J Fourier Anal Appl (2015) 21:137–160
DOI 10.1007/s00041-014-9363-0

A Family of Two Dimensionally Determined Ergodic Processes

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Received: 20 January 2014 / Revised: 26 August 2014 / Published online: 15 October 2014

Abstract We prove that for an ergodic rotation R and square integrable function f on a compact abelian group, the ergodic process $X = (f \circ R^n)_{n\in\mathbb{Z}}$ is uniquely determined by its two-dimensional laws if the same holds for the process $Y = (h \circ f \circ R^n)_{n\in\mathbb{Z}}$, for some real bounded function h, such that all Fourier-Stieltjes coefficients of $h \circ f$ are non null. Applied to the one or two dimensional torus, this result gives a large class of such processes, for instance any process given by non constant monotone continuous function, or having a discontinuity at an irrational point, on the unit interval, is in the corresponding class. We also prove that all Fourier coefficients of such a monotone function are non null.

Keywords Ergodic processes · Group rotation · Finite dimensional laws · Fourier-Stieltjes coefficients

Mathematics Subject Classification 42A16 · 37A05 · 37A50 · 60G10

1 Introduction

According to Kolmogorov theorem, the law of a stochastic process is determined by its family of all compatible finite dimensional distributions. It is natural to ask about the role of some fixed family of finite dimensional distributions in the complete determination of the law of that process. Here, we consider only discrete time stationary

Communicated by Hans G. Feichtinger.

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processes which are ergodic. The law μ of such a process $X = (X_n)_{n\in\mathbb{Z}}$, is, in general, completely determined by the family of the laws of all vectors $(X_0, X_1, \ldots, X_n)$, for all $n \geq 1$. For a given natural number $p \geq 1$, the family of p-dimensional distributions (or laws) of the process $X = (X_n)_{n\in\mathbb{Z}}$, is the family of distributions of all vectors $(X_{n_1}, \ldots, X_{n_p})$ for all integers $n_1 < n_2 < \ldots < n_p$. Let $Y = (Y_n)_{n\in\mathbb{Z}}$ be a stationary process. We say that the two processes X and Y have the same p-dimensional laws (or p-dimensional marginals) if, for all $n_1 < n_2 < \ldots < n_p$, the two vectors $(X_{n_1}, \ldots, X_{n_p})$ and $(Y_{n_1}, \ldots, Y_{n_p})$ have the same law.

We are concerned with the family of p-dimensional laws, so that the question above can be formulated as:

When is it true that the law μ of an ergodic process $X = (X_n)_{n\in\mathbb{Z}}$, is determined by its family of p-dimensional laws in the sense that if $Y = (Y_n)_{n\in\mathbb{Z}}$ is an ergodic process, with law ν, and if X and Y have the same p-dimensional laws, then we must have $\nu = \mu$? (In this case we will say then that X is p-dimensionally determined).

A more general question is: under what conditions the equalities $\int Y_{n_1} \ldots Y_{n_p} d\nu = \int X_{n_1} \ldots X_{n_p} d\mu$, for all $n_1 < \ldots < n_p$, will imply $\mu = \nu$? (In this case we will say then that X is p-spectrally determined).

A p-spectrally determined process is p-dimensionally determined.

Examples of pairwise independent non independent processes, hence not 2-dimensionally determined, are given in [7,13]. Ergodic examples are in [1,3–5,8,16].

More general examples of non 2-dimensionally determined ergodic processes are in [3,4].

Examples of ergodic processes which are not p-dimensionally determined for any p, are in [8,11,12].

In the continuous time case, examples of processes which are not p-dimensionally determined, for any p are in [10].

In [9] it is shown that if $(\Omega, \mathcal{A}, m, T)$ is an ergodic dynamical system and $f \in L^2(m)$ is such that its spectral measure is continuous and concentrated on a Kronecker set of the circle, then the process $(f \circ T^n)_{n\in\mathbb{Z}}$ is 2-spectrally determined. This result is generalized in [15] to the case of independent Helson set in place of Kronecker set, where also more general group actions are considered.

In [11] it is proved that if R_a is an ergodic rotation on a compact abelian group and $f \in L^\infty(m)$ has non null Fourier-Stieltjes coefficients, m being the Haar measure, then the process $(f \circ R_a^n)$ is 3-spectrally determined and not necessarily 2-spectrally determined.

In [6] two classes of two-valued 2-dimensionally determined processes are given.

In the present paper we prove, (Theorem 1, Sect. 2), that in the group rotation case, the process $(f \circ R_a^n)$ is 2-dimensionally determined if, for some real bounded function h, such that $h \circ f \in L^2(m)$ and has non null Fourier-Stieltjes coefficients, the process $(h \circ f \circ R_a^n)$ is 2-dimensionally determined.

This result enables us to prove (Proposition 1, Sect. 2), in the circle case, that for a large class $\mathcal{C}_1$ of functions f, the process $(f \circ R_a^n)$ is 2-dimensionally determined. This class $\mathcal{C}_1$ contains, for example the set $\mathcal{M}$ of all non constant monotone continuous functions on the unit interval, or having a discontinuity at an irrational point.

In the two dimensional torus, we establish a lemma (Lemma 2, Sect. 2) which implies (Corollary 1) that if A is a square with irrational length side less than one half, then the process $(1_A \circ R_a^n)$ is 2-dimensionally determined. With the help of this Corollary, the previous result shows (Proposition 1, Sect. 2) that the class $\mathcal{C}_2$ of functions f such that the process $(f \circ R_a^n)$ is 2-dimensionally determined, is large too, since $\mathcal{C}_2$ contains, for example, any function f which satisfies $f(A) \cap f(A^c) = \emptyset$, for some such square A.

In Sect. 3, we prove (Theorem 2) that, if $f \in \mathcal{M}$, then all Fourier coefficients of f are non null.

2 Two-Dimensionaly Determined Processes

In this section we prove that, for an ergodic rotation R on a compact abelian group and a large class of functions f, the process $(f \circ R^n)_{n \in \mathbb{Z}}$ is 2-dimensionally determined and give applications to the one or two dimensional torus.

We first recall some definitions and results which will be needed [2,17,18].

Let G be a locally compact abelian group. A continuous character ϕ of G is a continuous group homomorphism from G into the multiplicative group $\mathbb{T} = \{z \in \mathbb{C} :\mid z \mid = 1\}$ of complex numbers with modulus one. The character group $\hat{G}$ (the dual group) of G is the set of all continuous characters of G. If m is Haar measure on G, and $f \in L^1(m)$, the Fourier-Stieltjes coefficients of f are $\hat{f}(\phi) := \int f(x)\overline{\phi(x)}dm(x)$, for all $\phi \in \hat{G}$. More generally, if μ is a measure on G, then its Fourier-Stieltjes transforms $\hat{\mu}$, is defined by $\hat{\mu}(\phi) = \int_G \overline{\phi(x)}d\mu(x)$. If $f, g \in L^1(m)$, their convolution $f * g$ is defined by $f * g(y) = \int_G f(x)g(yx^{-1})dm(x)$ [17].

Also, $f \leq g$ means $f(x) \leq g(x)$ for m-almost all $x \in G$, and in particular, for measurable subsets A, B of G, $A \subset B$ means $1_A \leq 1_B$ which means then $m(A \cap B^c) = 0$, where B^c denotes the complement of B in G, and more particularly $A = B$ means then $m(A \Delta B) = 0$, where $A \Delta B$ denotes the symmetric difference set.

If $a \in G$, the rotation by a, will be denoted $R_a : R_a(x) = ax, \forall x \in G$. If G is compact and m is normalized then the dynamical system (G, R_a, m) is ergodic if and only if the set $\{a^n : n \in \mathbb{Z}\}$ is dense in G. Moreover, in this case, this dynamical system has discrete spectrum, the group Λ of eigenvalues is $\Lambda = \{\phi(a) : \phi \in \hat{G}\}$, and every character $\phi \in \hat{G}$ is a basis of the one dimensional eigenspace corresponding to the eigenvalue $\phi(a)$ [18].

