lambda)\,d\lambda\end{aligned}$$

on $\widehat{\mathbb{R}}^d$. If $\gamma\in\Lambda$ and $\lambda\in(\Lambda_\epsilon)^c$, then $\widehat{h}(\gamma-\lambda)=0$. Consequently, we obtain

$$\forall y\in\mathbb{R}^d \text{ and } \forall\gamma\in\Lambda,\quad(\eta_y)^\wedge(\gamma)=0.$$

Thus, since Λ is an S-set and $h(0)=1$, we obtain (6) from the definition of η_y. □

Example 2.20 Theorem 2.19 can be used to prove Beurling's sufficient condition for a Fourier frame in terms of balayage (Theorem 2.15), see part b. For convenience, let Λ be symmetric about $0\in\widehat{\mathbb{R}}^d$, i.e., $-\Lambda=\Lambda$. *a.* Using the notation of Theorem 2.19, we have the following estimate.

$$\sum_{x\in E}\left|\int a_x(y)h(x-y)f(y)\,dy\right|^2\le\sum_{x\in E}\int|a_x(y)||h(x-y)|^2\,dy\int|a_x(y)||f(y)|^2\,dy$$

$$\le C\,\|h\|_2^2 \int \left(\sum_{x\in E} |a_x(y)| \right) |f(y)|^2\, dy$$
$$\le C\,\|h\|_2^2\, K(E, \Lambda_\epsilon)\, \|f\|_2^2\,,$$

where C is a uniform bound of $\{|a_x(y)| : x \in E,\, y \in \mathbb{R}^d\}$.

b. It is sufficient to prove the lower frame bound. Let $F \in L^2(\Lambda)$ be considered as an element of $(PW_\Lambda)^\wedge$, i.e., $\widehat{f} = F$ vanishes off of Λ and $f \in L^2(\mathbb{R}^d)$. We shall show that

$$A\,\|F\|_{L^2(\Lambda)} \le \left(\sum_{x\in E} |f(x)|^2 \right)^{1/2}, \tag{8}$$

where A is independent of $F \in L^2(\Lambda)$.

$$\begin{aligned}
\|F\|^2_{L^2(\Lambda)} &= \int_\Lambda \overline{F(\lambda)} \left(\int f(y) e^{-2\pi i y\cdot\lambda}\, dy \right) d\lambda \\
&= \int_\Lambda \overline{F(\lambda)} \left(\int f(y) \left(\sum_{x\in E} a_x(y) h(x-y) e^{-2\pi i x\cdot\lambda} \right) dy \right) d\lambda \\
&= \sum_{x\in E} \overline{f(x)} \left(\int a_x(y) h(x-y) f(y)\, dy \right) \\
&\le \left(\sum_{x\in E} |f(x)|^2 \right)^{1/2} \left(\sum_{x\in E} \left| \int a_x(y) h(x-y) f(y)\, dy \right|^2 \right)^{1/2} \\
&\le \left[C\,\|h\|_2^2\, K(E, \Lambda_\epsilon) \right]^{1/2} \left(\sum_{x\in E} |f(x)|^2 \right)^{1/2} \|f\|_2\,,
\end{aligned}$$

and so we set $A = 1/[C\,\|h\|_2^2\, K(E, \Lambda_\epsilon)]^{1/2}$ to obtain (8).

3 Short Time Fourier Transform (STFT) Frame Inequalities

Definition 3.1 *a.* Let $f, g \in L^2(\mathbb{R}^d)$. The *short-time Fourier transform* (STFT) of f with respect to g is the function $V_g f$ on $\mathbb{R}^{2d}$ defined as

$$V_g f(x, \omega) = \int f(t) \overline{g(t-x)}\, e^{-2\pi i t\cdot\omega}\, dt,$$

see [40,41].

b. The STFT is uniformly continuous on $\mathbb{R}^{2d}$. Further, for a fixed "window" $g \in L^2(\mathbb{R}^d)$ with $\|g\|_2 = 1$, we can recover the original function $f \in L^2(\mathbb{R}^d)$ from its STFT $V_g f$ by means of the vector-valued integral inversion formula,

$$f = \int\!\!\int V_g f(x, \omega)\, e_\omega \tau_x g\, d\omega\, dx, \tag{9}$$

where modulation e_ω was defined earlier and translation τ_x is defined as $\tau_x g(t) = g(t-x)$. Explicitly, Equation (9) signifies that we have the vector-valued mapping, $(x,\omega) \mapsto e_\omega \tau_x g \in L^2(\mathbb{R}^d)$, and

$$\forall\, h \in L^2(\mathbb{R}^d),\ \langle f, h\rangle = \int\int \left[\int V_g f(x,\omega)(e_\omega \tau_x g(t))\overline{h(t)}\, dt\right] d\omega dx.$$

Also, if $\widehat{f} = F$ and $\widehat{g} = G$, where $f, g \in L^2(\mathbb{R}^d)$, then one obtains the *fundamental identity of time frequency analysis,*

$$V_g f(x,\omega) = e^{-2\pi i x\cdot\omega} V_G F(\omega, -x). \tag{10}$$

c. Let $g_0(x) = 2^{d/4} e^{-\pi\|x\|^2}$. Then $G_0(\gamma) = \widehat{g}_0(\gamma) = 2^{d/4}e^{-\pi\|\gamma\|^2}$ and $\|g_0\|_2 = 1$, see [8] for properties of g_0. The *Feichtinger algebra*, $\mathcal{S}_0(\mathbb{R}^d)$, is

$$\mathcal{S}_0(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) \colon \|f\|_{\mathcal{S}_0} = \|V_{g_0} f\|_1 < \infty\}.$$

For now it is useful to note that the Fourier transform of $\mathcal{S}_0(\mathbb{R}^d)$ is an isometric isomorphism onto itself, and, in particular, $f \in \mathcal{S}_0(\mathbb{R}^d)$ if and only if $F \in \mathcal{S}_0(\widehat{\mathbb{R}}^d)$.

Theorem 3.2 *Let $E = \{x_n\} \subseteq \mathbb{R}^d$ be a separated sequence, that is symmetric about $0 \in \mathbb{R}^d$; and let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be an S-set of strict multiplicity, that is compact, convex, and symmetric about $0 \in \widehat{\mathbb{R}}^d$. Assume balayage is possible for (E, Λ). Further, let $g \in L^2(\mathbb{R}^d)$, $\widehat{g} = G$, have the property that $\|g\|_2 = 1$.*

a. We have that

$$\exists\, A > 0, \quad \text{such that} \quad \forall f \in PW_\Lambda \backslash \{0\}, \quad \widehat{f} = F,$$

$$A\|f\|_2^2 = A\|F\|_2^2 \le \sum_{x\in E} \int |V_G F(\omega, x)|^2\, d\omega = \sum_{x\in E}\int |V_g f(x,\omega)|^2\, d\omega. \tag{11}$$

b. Let $g \in \mathcal{S}_0(\mathbb{R}^d)$. We have that

$$\exists\, B > 0, \quad \text{such that} \quad \forall f \in PW_\Lambda \backslash \{0\}, \quad \widehat{f} = F,$$

$$\sum_{x\in E}\int |V_g f(x,\omega)|^2\, d\omega = \sum_{x\in E}\int |V_G F(\omega,-x)|^2\, d\omega \le B\|F\|_2^2 = B\|f\|_2^2, \tag{12}$$

where B can be taken as $2^{d/2}\, C\|V_{g_0} g\|_1^2$ and where

$$C = \sup_{u\in\mathbb{R}^d} \sum_{x\in E} e^{-\|x-u\|^2}.$$

see the technique in [35], Lemma 3.2.15, cf. [34], Lemma 3.2.

Proof a.i. We first combine the $STFT$ and balayage to compute

$$\begin{aligned}
\|f\|_2^2 &= \int_\Lambda F(\gamma)\,\overline{F(\gamma)}\,d\gamma \\
&= \int_\Lambda F(\gamma)\,\left(\int\int \overline{V_G F(y,\omega)}\;\overline{e_\omega(\gamma)}\;\overline{G(\gamma-y)}\,d\omega\,dy\right)\,d\gamma \\
&= \int_\Lambda F(\gamma)\,\left(\int\int \overline{V_G F(y,\omega)}\;\overline{G(\gamma-y)}\left(\sum_{x\in E}\overline{a_x(\omega)}\;\overline{h(x-\omega)}\,e^{-2\pi i x\cdot\gamma}\right)d\omega\,dy\right)d\gamma \\
&= \int\int \overline{V_G F(y,\omega)}\left(\sum_{x\in E}\overline{a_x(\omega)}\;\overline{h(x-\omega)}\int F(\gamma)\,\overline{G(\gamma-y)}\,e^{-2\pi i x\cdot\gamma}\,d\gamma\right)d\omega\,dy \\
&= \int\int \overline{V_G F(y,\omega)}\left(\sum_{x\in E}\overline{a_x(\omega)}\;\overline{h(x-\omega)}\,V_G F(y,x)\right)d\omega\,dy \\
&= \int\left[\sum_{x\in E}\left(\int \overline{V_G F(y,\omega)}\;\overline{a_x(\omega)}\;\overline{h(x-\omega)}\,d\omega\right)\,V_G F(y,x)\right]dy \\
&\le \int\left(\sum_{x\in E}\left|\int a_x(\omega)\,h(x-\omega)\,V_G F(y,\omega)\,d\omega\right|^2\right)^{1/2}\left(\sum_{x\in E}|V_G F(y,x)|^2\right)^{1/2}dy. \qquad (13)
\end{aligned}$$

a.ii. We shall show that there is a constant $C > 0$, independent of $f \in PW_\Lambda$, such that

$$\forall\, y\in\mathbb{R}^d,\ \sum_{x\in E}\left|\int a_x(\omega)\,h(x-\omega)V_G F(y,\omega)\,d\omega\right|^2 \le C^2\int |V_G F(y,\omega)|^2\,d\omega. \qquad (14)$$

