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The Hardy Inequality for Hermite Expansions

Zhongkai Li · Yufeng Yu · Yehao Shi

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Abstract The purpose of the paper is to prove a sharp form of Hardy-type inequality, conjectured by Kanjin, for Hermite expansions of functions in the Hardy space $H^1(\mathbb{R})$, that is, $\sum_{n=1}^{\infty} n^{-\frac{3}{4}}|a_n(f)| \le A\|f\|_{H^1(\mathbb{R})}$ for all $f \in H^1(\mathbb{R})$, where A is a constant independent of f.

Keywords Hardy inequality · Hermite expansion · Hardy space

Mathematics Subject Classification 42B30 · 42C10 · 42B15

1 Introduction and Result

The Hermite polynomials $H_n(x)$ $(n \ge 0)$ are defined by the orthogonal relation (cf. [15,16])

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = \pi^{\frac{1}{2}} 2^n n! \delta_{nm},$$

and the Hermite functions are given by

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Z. Li · Y. Yu
Department of Mathematics, Capital Normal University, Beijing 100048, China
e-mail: lizk@mail.cnu.edu.cn

Y. Shi (✉)
Elementary Education College, Capital Normal University, Beijing 100048, China
e-mail: cnusyh@163.com

$$\mathcal{H}_n(x) = \left(\pi^{\frac{1}{2}} 2^n n!\right)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_n(x), \quad n = 0, 1, 2, \ldots,$$

which are orthonormal over $(-\infty, \infty)$ associated with the Lebesgue measure.

For $f \in L(\mathbb{R})$, its Hermite expansion is

$$f(x) \sim \sum_{n=0}^{\infty} a_n(f)\mathcal{H}_n(x), \quad a_n(f) = \int_{-\infty}^{\infty} f(x)\mathcal{H}_n(x)dx.$$

The Hardy type inequality for Hermite expansions of functions in the Hardy space $H^1(\mathbb{R})$ has been studied in several works. The first step was done by Kanjin [5] who proved the inequality

$$\sum_{n=1}^{\infty} n^{-\frac{29}{36}} |a_n(f)| \leq A\|f\|_{H^1(\mathbb{R})}, \quad f \in H^1(\mathbb{R}), \tag{1}$$

where $A > 0$ is a constant independent of f. Balasubramanian and Radha [2] extended Kanjin's result to $H^p(\mathbb{R})$, $0 < p \leq 1$. Radha and Thangavelu [10] (cf [17] also) obtained inequalities of Hardy type for d-dimensional Hermite and special Hermite expansions for $d \geq 2$, where the constant they determined, in place of $\frac{29}{36}$, is

$$\sigma = \left(\frac{d}{2} + 1\right)\left(\frac{2-p}{2}\right)$$

for $H^p(\mathbb{R}^d)$, $0 < p \leq 1$. In comparison with the case of d-dimension ($d \geq 2$), the Hardy inequality for one-dimensional Hermite expansions should be the one as (1) but with $\frac{3}{4}$ instead of $\frac{29}{36}$. However the method in [10] does not work for $d = 1$. An improved form of (1) with $\frac{3}{4} + \epsilon$ for $\epsilon > 0$ in place of $\frac{29}{36}$ was obtained by Kanjin [6]. Moreover Kanjin [6] proved his inequality for all $f \in L^1(\mathbb{R})$, that is,

$$\sum_{n=1}^{\infty} n^{-\frac{3}{4}-\epsilon} |a_n(f)| \leq A\|f\|_{L^1(\mathbb{R})}. \tag{2}$$

This again leads Kanjin to conjecture that the possible form of the Hardy inequality for Hermite expansions would be

$$\sum_{n=1}^{\infty} n^{-\frac{3}{4}} |a_n(f)| \leq A\|f\|_{H^1(\mathbb{R})}. \tag{3}$$

We shall give a positive answer to this conjecture in the present paper. Kanjin [6] also showed that there exists a function $f_0 \in L^1(\mathbb{R})$ such that

$$\sum_{n=1}^{\infty} n^{-\frac{3}{4}} |a_n(f_0)| = \infty,$$

so the Hardy space norm is required in (3).

The proofs of (1) and (2) in [5,6] were based on the pointwise estimate of the Hermite functions as follows: for given $\tau > 0$, there exist positive constants A, η and ξ such that

$$|\mathcal{H}_n(x)| \le A\left(|x|+\sqrt{n}\right)^{-\frac{1}{4}}\left(||x|-\sqrt{2n}|+n^{-\frac{1}{6}}\right)^{-\frac{1}{4}}\Psi_n(x) \tag{4}$$

holds for all $x \in \mathbb{R}$ and $n \ge 1$, where

$$\Psi_n(x) = \begin{cases} 1, & \text{for} \quad 0 \le |x| \le \sqrt{2n}; \\ \exp\left(-\eta n^{\frac{1}{4}}||x|-\sqrt{2n}|^{3/2}\right), & \text{for} \quad \sqrt{2n} \le |x| \le (1+\tau)\sqrt{2n}; \\ e^{-\xi x^2}, & \text{for} \quad (1+\tau)\sqrt{2n} \le |x|. \end{cases} \tag{5}$$

A separate description of (4) is given in [1] due to Skovgaard. The unified and simplified form as (4) is stated in [7], in virtue of the relation [16, (1.1.52), (1.1.53)] of Hermite polynomials and Laguerre polynomials and a unified description [9, (2.2)] of Laguerre polynomials based on [8] and the table in [1, p. 699].