We shall use also the result that two dynamical systems on Lebesgue spaces with pure point spectrum and having the same group of eigenvalues are isomorphic [2,18].

The image measure of a measure μ on G, by a measurable function $h : G \to \mathbb{C}$, is denoted $\mu \circ h^{-1}$. The measure with density $f \in L^1(m)$ with respect to m is denoted fm.

The circle group $\mathbb{T}$ will be identified with $[0, 1[$ mod(1) and similarly, $\mathbb{T}^2$ with $[0, 1[\times[0, 1[$. The imaginary part of a complex number z is denoted $\mathrm{Im}(z)$.

We begin with

Lemma 1 *Let m be the normalized Haar measure on the compact abelian group G. Let $f, g \in L^2(G, m)$ be real valued. Let $a \in G$ and R_a be the ergodic rotation by a. Set $X_n := f \circ R_a^n$, $Y_n := g \circ R_a^n$, for $n \in \mathbb{Z}$. Then the two processes (X_n) and (Y_n)*

have the same two dimensional laws if and only if for every $\phi \in \hat{G}$*, there is* $c(\phi) \in \mathbb{C}$*, with* $| c(\phi) |= 1$*, such that for any* $m \circ f^{-1}$*-integrable function* h*,*

$$\int h(f(x))\overline{\phi(x)}dm(x) = c(\phi) \int h(g(x))\overline{\phi(x)}dm(x).$$

Proof For $t_1, t_2 \in \mathbb{R}$, we have

$$\int e^{i(t_1 X_0(x)+t_2 X_n(x))} dm(x) = \int e^{i(t_1 f(x)+t_2 f(a^n x))} dm(x),$$

so that if $f_1(x) := e^{it_1 f(x)}$, $f_2(x) := e^{it_2 f(x)}$, and $f_3(x) := f_2(x^{-1})$, we obtain

$$\begin{aligned}\int e^{i(t_1 X_0(x)+t_2 X_n(x))} dm(x) &= \int f_1(x) f_2(a^n x) dm(x) = \int f_1(x) f_3(x^{-1} a^{-n}) dm(x)\\ &= f_1 * f_3(a^{-n}).\end{aligned}$$

Then, with similar notation,

$$\int e^{i(t_1 Y_0(x)+t_2 Y_n(x))} dm(x) = g_1 * g_3(a^{-n}).$$

Since $\{a^n : n \in \mathbb{Z}\}$ is dense in G, by continuity of $g_1 * g_3$ (the convolution product $f * g$ of two functions $f, g \in L^2(m)$ is a continuous function on G) it follows that (X_0, X_n) and (Y_0, Y_n) have the same law for all n if and only if $f_1 * f_3 = g_1 * g_3$, for all $t_1, t_2 \in \mathbb{R}$, and thus, taking Fourier-Stieltjes coefficients, if and only if $\hat{f_1}(\phi)\hat{f_3}(\phi) = \hat{g_1}(\phi)\hat{g_3}(\phi)$, for all $\phi \in \hat{G}$, and all $t_1, t_2 \in \mathbb{R}$.

If we set $\nu = \nu_\phi := (\overline{\phi} m) \circ f^{-1}$, and $\eta = \eta_\phi := (\overline{\phi} m) \circ g^{-1}$, we get

$$\begin{aligned}\hat{f_1}(\phi) &= \int_G f_1(x)\overline{\phi(x)}dm(x) = \int_G e^{it_1 f(x)}\overline{\phi(x)}dm(x) = \int_G e^{it_1 f(x)} d(\overline{\phi} m)(x)\\ &= \int_{\mathbb{R}} e^{it_1 y} d\nu_\phi(y),\end{aligned}$$

and

$$\begin{aligned}\hat{f_3}(\phi) &= \int_G f_3(x)\overline{\phi(x)}dm(x) = \int_G e^{it_2 f(x^{-1})}\overline{\phi(x)}dm(x)\\ &= \int_G e^{it_2 f(x)}\phi(x)dm(x) = \overline{\int_G e^{-it_2 f(x)}\overline{\phi(x)}dm(x)} = \overline{\int_{\mathbb{R}} e^{-it_2 y} d\nu_\phi(y)}.\end{aligned}$$

Then the equality

$$\hat{f_1}(\phi)\hat{f_3}(\phi) = \hat{g_1}(\phi)\hat{g_3}(\phi),$$

means

$$\hat{\nu}(-t_1)\overline{\hat{\nu}(t_2)} = \hat{\eta}(-t_1)\overline{\hat{\eta}(t_2)}. \tag{1}$$

In particular, for $t = -t_1 = t_2$, we get $|\ \hat{\nu}(t)\ |=|\ \hat{\eta}(t)\ |=: \rho(t)$. Then if $\hat{\nu}(t) = \rho(t)e^{i\alpha(t)}$ and $\hat{\eta}(t) = \rho(t)e^{i\beta(t)}$, we obtain

$$\rho(-t_1)\rho(t_2)e^{i(\alpha(-t_1)-\alpha(t_2))} = \rho(-t_1)\rho(t_2)e^{i(\beta(-t_1)-\beta(t_2))},$$

equivalently

$$\rho(-t_1)\rho(t_2)e^{i(\alpha(-t_1)-\beta(-t_1))} = \rho(-t_1)\rho(t_2)e^{i(\alpha(t_2)-\beta(t_2))},$$

so that, since this holds for all t_1, t_2,

$$\rho(t_1)\rho(t_2)e^{i(\alpha(t_1)-\beta(t_1))} = \rho(t_1)\rho(t_2)e^{i(\alpha(t_2)-\beta(t_2))}. \tag{2}$$

Now (2) implies that there exists a constant $c = c(\phi)$, with absolute value one, such that $\nu_\phi = c(\phi) \times \eta_\phi$. For in fact this equality holds trivially if $\rho(t) = 0, \forall t$, while, if for some t_2, $\rho(t_2) \neq 0$, then for t with $\rho(t) \neq 0$, (2) implies

$$e^{i(\alpha(t)-\beta(t))} = e^{i(\alpha(t_2)-\beta(t_2))} \tag{3}$$

which proves that $e^{i(\alpha(t)-\beta(t))}$ is independent of t when $\rho(t) \neq 0$. Denoting the common value by $c(\phi)$, (3) implies then

$$e^{i\alpha(t)} = c(\phi)e^{i\beta(t)},$$

from which follows the implication

$$\rho(t) \neq 0 \Rightarrow \hat{\nu}(t) = c(\phi)\hat{\eta}(t).$$

Since trivially

$$\rho(t) = 0 \Rightarrow \hat{\nu}(t) = \hat{\eta}(t) = 0 = c(\phi)\hat{\eta}(t),$$

we conclude that, $\hat{\nu}(t) = c(\phi)\hat{\eta}(t), \forall t$, which means $\nu_\phi = c(\phi)\eta_\phi$. It follows then that, for any Borel function $h : \mathbb{R} \to \mathbb{R}$, which is integrable with respect to ν,

$$\int h d\nu_\phi = c(\phi) \int h d\eta_\phi$$

which, by definitions, means

$$\int h(f(x))\overline{\phi(x)}dm(x) = c(\phi) \int h(g(x))\overline{\phi(x)}dm(x). \tag{4}$$

Conversely, the last equality (4) means $\nu_\phi = c(\phi)\eta_\phi$, which implies then

$$\hat{\nu_\phi}(-t_1)\overline{\hat{\nu_\phi}(t_2)} = c(\phi)\hat{\eta_\phi}(-t_1)\overline{c(\phi)\hat{\eta_\phi}(t_2)} = \hat{\eta_\phi}(-t_1)\overline{\hat{\eta_\phi}(t_2)}$$

and (1) holds, so that (X_n) and (Y_n) have the same two-dimensional laws. This achieves the proof. □

Definition 1 (1) For a stationary ergodic process $(X_n)_{n\in\mathbb{Z}}$, with law ν, we can suppose that ν is a probability measure on $\mathbb{R}^{\mathbb{Z}}$, invariant by the shift transformation T on $\mathbb{R}^{\mathbb{Z}}$ and that, for all n, X_n is the n-th coordinate function. We say that (X_n) has pure point spectrum if the associated ergodic dynamical system $(\mathbb{R}^{\mathbb{Z}}, T, \nu)$ has pure point spectrum.