The left side of (14) is bounded above by

$$\begin{aligned}
&\sum_{x\in E}\left(\int |a_x(\omega)|\,|h(x-\omega)|^2\,d\omega\right)\left(\int |a_x(\omega)|\,|V_G F(y,\omega)|^2\,d\omega\right) \\
&\quad\le \sum_{x\in E}\left(K_1\int |h(x-\omega)|^2\,d\omega\right)\left(\int |a_x(\omega)|\,|V_G F(y,\omega)|^2\,d\omega\right) \\
&\quad= K_1\,\|h\|_2^2\sum_{x\in E}\int |a_x(\omega)|\,|V_G F(y,\omega)|^2\,d\omega \\
&\quad= K_1\,\|h\|_2^2\int\left(\sum_{x\in E}|a_x(\omega)|\right)|V_G F(y,\omega)|^2\,d\omega \\
&\quad\le K_1\,K_2\,\|h\|_2^2\int |V_G F(y,\omega)|^2\,d\omega,
\end{aligned}$$

where we began by using Hölder's inequality and where K_1 and K_2 exist because of (7) in Theorem 2.19. Let $C^2 = K_1K_2\,\|h\|_2^2$.

a.iii. Combining parts *a.i* and *a.ii*, we have from (13) and (14) that

$$\begin{aligned}\|f\|_2^2 &= \int_\Lambda F(\gamma)\,\overline{F(\gamma)}\,d\gamma \\ &\le \int C\left(\int |V_G F(y,\omega)|^2\,d\omega\right)^{1/2}\left(\sum_{x\in E}|V_G F(y,x)|^2\right)^{1/2} dy \\ &\le C\left(\int\int |V_G F(y,\omega)|^2\,d\omega\,dy\right)^{1/2}\left(\int\sum_{x\in E}|V_G F(y,x)|^2\,dy\right)^{1/2} \\ &= C\left(\int_\Lambda |F(\gamma)|^2\,d\gamma\right)^{1/2}\left(\int\sum_{x\in E}|V_G F(y,x)|^2\,dy\right)^{1/2},\end{aligned}$$

where we have used Hölder's inequality and the fact that the STFT is an isometry from $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$. Consequently, by the symmetry of E, we have

$$\begin{aligned}\frac{1}{C^2}\|f\|_2^2 &= \frac{1}{C^2}\int_\Lambda |F(\gamma)|^2\,d\gamma \\ &\le \int\sum_{x\in E}|V_G F(\omega,-x)|^2\,d\omega = \int\sum_{x\in E}|V_g f(x,\omega)|^2 d\omega,\end{aligned}$$

where we have used (10). Part *a* is completed by setting $A = 1/C^2$.

b.i. The proof of (12) will require the reproducing formula [34], p. 412:

$$V_g f(y,\gamma) = \langle V_{g_0} f, V_{g_0}(e_\gamma \tau_y g)\rangle, \tag{15}$$

where $\widehat{g_0} = G_0$. Equation (15) is a consequence of the inversion formula,

$$f = \int\int V_{g_0} f(x,\omega) e_\omega \tau_x g_0\,d\omega\,dx,$$

and substituting the right side into the definition $\langle f, e_\gamma \tau_y g\rangle$ of $V_g f(y,\gamma)$. Equation (15) is valid for all $f, g \in L^2(\mathbb{R}^d)$.

b.ii. Using Eq. (15) from part *b.i* we compute

$$\begin{aligned}&\sum_{x\in E}\int |V_g f(x,\omega)|^2 d\omega \\ &= \int\sum_{x\in E}|\langle V_{g_0} f, V_{g_0}(e_\omega \tau_x g)\rangle|^2 d\omega \\ &= \int\sum_{x\in E}\left|\int\int \overline{V_{g_0} f(y,\gamma)}\,V_{g_0}(e_\omega\tau_x g)(y,\gamma)\,dy\,d\gamma\right|^2 d\omega \\ &\le \int\sum_{x\in E}\left(\left(\int\int |V_{g_0} f(y,\gamma)|^2 |V_{g_0}(e_\omega\tau_x g)(y,\gamma)|\,dy\,d\gamma\right)\right.\end{aligned}$$

$$\left(\int\int |V_{g_0}(e_\omega \tau_x g)(y,\gamma)|\, dy\, d\gamma\right)\Bigg)\, d\omega.$$

b.iii. Since

$$\begin{aligned}V_{g_0}(e_\omega \tau_x)g)(y,\gamma) &= \int g(t-x)\,\overline{g_0(t-y)}\,e^{-2\pi i t\cdot(\gamma-\omega)}dt\\ &= e^{-2\pi i x\cdot(\gamma-\omega)}\int g(u)\,\overline{g_0(u+(x-y))}\,e^{-2\pi i u\cdot(\gamma-\omega)}du,\end{aligned}$$

we have

$$|V_{g_0}(e_\omega \tau_x)g)(y,\gamma)| \le |V_{g_0}g(y-x,\gamma-\omega).$$

Inserting this inequality into the last term of part *b.ii*, the inequality of part *b.ii* becomes

$$\begin{aligned}&\sum_{x\in E}\int |V_g f(x,\omega)|^2\, d\omega\\ &\le \int\sum_{x\in E}\Bigg(\left(\int\int |V_{g_0} f(y,\gamma)|^2 |V_{g_0}g(y-x,\gamma-\omega)|\, dy\, d\gamma\right)\\ &\quad \left(\int\int |V_{g_0}g)(y-x,\gamma-\omega)|\, dy\, d\gamma\right)\Bigg)\, d\omega\\ &= \|V_{g_0}g\|_1 \int\sum_{x\in E}\left(\int\int |V_{g_0} f(y,\gamma)|^2 |V_{g_0}g(y-x,\gamma-\omega)|\, dy\, d\gamma\right)\, d\omega\\ &\le \|V_{g_0}g\|_1 \int\int |V_{g_0} f(y,\gamma)|^2 \left(\int\sum_{x\in E}|V_{g_0}g(y-x,\gamma-\omega)|\, d\omega\right)\, dy\, d\gamma.\end{aligned}$$

b.iv. By the reproducing formula, Eq. (15), the integral-sum factor in the last term of part *b.iii* is

$$\begin{aligned}&\int\sum_{x\in E}|V_{g_0}g(y-x,\gamma-\omega)|\, d\omega\\ &= \int\sum_{x\in E}|\int\int V_{g_0}g(z,\zeta)\,\overline{V_{g_0}(e_{\gamma-\omega}\tau_{y-x}g_0)(z,\zeta)}\, dz\, d\zeta|d\omega\\ &= \int\sum_{x\in E}|\int\int V_{g_0}g(z,\zeta)\left(\overline{\int g_0(u)\overline{g_0(u-(z+x-y))}\,e^{-2\pi i u\cdot(\zeta-\gamma+\omega)}\, du}\right) dz\, d\zeta|\, d\omega\\ &= \int\sum_{x\in E}|\int\int V_{g_0}g(z,\zeta)\,\overline{V_{g_0}g_0(z+(x-y),\zeta+(\omega-\gamma))}\, dz\, d\zeta|d\omega\\ &\le \int\int |V_{g_0}g(z,\zeta)|\left(\int\sum_{x\in E}|V_{g_0}g_0(z+(x-y),\zeta+(\omega-\gamma))|d\omega\right) dz\, d\zeta.\end{aligned}$$

b.v. Substituting the last term of part *b.iv* in the last term of part *b.iii*, the inequality of part *b.ii* becomes

$$\sum_{x\in E}\int |V_g f(x,\omega)|^2\,d\omega \le \|V_{g_0}g\|_1 \int\int |V_{g_0}f(y,\gamma)|^2$$
$$\times\left(\int\int |V_{g_0}g(z,\zeta)|\left(\sum_{x\in E}\left(\int |V_{g_0}g_0(z+(x-y)),\zeta+(\omega-\gamma))|d\omega\right)\right)dz\,d\zeta\right)dy\,d\gamma$$
$$= \|V_{g_0}g\|_1 \int\int |V_{g_0}f(y,\gamma)|^2\left(\int\int |V_{g_0}g(z,\zeta)|\left(\sum_{x\in E}K(x,y,z,\gamma,\zeta)\right)dz\,d\zeta\right)dy\,d\gamma,$$

where

$$K(x,y,z,\gamma,\zeta) = e^{-\frac{\pi}{2}\|z+(x-y)\|^2}\int e^{-\frac{\pi}{2}\|\zeta+(\omega-\gamma\|^2}d\omega.$$

Hence,

$$\sum_{x\in E}\int |V_g f(x,\omega)|^2\,d\omega \le 2^{\frac{d}{2}}\,C\,\|V_{g_0}g\|_1^2\,\|V_{g_0}f\|^2,$$

where

$$C = \sup_{u\in\mathbb{R}^d}\sum_{x\in E}e^{-\|x-u\|^2}.$$

The fact, $C < \infty$, is straightforward to verify, but see [67] and [66], Lemma 2.1, for an insightful, refined estimate of C. The proof of part b is completed by a simple application of Eq. (22). □

We now recall a special case of a fundamental theorem of Gröchenig for non-uniform Gabor frames, see [38], Theorem S, and [40], Theorem 13.1.1, cf. [30] and [31] for a precursor of this result, presented in an almost perfectly disguised way for the senior author to understand. The general case of Gröchenig's theorem is true for the class of modulation spaces, $M_v^1(\mathbb{R}^d)$, where the Feichtinger algebra, $\mathcal{S}_0(\mathbb{R}^d)$, is the case that the weight v is identically 1 on $\mathbb{R}^d$. The author's proof at all levels of generalization involves a significant analysis of convolution operators on the Heisenberg group. See [40] for an authoritative exposition of modulation spaces as well as their history.