A direct consequence of (4) and (5) is

$$|\mathcal{H}_n(x)| \le An^{-\frac{1}{12}},$$

and $\mathcal{H}_n(x)$ attains this bound near the point $x = \sqrt{2n}$. But for most x it has a much smaller bound as a multiple of $n^{-\frac{1}{4}}$. It is a very hard work to apply such a non-proportional property of the Hermite functions as Kanjin did in [5,6], and certainly, it is also difficult to achieve the best result for related problems. However, if for $d \ge 2$, we denote by Φ_α, $\alpha \in \mathbb{N}^d$, the d-dimensional Hermite functions, namely,

$$\Phi_\alpha(x_1, \ldots, x_d) = \mathcal{H}_{\alpha_1}(x_1)\cdots\mathcal{H}_{\alpha_d}(x_d), \quad \alpha = (\alpha_1, \ldots, \alpha_d),$$

then there exists a constant $A > 0$ independent of n and $(x_1, \ldots, x_d)$ such that (see [16, Lemma 3.2.2])

$$\sum_{|\alpha|=n} |\Phi_\alpha(x_1, \ldots, x_d)|^2 \le A(n+1)^{\frac{d}{2}-1}. \tag{6}$$

Obviously this is not true for $d = 1$. The bound in (6) has been used in research of various problems for $d \ge 2$, as in [16] for example; it is also the key in the proof of the inequalities of Hardy type in [10] for d-dimensional Hermite expansions for $d \ge 2$.

In order to prove the Hardy inequality (3), we shall follow a different approach, by evaluating the square integration of the Poisson integral associated to Hermite expansions of functions in $H^1(\mathbb{R})$.

Indeed, we shall work with the generalized Hermite expansions of functions in $H^1(\mathbb{R})$. If $\lambda > -1/2$, the generalized Hermite polynomials $H_n^{(\lambda)}(x)$ $(n \ge 0)$ are defined by (see [3])

$$H_{2k}^{(\lambda)}(x) = \left(\frac{k!}{\Gamma(k+\lambda+1/2)}\right)^{1/2} L_k^{(\lambda-1/2)}(x^2), \tag{7}$$

$$H_{2k-1}^{(\lambda)}(x) = \left(\frac{(k-1)!}{\Gamma(k+\lambda+1/2)}\right)^{1/2} x L_{k-1}^{(\lambda+1/2)}(x^2), \tag{8}$$

where $L_n^{(\alpha)}(x)$ ($\alpha > -1$, $n \geq 0$) are the Laguerre polynomials determined by the orthogonal relation (see [15,16])

$$\int_0^\infty e^{-x} x^\alpha L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) dx = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{mn}.$$

The system $\left\{H_n^{(\lambda)}(x)\right\}$ is orthonormal over $(-\infty, +\infty)$ with respect to the weight $|x|^{2\lambda} e^{-x^2}$, and $H_n(x) = H_n^{(0)}(x)$ ($n \geq 0$) are the usual Hermite polynomials (up to constants).

The generalized Hermite functions $\mathcal{H}_n^{(\lambda)}(x)$ ($n \geq 0$) are given by

$$\mathcal{H}_n^{(\lambda)}(x) = e^{-\frac{x^2}{2}} |x|^\lambda H_n^{(\lambda)}(x),$$

which are orthonormal over $(-\infty, \infty)$ associated with Lebesgue measure. For a function $f \in L(\mathbb{R})$, its generalized Hermite expansion is

$$f \sim \sum_{n=0}^\infty a_n^{(\lambda)}(f) \mathcal{H}_n^{(\lambda)}(x), \qquad a_n^{(\lambda)}(f) = \int_{-\infty}^\infty f(t) \mathcal{H}_n^{(\lambda)}(t) dt. \tag{9}$$

In what follows we assume that $\lambda \geq 0$. Our main result is stated as follows.

Theorem 1.1 *Let $\lambda \geq 0$. Then there exists a constant $A > 0$ such that for all $f \in H^1(\mathbb{R})$,*

$$\sum_{n=1}^\infty n^{-\frac{3}{4}} |a_n^{(\lambda)}(f)| \leq A \|f\|_{H^1(\mathbb{R})}. \tag{10}$$

The generalized Hermite polynomials $H_n^{(\lambda)}(x)$ ($n \geq 0$) were used in a Bose-like oscillator calculus in [11]; their further generalizations to the Dunkl setting in several variables can be found in [13] and those in the A_{d-1} case are eigenfunctions of the Hamiltonian of the linear quantum Calogero-Moser-Sutherland model (with some modification) in $\mathbb{R}^d$ (see [13] and references therein). It would be interesting to extend various results on the usual Hermite expansions such as in [10] and others to orthogonal expansions associated to these generalizations. We remark that in [4], the Hermite functions were used in a description of Feichtinger's space S_0.