(2) Let $\mathcal{P}$, $\mathcal{Q}$ be families of stationary proceeses, such that $\mathcal{P} \subset \mathcal{Q}$. We say that a stationary process $X = (X_n)_{n\in\mathbb{Z}}$ in $\mathcal{P}$, is two dimensionally determined in $\mathcal{Q}$, if for any process $Y = (Y_n)_{n\in\mathbb{Z}}$ in $\mathcal{Q}$, the condition

$$(Y_0, Y_n) \text{ has the same law as } (X_0, X_n) \text{ for each } n \in \mathbb{Z}$$

implies that the whole process $(Y_n)_{n\in\mathbb{Z}}$ has the same law as $(X_n)_{n\in\mathbb{Z}}$.

Clearly if $\mathcal{Q}$ and $\mathcal{R}$ are such families with $\mathcal{P} \subset \mathcal{Q} \subset \mathcal{R}$, then a process in $\mathcal{P}$ which is two dimensionally determined in $\mathcal{R}$ is two dimensionally determined in $\mathcal{Q}$.

Theorem 1 *Let R_a be an ergodic rotation on a compact abelian group G, m the Haar probability measure on G, $f \in L^2(G, m)$, and $X_n = f \circ R_a^n, \forall n \in \mathbb{Z}$. Suppose that there exists a real bounded function h, such that $h \circ f$ belongs to $L^2(m)$ and has non null Fourier-Stieltjes coefficients. Let $Y_n = h \circ f \circ R_a^n, \forall n \in \mathbb{Z}$. Then if the process $Y = (Y_n)$ is two dimensionally determined in the class of ergodic processes with pure point spectrum, the process $X = (X_n)$ is two dimensionally determined in the class of stationary ergodic processes.*

Proof Suppose that the property holds for the process Y. Let $Z = (Z_n)$ be a stationary ergodic process, such that for every n, (X_0, X_n) and (Z_0, Z_n) have the same law. We can suppose that the law ν of Z, is a probability measure on $\Omega = \mathbb{R}^{\mathbb{Z}}$, Z_n is the n-th coordinate function on $\mathbb{R}^{\mathbb{Z}}$, so that if T denotes the shift transformation on Ω, the system (Ω, T, ν) is ergodic and we have $Z_n = Z_0 \circ T^n$.

Then $(h(X_0), h(X_n))$ and $(h(Z_0), h(Z_n))$ have the same law, and in particular

$$\int_\Omega h(Z_0)h(Z_0) \circ T^n d\nu = \int_\Omega h(Z_0)h(Z_n) d\nu = \int_G h(X_0)h(X_n) dm$$
$$= \int_G h(X_0)h(X_0) \circ R_a^n dm,$$

and this implies that the closed linear space H_1 generated in $L^2(G, m)$, by $\{h(X_0) \circ R_a^n : n \in \mathbb{Z}\}$ is unitarily equivalent to H_2, the one generated in $L^2(\Omega, \nu)$, by $\{h(Z_0) \circ T^n : n \in \mathbb{Z}\}$. The non-nullity of the Fourier-Stieltjes coefficients of $h(X_0)$, implies that $H_1 = L^2(G, m)$. We shall prove that $H_2 = L^2(\nu)$. For this, we prove first that there exists a T- invariant sigma algebra $\mathcal{F} \subset \mathcal{B}(\Omega)$, such that $H_2 = L^2(\Omega, \mathcal{F}, \nu)$, and that the two dynamical systems (G, R_a, m) and $(\Omega, \mathcal{F}, T, \nu)$ are isomorphic. In fact, let $\Delta = \{\phi(a) : \phi \in \hat{G}\}$ denote the group of eigenvalues of R_a. By ergodicity of (G, R_a, m), as in [18] (pp.70–71, Proof of $(ii) \Rightarrow (iii)$ in Theorem 3.4), we can find an orthonormal basis in $L^2(m)$ of eigenvectors $(f_\phi)_{\phi\in\hat{G}}$ for R_a, such that f_ϕ has modulus one, f_ϕ corresponds to the eigenvalue $\phi(a)$, and $f_\phi f_\psi = f_{\phi\psi}$, for $\phi, \psi \in \hat{G}$. Also, since R_a is unitarily equivalent to the restriction of T to H_2, and because T is ergodic, there exists an orhtonormal basis $(g_\phi)_{\phi\in\hat{G}}$ in H_2, of eigenvectors for T, with g_ϕ corresponding to the eigenvalue $\phi(a)$, and $g_\phi g_\psi = g_{\phi\psi}$. We define $W : H_1 \to H_2$ by setting first $W f_\phi = g_\phi$, for all $\phi \in \hat{G}$, and extending by linearity and then by density to all H_1. Clearly W is thus an isomorphism of H_1 onto H_2, and satisfies $W R_a = T W$, and $W(f_{\phi\psi}) = W f_\phi W f_\psi$, for $\phi, \psi \in \hat{G}$. Let $\mathcal{F}$ be the family of Borel subsets B of Ω such that there is some Borel subset A of G with $1_B = W(1_A)$. We shall prove that $\mathcal{F}$ is a sigma-algebra. For this let $A, A' \in \mathcal{B}(G)$. There is a sequence of finite linear combinations $\sum_{\phi\in J_n} x_\phi f_\phi$ converging to 1_A and a sequence $\sum_{\phi\in K_n} y_\phi f_\phi$ which converges to $1_{A'}$ (J_n and K_n are finite subsets of $\hat{G}$, for each n). Then $1_A \sum_{\phi\in K_n} y_\phi f_\phi$ converges to $1_A 1_{A'}$, since 1_A is bounded. It follows, by linearity and continuity, that $\sum_{\phi\in K_n} y_\phi W(1_A f_\phi)$ to $W(1_A 1_{A'})$. But for every p and $\psi \in K_p$, $\sum_{\phi\in J_n} x_\phi f_\phi f_\psi$ converges to $1_A f_\psi$, since f_ψ is bounded, so we have

$$W(1_A f_\psi) = \lim_n \sum_{\phi\in J_n} x_\phi W(f_\phi f_\psi) = \lim_n \sum_{\phi\in J_n} x_\phi W(f_\phi) W(f_\psi) = W(1_A) W(f_\psi),$$

and thus

$$W(1_A 1_{A'}) = \lim_n \sum_{\phi\in K_n} y_\phi W(1_A f_\phi) = W(1_A) \lim_n \sum_{\phi\in K_n} y_\phi W(f_\phi),$$

hence, since some subsequence of $\sum_{\phi\in K_n} y_\phi W(f_\phi)$ converges almost everywhere to $W(1_{A'})$, we obtain

$$W(1_A 1_{A'}) = W(1_A) W(1_{A'}).$$

This equality shows that $\mathcal{F}$ is closed under finite intersections. Taking $A' = A$, we see that $W(1_A)$ is an indicator of some measurable subset B of Ω, and that A and B have the same measure: $m(A) = \nu(B)$. This implies in particular that $W(1_G) = 1_\Omega$, so $\Omega \in \mathcal{F}$ and by linearity $\mathcal{F}$ is then closed by taking complements. On the other hand, the continuity of W implies that $\mathcal{F}$ is closed by passing to the limit and henceforth that it is a monotone class. So $\mathcal{F}$ is a sigma-algebra. Now, from the definition of $\mathcal{F}$, the inclusion $L^2(\Omega, \mathcal{F}, \nu) \subset H_2$ holds. As we have seen before, $\forall A \in \mathcal{B}(G)$, $W(1_A) = 1_B$, for some B, so that $W(H_1) \subset L^2(\Omega, \mathcal{F}, \nu)$ (recall that $H_1 = L^2(m)$), we obtain $H_2 = L^2(\Omega, \mathcal{F}, \nu)$. The equality $T W 1_A = W R_a 1_A = W 1_{R_a^{-1} A}$ implies $T^{-1}\mathcal{F} \subset \mathcal{F}$, and the equality $T^{-1} W 1_A = W R_a^{-1} 1_A = W 1_{R_a A}$ implies $T\mathcal{F} \subset \mathcal{F}$. Then $T^{-1}\mathcal{F} = \mathcal{F} = T\mathcal{F}$.