Theorem 3.3 *Given any $g \in \mathcal{S}_0(\mathbb{R}^d)$. There is $r = r(g) > 0$ such that if $E = \{(s_n,\sigma_n)\} \subseteq \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ is a separated sequence with the property that*

$$\bigcup_{n=1}^{\infty}\overline{B((s_n,\sigma_n),r(g))} = \mathbb{R}^d \times \widehat{\mathbb{R}}^d,$$

then the frame operator, $S = S_{g,E}$, defined by

$$S_{g,E}\,f = \sum\nolimits_{n=1}^{\infty}\langle f,\tau_{s_n}e_{\sigma_n}g\rangle\,\tau_{s_n}e_{\sigma_n}g,$$

is invertible on $\mathcal{S}_0(\mathbb{R}^d)$.

Moreover, every $f \in \mathcal{S}_0(\mathbb{R}^d)$ has a non-uniform Gabor expansion,

$$f = \sum_{n=1}^{\infty} \langle f, \tau_{s_n} e_{\sigma_n} g \rangle S_{g,E}^{-1}(\tau_{s_n} e_{\sigma_n} g),$$

where the series converges unconditionally in $\mathcal{S}_0(\mathbb{R}^d)$.
(E depends on g.)

The following result can be compared with Theorem 3.3. It is also a theorem about Gabor expansions of certain band-limited functions with respect to a band-limited window, and as such can also be compared to results about Gabor frames for subspaces, see Example 3.5 as well as earlier work of Gröchenig [39] relating sampling theorems for band-limited functions with Gabor frames.

Theorem 3.4 *Let $E = \{(s_n, \sigma_n)\} \subseteq \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ be a separated sequence; and let $\Lambda \subseteq \widehat{\mathbb{R}}^d \times \mathbb{R}^d$ be an S-set of strict multiplicity that is compact, convex, and symmetric about $0 \in \widehat{\mathbb{R}}^d \times \mathbb{R}^d$. Assume balayage is possible for (E, Λ). Further, let $g \in L^2(\mathbb{R}^d)$, $\widehat{g} = G$, have the property that $\|g\|_2 = 1$. We have that*

$$\exists\, A,\ B > 0, \quad \textit{such that} \quad \forall f \in \mathcal{S}_0(\mathbb{R}^d), \quad \textit{for which} \quad \mathrm{supp}(\widehat{V_g f}) \subseteq \Lambda,$$

$$A\, \|f\|_2^2 \leq \sum_{n=1}^{\infty} |V_g f(s_n, \sigma_n)|^2 \leq B\, \|f\|_2^2. \tag{16}$$

Consequently, the frame operator, $S = S_{g,E}$, is invertible in $L^2(\mathbb{R}^d)$–norm on the subspace of $\mathcal{S}_0(\mathcal{R}^d)$, whose elements f have the property, supp $(\widehat{V_g f}) \subseteq \Lambda$.

Moreover, every $f \in \mathcal{S}_0(\mathbb{R}^d)$ satisfying the support condition, $\mathrm{supp}(\widehat{V_g f}) \subseteq \Lambda$, *has a non-uniform Gabor expansion,*

$$f = \sum_{n=1}^{\infty} \langle f, \tau_{s_n} e_{\sigma_n} g \rangle S_{g,E}^{-1}(\tau_{s_n} e_{\sigma_n} g),$$

where the series converges unconditionally in $L^2(\mathbb{R}^d)$.
(E does not depend on g.)

Proof a. Using Theorem 2.19 for the setting $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, where $h \in L^1(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ from Ingham's theorem has the property that $\mathrm{supp}(\widehat{h}) \subseteq \overline{B(0,\epsilon)} \subseteq \widehat{\mathbb{R}}^d \times \mathbb{R}^d$, we compute

$$\int |f(x)|^2\, dx = \int\int |V_g f(y,\omega)|^2\, dy\, d\omega \tag{17}$$
$$= \int\int \overline{V_g f(y,\omega)} \sum_{n=1}^{\infty} a_{s_n,\sigma_n}(y,\omega) h(s_n - y, \sigma_n - \omega) V_g f(s_n, \sigma_n)\, dy\, d\omega,$$

where

$$V_g f(y,\omega) = \sum_{n=1}^{\infty} a_{s_n,\sigma_n}(y,\omega) h(s_n - y, \sigma_n - \omega) V_g f(s_n, \sigma_n)$$

and

$$\sup_{(y,\omega)\in\mathbb{R}^d\times\widehat{\mathbb{R}}^d}\sum_{n=1}^{\infty}|a_{s_n,\sigma_n}(y,\omega)| \leq K(E,\Lambda_\epsilon) < \infty.$$

Interchanging summation and integration on the right side of Equation (17), we use Hölder's inequality to obtain

$$\begin{aligned} &\int |f(x)|^2\,dx \leq \\ &\quad \left(\sum_{n=1}^{\infty}|V_g f(s_n,\sigma_n)|^2\right)^{1/2} \\ &\quad \left(\sum_{n=1}^{\infty}\left|\int\int a_{s_n,\sigma_n}(y,\omega)h(s_n-y,\sigma_n-\omega)\,\overline{V_g f(y,\omega)}\,dy\,d\omega\right|^2\right)^{1/2} \qquad (18)\\ &\leq {S_1}^{1/2}\,{S_2}^{1/2}. \end{aligned}$$

We bound the second sum S_2 using Hölder's inequality for the integrand,

$$[(a_{s_n,\sigma_n}(y,\omega))^{1/2}h(s_n-y,\sigma_n-\omega)][(a_{s_n,\sigma_n}(y,\omega))^{1/2}\overline{V_g f(y,\omega)}],$$

as follows:

$$\begin{aligned} S_2 &\leq \sum_{n=1}^{\infty}\left(\int\int|a_{s_n,\sigma_n}(y,\omega)||h(s_n-y,\sigma_n-\omega)|^2\,dy\,d\omega\int\int|a_{s_n,\sigma_n}(y,\omega)||V_g f(y,\omega)|^2\,dy\,d\omega\right)\\ &\leq K_1\sum_{n=1}^{\infty}\left(\int\int|h(s_n-y,\sigma_n-\omega)|^2\,dy\,d\omega\int\int|a_{s_n,\sigma_n}(y,\omega)||V_g f(y,\omega)|^2\,dy\,d\omega\right) \qquad (19)\\ &= K_1\,\|h\|_2^2\int\int\left(\sum_{n=1}^{\infty}|a_{s_n,\sigma_n}(y,\omega)||V_g f(y,\omega)|^2\right)dy\,d\omega \;\leq\; K_1K_2\,\|h\|_2^2\,\|f\|_2^2, \end{aligned}$$

where K_1 is a uniform bound on $\{a_{s_n,\sigma_n}(y,\omega)\}$, K_2 invokes the full power of Theorem 2.19, and $\|f\|_2^2 = \left\|V_g f\right\|_2^2$.

Combining (18) and (19), we obtain

$$\|f\|_2^2 \leq (S_1K_1K_2)^{1/2}\,\|h\|_2\,\|f\|_2,$$

and so the left hand inequality of (16) is valid for $1/(K_1K_2\,\|h\|_2^2)$.

b. The right hand inequality of (16) follows directly from the Pólya-Plancherel theorem, cf. Theorem 3.2*b*. □

Example 3.5 *a.* In comparing Theorem 3.3 with Theorem 3.4, a possible weakness of the former is the dependence of E on g, whereas a possible weakness of the latter is the hypothesis that $\mathrm{supp}(\widehat{V_g f}) \subseteq \Lambda$. Let us look at this latter possibility more closely.

a.i. Let $f, g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. We know that $V_g f \in L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, and

$$\widehat{V_g f}(\zeta, z) = \int\int \left(\int f(t)\, g(t-x)\, e^{-2\pi i t\cdot\omega}\, dt\right) e^{-2\pi i(x\cdot\zeta + z\cdot\omega)}\, dx\, d\omega.$$

The right side is

$$\int\int f(t) \left(\int g(t-x)\, e^{-2\pi i x\cdot\zeta}\, dx\right) e^{-2\pi i t\cdot\omega}\, e^{-2\pi i z\cdot\omega}\, dt\, d\omega,$$

where the interchange in integration follows from the Fubini-Tonelli theorem and the hypothesis that $f, g \in L^1(\mathbb{R}^d)$. This, in turn, is

$$\hat{g}(-\zeta) \int \left(\int f(t)\, e^{-2\pi i t\cdot\zeta}\, e^{-2\pi i t\cdot\omega}\, dt\right) e^{-2\pi i z\cdot\omega}\, d\omega$$

$$= \hat{g}(-\zeta) \int \hat{f}(\zeta + \omega)\, e^{-2\pi i z\cdot\omega}\, d\omega = e^{-2\pi i z\cdot\zeta}\, f(-z)\, \hat{g}(-\zeta).$$

Consequently, we have shown that if $f, g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, then

$$f, g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d), \quad \widehat{V_g f}(\zeta, z) = e^{-2\pi i z\cdot\zeta}\, f(-z)\, \hat{g}(-\zeta). \tag{20}$$

The *Rihaczek distribution* of $f, g \in L^2(\mathbb{R}^d)$ is the function $R(f, g)$ defined on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ as

$$R(f, g)(x, \omega) = e^{-2\pi i x\cdot\omega}\, f(x)\, \overline{\hat{g}(\omega)},$$

see [41], pp. 142–148.

a.ii. Let $\Lambda \subseteq \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ be compact, convex, and symmetric, and suppose that $\operatorname{supp}(\widehat{V_g f}) \subseteq \Lambda$ as in Theorem 3.4. From this assumption we can conclude that f *and* g *have compact support*. In fact, if $\Lambda \subseteq [\Omega, \Omega]^d \times [\Omega, \Omega]^d$, then $\operatorname{supp}(f) \subseteq [\Omega, \Omega]^d$ and $\operatorname{supp}(\hat{g}) \subseteq [\Omega, \Omega]^d$.