Throughout the paper, $U \lesssim V$ means that $U \le cV$ for some positive constant c independent of variables, functions, n, k, etc., but possibly dependent of the parameter λ.

2 Some Facts on the Poisson kernel

The Poisson kernel of the generalized Hermite polynomials $H_n^{(\lambda)}(x)$ $(n \ge 0)$ is, for $0 \le r < 1$,

$$P^{(\lambda)}(r; x, y) = \sum_{n=0}^{\infty} r^n H_n^{(\lambda)}(x) H_n^{(\lambda)}(y).$$

If we write $P^{(\lambda)}(r; x, y)$ into two parts, one being the summation for even n and the other for odd n, then from (7), (8), and by [15] or [16, (1.1.47)], we have

$$P^{(\lambda)}(r; x, y) = \frac{(1-r^2)^{-\lambda-1/2}}{\Gamma(\lambda+1/2)} \exp\left(-\frac{r^2}{1-r^2}(x^2+y^2)\right) E_\lambda\left(\frac{2rxy}{1-r^2}\right),$$

where $E_\lambda(z)$ is the one-dimensional Dunkl kernel

$$E_\lambda(z) = j_{\lambda-1/2}(iz) + \frac{z}{2\lambda+1} j_{\lambda+1/2}(iz),$$

and $j_\alpha(z)$ is the normalized Bessel function

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n! \Gamma(n+\alpha+1)}.$$

If $\lambda = 0$, $E_0(z) = e^z$, and for $\lambda > 0$, a Laplace-type representation of $E_\lambda(z)$ is (cf. [12, Lemma 2.1])

$$E_\lambda(z) = c'_\lambda \int_{-1}^{1} e^{zt}(1+t)(1-t^2)^{\lambda-1} dt, \quad c'_\lambda = \frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda)\Gamma(1/2)}. \tag{11}$$

By (9), the Poisson kernel $\mathcal{P}^{(\lambda)}(r; x, y) = \sum_{n=0}^{\infty} r^n \mathcal{H}_n^{(\lambda)}(x) \mathcal{H}_n^{(\lambda)}(y)$ of the generalized Hermite functions $\mathcal{H}_n^{(\lambda)}$ $(n \ge 0)$ can be written as

$$\mathcal{P}^{(\lambda)}(r; x, y) = |xy|^\lambda \frac{(1-r^2)^{-\lambda-1/2}}{\Gamma(\lambda+1/2)} \exp\left(-\frac{1+r^2}{1-r^2} \cdot \frac{x^2+y^2}{2}\right) E_\lambda\left(\frac{2rxy}{1-r^2}\right).$$

The Poisson integral of the generalized Hermite expansion of a function $f \in L^1(\mathbb{R})$ is defined by

$$f_\lambda(r; x) = \int_{-\infty}^{\infty} \mathcal{P}^{(\lambda)}(r; x, y) f(y) dy. \tag{12}$$

For $\lambda = 0$, $\mathcal{P}^{(0)}(r; x, y)$ is consistent with the Poisson kernel of the usual Hermite functions $\mathcal{H}_n$ $(n \geq 0)$ (see [16])

$$\mathcal{P}(r; x, y) = \frac{(1-r^2)^{-\frac{1}{2}}}{\sqrt{\pi}} \exp\left(-\frac{2r(x-y)^2 + (1-r)^2(x^2+y^2)}{2(1-r^2)}\right).$$

In order to give some necessary estimates of the Poisson kernel $\mathcal{P}^{(\lambda)}(r; x, y)$ we rewrite it into

$$\mathcal{P}^{(\lambda)}(r; x, y) = \frac{(1-r^2)^{-\lambda-1/2}}{\Gamma(\lambda+1/2)} |xy|^{\lambda} \psi_{r,x}(y) e^{-|\xi y|} E_\lambda(\xi y), \quad \xi = \frac{2rx}{1-r^2}, \quad (13)$$

where

$$\psi_{r,x}(y) = \exp\left(-\frac{1+r^2}{1-r^2}\frac{x^2+y^2}{2} + \frac{2r|xy|}{1-r^2}\right).$$

We shall need several preliminary lemmas.

Lemma 2.1 *For $y \neq 0$,*

(i) $\left|e^{-|\xi y|} E_\lambda(\xi y)\right| \lesssim (1 + |\xi y|)^{-\lambda}$;
(ii) $\left|\frac{\partial}{\partial y}\left(e^{-|\xi y|} E_\lambda(\xi y)\right)\right| \lesssim |\xi|(1 + |\xi y|)^{-\lambda-1}$.

Proof If $\lambda = 0$, both (i) and (ii) are trivial. In what follows we assume that $\lambda > 0$. From (11) we have

$$\left|e^{-|\xi y|} E_\lambda(\xi y)\right| \leq 2c'_\lambda \int_{-1}^{1} e^{-|\xi y|(1-|t|)}(1-t^2)^{\lambda-1} dt \lesssim \int_0^1 e^{-|\xi y|(1-t)}(1-t)^{\lambda-1} dt$$
$$= |\xi y|^{-\lambda} \int_0^{|\xi y|} e^{-s} s^{\lambda-1} ds,$$

which is bounded by a multiple of $(1+|\xi y|)^{-\lambda}$, since $\int_0^A e^{-s}s^{\lambda-1}ds \asymp (A/(A+1))^\lambda$.