Since the two systems (G, R_a, m) and $(\Omega, \mathcal{F}, T, \nu)$ have pure point spectrum and the same group of eigenvalues, they are conjugate and thus isomorphic because they are Lebesgue spaces. Let $\theta_1 : (G, R_a, m) \to (\Omega, \mathcal{F}, T, \nu)$ be an isomorphism, and set $V_0 := X_0 \circ \theta_1^{-1}$, and $V_n = V_0 \circ T^n$, so that the two processes $(X_n)_{n\in\mathbb{Z}}$ and $(V_n)_{n\in\mathbb{Z}}$ have the same law, and in particular they have the same two dimensional laws, and hence, for each n, (V_0, V_n) and (Z_0, Z_n) have the same law.

We continue the proof by establishing first the statement when the system (Ω, T, ν) is ergodic and have pure point spectrum, from which the proof in the general case will then follows using the Spectral mixing theorem.

Suppose then that (Ω, T, ν) is ergodic and have pure point spectrum so that it is isomorphic to an ergodic rotation on a compact abelian group G_1, so we can suppose that T is an ergodic rotation on G_1 and ν is the Haar measure, so that V_0 and Z_0 satisfy the condition of Lemma 1, and it follows then that for every $\phi \in \hat{G}_1$ there exists $c(\phi) \in \mathbb{T}$ such that for any f which is $\nu \circ Z_0^{-1}$-integrable,

$$\int_{G_1} f(V_0)\overline{\phi} d\nu = c(\phi) \int_{G_1} f(Z_0)\overline{\phi} d\nu. \tag{E_1}$$

Since, for each n, $(h(V_0), h(V_n)$, $(h(Z_0), h(Z_n)$ and $(h(X_0), h(X_n))$ have the same law, and because the process $(h(V_n))_{n\in\mathbb{Z}}$ is ergodic and have pure point spectrum, we obtain from the hypothesis that $(h(V_n))_{n\in\mathbb{Z}}$ and $(h(Z_n))_{n\in\mathbb{Z}}$ have the same law, and then, by Lemma A1 below (a modified version of Lemma 2.2 in [11]), there exists $t \in G_1$ such that for almost all $x \in G_1$,

$$h(V_0)(x) = h(Z_0)(x + t),$$

which implies

$$\begin{aligned} \int h(V_0)(x)\overline{\phi(x)} d\nu(x) &= \int h(Z_0)(x + t)\overline{\phi(x)} d\nu(x) \\ &= \phi(t) \int h(Z_0(x))\overline{\phi(x)} d\nu(x). \end{aligned} \tag{E_2}$$

(E_1) and (E_2) implie then the equalities

$$[c(\phi) - \phi(t)] \int h(Z_0)\overline{\phi} d\nu = 0, \phi \in \hat{G}_1. \tag{E_3}$$

Now, since ϕ is an eigenvector for T, corresponding to the eigenvalue λ_ϕ, the following equalities, which hold for all $k \in \mathbb{Z}$,

$$\int h(Z_0)\overline{\phi} d\nu = \int h(Z_0) \circ T^k \overline{\phi \circ T^k} d\nu = \overline{\lambda_\phi^k} \int h(Z_0) \circ T^k \overline{\phi} d\nu,$$

show the following

$$\int h(Z_0)\overline{\phi} d\nu = 0 \iff \phi \in H_2^{\perp}. \qquad (E_4)$$

(Here $H_2^{\perp}$ denotes the orthocomplement of H_2 in $L^2(\nu)$).
Let $\Lambda := \{\phi \in \hat{G}_1 : \int h(Z_0)\overline{\phi} d\nu \neq 0\}$, and Λ^c be the complement of Λ in $\hat{G}_1$. Then

$$Z_0 = \sum_{\phi \in \Lambda} \left(\int Z_0 \overline{\phi} d\nu \right) \phi + \sum_{\phi \in \Lambda^c} \left(\int Z_0 \overline{\phi} d\nu \right) \phi,$$

so that by (E_1), with $f = 1$,

$$Z_0 = \sum_{\phi \in \Lambda} (\overline{c(\phi)} \int V_0 \overline{\phi} d\nu)\phi + \sum_{\phi \in \Lambda^c} (\overline{c(\phi)} \int V_0 \overline{\phi} d\nu)\phi,$$

and thus, since by (E_4), $\int V_0 \overline{\phi} d\nu = 0$, $\forall \phi \in \Lambda^c$, we get the equalities

$$V_0 = \sum_{\phi \in \Lambda} (\int V_0 \overline{\phi} d\nu)\phi$$

and

$$Z_0 = \sum_{\phi \in \Lambda} (\overline{c(\phi)} \int V_0 \overline{\phi} d\nu)\phi. \qquad (E_5)$$

Now, by (E_3), $c(\phi) = \phi(t)$, for each ϕ in Λ and then (E_5) reads

$$Z_0 = \sum_{\phi \in \Lambda} (\overline{\phi(t)} \int V_0 \overline{\phi} d\nu)\phi,$$

which means that $Z_0(x) = V_0(x - t)$ for almost all $x \in G_1$, so that $(Z_n)_{n \in \mathbb{Z}}$ and $(V_n)_{n \in \mathbb{Z}}$ have the same law which is the law of the process $(X_n)_{n \in \mathbb{Z}}$.

Now, suppose that the ergodic system is not necessarily with pure point spectrum and let $\mathcal{G}$ be the T- invariant sigma-algebra generated by all eigenvectors for T. Then the system $(\Omega, \mathcal{G}, T, \nu)$ is ergodic and have pure point spectrum. Let $H := L^2(\Omega, \mathcal{G}, \nu)$ and $H^{\perp}$ its orthogonal complement in $L^2(\Omega, \mathcal{B}(\Omega), \nu)$. Clearly H and $H^{\perp}$ are invarint under T. By the Spectral mixing theorem of Koopman-von Neumann ([14], Theorem 3.4, p. 96), for any $x \in H^{\perp}$, the specral measure σ_x (the measure on the circle, defined by $\hat{\sigma_x}(n) = \langle T^n x, x \rangle$, $\forall n \in \mathbb{Z}$) has no atoms. Notice that for any $x \in H$, σ_x is atomic. Let $Z_0 = Z_0^{(1)} + Z_0^{(2)}$, $Z_0^{(1)} \in H$, $Z_0^{(2)} \in H^{\perp}$ be the decomposition of Z_0. Then for each $n \in \mathbb{Z}$,

$$\langle T^n Z_0, Z_0 \rangle = \langle T^n Z_0^{(1)}, Z_0^{(1)} \rangle + \langle T^n Z_0^{(2)}, Z_0^{(2)} \rangle,$$

so that

$$\sigma_{Z_0} = \sigma_{Z_0^{(1)}} + \sigma_{Z_0^{(2)}}. \tag{E_6}$$

The hypothesis (X_0, X_n) and (Z_0, Z_n) have the same law for each $n \in \mathbb{Z}$ implies the equality $\sigma_{X_0} = \sigma_{Z_0}$, so that σ_{Z_0} is atomic, and (E_6) implies then that $\sigma_{Z_0^{(2)}}$ is zero and thus $Z_0^{(2)} = 0$, which proves that Z_0 belongs to H. It follows then, from the first part of the proof, that $(Z_n)_{n\in\mathbb{Z}}$ has the same law of $(X_n)_{n\in\mathbb{Z}}$. □