This claim, that f and g have compact support, is a consequence of the fact,

$$\widehat{V_g f}(x, \omega) = R(f, g)(x, \omega), \tag{21}$$

since Equation (21) implies that

$$\operatorname{supp}(\widehat{V_g f}) = \operatorname{supp}(f) \times \operatorname{supp}(\hat{g}).$$

In particular, if $\Lambda = \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, then $\operatorname{supp}(f) \subseteq [\Omega, \Omega]^d$ and $\operatorname{supp}(\hat{g}) \subseteq [\Omega, \Omega]^d$; and if $\Lambda \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ properly, then the supports of f and $\hat{g}$ must be even smaller to ensure that $\operatorname{supp}(f) \times \operatorname{supp}(\hat{g})$ is contained in Λ.

a.iii Thus, Theorem 3.4 provides the construction of a Gabor frame for subspaces of $L^2(\mathbb{R}^d)$. In this context, the coorbit theory of Feichtinger and Gröchenig yields Gabor expansions for all of $L^2(\mathbb{R}^d)$, e.g., see [32].

b. Theorems 3.3 and 3.4 give non-uniform Gabor frame expansions. Generally, for $g \in L^2(\mathbb{R})$, if $\{e_{\sigma_n}\tau_{s_n}g\}$ is a frame for $L^2(\mathbb{R})$, then $E = \{s_n, \sigma_n\} \subseteq \mathbb{R} \times \widehat{\mathbb{R}}$ is a finite union of separated sequences and $D^-(E) \geq 1$, where D^- denotes the lower Beurling density, [22]. (Beurling density has been analyzed deeply in terms of Fourier frames, e.g., [17,47,56,74], and it is defined as

$$D^-(E) = \lim_{r\to\infty} \frac{n^-(r)}{r^2},$$

where $n^-(r)$ is the minimal number of points from $E \subseteq \mathbb{R} \times \widehat{\mathbb{R}}$ in a ball of radius $r/2$.) For perspective, in the case of $\{e_{mb}\tau_{na}g : m, n \in \mathbb{Z}\}$, this necessary condition is equivalent to the condition $ab \leq 1$. It is also well-known that if $ab > 1$, then $\{e_{mb}\tau_{na}g : m, n \in \mathbb{Z}\}$ is not complete in $L^2(\mathbb{R})$. As such, it is not unexpected that $\{e_{\sigma_n}\tau_{s_n}g\}$ is incomplete if $D^-(E) < 1$; however, this is not the case as has been shown by explicit construction, see [11], Theorem 2.6. Other sparse complete Gabor systems have been constructed in [72] and [80].

Example 3.6 *a.* Let $(X, \mathcal{A}, \mu)$ be a measure space, i.e., X is a set, $\mathcal{A}$ is a σ−algebra in the power set $\mathcal{P}(X)$, and μ is a measure on $\mathcal{A}$, see [8]. Let H be a complex, separable Hilbert space. Assume

$$\mathcal{F}\colon X \to H$$

is a weakly measurable function in the sense that for each $f \in H$, the complex-valued mapping $x \mapsto \langle f, \mathcal{F}(x)\rangle$ is measurable. $\mathcal{F}$ is a $(X, \mathcal{A}, \mu)$*–frame* for H if

$$\exists\, A, B > 0 \text{ such that } \forall\, f \in H, \quad A\|f\|^2 \leq \int_X |\langle f, \mathcal{F}(x)\rangle|^2\, d\mu(x) \leq B\|f\|^2.$$

Typically, $\mathcal{A}$ is the Borel algebra $\mathcal{B}(\mathbb{R}^d)$ for $X = \mathbb{R}^d$ and $\mathcal{A} = \mathcal{P}(\mathbb{Z})$ for $X = \mathbb{Z}$. In these cases we use the terminology, (X, μ)-frame.

b. Continuous and discrete wavelet and Gabor frames are special cases of $(X, \mathcal{A}, \mu)$-frames and could have been formulated as such from the time of [23,43]. In mathematical physics the idea was introduced in [2,3,50]. Recent mathematical contributions are found in [36,37]. $(X, \mathcal{A}, \mu)$-frames are sometimes referred to as *continuous frames.* Also, in a slightly more concrete way we could have let X be a locally compact space and μ a positive Radon measure on X.

c. Let $X = \mathbb{Z}, \mathcal{A} = \mathcal{P}(\mathbb{Z})$, and $\mu = c$, where c is counting measure, $c(Y) = \text{card}(Y)$. Define $\mathcal{F}(n) = x_n \in H, n \in \mathbb{Z}$, for a given complex, separable Hilbert space, H. We have

$$\forall\, f \in H, \ \int_{\mathbb{Z}} |\langle f, x_n\rangle|^2\, d\,c(n) = \sum_{n\in\mathbb{Z}} \int_{\{n\}} |\langle f, x_n\rangle|^2\, d\,c(n) = \sum_{n\in\mathbb{Z}} |\langle f, x_n\rangle|^2.$$

Thus, *frames* $\{x_n\}$ for H, as defined in Definition 2.1, are $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), c)$–frames. For the present discussion we also refer to them as *discrete frames.*

d. Let $X = \mathbb{R}^d, \mathcal{A} = \mathcal{B}(\mathbb{R}^d)$, and $\mu = p$ a probability measure, i.e. $p(\mathbb{R}^d) = 1$; and let $H = \mathbb{R}^d$. The measure p is a *probabilistic frame* for $H = \mathbb{R}^d$ if

$$\exists\, A, B > 0 \text{ such that } \forall\, x \in \mathbb{R}^d\ (= H), \quad A\|x\|^2 \leq \int_X |\langle x, y\rangle|^2\, d\, p(y) \leq B\|x\|^2,$$

see [28,29]. Define

$$\mathcal{F}\colon X = \mathbb{R}^d \to H = \mathbb{R}^d$$

by $\mathcal{F}(x) = x \in \mathbb{R}^d$. Suppose $\mathcal{F}$ is a $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), p)$-frame for $H = \mathbb{R}^d$. Then

$$\forall\, x \in H, \quad A\|x\|^2 \leq \int_X |\langle x, y\rangle|^2\, d\, p(y) \leq B\|x\|^2,$$

and this is precisely the same as saying that p is a probabilistic frame for $H = \mathbb{R}^d$.

Suppose we try to generalize probabilistic frames to the setting that X is locally compact, as well as being a vector space because of probabilistic applications. This simple extension can not be effected since Hausdorff, locally compact vector spaces are, in fact, finite dimensional (F. Riesz).

e. Let $(X, \mathcal{A}, \mu)$ be a measure space and let H be a complex, separable Hilbert space. A *positive operator-valued measure* $(POVM)$ is a function $\pi : \mathcal{A} \to \mathcal{L}(H)$, where $\mathcal{L}(H)$ is the space of the bounded linear operators on H, such that $\pi(\emptyset) = 0$, $\pi(X) = I$ (Identity), $\pi(A)$ is a positive, and therefore self-adjoint (since H is a complex vector space), operator on H for each $A \in \mathcal{A}$, and

$$\forall \text{ disjoint } \{A_j\}_{j=1}^{\infty} \subseteq \mathcal{A}, \quad x, y \in H \implies \langle \pi\left(\cup_{j=1}^{\infty} A_j\right) x, y\rangle = \sum_{j=1}^{\infty} \langle \pi(A_j)x, y\rangle.$$

$POVMs$ are a staple in quantum mechanics, see [3,12] for rationale and references. If $\{x_n\} \subseteq H$ is a 1-tight discrete frame for H, then it is elementary to see that the formula,

$$\forall\, x \in H \text{ and } \forall\, A \in \mathcal{P}(\mathbb{Z}), \;\; \pi(A)x = \sum_{n\in A} \langle x, x_n\rangle x_n,$$

defines a $POVM$. Conversely, if $H = \mathbb{C}^d$ and π is a $POVM$ for X countable, then by the spectral theorem there is a corresponding 1-tight discrete frame. This relationship between tight frames and $POVMs$ extends to more general $(X, \mathcal{A}, \mu)$-frames, e.g., [3], Chapter 3.