Furthermore, from (11) we have

$$\frac{\partial}{\partial y}\left(e^{-|\xi y|} E_\lambda(\xi y)\right) = -c'_\lambda \xi \int_{-1}^{1} e^{-\xi y(1-t)}(1+t)^{\lambda}(1-t)^{\lambda} dt \quad \text{for} \quad \xi y > 0;$$
$$= c'_\lambda \xi \int_{-1}^{1} e^{\xi y(1-t)}(1+t)^{\lambda-1}(1-t)^{\lambda+1} dt \quad \text{for} \quad \xi y < 0.$$

In the both cases we split the integral into two parts as $\int_{-1}^1 = \int_{-1}^0 + \int_0^1$, to get

$$\left|\frac{\partial}{\partial y}\left(e^{-|\xi y|} E_\lambda(\xi y)\right)\right| \lesssim |\xi| \left[e^{-|\xi y|} + \int_0^1 e^{-|\xi y|(1-t)}(1-t)^{\lambda} dt\right]$$
$$= |\xi| \left[e^{-|\xi y|} + |\xi y|^{-\lambda-1} \int_0^{|\xi y|} e^{-s} s^{\lambda} ds\right]$$

which implies the desired estimate in part (ii), since $\int_0^A e^{-s}s^\lambda ds \asymp (A/(A+1))^{\lambda+1}$.

□

Lemma 2.2 (i) *For* $0 \le r < 1$,

$$\psi_{r,x}(y) = \exp\left(-\frac{r(|x|-|y|)^2}{1-r^2}\right)\exp\left(-\frac{1-r}{1+r}\frac{x^2+y^2}{2}\right); \tag{14}$$

(ii) *for* $0 \le r \le \frac{1}{2}$, $\psi_{r,x}(y) \lesssim \exp\left(-c_1(x^2+y^2)\right)$, *and for* $\frac{1}{2} \le r < 1$, $\psi_{r,x}(y) \lesssim \exp\left(-c_2\frac{(|x|-|y|)^2}{1-r}\right)$, *where* $c_1, c_2 > 0$ *are independent of* x, y *and* r.

Part (i) is obvious and implies part (ii) immediately.

Lemma 2.3 *For* $x, y \in (-\infty, \infty)$ *and* $r \in [0, 1)$,

$$\frac{(1-r)^{1/2}|x|}{1-r+|xy|} \lesssim 1 + \frac{||x|-|y||}{(1-r)^{1/2}}.$$

Indeed, for $|y| \le |x|/2$ the left hand side is bounded by $|x|/(1-r)^{1/2} \le 2\,||x|-|y||\,/(1-r)^{1/2}$, and for $|y| \ge |x|/2$, by $|x|/\left(2|xy|^{1/2}\right) \le 1$.

Proposition 2.4 (i) *If* $0 \le r \le \frac{1}{2}$, *then for* $y \ne 0$,

$$\left|\frac{\partial}{\partial y}\mathcal{P}^{(\lambda)}(r; x, y)\right| \lesssim |x|^\lambda |y|^{\lambda-1}\left(\lambda + |xy| + |y|^2\right)\exp\left(-c_1(x^2+y^2)\right); \tag{15}$$

(ii) *if* $\frac{1}{2} \le r < 1$, *then for* $y \ne 0$,

$$\left|\frac{\partial}{\partial y}\mathcal{P}^{(\lambda)}(r; x, y)\right| \lesssim \lambda\frac{(1-r)^{-1/2}|x|^\lambda|y|^{\lambda-1}}{(1-r+|xy|)^\lambda}\psi_{r,x}(y) + (1-r)^{-1}\psi_{r,x}(y)^{\frac{1}{2}}. \tag{16}$$

Proof For $y \ne 0$, from (13) we have

$$\frac{\partial}{\partial y}\mathcal{P}^{(\lambda)}(r; x, y) = \frac{(1-r^2)^{-\lambda-1/2}}{\Gamma(\lambda+1/2)}\left(U_1(x,y) + U_2(x,y)\right),$$

where

$$U_1(x,y) = \lambda|x|^\lambda|y|^{\lambda-2}y\left(\psi_{r,x}(y)e^{-|\xi y|}E_\lambda(\xi y)\right),$$
$$U_2(x,y) = |xy|^\lambda\left(\psi_{r,x}(y)e^{-|\xi y|}E_\lambda(\xi y)\right)'_y.$$

For $0 \le r \le \frac{1}{2}$, it follows from Lemmas 2.1(i) and 2.2(ii) that $(1-r^2)^{-\lambda-1/2}U_1(x,y)$ is bounded by a multiple of $\lambda|x|^\lambda|y|^{\lambda-1}\exp\left(-c_1(x^2+y^2)\right)$, and for $\frac{1}{2} \le r < 1$, by a multiple of

$$\lambda\frac{(1-r)^{-\lambda-1/2}|x|^\lambda|y|^{\lambda-1}}{(1+|\xi y|)^\lambda}\psi_{r,x}(y) \lesssim \lambda\frac{(1-r)^{-1/2}|x|^\lambda|y|^{\lambda-1}}{(1-r+|xy|)^\lambda}\psi_{r,x}(y).$$