Lemma A1 *Let G be a compact abelian group, m the normalized Haar measure on G, $R := R_a$ an ergodic rotation by $a \in G$, $f, g \in L^\infty(m)$ be real valued. Then the following properties are equivalent:*

- (i) $\int f f\circ R^{p_1} f\circ R^{p_2} ... f\circ R^{p_n} dm = \int g g\circ R^{p_1} g\circ R^{p_2} ... g\circ R^{p_n} dm$, $\forall n \geq 1$ and all $p_1, p_2, ..., p_n \in \mathbb{Z}$.
- (ii) *The two processes* $(f\circ R^n)_{n\in\mathbb{Z}}$ *and* $(g\circ R^n)_{n\in\mathbb{Z}}$ *have the same law.*
- (iii) *There is* $t \in G$ *such that* $g(x) = f(t + x)$ for *m almost all $x \in G$.*

Proof (iii) implies (ii), and (ii) implies (i) are trivial. To prove that (i) implies (iii), define two functions F and H on G^n by:

$$F(x_1, ..., x_n) = \int f(t) f(t + x_1) ... f(t + x_n) dm(t),$$
$$H(x_1, ..., x_n) = \int g(t) g(t + x_1) ... g(t + x_n) dm(t).$$

The boundedness of f and, for fixed $h \in L^2(m)$, the continuity of the map $\tau : G \to L^2(m)$, defined by $\tau(x) = h_x$, where $h_x(t) = h(x + t)$, imply, using Cauchy-Schwarz inequality, the continuity of F, and similarily for H. Now, (i) means that $F = H$ on the set $\{(a^{p_1}, ..., a^{p_n}) : p_1, ..., p_n \in \mathbb{Z}\}$, which is dense in G^n. Then $F = H$ on G^n. Then, for every $\phi_1, ..., \phi_n \in \hat{G}$

$$\int F(x_1, ..., x_n)\phi_1(x_1)...\phi_n(x_n) dm(x_1)...dm(x_n)$$
$$= \int H(x_1, ..., x_n)\phi_1(x_1)...\phi_n(x_n) dm(x_1)...dm(x_n).$$

But, by Fubini,

$$\int F(x_1, ..., x_n)\phi_1(x_1)...\phi_n(x_n) dm(x_1)...dm(x_n)$$
$$= \int f(t) f(t + x_1) ... f(t + x_n)\phi_1(x_1)...\phi_n(x_n) dm(t) dm(x_1)...dm(x_n)$$
$$= \hat{f}(\phi_1^{-1})...\hat{f}(\phi_n^{-1})\hat{f}(\phi_1...\phi_n);$$

and similarily for H, so that (i) holds if and only if , for every n and every $\phi_1, ..., \phi_n \in \hat{G}$,

$$\hat{f}(\phi_1^{-1})...\hat{f}(\phi_n^{-1})\hat{f}(\phi_1...\phi_n) = \hat{g}(\phi_1^{-1})...\hat{g}(\phi_n^{-1})\hat{g}(\phi_1...\phi_n). \qquad (a_1)$$

In particular for $n = 1$,

$$\hat{f}(\phi_1^{-1})\hat{f}(\phi_1) = \hat{f}(\phi_1^{-1})\hat{f}(\phi_1)$$

means that for every ϕ, $| \hat{f}(\phi) |=| \hat{f}(\phi) |$, so that for some $c(\phi) \in \mathbb{T}$, $\hat{g}(\phi) = c(\phi)\hat{f}(\phi)$. Replacing in (a_1), we obtain

$$\hat{f}(\phi_1^{-1})...\hat{f}(\phi_n^{-1})\hat{f}(\phi_1...\phi_n)[1 - c(\phi_1^{-1})...c(\phi_n^{-1})c(\phi_1...\phi_n)] = 0,$$

so that if $\Lambda := \{\phi : \hat{f}(\phi) \neq 0\}$, then $\Lambda = \Lambda^{-1}$, and

$$\phi_1, ..., \phi_n, \phi_1...\phi_n \in \Lambda \Rightarrow c(\phi_1...\phi_n) = c(\phi_1)...c(\phi_n). \qquad (a_2)$$

We conclude that (i) is satisfied if and only if (a_2) holds, for every n.
Also, for $n = 2$, (a_1) becomes

$$\hat{f}(\phi_1^{-1})\hat{f}(\phi_2^{-1})\hat{f}(\phi_1\phi_2) = \hat{g}(\phi_1^{-1})\hat{g}(\phi_2^{-1})\hat{g}(\phi_1\phi_2)$$

which, with $\phi_2 = \phi_1^{-1}$, yields

$$| \hat{f}(\phi_1^{-1}) | \hat{f}(1) =| \hat{g}(\phi_1^{-1}) | \hat{g}(1).$$

Then, since for some ϕ, $| \hat{f}(\phi) |=| \hat{g}(\phi) |\neq 0$, we get $\hat{f}(1) = \hat{g}(1)$. Now, if $r \in \mathbb{R}$, then condition (i)(resp. (iii)) holds for f and g if and only if it holds for $f + r$ and $g + r$. Then we can suppose $\hat{f}(1) \neq 0$, hence $1 \in \Lambda$ and $c(1) = 1$.
We will show now that if $\tilde{\Lambda}$ is the subgroup generated by Λ then there is a group homomorphism $\tilde{c} : \tilde{\Lambda} \to \mathbb{T}$ with the properties

1. $\tilde{c}(\phi) = c(\phi), \forall \phi \in \Lambda$,
2. $\hat{g}(\phi) = \tilde{c}(\phi)\hat{f}(\phi), \forall \phi \in \tilde{\Lambda}$.

In fact, if $\phi \in \tilde{\Lambda}$ is given by

$$\phi = \phi_1...\phi_n = \psi_1...\psi_k,$$

with $\phi_1, ..., \phi_n \in \Lambda$ and also $\psi_1, ..., \psi_k \in \Lambda$, then we must prove the following equality

$$c(\phi_1)...c(\phi_n) = c(\psi_1)...c(\psi_k).$$

For this, note that the equality $\phi_1...\phi_n = \psi_1...\psi_k$ means $1 = \phi_1...\phi_n\psi_1^{-1}...\psi_k^{-1}$, which, by (a_2), implies

$$1 = c(1) = c(\phi_1)...c(\phi_n)c(\psi_1^{-1})...c(\psi_k^{-1}),$$

which means

$$c(\phi_1)...c(\phi_n) = c(\psi_1)...c(\psi_k).$$

In conclusion $\tilde{c}$ is a group homomorphism on $\tilde{\Lambda}$, which coincides with c on Λ, and it is such that

$$\hat{g}(\phi) = \tilde{c}(\phi)\hat{f}(\phi), \forall \phi \in \Lambda,$$

and then, because $\hat{f}(\phi) = \hat{g}(\phi) = 0$ if $\phi \in \tilde{\Lambda}$ and $\phi \notin \Lambda$,

$$\hat{g}(\phi) = \tilde{c}(\phi)\hat{f}(\phi), \forall \phi \in \tilde{\Lambda}.$$

We use now the following

Lemma A2 *Let H be a subgroup of a commutative group G, and $\phi : H \to \mathbb{T}$ be a group homomorphism. Then there is a group homomorphism $\psi : G \to \mathbb{T}$ which is an extension of ϕ.*

Proof Let X be the set of all (M, ψ) such that M is a subgroup of G which contains H, and $\psi : M \to \mathbb{T}$ is a group homomorphism such that $\psi\ |_H = \phi$. Define on X the order relation $\lesssim$ by declaring $(M, \psi) \lesssim (M', \psi')$ if $M \subset M'$ and $\psi'\ |_M = \psi$. Clearly $\lesssim$, thus defined, is an order relation on X. Let $\{(M_j, \psi_j) : j \in J\}$ be a totally ordered subset of X. Then $M = \cup_{j \in J} M_j$ is a subgroup of X which contains H, and ψ defined on M by $\psi(x) = \psi_j(x)$ if $x \in M_j$, is a group homomorphism from M to $\mathbb{T}$, which satisfies $\psi\ |_{M_j} = \psi_j$, for all $j \in J$, so that $(M, \psi) \in X$ and $(M_j, \psi_j) \lesssim (M, \psi)$ for all j. This say that (M, ψ) is an upper bound of $\{(M_j, \psi_j) : j \in J\}$. Let (N, ξ) be an upper bound of $\{(M_j, \psi_j) : j \in J\}$. Then for all j, $M_j \subset N$ and $\xi\ |_{M_j} = \psi_j$. It follows that $M = \cup_{j \in J} M_j \subset N$ and $\xi\ |_{M_j} = \psi_j = \psi\ |_{M_j}$, and thus $\xi\ |_M = \psi\ |_M = \psi$, so that $(M, \psi) \lesssim (N, \xi)$, which proves that (M, ψ) is the least upper bound of $\{(M_j, \psi_j) : j \in J\}$. It follows then, from Zorn's Lemma, that X has a maximal element, say, (L, α).