In this setting, and related to *probability of quantum detection error,* P_e, which is defined in terms of $POVMs$, Kebo and one of the authors have proved the following for $H = \mathbb{C}^d, \{y_j\}_{j=1}^N \subseteq H$, and $\{\rho_j > 0\}_{j=1}^N, \sum_{j=1}^N \rho_j = 1$: there is a 1-tight discrete frame $\{x_n\}_{n=1}^N \subseteq H$ for H that minimizes P_e, [12], Theorem A.2.

f. Let $X = \mathbb{R}^{2d}$ and let $H = L^2(\mathbb{R}^d)$. Given $g \in L^2(\mathbb{R}^d)$ and define the function

$$\begin{aligned}\mathcal{F}\colon \mathbb{R}^{2d} &\to L^2(\mathbb{R}^d)\\ (x,\omega) &\mapsto e^{2\pi i t\cdot\omega}\, g(t-x).\end{aligned}$$

$\mathcal{F}$ is a $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^{2d}), m)$-frame for $L^2(\mathbb{R}^{2d})$, where m is Lebesgue measure on $\mathbb{R}^{2d}$; and, in fact, it is a tight frame for $L^2(\mathbb{R}^d)$ with frame constant $A = B = \|g\|_2^2$. To see this we need only note the following consequence of the orthogonality relations for the $STFT$:

$$\|V_g f\|_2 = \|g\|_{L^2(\mathbb{R}^d)}\|f\|_{L^2(\mathbb{R}^d)}. \tag{22}$$

Equation (22) is also used in the proof of (9).

g. Clearly, Theorems 3.2, 3.3, and 3.4 can be formulated in terms of (X,μ)–frames.

4 Examples and Modifications of Beurling's Method

4.1 Generalizations of Beurling's Fourier Frame Theorem

Using more than one measure, we can extend Theorem 2.15 to more general types of Fourier frames. For clarity we give the result for three simple measures.

Lemma 4.1 *Given the notation and hypotheses of Theorems 2.18 and 2.19. Then,*

$$\forall f \in PW_\Lambda \setminus \{0\},\ \widehat{f} = F,$$

$$\sum_{x\in E}\left|\int a_x(y)h(x-y)f(y)\,dy\right|^2 \le [K(E,\Lambda_\epsilon)\,\|h\|_2]^2\int_\Lambda |F(\gamma)|^2\,d\gamma.$$

Proof We compute:

$$\begin{aligned}&\sum_{x\in E}\left|\int a_x(y)h(x-y)f(y)\,dy\right|^2\\
&\quad\le \sum_{x\in E}\left|\left(\int |a_x(y)^{1/2}h(x-y)|^2\,dy\right)^{1/2}\left(\int |a_x(y)^{1/2}f(y)|^2\,dy\right)^{1/2}\right|^2\\
&\quad\le \sup_{x\in E}\left(\int |a_x(y)||h(x-y)|^2\,dy\right)\left(\sum_{x\in E}\int |a_x(y)||f(y)|^2\,dy\right)\\
&\quad\le K(E,\Lambda_\epsilon)\sup_{x\in E}\left(\int |a_x(y)||h(x-y)|^2\,dy\right)\int_\Lambda |F(\gamma)|^2\,d\gamma\\
&\quad\le K(E,\Lambda_\epsilon)^2\,\|h\|_2^2\int_\Lambda |F(\gamma)|^2\,d\gamma,\end{aligned}$$

where we have used the Plancherel theorem to obtain the third inequality. ◻

Theorem 4.2 *Let $E = \{x_n\} \subseteq \mathbb{R}^d$ be a separated sequence, and let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a compact S-set of strict multiplicity. Assume that Λ is a compact, convex set, that is symmetric about $0 \in \widehat{\mathbb{R}}^d$. If balayage is possible for (E, Λ), then*

$$\exists\, A, B > 0 \text{ such that} \forall f \in PW_\Lambda \setminus \{0\},\ F = \widehat{f},$$

$$\begin{aligned} A^{1/2} \frac{\int_\Lambda |F(\gamma) + F(2\gamma) + F(3\gamma)|^2 \, d\gamma}{\left(\int_\Lambda |F(\gamma)|^2 \, d\gamma\right)^{1/2}} &\leq \left(\sum_{x \in E} |f(x)|^2 \right)^{1/2} + \frac{1}{2} \left(\sum_{x \in E} |f(\tfrac{1}{2}x)|^2 \right)^{1/2} + \frac{1}{3} \left(\sum_{x \in E} |f(\tfrac{1}{3}x)|^2 \right)^{1/2} \\ &\leq B^{1/2} \left(\int_\Lambda |F(\gamma)|^2 \, d\gamma \right)^{1/2}. \end{aligned} \tag{23}$$

Proof By hypothesis, we can invoke Theorem 2.13 to choose $\epsilon > 0$ so that balayage is possible for (E, Λ_ϵ), i.e., $K(E, \Lambda_\epsilon) < \infty$. For this $\epsilon > 0$ and appropriate Ω, we use Theorem 2.18 to choose $h \in L^1(\mathbb{R}^d)$ for which $h(0) = 1$, $\operatorname{supp}(\widehat{h}) \subseteq \overline{B(0, \epsilon)}$, and $|h(x)| = O(e^{-\Omega(\|x\|)})$, $\|x\| \to \infty$.

Therefore, for a fixed $y \in \mathbb{R}^d$ and $g \in \mathcal{C}(\Lambda)$, Theorem 2.19 allows us to assert that

$$\begin{aligned} &g(y) + g(2y) + g(3y) \\ &\quad = \sum_{x \in E} g(x)\, (a_x(y)h(x-y) + a_x(2y)h(x-2y) + a_x(3y)h(x-3y)) \end{aligned}$$

and

$$\sum_{x \in E} |a_x(jy)| \leq K(E, \Lambda_\epsilon),\ \ j = 1, 2, 3.$$

Hence, if $\gamma \in \Lambda$ is fixed and $g(w) = e^{-2\pi i w \cdot \gamma}$, then

$$\begin{aligned} &e^{-2\pi i y \cdot \gamma} + e^{-2\pi i (2y) \cdot \gamma} + e^{-2\pi i (3y) \cdot \gamma} \\ &\quad = \sum_{x \in E} (a_x(y)h(x-y) + a_x(2y)h(x-2y) + a_x(3y)h(x-3y))\ e^{-2\pi i x \cdot \gamma}, \end{aligned}$$

which we write as

$$\sum_{x \in E} b_x(y) e^{-2\pi i x \cdot \gamma}.$$

Since $L^1(\mathbb{R}^d) \cap PW_\Lambda$ is dense in PW_Λ, we take $f \in L^1(\mathbb{R}^d) \cap PW_\Lambda$ in the following argument without loss of generality. We compute

$$\begin{aligned}
&\sum_{x\in E} e^{-2\pi i x\cdot\gamma} \int b_x(y) f(y)\, dy \\
&\quad = \int f(y) \left(\sum_{x\in E} b_x(y) e^{-2\pi i x\cdot\gamma} \right) dy \\
&\quad = \int f(y) \left(e^{-2\pi i y\cdot\gamma} + e^{-2\pi i (2y)\cdot\gamma} + e^{-2\pi i (3y)\cdot\gamma} \right) dy \\
&\quad = F(\gamma) + F(2\gamma) + F(3\gamma) = J_F(\gamma).
\end{aligned}$$

As such, we have

$$J_F(\gamma) = \sum_{x\in E} \widetilde{f}(x) e^{-2\pi i x\cdot\gamma}, \quad \text{where } \widetilde{f}(x) = \int b_x(y) f(y)\, dy.$$

Next, we compute the following inequality for the inner product $\langle J_F, J_F \rangle_\Lambda$:

$$\begin{aligned}
&\int_\Lambda J_F(\gamma)\overline{J_F(\gamma)}\, d\gamma = \int_\Lambda J_F(\gamma) \left(\sum_{x\in E} \overline{\widetilde{f}(x)} e^{2\pi i x\cdot\gamma} \right) d\gamma \\
&= \sum_{x\in E} \overline{\widetilde{f}(x)} \left(\int_\Lambda J_F(\gamma) e^{2\pi i x\cdot\gamma}\, d\gamma \right) = \sum_{x\in E} \overline{\widetilde{f}(x)} \left(f(x) + \frac{1}{2} f(\frac{x}{2}) + \frac{1}{3} f(\frac{x}{3}) \right) \qquad (24)\\
&\le \left(\sum_{x\in E} |\widetilde{f}(x)|^2 \right)^{1/2} \left(\sum_{x\in E} \left| f(x) + \frac{1}{2} f(\frac{x}{2}) + \frac{1}{3} f(\frac{x}{3}) \right|^2 \right)^{1/2} \\
&\le \left(\sum_{x\in E} |\widetilde{f}(x)|^2 \right)^{1/2} \left[\left(\sum_{x\in E} |f(x)|^2 \right)^{1/2} + \frac{1}{2} \left(\sum_{x\in E} |f(\frac{x}{2})|^2 \right)^{1/2} + \frac{1}{3} \left(\sum_{x\in E} |f(\frac{x}{3})|^2 \right)^{1/2} \right]
\end{aligned}$$

by Hölder's and Minkowski's inequalities. Further, there is $A > 0$ such that

$$\sum_{x\in E} |\widetilde{f}(x)|^2 \le \frac{1}{A} \int_\Lambda |F(\gamma)|^2\, d\gamma. \qquad (25)$$

This is a consequence of Lemma 4.1. Combining the definition of J_F with the inequalities (24) and (25) yield the first inequality of (23).