In order to verify that $(1-r^2)^{-\lambda-1/2}U_2(x,y)$ has the desired estimates as in (15) and (16), we shall show that for $0 \le r \le \frac{1}{2}$,

$$\left|\left(\psi_{r,x}(y)e^{-|\xi y|}E_\lambda(\xi y)\right)'_y\right| \lesssim (|x|+|y|)\exp\left(-c_1(x^2+y^2)\right); \tag{17}$$

and for $\frac{1}{2} \le r < 1$,

$$\left|\left(\psi_{r,x}(y)e^{-|\xi y|}E_\lambda(\xi y)\right)'_y\right| \lesssim \frac{\psi_{r,x}(y)}{(1-r)^{1-\lambda}}\left[\frac{||x|-|y||+(1-r)^2|y|}{(1-r+|xy|)^\lambda}+\frac{(1-r)|x|}{(1-r+|xy|)^{\lambda+1}}\right]. \tag{18}$$

Indeed, since from (14),

$$\left(\psi_{r,x}(y)e^{-|\xi y|}E_\lambda(\xi y)\right)'_y = \left[\frac{2r(|x|-|y|)(\operatorname{sgn} y)-(1-r)^2y}{1-r^2}\left(e^{-|\xi y|}E_\lambda(\xi y)\right)+\left(e^{-|\xi y|}E_\lambda(\xi y)\right)'_y\right]\psi_{r,x}(y),$$

by Lemma 2.1 we have

$$\left|\left(\psi_{r,x}(y)e^{-|\xi y|}E_\lambda(\xi y)\right)'_y\right| \lesssim \left[\frac{||x|-|y||+(1-r)^2|y|}{(1-r)(1+|\xi y|)^\lambda}+\frac{|\xi|}{(1+|\xi y|)^{\lambda+1}}\right]\psi_{r,x}(y).$$

Thus (17) and (18) follow immediately.

For $0 \le r \le \frac{1}{2}$, from (17) it is obvious that

$$\left|(1-r^2)^{-\lambda-1/2}U_2(x,y)\right| \lesssim |xy|^\lambda(|x|+|y|)\exp\left(-c_1(x^2+y^2)\right);$$

and for $\frac{1}{2} \le r < 1$, from (18) we have

$$\left|(1-r^2)^{-\lambda-1/2}U_2(x,y)\right| \lesssim \frac{\psi_{r,x}(y)}{1-r}\left[\frac{||x|-|y||}{(1-r)^{1/2}}+(1-r)^{3/2}|y|+\frac{(1-r)^{1/2}|x|}{1-r+|xy|}\right].$$

By Lemma 2.3, it is easy to see that the expression in the brackets above is bounded by a multiple of $\psi_{r,x}(y)^{-1/2}$, and so $\left|(1-r^2)^{-\lambda-1/2}U_2(x,y)\right| \lesssim \psi_{r,x}(y)^{1/2}/(1-r)$. The proof of the proposition is finished. □

3 Proof of the Main Result

The following theorem is crucial in the proof of Theorem 1.1.

Theorem 3.1 *Assume that* $\lambda \geq 0$*. There is a positive constant A such that for all* $f \in H^1(\mathbb{R})$,

$$\int_0^1 (1-r)^{-\frac{3}{4}} \left(\int_{-\infty}^{\infty} |f_\lambda(r;y)|^2 dy \right)^{\frac{1}{2}} dr \leq A \|f\|_{H^1(\mathbb{R})}, \tag{19}$$

where $f_\lambda(r;y)$ *is the Poisson integral (12) associated to the generalized Hermite expansion of* f.

The proof of Theorem 3.1 is based upon some norm estimates stated below.

Lemma 3.2 (i) *If* $\lambda = 0$ *or* $\lambda > 1$*, then for* $y, \bar{y} \in (-\infty, \infty)$,

$$\left(\int_{-\infty}^{\infty} \left| \mathcal{P}^{(\lambda)}(r;x,y) - \mathcal{P}^{(\lambda)}(r;x,\bar{y}) \right|^2 dx \right)^{\frac{1}{2}} \lesssim \frac{|y-\bar{y}|}{(1-r)^{3/4}}; \tag{20}$$

(ii) *If* $0 < \lambda \leq 1$*, then for* $y, \bar{y} \in (-\infty, \infty)$,

$$\left(\int_{-\infty}^{\infty} \left| \mathcal{P}^{(\lambda)}(r;x,y) - \mathcal{P}^{(\lambda)}(r;x,\bar{y}) \right|^2 dx \right)^{\frac{1}{2}} \lesssim \frac{|y-\bar{y}|^\lambda}{(1-r)^{(2\lambda+1)/4}} \left(1 + \frac{|y-\bar{y}|}{(1-r)^{1/2}} \right).$$