Suppose that $L \neq G$, and let $x \in G$, $x \notin L$. Let L_1 be the subgroup generated by L and x. Then $L_1 = \{yx^k : y \in L, k \in \mathbb{Z}\}$. Two cases can occur:

1. $x^k \notin L$ for every $k \in \mathbb{Z}$, with $k \neq 0$,
2. There exists $k \geq 2$ such that $x^k \in L$.

In the first case, note that the equivalence $yx^k = zx^l \iff z^{-1}y = x^{l-k}$, implies that every element $x_1 \in L_1$ has a unique expression as $x_1 = yx^k$, with $y \in L$ and $k \in \mathbb{Z}$. We define $\tilde{\alpha} : L_1 \to \mathbb{T}$, by

$$\tilde{\alpha}(yx^k) = \alpha(y).$$

Clearly $\tilde{\alpha}$ is a homomorphism which extends α, so that $(L, \alpha) \lesssim (L_1, \tilde{\alpha})$, which implies the equality $L_1 = L$, which contradicts the assumption $x \notin L$.

In the second case let k_0 be the least integer ≥ 2, such that $y_0 = x^{k_0} \in L$. Let θ_0 be such that $\alpha(y_0) = e^{i\theta_0}$. Set $\beta_0 = \frac{\theta_0}{k_0}$. Define , for $z \in L$, and $k \in \mathbb{Z}$, $\tilde{\alpha}(zx^k) = \alpha(z)e^{ik\beta_0}$.
Then, in order for $\tilde{\alpha}$ to be well deined, we must prove the following implication

$$z, u \in L, k, l \in \mathbb{Z}, zx^k = ux^l \Rightarrow \alpha(z)e^{ik\beta_0} = \alpha(u)e^{il\beta_0}.$$

Now, if $k = l$, it is verified. If $k \neq l$, suppose that $l < k$. Writing $kl = qk_0 + r$, with $k, r \in \mathbb{N}$, $0 \leq r < k_0$, the following

$$zx^k = ux^l \iff uz^{-1} = x^{k-l} \iff uz^{-1} = x^{qk_0}x^r \iff uz^{-1} = y_0^q x^r$$

implies $r = 0$, so that $k - l = qk_0$ and $uz^{-1} = y_0^q$, that is $z = uy_0^{-q}$.
Then

$$\alpha(z)e^{ik\beta_0} = \alpha(u)\alpha(y_0)^{-q}e^{ik\beta_0} = \alpha(u)e^{-iq\theta_0}e^{ik\beta_0}$$
$$= \alpha(u)e^{-iqk_0\beta_0}e^{ik\beta_0} = \alpha(u)e^{i(kqk_0)\beta_0} = \alpha(u)e^{il\beta_0}.$$

Then $\tilde{\alpha}$ is well defined and it is clearly a homomorphism from L_1 to $\mathbb{T}$, whose restriction to L coincides with α, so that $(L, \alpha) \lesssim (L_1, \tilde{\alpha})$, and hence, by maximality of (L, α), $L_1 = L$, which coontradicts the assumption $x \notin L$, and this ends the proof of the Lemma A2. □

It follows, from the preceding Lemma A2, that there is a group homomorphism $\gamma : \hat{G} \to \mathbb{T}$, which extends $\tilde{c}$. Now, since $\hat{g}(\phi) = \hat{f}(\phi) = 0$ for any $\phi \in \hat{G}$, with $\phi \notin \tilde{\Lambda}$, the equality $\hat{g}(\psi) = \gamma(\psi)\hat{f}(\psi)$ holds for all $\psi \in \hat{G}$. By Pontrytagin's duality theorem, γ is given by evaluation at some point $t \in G$:
$\gamma(\psi) = \psi(t), \forall \psi \in \hat{G}$, so that

$$\hat{g}(\psi) = \psi(t)\hat{f}(\psi), \forall \psi \in \hat{G},$$

which means that g is the translate of f by t, so that (iii) holds, and this ends the proof of Lemma A1. □

Remark 1 The equalities $\int f f \circ R_a^n dm = f * \tilde{f}(a^{-n})$, $n \in \mathbb{Z}$, where $\tilde{f}(x) = f(x^{-1})$, show that $\int f f \circ R_a^n dm = \int g g \circ R_a^n dm, \forall n$ if and only if $| \hat{f}(\phi) |=| \hat{g}(\phi) |, \forall \phi \in \hat{G}$. It follows that, for every $f \in L^2(m)$, the process $(f \circ R_a^n)_{n \in \mathbb{Z}}$ is not 2-spectrally determined.

The following Lemma will be useful in order to obtain functions satisfying the condition in Theorem 1, in the case where the group is the two-dimensional torus.

Lemma 2 *Let* $\delta \in]0, \frac{1}{2}[$, *and* $A = [s, s+\delta] \times [t, t+\delta]$ *be a square with length side* δ, $A \subset [0, 1[\times[0, 1[= \mathbb{T}^2$. *Let* $a = (\alpha, \beta) \in]0, 1[\times]0, 1[$, *where* $1, \alpha, \beta$ *are linearly independent over the rationals. Let* $B \subset [0, 1[\times[0, 1[$ *be a measurable subset. Let* m *be the normalized Lebesgue measure on* $\mathbb{T}^2$.*Then the following properties are equivalent:*

(1) $m(A \cap (na + A)) = m(B \cap (na + B)), \forall n \in \mathbb{Z}^2$.

(2) $m(A \cap (x + A)) = m(B \cap (x + B)), \forall x \in \mathbb{T}^2$.
(3) *The Fourier Coefficients of* 1_A *and* 1_B *have the same absolute values.*
(4) *B is a translate of A.*

Proof The proof of (1) $\iff$ (2) $\iff$ (3) is the same as in [6]. Since (4) $\Rightarrow$ (3) is trivial, it remains only to prove (2) $\Rightarrow$ (4). For this, note first that we can and do suppose that $A = [0, \delta] \times [0, \delta]$.

Let

$$A_1 = (1 - \delta, 0) + A, A_2 = A, A_3 = (1 - \delta, 1 - \delta) + A, A_4 = (0, 1 - \delta) + A.$$

Then A_1, A_2, A_3 and A_4 are disjoint since $\delta < \frac{1}{2}$. Let

$$D = A_1 \cup A_2 \cup A_3 \cup A_4.$$

Then there is a translate B' of B such that , almost everywhere

$$B' \subset D.$$

In fact, since for every $x \in D^c$, $m(A \cap (x + A)) = 0$, we have

$$0 = \int_{D^c} m(B \cap (x + B))dm(x) = \int 1_B(y)dy \int 1_B(y - x)1_{D^c}(x)dx$$
$$= \int 1_B(y)dy \int 1_B(t)1_{D^c}(y - t)dt = \int 1_B(t)dt \int 1_B(y)1_{D^c}(y - t)dt,$$

so that for almost all $t \in B$, $y \in t + D$, for almost all $y \in B$, in particular, for some t_0, $B \subset t_0 + D$. Take $B' = B - t_0$.