The second inequality of (23) only requires the assumption that E be separated, and, as such, it is a consequence of the Plancherel-Pólya theorem, which asserts that if E is separated, then

$$\exists\, B_j \text{ such that } \forall\, f \in PW_\Lambda,$$

$$\sum_{x \in E} \left| f\left(\frac{x}{j}\right) \right|^2 \leq B_j \, \|f\|_2^2, \ j = 1, 2, 3,$$

see [6], pp. 474–475, [56,79], pp. 109–113. □

Theorem 4.2 can be generalized extensively.

Example 4.3 Given the setting of Theorem 4.2.

a. Define the set $\{e_{j,x}^{\vee} : j = 1, 2, 3 \text{ and } x \in E\}$ of functions on $\mathbb{R}^d$ by

$$e_{j,x}(\gamma) = \frac{1}{j} \mathbb{1}_{\Lambda}(\gamma) e^{-2\pi i (1/j) x \cdot \gamma},$$

and define the mapping $S : PW_{\Lambda} \to PW_{\Lambda}$ by

$$Sf = \sum_{j=1}^{3} \sum_{x \in E} \left\langle f, e_{j,x}^{\vee} \right\rangle e_{j,x}^{\vee}.$$

We compute

$$\forall f \in PW_{\Lambda}, \quad \langle Sf, f \rangle = \sum_{j=1}^{3} \frac{1}{j^2} \sum_{x \in E} \left| f\left(\frac{x}{j}\right) \right|^2.$$

b. Let $f \in PW_{\Lambda}$, $\widehat{f} = F$, and define $J_F(\gamma) = F(\gamma) + F(2\gamma) + F(3\gamma)$. Since $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ for $a, b, c \in \mathbb{R}$, Theorem 4.2 and part *a* allow us to write the frame-type inequality,

$$\frac{A}{3} \frac{\langle J_F, J_F \rangle^2}{\|F\|_2} \leq \langle Sf, f \rangle = \|Lf\|_{\ell^2}^2 \leq B \, \|f\|_2^2, \tag{26}$$

where $Lf = \{\left\langle f, e_{j,x}^{\vee} \right\rangle : j = 1, 2, 3 \text{ and } x \in E\}$ so that $S = L^* L$. The inequalities (26) do not a priori define a frame for PW_{Λ}. However, $\{e_{j,x} : j = 1, 2, 3 \text{ and } x \in E\}$ is a frame for PW_{Λ} with frame operator S. This is a consequence of Theorem 2.15.

Theorem 4.4 *Let $E = \{x_n\} \subseteq \mathbb{R}^d$ be a separated sequence, and let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be an S-set of strict multiplicity. Assume that Λ is a compact, convex set, that is symmetric about $0 \in \widehat{\mathbb{R}}^d$. Further, let $G \in L^{\infty}(\mathbb{R}^d)$ be non-negative on $\widehat{\mathbb{R}}^d$. If balayage is possible for (E, Λ), then*

$$\exists \, A, B > 0, \ \textit{such that} \ \forall \ f \in PW_{\Lambda} \setminus \{0\}, F = \widehat{f},$$

$$A\frac{\left(\int_\Lambda |F(\gamma)|^2\, G(\gamma)\, d\gamma\right)^2}{\int_\Lambda |F(\gamma)|^2\, d\gamma} \le \sum_{x\in E} |\,(F\ G)^\vee\,(x)|^2$$
$$\le B\int_\Lambda |F(\gamma)|^2\ d\gamma. \qquad (27)$$

We can take $A = 1/\left(K(E,\Lambda_\epsilon)\,\|h\|_2^2\right)$ *and* $B = B_1\,\|G\|_\infty^2$, *where* B_1 *is the Bessel bound in the Plancherel-Pólya theorem for* PW_Λ.

Proof By hypothesis, we can invoke Theorem 2.13 to choose $\epsilon > 0$ so that balayage is possible for (E, Λ_ϵ), i.e., $K(E,\Lambda_\epsilon) < \infty$. For this $\epsilon > 0$ and appropriate Ω, we use Theorem 2.18 to choose $h \in L^1(\mathbb{R}^d)$ for which $h(0) = 1$, $\operatorname{supp}\widehat{h} \subseteq \overline{B(0,\epsilon)}$, and $|h(x)| = O(e^{-\Omega(\|x\|)})$, $\|x\| \to \infty$. Consequently, we have

$$\forall\ y \in \mathbb{R}^d \text{ and } \forall\ \gamma \in \Lambda,$$
$$e^{-2\pi i y\cdot\gamma} = \sum_{x\in E} a_x(y)h(x-y)e^{-2\pi i x\cdot\gamma},\ \text{ where } \sum_{x\in E}|a_x(y)| \le K(E,\Lambda_\epsilon).$$

If $f \in PW_\Lambda$, $\widehat{f} = F$, and noting that $F \in L^1(\widehat{\mathbb{R}}^d)$, we have the following computation:

$$\begin{aligned}
&\int_\Lambda |F(\gamma)|^2 G(\gamma)\, d\gamma\\
&= \int_\Lambda F(\gamma)G(\gamma)\left(\int \overline{f(w)}\left(\sum_{x\in E} a_x(w)h(x-w)e^{2\pi i x\cdot\gamma}\right) dw\right) d\gamma\\
&= \sum_{x\in E}\left(\int_\Lambda F(\gamma)G(\gamma)e^{2\pi i x\cdot\gamma}\ d\gamma\right)\left(\int \overline{f(w)}a_x(w)h(x-w)\, dw\right)\\
&\le \left(\sum_{x\in E}|(FG)^\vee(x)|^2\right)^{1/2}\left(\sum_{x\in E}\left|\int \overline{f(w)}a_x(w)h(x-w)\, dw\right|^2\right)^{1/2}\\
&\le K(E,\Lambda_\epsilon)\,\|h\|_2\left(\int_\Lambda |F(\gamma)|^2\, d\gamma\right)^{1/2}\left(\sum_{x\in E}\left|(FG)^\vee(x)\right|^2\right)^{1/2}, \qquad (28)
\end{aligned}$$

where the last step is a consequence of Lemma 4.1. Clearly, (28) gives the first inequality of (27). As in Theorem 4.2, the second inequality of (27) only requires the assumption that E be separated, and, as such, it is a consequence of the Plancherel-Pólya theorem for PW_Λ. □

Theorem 4.4 is an elementary generalization of the classical result for the case $G = 1$ on $\mathbb{R}$, and itself has significant generalizations to other weights G. We have not written $(FG)^\vee$ as a convolution since for such generalizations there are inherent subtleties in defining the convolution of distributions, e.g., [73], Chapitre VI, [63], see [7], pp. 99–102, for contributions of Hirata and Ogata, Colombeau, et al. Even in the case of Theorem 4.4, $G^\vee = g$ is in the class of pseudo-measures, which themselves play a basic role in spectral synthesis [5].

4.2 A Bounded Operator $B : L^p(\mathbb{R}^d) \to l^p(E),\ p > 1$

a. In Example 2.20*b* we proved the lower frame bound assertion of Theorem 2.15. This can also be achieved using Beurling's generalization of balayage to so-called linear balayage operators B, see [17], pp. 348–350.

In fact, with this notion and assuming the hypotheses of Theorem 2.19, Beurling proved that the mapping,

$$\begin{aligned} L^p(\mathbb{R}^d) &\longrightarrow l^p(E), \quad p > 1, \\ k &\mapsto \{k_x\}_{x \in E}, \end{aligned}$$

where

$$\forall\, x \in E, \quad k_x = \int_{\mathbb{R}^d} a_x(y) h(x - y) k(y)\, dy,$$

has the property that

$$\exists\, C_p > 0 \text{ such that } \forall\, k \in L^p(\mathbb{R}^d),$$

$$\sum_{x \in E} |k_x|^p \leq C_p \int |k(y)|^p dy. \tag{29}$$

Let $p = 2$ and fix $f \in PW_\Lambda$. We shall use (29) and the definition of norm to obtain the desired lower frame bound. This is Landau's idea. Set

$$I_k = \int_\Lambda F(\gamma) \overline{K(\gamma)} d\gamma, \qquad \widehat{f} = F,$$

where $K^\vee = k \in L^2(\mathbb{R}^d)$. By balayage, we have

$$K(\gamma) = \sum_{x \in E} k_x e^{-2\pi i x \cdot \gamma} \text{ on } \Lambda;$$

and so,

$$I_k = \sum_{x \in E} f(x) \overline{k_x},$$

allowing us to use (29) to make the estimate,

$$|I_k|^2 \leq C \|K\|_2^2 \sum_{x \in E} |f(x)|^2.$$

By definition of $\|f\|_2$, we have

$$\|f\|_2 = \sup_K \frac{|I_K|}{\|K\|_2} \leq C\left(\sum_{x\in E}|f(x)|^2\right)^{1/2},$$

and this is the lower frame bound inequality with bound $A = 1/C^2$.