Proof For $\lambda > 1$ and $y > \bar{y}$ we have

$$\begin{aligned} \left\| \mathcal{P}^{(\lambda)}(r;\cdot,y) - \mathcal{P}^{(\lambda)}(r;\cdot,\bar{y}) \right\|_{L^2(\mathbb{R})} &= \left\| \int_{\bar{y}}^{y} \frac{\partial}{\partial z} \mathcal{P}^{(\lambda)}(r;\cdot,z) dz \right\|_{L^2(\mathbb{R})} \\ &\leq \int_{\bar{y}}^{y} \left\| \frac{\partial}{\partial z} \mathcal{P}^{(\lambda)}(r;\cdot,z) \right\|_{L^2(\mathbb{R})} dz. \end{aligned}$$

If $0 \leq r \leq \frac{1}{2}$, then by Proposition 2.4(i), $\left\| \frac{\partial}{\partial z} \mathcal{P}^{(\lambda)}(r;\cdot,z) \right\|_{L^2(\mathbb{R})} \lesssim 1$, so that

$$\left\| \mathcal{P}^{(\lambda)}(r;\cdot,y) - \mathcal{P}^{(\lambda)}(r;\cdot,\bar{y}) \right\|_{L^2(\mathbb{R})} \lesssim |y-\bar{y}| \lesssim \frac{|y-\bar{y}|}{(1-r)^{3/4}}.$$

Since for $\lambda > 1$ and $\frac{1}{2} \leq r < 1$, by Lemma 2.3 we have

$$\frac{(1-r)^{-1/2}|x|^\lambda |y|^{\lambda-1}}{(1-r+|xy|)^\lambda} \leq \frac{(1-r)^{-1/2}|x|}{1-r+|xy|} \lesssim \frac{1}{1-r} \left(1 + \frac{||x|-|y||}{(1-r)^{1/2}} \right) \lesssim \frac{\psi_{r,x}(y)^{-\frac{1}{2}}}{1-r},$$

inserting this into (16) yields that $\left|\frac{\partial}{\partial y}\mathcal{P}^{(\lambda)}(r;x,y)\right| \lesssim (1-r)^{-1}\psi_{r,x}(y)^{\frac{1}{2}}$ and hence

$$\left\|\mathcal{P}^{(\lambda)}(r;\cdot,y)-\mathcal{P}^{(\lambda)}(r;\cdot,\bar{y})\right\|_{L^2(\mathbb{R})} \le \frac{1}{1-r}\int_{\bar{y}}^{y}\left\|\psi_{r,x}(z)^{\frac{1}{2}}\right\|_{L^2(\mathbb{R})}dz \lesssim \frac{|y-\bar{y}|}{(1-r)^{3/4}}.$$

This proves part (i) for $\lambda > 1$.

If $\lambda = 0$, by Proposition 2.4 $\left|\frac{\partial}{\partial y}\mathcal{P}^{(0)}(r;x,y)\right|$ is bounded by a multiple of $(|x| + |y|)e^{-c_1(x^2+y^2)}$ for $0 \le r \le \frac{1}{2}$, and $(1-r)^{-1}\psi_{r,x}(y)^{\frac{1}{2}}$ for $\frac{1}{2} \le r < 1$. Hence the above process still works for $\lambda = 0$. This completes the proof of part (i).

Now we turn to the proof of part (ii). For $0 < \lambda \le 1$ and $y > \bar{y}$, we write

$$\begin{aligned}
&\mathcal{P}^{(\lambda)}(r;x,y)-\mathcal{P}^{(\lambda)}(r;x,\bar{y})\\
&\quad=\frac{(1-r^2)^{-\lambda-1/2}}{\Gamma(\lambda+1/2)}\left[\int_{\bar{y}}^{y}U_2(x,z)dz+|x|^{\lambda}(|y|^{\lambda}-|\bar{y}|^{\lambda})\psi_{r,x}(y)e^{-|\xi y|}E_{\lambda}(\xi y)\right.\\
&\qquad\left.+\int_{\bar{y}}^{y}|x|^{\lambda}(|\bar{y}|^{\lambda}-|z|^{\lambda})\left(\psi_{r,x}(z)e^{-|\xi z|}E_{\lambda}(\xi z)\right)'_z dz\right],
\end{aligned}$$

where $U_2(x,z)$ is defined as the same as in the proof of Proposition 2.4. There we have shown that (for all $\lambda \ge 0$) $(1-r^2)^{-\lambda-1/2}U_2(x,z)$ is bounded by a multiple of $|xz|^{\lambda}(|x|+|z|)\exp\left(-c_1(x^2+z^2)\right)$ for $0 \le r \le 1/2$, and of $\psi_{r,x}(z)^{1/2}/(1-r)$ for $1/2 \le r < 1$. From the proof of part (i) above it follows that

$$\left\|\mathcal{P}^{(\lambda)}(r;\cdot,y)-\mathcal{P}^{(\lambda)}(r;\cdot,\bar{y})\right\|_{L^2(\mathbb{R})} \lesssim \frac{|y-\bar{y}|}{(1-r)^{3/4}} + V_1 + V_2, \tag{21}$$

where

$$\begin{aligned}
V_1 &= \frac{|y-\bar{y}|^{\lambda}}{(1-r)^{\lambda+1/2}}\left(\int_{-\infty}^{\infty}\left||x|^{\lambda}\psi_{r,x}(y)e^{-|\xi y|}E_{\lambda}(\xi y)\right|^2 dx\right)^{1/2},\\
V_2 &= \frac{|y-\bar{y}|^{\lambda}}{(1-r)^{\lambda+1/2}}\int_{\bar{y}}^{y}\left(\int_{-\infty}^{\infty}\left||x|^{\lambda}\left(\psi_{r,x}(z)e^{-|\xi z|}E_{\lambda}(\xi z)\right)'_z\right|^2 dx\right)^{1/2}dz.
\end{aligned}$$