Since $m(B \cap (x + B)) = m(B' \cap (x + B'))$ for all x, we can and do suppose that B itself is contained in D. Let

$$B_j = B \cap A_j, j = 1, \ldots, 4.$$

Then $(B_j)_{j=1}^4$ is a partition of B, so that for all (x, y),

$$m(A + (x, y) \cap A) = m(B + (x, y) \cap B) = \sum_{j,k=1}^{4} m(B_j + (x, y) \cap B_k) \qquad (E)$$

We shall prove that$\{(\delta, 0) + B_1, B_2, (\delta, \delta) + B_3, (0, \delta) + B_4\}$ is a partition of A. Since, clearly all the sets $(\delta, 0) + B_1, B_2, (\delta, \delta) + B_3, (0, \delta) + B_4$ are subsets of A, the following equalities

$$m((\delta, 0) + B_1 \cap B_2) = 0, m((\delta, 0) + B_1) \cap (\delta, \delta) + B_3) = m(B_1 \cap (0, \delta) + B_3) = 0,$$
$$m((\delta, 0) + B_1 \cap (0, \delta) + B_4) = m(B_1 \cap (-\delta, \delta) + B_4) = m(B_1 \cap (1 - \delta, \delta) + B_4) = 0,$$
$$m(B_2 \cap (\delta, \delta) + B_3) = 0, m(B_2 \cap (0, \delta) + B_4) = 0,$$
$$m((\delta, \delta) + B_3 \cap (0, \delta) + B_4) = m(\delta, 0) + B_3 \cap B_4) = 0,$$

imply the disjoint equality

$$A = B_2 \cup (\delta, \delta) + B_3 \cup (\delta, 0) + B_1 \cup (0, \delta) + B_4 \tag{5}$$

Now, if λ is the Lebesgue measure on $[0, 1]$, and if $J_k(x) = \{y : (x, y) \in B_k\}$, then, for $0 < u < 1, \delta < v < 1$, we obtain

$$m(B_1 \cap ((u, v) + B_3) = 0 \Rightarrow \int dx \int 1_{B_1}((x, y)1_{B_3}(x - u, y - v)dy = 0 \Rightarrow$$
$$\int dx \int 1_{B_1}(x, t + v)1_{B_3}(x - u, t)dt = 0 \Rightarrow \int \lambda(-v + J_1(x) \cap (J_3(x - u))dx = 0$$
$$\Rightarrow \int \lambda(J_1(x) \cap (v + J_3(x - u))dx = 0.$$

The last equality, which holds for all u, v, as before, implies that there is a horizontal line between B_1 and all the translates of $(0, \delta) + B_3$, by $(t, 0), 0 < t < 1$. Similarily, from

$$m(B_2 \cap ((u, v) + B_3) = 0 \Rightarrow \int dx \int 1_{B_2}((x, y)1_{B_3}(x - u, y - v)dy = 0 \Rightarrow$$
$$\int dx \int 1_{B_2}(x, t + v)1_{B_3}(x - u, t)dt = 0 \Rightarrow \int \lambda(-v + J_2(x) \cap (J_3(x - u))dx = 0$$
$$\Rightarrow \int \lambda(J_2(x) \cap (v + J_3(x - u))dx = 0,$$

we obtain that the same horizontal line separates B_2 and the transllates of $(0, \delta) + B_3$ by $(t, 0), 0 < t < 1$. Moreover, if the equation of this line is $y = y_1$, then $0 \leq y_1 \leq \delta$, and B_1 and B_2 are (as any one of their horizontal translates) contained in the (lower) rectangle with vertices $(0, 0)$, $(0, y_1)$, $(\delta, 0)$ and (δ, y_1), while all the horizontal translates of $(0, \delta) + B_3$ are contained in the (upper) rectangle with vertices $(0, y_1)$, $(0, \delta)$, (δ, y_1) and (δ, δ). In particular $(\delta, \delta) + B_3$ is contained in the upper rectangle.

The same properties hold also for B_4 in place of B_3. In particular $(0, \delta) + B_4$ is also in the upper rectangle.

Also, by Fubini, reversing the order of integration, we can see that there is a vertical line between $(\delta, 0) + B_1$ and the vertical translations of B_2 by $(0, t)$, for $0 < t < \delta$, and also the same properties hold for B_4 in place of B_2, with the same vertical line. In the same way, if $x = x_1$ is the equation of this line, then $0 \leq x_1 \leq \delta$, and B_2 and $(0, \delta) + B_4$ are contained in the (left) rectangle whose vertices are $(0, 0)$, $(0, \delta)$, $(x_1, 0)$ and (x_1, δ), while $(\delta, 0) + B_1$ and $(\delta, \delta) + B_3$ are contained in the right rectangle with vertices $(x_1, 0)$, (x_1, δ), (δ, δ) and $(\delta, 0)$.

We get then, by the disjoint equality (5), the following

$$B_2 = [0, x_1] \times [0, y_1],\ B_1 + (\delta, 0) = [x_1, \delta] \times [0, y_1],$$
$$B_3 + (\delta, \delta) = [x_1, \delta] \times [y_1, \delta],\ B_4 + (0, \delta) = [0, x_1] \times [y_1, \delta],$$

which mean

$$B_1 = [x_1 - \delta, 0] \times [0, y_1],\ B_3 = [x_1 - \delta, 0] \times [y_1 - \delta, 0],\ B_4 = [0, x_1] \times [y_1 - \delta, 0],$$

so that $B = B_1 \cup B_2 \cup B_3 \cup B_4$ is the square with vertices $(x_1 - \delta, y_1 - \delta)$, $(x_1 - \delta, y_1)$, (x_1, y_1) and $(x_1, y_1 - \delta)$, which is clearly a translate of A. □

Corollary 1 *Let A be a square contained in* $\mathbb{T}^2 = [0, 1[\times[0, 1[$, *with irrational length side less than* $\frac{1}{2}$, *and* $a = (\alpha, \beta) \in \mathbb{T}^2$, *with* $1, \alpha, \beta$ *linearly independent over the rationals. Then the process* $(1_A \circ R_a^n)_{n\in\mathbb{Z}}$ *is determined by its two-dimensional laws.*

Proof Let $Y = (Y_n)$ be an ergodic process, with law ν, such that for all n, $(1_A, 1_A \circ R_a^n)$ and (Y_0, Y_n) have the same law. Then, in particular there exists a measurable set $E \subset \Omega$, such that ν-almost everywhere $Y_0 = 1_E$, We can and do suppose that the law ν of the process Y is a probability measure on $\Omega := \{0, 1\}^{\mathbb{Z}}$, Y_n is the n-th coordinate function, T is the shift transformation and that (Ω, T, ν) is ergodic. Since, the Fourier coefficients of 1_A are all non zero, an argument, exactly as in the proof of Theorem 1, yields that the two dynamical systems $(\mathbb{T}^2, R_a, m)$ and (Ω, T, ν) are isomorphic. Let $\theta : \mathbb{T}^2 \to \Omega$ be such an isomorphism. Then, if $B = \theta^{-1}E$, we obtain that the two processes $(1_B \circ R_a^n)$ and (Y_n) have the same law. Hence for all n, $(1_B, 1_B \circ R_a^n)$ and $(1_A, 1_A \circ R_a^n)$ have the same law, so that, in particular

$$\int 1_B 1_B \circ R_a^n dm = \int 1_A 1_A \circ R_a^n dm,$$

which means

$$m(B \cap (na + B)) = m(A \cap (na + A)),$$

so, by Lemma 2, B is a translate of A, and then the two processes $(1_B \circ R_a^n)$ and $(1_A \circ R_a^n)$ have the same law and this achieves the proof. □

Remark 2 (1) Theorem 1 and Corollary 1 enable us to obtain examples of functions f on $\mathbb{T}^2$ for which the process $(f \circ R_a^n)$ is two dimensionally determined, for any $a = (\alpha, \beta)$, where $1, \alpha, \beta$ are linearly independent over the rationals. In fact, for example, any function for which there is some colsed square V with irrational side length less than $\frac{1}{2}$, such that

$$\sup_{y\in V} f(y) < f(z), \forall z \notin V,$$

satisfies the condition of the theorem because if $B = f(V)$, then $1_B(f) = 1_V$ will have all its Fourier-Stieltjes non zero and the process $(1_B(f) \circ R_a^n)$ is, by the Corollary 1, uniquely determined by its two-dimensional laws.
Also, if V is such a square and if $\{A_1, \ldots, A_n\}$ is a partition of V^c in $\mathbb{T}^2$, and $f = x_0 1_V + \sum_{j=1}^n x_j 1_{A_j}$ is a simple function, where $x_j \neq x_0$, for all j, so that $1_{\{x_0\}}(f) = 1_V$, then the process $(f \circ R_{(\alpha,\beta)}^n)$ is determined by its two dimensional laws.