Because of this approach we can think of balayage as "$l^2 - L^2$ balayage".

b. Motivated by part *a*, we shall say that $l^1 - L^2$ *balayage is possible* for (E, Λ), where E is separated and Λ is a compact set of positive measure $|\Lambda|$, if

$$\exists\, C > 0 \text{ such that } \forall\, k \in L^2(\mathbb{R}^d), \widehat{k} = K,$$

$$\sum_{x\in E}|k_x| \leq C\int_\Lambda |K(\gamma)|^2 d\gamma$$

and

$$K(\gamma) = \sum_{x\in E} k_x e^{-2\pi i x\cdot\gamma} \text{ on } \Lambda.$$

For fixed $f \in PW_\Lambda$ and using the notation of part *a,* we have

$$|I_k|^2 \leq \sum_{x\in E}|k_x|^2 \sum_{x\in E}|f(x)|^2. \tag{30}$$

An elementary calculation gives

$$\sum_{x\in E}|k_x|^2 \leq C^2|\Lambda| \int_\Lambda |K(\gamma)|^2 d\gamma,$$

which, when substituted into (30), gives

$$\frac{1}{C^2|\Lambda|}\left(\frac{|I_K|^2}{\int_\Lambda |K(\gamma)|^2 d\gamma}\right) \leq \sum_{x\in E}|f(x)|^2.$$

We obtain the desired lower frame inequality with bound $A = 1/(C^2|\Lambda|)$.

5 Pseudo-differential Operator Frame Inequalities

Let $\sigma \in \mathcal{S}'(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. The operator, K_σ, formally defined as

$$(K_\sigma f)(x) = \int \sigma(x,\gamma)\widehat{f}(\gamma)e^{2\pi i x\cdot\gamma}\, d\gamma,$$

is the *pseudo-differential operator* with Kohn-Nirenberg symbol, σ, see [40] Chapter 14, [41] Chapter 8, [45], and [78], Chapter VI. For consistency with the notation of the previous sections, we shall define pseudo-differential operators, K_s, with tempered distributional Kohn-Nirenberg symbols, $s \in \mathcal{S}'(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, as

$$(K_s \widehat{f})(\gamma) = \int s(y, \gamma) f(y) e^{-2\pi i y \cdot \gamma}\, dy.$$

Further, we shall actually deal with Hilbert–Schmidt operators, $K : L^2(\widehat{\mathbb{R}}^d) \to L^2(\widehat{\mathbb{R}}^d)$; and these, in turn, can be represented as $K = K_s$, where $s \in L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. Recall that $K : L^2(\widehat{\mathbb{R}}^d) \to L^2(\widehat{\mathbb{R}}^d)$ is a *Hilbert–Schmidt operator* if

$$\sum\nolimits_{n=1}^{\infty} \|K e_n\|_2^2 < \infty$$

for some orthonormal basis, $\{e_n\}_{n=1}^{\infty}$, for $L^2(\widehat{\mathbb{R}}^d)$, in which case the *Hilbert–Schmidt norm* of K is defined as

$$\|K\|_{HS} = \left(\sum_{n=1}^{\infty} \|K e_n\|_2^2 \right)^{1/2},$$

and $\|K\|_{HS}$ is independent of the choice of orthonormal basis. The first theorem about Hilbert-Schmidt operators is the following [71]:

Theorem 5.1 *If $K : L^2(\widehat{\mathbb{R}}^d) \to L^2(\widehat{\mathbb{R}}^d)$ is a bounded linear mapping and $(K \widehat{f})(\gamma) = \int m(\gamma, \lambda) \widehat{f}(\lambda)\, d\lambda$, for some measurable function m, then K is a Hilbert-Schmidt operator if and only if $m \in L^2(\widehat{\mathbb{R}}^{2d})$ and, in this case, $\|K\|_{HS} = \|m\|_{L^2(\widehat{\mathbb{R}}^{2d})}$.*

The following is our result about pseudo-differential operator frame inequalities.

Theorem 5.2 *Let $E = \{x_n\} \subseteq \mathbb{R}^d$ be a separated sequence, that is symmetric about $0 \in \mathbb{R}^d$; and let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be an S-set of strict multiplicity, that is compact, convex, and symmetric about $0 \in \widehat{\mathbb{R}}^d$. Assume balayage is possible for (E, Λ). Further, let K be a Hilbert-Schmidt operator on $L^2(\widehat{\mathbb{R}}^d)$ with pseudo-differential operator representation,*

$$(K \widehat{f})(\gamma) = (K_s \widehat{f})(\gamma) = \int s(y, \gamma) f(y) e^{-2\pi i y \cdot \gamma}\, dy,$$

where $s_\gamma(y) = s(y, \gamma) \in L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ is the Kohn-Nirenberg symbol and where we make the further assumption that

$$\forall \gamma \in \widehat{\mathbb{R}}^d, \quad s_\gamma \in C_b(\mathbb{R}^d) \quad \textit{and} \quad \operatorname{supp}\, (s_\gamma e_{-\gamma})\widehat{\ } \subseteq \Lambda. \tag{31}$$

Then,

$$\exists A,\ B > 0 \quad \textit{such that} \quad \forall f \in L^2(\mathbb{R}^d) \backslash \{0\},$$

$$A\frac{\|K_s\widehat{f}\|_2^4}{\|f\|_2^2} \leq \sum_{x\in E} |\langle (K_s\widehat{f})(\cdot), \overline{s(x,\cdot)}\, e_x(\cdot)\rangle|^2 \leq B\ \|s\|^2_{L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}\|K_s\widehat{f}\|_2^2. \qquad (32)$$

Proof a. In order to prove the assertion for the lower frame bound, we first combine the pseudo-differential operator K_s, with Kohn-Nirenberg symbol s, and balayage to compute

$$\begin{aligned}\int |(K_s\widehat{f})(\gamma)|^2\, d\gamma &= \int \overline{(K_s\widehat{f})(\gamma)}(K_s\widehat{f})(\gamma)\, d\gamma\\ &= \int \overline{(K_s\widehat{f})(\gamma)}\left(\int s(y,\gamma) f(y)e^{-2\pi i y\cdot\gamma}\, dy\right) d\gamma\\ &= \int \overline{(K_s\widehat{f})(\gamma)}\left(\int f(y)k(y,\gamma)\, dy\right) d\gamma\\ &= \int \overline{(K_s\widehat{f})(\gamma)}\left(\int f(y)\left(\sum_{x\in E} k(x,\gamma)a_x(y,\gamma)h(x-y)\right)dy\right)d\gamma, \qquad (33)\end{aligned}$$

where $k_\gamma(y) = k(y,\gamma) = s(y,\gamma)e^{-2\pi i y\cdot\gamma}$ on $\mathbb{R}^d$ and $k_\gamma \in \mathcal{C}(\Lambda)$ for each fixed $\gamma \in \widehat{\mathbb{R}}^d$, and where

$$\sup_{\gamma\in\widehat{\mathbb{R}}^d}\ \sup_{y\in\mathbb{R}^d} \sum_{x\in E} |a_x(y,\gamma)| \leq K(E,\Lambda_\epsilon) = C < \infty. \qquad (34)$$

Because of Theorems 2.18 and 2.19, we do not need to have the function h depend on $\gamma \in \widehat{\mathbb{R}}^d$. Further, because of (34) and estimates we shall make, we can write $a_x(y,\gamma) = a_x(y)$.

Thus, the right side of (33) is

$$\begin{aligned}&\int f(y)\left[\sum_{x\in E} a_x(y)h(x-y)\left(\int \overline{(K_s\widehat{f})(\gamma)}k(x,\gamma)\, d\gamma\right)\right]dy\\ &= \sum_{x\in E}\left(\int f(y)a_x(y)h(x-y)\, dy \int \overline{(K_s\widehat{f})(\gamma)}k(x,\gamma)\, d\gamma\right)\\ &\leq \left(\sum_{x\in E}\left|\int f(y)a_x(y)h(x-y)\, dy\right|^2\right)^{1/2}\left(\sum_{x\in E}\left|\overline{(K_s\widehat{f})(\gamma)}k(x,\gamma)\right|^2\right)^{1/2}. \qquad (35)\end{aligned}$$

Note that, by Hölder's inequality applied to the integral, we have

$$\begin{aligned}&\sum_{x\in E}\left|\int f(y)a_x(y)h(x-y)\, dy\right|^2\\ &\quad\leq \sum_{x\in E}\left|\left(\int |a_x(y)||h(x-y)|^2\, dy\right)^{1/2}\left(\int |f(y)|^2|a_x(y)|\, dy\right)^{1/2}\right|^2\end{aligned}$$

$$\leq \sum_{x\in E}\left(C\int |h(x-y)|^2\,dy\right)\left(\int |f(y)|^2|a_x(y)|\,dy\right)$$
$$\leq C\|h\|_2^2\int\left(\left(\sum_{x\in E}|a_x(y)|\right)|f(y)|^2\,dy\right)$$
$$\leq C^2\|h\|_2^2\|f\|_2^2. \tag{36}$$

Combining (33), (35), and (36), we obtain

$$\|K_s\widehat{f}\|_2^2 \leq C\|h\|_2\|f\|_2\left(\sum_{x\in E}\left|\int (K_s\widehat{f})(\gamma)k(x,\gamma)\,d\gamma\right|^2\right)^{1/2}.$$

Consequently, setting $A = 1/(C\|h\|_2)^2$, we have

$$\forall f\in L^2(\mathbb{R}^d)\backslash\{0\},\quad A\frac{\|K_s\widehat{f}\|_2^4}{\|f\|_2^2} \leq \sum_{x\in E}\left|\int (K_s\widehat{f})(\gamma)s(x,\gamma)e^{-2\pi i x\cdot\gamma}\,d\gamma\right|^2$$
$$= \sum_{x\in E}|\langle (K_s\widehat{f})(\cdot), \overline{s(x,\cdot)}e_x(\cdot)\rangle|^2$$

and the assertion for the lower frame bound is proved.

b.i. In order to prove the assertion for the upper frame bound, we begin by formally defining

$$\forall f\in L^2(\mathbb{R}^d),\quad (I_s\widehat{f})(x) = \int s(x,\gamma)(K_s\widehat{f})(\gamma)e^{-2\pi i x\cdot\gamma}\,d\gamma,$$

which is the inner product in (32).