If $0 \le r \le \frac{1}{2}$, then by Lemmas 2.1(i) and 2.2(ii),

$$\begin{aligned}
V_1 &\lesssim |y-\bar{y}|^{\lambda}\left(\int_{-\infty}^{\infty}|x|^{2\lambda}\exp\left(-2c_1(x^2+y^2)\right)dx\right)^{1/2}\\
&\lesssim |y-\bar{y}|^{\lambda} \lesssim \frac{|y-\bar{y}|^{\lambda}}{(1-r)^{(2\lambda+1)/4}};
\end{aligned}\tag{22}$$

and from (17),

$$V_2 \lesssim |y-\bar{y}|^{\lambda} \int_{\bar{y}}^{y} \left(\int_{-\infty}^{\infty} |x|^{2\lambda} (|x|+|z|)^2 \exp\left(-2c_1(x^2+z^2)\right) dx \right)^{1/2} dz$$
$$\lesssim |y-\bar{y}|^{\lambda+1} \lesssim \frac{|y-\bar{y}|^{\lambda+1}}{(1-r)^{(2\lambda+3)/4}}. \tag{23}$$

If $\frac{1}{2} \le r < 1$, then by Lemmas 2.1(i), 2.2(ii) and 2.3,

$$V_1 \lesssim \frac{|y-\bar{y}|^{\lambda}}{(1-r)^{1/2}} \left(\int_{-\infty}^{\infty} \left| \psi_{r,x}(y) \frac{|x|^{\lambda}}{(1-r+|xy|)^{\lambda}} \right|^2 dx \right)^{1/2}$$
$$\lesssim \frac{|y-\bar{y}|^{\lambda}}{(1-r)^{(\lambda+1)/2}} \left(\int_{-\infty}^{\infty} \left(1+\frac{||x|-|y||}{(1-r)^{1/2}}\right)^{2\lambda} \exp\left(-2c_2 \frac{(|x|-|y|)^2}{1-r}\right) dx \right)^{1/2}$$
$$\lesssim \frac{|y-\bar{y}|^{\lambda}}{(1-r)^{(2\lambda+1)/4}}; \tag{24}$$

and by (18), Lemmas 2.2(ii) and 2.3,

$$V_2 \lesssim \frac{|y-\bar{y}|^{\lambda}}{(1-r)^{(\lambda+2)/2}} \int_{\bar{y}}^{y} \left(\int_{-\infty}^{\infty} \left[\left(\frac{(1-r)^{1/2}|x|}{1-r+|xz|} \right)^{\lambda} \left(\frac{||x|-|z||}{(1-r)^{1/2}} + (1-r)^{\frac{3}{2}}|z| \right) \right.\right.$$
$$\left.\left. + \left(\frac{(1-r)^{1/2}|x|}{1-r+|xz|} \right)^{\lambda+1} \right]^2 \psi_{r,x}(z)^2 dx \right)^{1/2} dz$$
$$\lesssim \frac{|y-\bar{y}|^{\lambda}}{(1-r)^{(\lambda+2)/2}} \int_{\bar{y}}^{y} \left(\int_{-\infty}^{\infty} \left[\left(1+\frac{||x|-|z||}{(1-r)^{1/2}} \right)^{\lambda} \left(\frac{||x|-|z||}{(1-r)^{1/2}} + (1-r)^{\frac{3}{2}}|z| \right) \right.\right.$$
$$\left.\left. + \left(1+\frac{||x|-|z||}{(1-r)^{1/2}} \right)^{\lambda+1} \right]^2 \psi_{r,x}(z)^2 dx \right)^{1/2} dz$$
$$\lesssim \frac{|y-\bar{y}|^{\lambda}}{(1-r)^{(\lambda+2)/2}} \int_{\bar{y}}^{y} \left(\int_{-\infty}^{\infty} \psi_{r,x}(z) dx \right)^{1/2} dz$$
$$\lesssim \frac{|y-\bar{y}|^{\lambda+1}}{(1-r)^{(2\lambda+3)/4}}. \tag{25}$$

Collecting the estimates of (22), (23), (24) and (25) into (21), we finish the proof of part (ii) of the lemma. □

Now we come to the proof of Theorem 3.1.

Let us first recall the atom characterization of the Hardy space $H^1(\mathbb{R})$ (cf. [14,18]). An H^1 atom is a measurable function a on $\mathbb{R}$ satisfying

(i) $\operatorname{supp} a \subseteq I$ for some interval $I \subset \mathbb{R}$;

(ii) $\|a\|_{L^\infty} \le |I|^{-1}$, $|I|$ denoting the length of I;
(iii) $\int_{-\infty}^{\infty} a(x)dx = 0$.

For a function $f \in H^1(\mathbb{R})$, there are a sequence $\{a_k\}$ of H^1 atoms and a sequence $\{\lambda_k\}$ of complex numbers satisfying $\sum |\lambda_k| < \infty$, such that $f = \sum_{k=1}^{\infty} \lambda_k a_k$ and

$$A_1\|f\|_{H^1(\mathbb{R})} \le \sum |\lambda_k| \le A_2\|f\|_{H^1(\mathbb{R})},$$

where A_1, A_2 are positive constants independent of f.

In order to prove Theorem 3.1, it suffices to verify the inequality (19) for each H^1 atom $f = a$, namely, there exists a absolute constant $A > 0$, such that

$$\int_0^1 (1-r)^{-\frac{3}{4}} \|a_\lambda(r;\cdot)\|_{L^2(\mathbb{R})} dr \le A \tag{26}$$

holds for all H^1 atoms a satisfying (i), (ii) and (iii) above, where $a_\lambda(r; y)$ is the Poisson integral (12) of a associated to its generalized Hermite expansion.

By Parseval's inequality, from (i) and (ii) we have

$$\|a_\lambda(r;\cdot)\|_{L^2(\mathbb{R})} \le \|a\|_{L^2(\mathbb{R})} \le |I|^{-\frac{1}{2}}. \tag{27}$$

(27) is useful for atoms with larger supports, but we also need a more accurate estimate for atoms with smaller supports.

For an atom a satisfying (i), (ii) and (iii), let $\bar{y}$ be the left endpoint of the interval I. Applying the cancelation property (iii) and Minkowski's inequality, we have

$$\begin{aligned}\|a_\lambda(r;\cdot)\|_{L^2(\mathbb{R})} &= \left(\int_{-\infty}^{\infty}\left[\int_I \left(\mathcal{P}^{(\lambda)}(r;x,y) - \mathcal{P}^{(\lambda)}(r;x,\bar{y})\right) a(y)dy\right]^2 dx\right)^{\frac{1}{2}} \\ &\le \int_I |a(y)| \left(\int_{-\infty}^{\infty} \left|\mathcal{P}^{(\lambda)}(r;x,y) - \mathcal{P}^{(\lambda)}(r;x,\bar{y})\right|^2 dx\right)^{1/2} dy.\end{aligned}$$

By means of Lemma 3.2 we obtain

$$\|a_\lambda(r;\cdot)\|_{L^2(\mathbb{R})} \lesssim \frac{|I|}{(1-r)^{3/4}} + \frac{|I|^\lambda}{(1-r)^{(2\lambda+1)/4}} + \frac{|I|^{\lambda+1}}{(1-r)^{(2\lambda+3)/4}}. \tag{28}$$

If $\lambda = 0$ or $\lambda > 1$, the first term on the right hand side appears only.

If $|I| \ge 1$, from (27) it is trivial that

$$\int_0^1 (1-r)^{-\frac{3}{4}} \|a_\lambda(r;\cdot)\|_{L^2(\mathbb{R})} dr \le 4|I|^{-\frac{1}{2}} \le 4;$$

and if $|I| < 1$, we split the integral into two parts, over $[1-|I|^2, 1)$ and $[0, 1-|I|^2]$ respectively, and apply (27) and (28) to obtain

$$\int_0^1 (1-r)^{-\frac{3}{4}} \|a_\lambda(r;\cdot)\|_{L^2(\mathbb{R})} dr \lesssim \int_{1-|I|^2}^1 |I|^{-1/2}(1-r)^{-3/4} dr$$
$$+ \int_0^{1-|I|^2} \left(\frac{|I|}{(1-r)^{3/2}} + \frac{|I|^\lambda}{(1-r)^{(\lambda+2)/2}} + \frac{|I|^{\lambda+1}}{(1-r)^{(\lambda+3)/2}} \right) dr.$$

It is easy to see that the values of the two integrals above are independent of $|I| < 1$. This proves (26), and hence Theorem 3.1.

Proof of Theorem 1.1 By Hölder's inequality one has

$$\sum_{n=0}^\infty r^{2n} |a_n^{(\lambda)}(f)| \le \left(\sum_{n=0}^\infty r^{2n} \right)^{1/2} \left(\sum_{n=0}^\infty r^{2n} |a_n^{(\lambda)}(f)|^2 \right)^{1/2}$$
$$\le (1-r)^{-1/2} \|f_\lambda(r;\cdot)\|_{L^2}.$$

Multiplying $(1-r)^{-1/4}$ on both sides and taking integration over $[0, 1)$ respectively, then by Theorem 3.1 we get

$$\sum_{n=0}^\infty B(2n+1, 3/4) |a_n^{(\lambda)}(f)| \le A \|f\|_{H^1},$$

where $B(\alpha, \beta)$ denotes the beta function. Finally, the Hardy inequality (10) follows from the fact that $B(2n+1, 3/4) \sim (n+1)^{-3/4}$. The proof of Theorem 1.1 is completed. □

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