Note that $I_s\widehat{f}\in L^2(\mathbb{R}^d)$. In fact, we know $K_s\widehat{f}\in L^2(\widehat{\mathbb{R}}^d)$ and $s\in L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)$ so that

$$|I_s\widehat{f}(x)|^2 \leq \int |s(x,\gamma)|^2\,d\gamma\int |K_s\widehat{f}(\gamma)|^2\,d\gamma$$

by Hölder's inequality, and, hence,

$$\|I_s\widehat{f}\|_2^2 \leq \|s\|^2_{L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}\|K_s\widehat{f}\|_2^2. \tag{37}$$

b.ii. We shall now show that $\mathrm{supp}((I_s\widehat{f})\widehat{\ }) \subseteq \Lambda$, and to this end we use (31). We begin by computing

$$\begin{aligned}(I_s\widehat{f})\widehat{\ }(\omega) &= \int \left(\int s(y,\gamma)(K_s\widehat{f})(\gamma)\, e^{-2\pi i y\cdot\gamma}\, d\gamma \right) e^{-2\pi i y\cdot\omega}\, dy \\ &= \int (K_s\widehat{f})(\gamma) \left(\int k_\gamma(y) e^{-2\pi i y\cdot\omega}\, dy \right) d\gamma \\ &= \int (K_s\widehat{f})(\gamma)(k_\gamma)\widehat{\ }(\omega)\, d\gamma,\end{aligned}$$

where

$$k_\gamma(y) = k(y,\gamma) = s(y,\gamma)e^{-2\pi i y\cdot\gamma} = (s_\gamma e_{-\gamma})(y),$$

as in part a. Also, $\text{supp}(k_\gamma)\widehat{\ } \subseteq \Lambda$ by our assumption, (31); that is, for each $\gamma \in \widehat{\mathbb{R}}^d$, $(k_\gamma)\widehat{\ } = 0$ a.e. on $\widehat{\mathbb{R}}^d \backslash \Lambda$.

Since $\text{supp}(I_s\widehat{f})\widehat{\ }$ is the smallest closed set outside of which $(I_s\widehat{f})\widehat{\ }$ is 0 a.e., we need only show that if $\text{supp}(L) \subseteq \widehat{\mathbb{R}}^d \backslash \Lambda$ then

$$\int L(\omega)(I_s\widehat{f})\widehat{\ }(\omega)\, d\omega = 0.$$

This follows because

$$\int L(\omega)(I_s\widehat{f})\widehat{\ }(\omega)\, d\omega = \int (K_s\widehat{f})(\gamma) \left(\int L(\omega)(k_\gamma)\widehat{\ }(\omega)\, d\omega \right) d\gamma$$

and $(k_\gamma)\widehat{\ } = 0$ on $\widehat{\mathbb{R}}^d \backslash \Lambda$.

b.iii. Because of parts *b.i* and *b.ii*, we can invoke the Pólya-Plancherel theorem to assert the existence of $B > 0$ such that

$$\forall f \in L^2(\mathbb{R}^d), \quad \sum_{x\in E} |(I_s\widehat{f})(x)| \leq B\|I_s\widehat{f}\|_2^2,$$

and the upper frame inequality of (32) follows from (37). □

Example 5.3 We shall define a Kohn–Nirenberg symbol class whose elements s satisfy the hypotheses of Theorem 5.2, cf. the discrete symbol calculus of Demanet and Ying [25].

Choose $\{\lambda_j\} \subseteq \text{int}(\Lambda)$, $a_j \in C_b(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, and $b_j \in C_b(\widehat{\mathbb{R}}^d) \cap L^2(\widehat{\mathbb{R}}^d)$ with the following properties:

i. $\sum_{j=1}^{\infty} |a_j(y)b_j(\gamma)|$ is uniformly bounded and converges uniformly on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$;

ii. $\sum_{j=1}^{\infty} \|a_j\|_2\|b_j\|_2 < \infty$;

iii. $\forall j = 1, \ldots, \exists \epsilon_j > 0$ such that $\overline{B(\lambda_j, \epsilon_j)} \subseteq \Lambda$ and $\text{supp}(\widehat{a}_j) \subseteq \overline{B(0, \epsilon_j)}$.

These conditions are satisfied for a large class of functions a_j and b_j.

The Kohn-Nirenberg symbol class consisting of functions, s, defined as

$$s(y, \gamma) = \sum_{j=1}^{\infty} a_j(y) b_j(\gamma) e^{-2\pi i y\cdot\lambda_j}$$

satisfy the hypotheses of Theorem 5.2. To see this, first note that condition i tells us that, if we set $s_\gamma(y) = s(y, \gamma)$, then

$$\forall \gamma \in \widehat{\mathbb{R}}^d, \quad s_\gamma \in C_b(\mathbb{R}^d).$$

Condition ii allows us to assert that $s \in L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ since we can use Minkowski's inequality to make the estimate,

$$\|s\|_{L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} \le \sum_{j=1}^{\infty} \left(\int\int \left| b_j(\gamma) a_j(y) e^{-2\pi i y\cdot(\lambda_j-\gamma)} \right|^2 dy\, d\gamma \right)^{1/2} = \sum_{j=1}^{\infty} \|a_j\|_2 \|b_j\|_2.$$

Finally, using condition iii, we obtain the support hypothesis, $\mathrm{supp}(s_\gamma e_{-\gamma})\widehat{\ } \subseteq \Lambda$, of Theorem 5.2 for each $\gamma \in \widehat{\mathbb{R}}^d$, because of the following calculations:

$$(s_\gamma e_{-\gamma})\widehat{\ }(\omega) = \sum_{j=1}^{\infty} b_j(\gamma)(\widehat{a}_j * \delta_{-\lambda_j})(\omega)$$

and, for each j,

$$\mathrm{supp}(\widehat{a}_j * \delta_{-\lambda_j}) \subseteq \overline{B(0, \epsilon_j)} + \{\lambda_j\} \subseteq \overline{B(\lambda_j, \epsilon_j)} \subseteq \Lambda.$$

6 The Beurling Covering Theorem

Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a convex, compact set which is symmetric about the origin and has non-empty interior. Then $\|\cdot\|_\Lambda$, defined by

$$\forall \gamma \in \widehat{\mathbb{R}}^d, \quad \|\gamma\|_\Lambda = \inf\{\rho > 0 : \gamma \in \rho\Lambda\},$$

is a norm on $\widehat{\mathbb{R}}^d$ equivalent to the Euclidean norm. The polar set $\Lambda^* \subseteq \mathbb{R}^d$ of Λ is defined as

$$\Lambda^* = \{x \in \mathbb{R}^d : x \cdot \gamma \le 1, \text{ for all } \gamma \in \Lambda\}.$$

It is elementary to check that Λ^* is a convex, compact set which is symmetric about the origin, and that it has non-empty interior.

Example 6.1 Let $\Lambda = [-1, 1] \times [-1, 1]$. Then, for $(\gamma_1, \gamma_2) \in \widehat{\mathbb{R}}^2$,

$$\|(\gamma_1, \gamma_2)\|_\Lambda = \inf\{\rho > 0 : |\gamma_1| \le \rho, |\gamma_2| \le \rho\} = \|(\gamma_1, \gamma_2)\|_\infty .$$

The polar set of Λ is

$$\Lambda^* = \{(x_1, x_2) : |x_1| + |x_2| \leq 1\} = \{(x_1, x_2) : \|(x_1, x_2)\|_1 \leq 1\}.$$

Theorem 6.2 (Beurling covering theorem) *Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a convex, compact set which is symmetric about the origin and has non-empty interior, and let $E \subseteq \mathbb{R}^d$ be a separated set satisfying the covering property,*

$$\bigcup_{y \in E} \tau_y \Lambda^* = \mathbb{R}^d.$$

If $\rho < 1/4$, then $\{(e_{-x}\mathbb{1}_\Lambda)^\vee : x \in E\}$ is a Fourier frame for $PW_{\rho\Lambda}$.

Theorem 6.2 [13,14] involves the Paley–Wiener theorem and properties of balayage, and it depends on the theory developed in [17], pp. 341–350, [15], and [56]. For a recent development, see [69].

7 Epilogue

This paper is rooted in Beurling's deep ideas and techniques dealing with balayage, that themselves have spawned wondrous results in a host of areas ranging from Kahane's creative formulation and theory exposited in [48] to the setting of various locally compact abelian groups with surprising twists and turns and many open problems, e.g., [75,76], to the new original chapter on quasi-crystals led by by Yves Meyer, e.g., [44,55,60–62,64,65] as well as the revisiting by Beurling [16].

Even with the focused theme of this paper, there is the important issue, as emphasized in the Abstract and Introduction, of completing our program of implementation and computation vis a vis balayage for explicit and genuine applications of non-uniform sampling.

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