More generally, for every function with the property

$$f(V) \cap f(V^c) = \emptyset, \tag{6}$$

(or $m(\{x \in V^c : f(x) \in f(V)\}) = 0$, or $m(f^{-1}(f(V)) = m(V))$ the process $(f \circ R^n_{(\alpha,\beta)})$ will be determined by its two-dimensional laws.

(2) For a function of one variable, Theorem 1 and Theorem 2.1 in [6] implie that, if $\alpha \notin \mathbb{Q}$, $0 \leq a < b < 1$, and $b - a \notin \mathbb{Q}$, and given three square integrable functions

$$f_1 : [0, a] \to \mathbb{R},\ f_2 : [a, b] \to \mathbb{R},\ f_3 : [b, 1[\to \mathbb{R}$$

such that

$$f_1(x) < f_2(y < f_3(z), \forall x \in [0, a], \forall y \in [a, b], \forall z \in [b, 1[,$$

and if

$$f = 1_{[0,a[} f_1 + 1_{[a,b]} f_2 + 1_{]b,1[} f_3,$$

then the law of the process $X = (f \circ R^n_\alpha)_{n\in\mathbb{Z}}$ is determined by the family of its two-dimensional laws. In fact, if $B = f_2([a, b])$, then $1_B(f) = 1_{[a,b]}$ and the process $Y = (1_{[a,b]} \circ R^n_\alpha)_{n\in\mathbb{Z}}$ is determined by its two-dimensonal laws so the corollary gives the assertion.

Also, we can replace the preceding condition by the following more general one

$$f_2([a, b]) \cap (f_1([0, a[) \cup f_3(]b, 1])) = \emptyset,$$

which can be written as

$$f([a, b]) \cap f([a, b]^c) = \emptyset. \tag{7}$$

Proposition 1 *For $j = 1, 2$, let $\mathcal{C}_j$ be the set of real measurable functions f on $\mathbb{T}^j$, such that the process $(f \circ R^n_a)$ be 2-dimensionally determined. Then*

(1) *$\mathcal{C}_1$ contains any real square integrable f such that for some arc A with irrational length, $f(A) \cap f(A^c) = \emptyset$.*
(2) *$\mathcal{C}_2$ contains any real square integrable f such that for some square A with irrational length side less than one half, $f(A) \cap f(A^c) = \emptyset$.*

Remark 3 In the proposition above, the set $\mathcal{C}_1$ contains in particular any monotone continuous non constant function on $[0, 1]$. It contains also any monotone function on this interval having a discontinuity at an irrational point, and any strictly monotone function as well.

Notice that condition (6) (resp. 7) is very helpful, since if it holds the process $(f \circ R^n)$ will be two dimensionally determined in the class of stationary ergodic processes, regardless of whether or not all the Fourier coefficients of f are non null. Also, in the case where the graph of the function f is known, condition (6) (resp. 7) can be, easily cheked. For example, for the function $f(x) := \sin(2\pi x)$, (7) holds and then the process $(f \circ R_\alpha^n)$ is two dimensionally determined in the class of stationary ergodic processes even though all but a finite number of the Fourier coefficients of f are null.

The following proposition shows that Riemann integrable functions on $\mathbb{T}$, or on $\mathbb{T}^2$, can be uniformly approximated by functions f such that the process $(f \circ R_a^n)_{n\in\mathbb{Z}}$ is determined by its two-dimensional laws.

Proposition 2 *Let $G = \mathbb{T}$, or $G = \mathbb{T}^2$. Let m be the Haar measure and R_a be an ergodic rotation on G. Let H be the set of functions $f \in L^\infty(m)$ such that the process $(f \circ R_a^n)$ is determined by its two dimensional laws. Then $C(G)$ is contained in the $L^\infty(m)$- closure $\overline{H}$ of H.*

Proof Let $f \in C(G)$, and $\epsilon > 0$. Let $\delta \in]0, \frac{1}{2}[$ be irrational. In case $G = \mathbb{T}$ let $A = A_\delta = [0, \delta]$, and in case $G = \mathbb{T}^2$ let $A = A_\delta = [0, \delta] \times [0, \delta]$. The set $B := \{x \in G : f(x) >|| f ||_\infty -\epsilon\}$ is open and not empty. Then there exists δ, such that a translate J of A_δ, be contained in B. Let $g = f + 2\epsilon 1_J$. Then

$$x \in J \Rightarrow g(x) = f(x) + 2\epsilon >|| f ||_\infty +\epsilon,$$
$$x \notin J \Rightarrow g(x) = f(x) \leq|| f ||_\infty,$$

so that

$$g(J) \subset [|| f ||_\infty +\epsilon, || f ||_\infty +2\epsilon],$$
$$g(J^c) \subset] - \infty, || f ||_\infty],$$

which implie

$$g(J) \cap g(J^c) = \emptyset.$$

By (6) or (7) in the preceding remarks, this last relationship implies that g belongs to H. Since $|| f - g ||_\infty = 2\epsilon$, and ϵ is arbitrary, the proof is complete. □

3 Fourier Coefficients of Monotone Functions on the Unit Interval

Even though Theorem 1 weakens the condition that the function f has all Fourier-Stieltjes coefficients non zero, it can be interesting to give simple condition on f in order to have this property. In the circle case, Theorem 2 below, shows, for instance, that non constant monotone continuous functions have the property. For the proof, we begin with

Lemma 3 *Let f be a real monotone Lebesgue integrable function on the interval $[0, 1]$, with the property $f(\frac{1}{2} - x) = -f(\frac{1}{2} + x)$, for almost all $-\frac{1}{2} < x < \frac{1}{2}$, and suppose that f is continuous at the points $\frac{k}{2q+1}$, $\frac{2k-1}{4q}$ for all $q \geq 1$ and $k = 1, \dots, q$.*

Then either

1. $f = c(21_{[\frac{1}{2},1]} - 1)$ *for some* $c \in \mathbb{R}$, *or*
2. $\int_0^1 f(x)e^{i2\pi nx}dx \neq 0, \forall n \neq 0.$

Proof Suppose that $f : [0, 1] \to \mathbb{R}$, is non-decreasing (if f is non-increasing, consider $g = -f$) and satisfies also $f(\frac{1}{2} - x) = -f(f\frac{1}{2} + x), \forall x \in [0, 1]$, and let, for $n \geq 1$,

$$I_n := \int_0^1 f(x)e^{2i\pi nx}dx.$$

Then, from

$$I_n = \int_0^{\frac{1}{2}} f(x)e^{in2\pi x}dx + \int_{\frac{1}{2}}^1 f(x)e^{in2\pi x}dx,$$

together with

$$\int_{\frac{1}{2}}^1 f(x)e^{in2\pi x}dx = -\int_0^{\frac{1}{2}} f(x)e^{-i2\pi nx}dx,$$

due the equality $f(\frac{1}{2} - x) = -f(\frac{1}{2} + x)$, we obtain

$$\begin{aligned} I_n &= 2i\int_0^{\frac{1}{2}} f(x)\sin(2\pi nx)dx = 2i\int_0^{n\pi} f\left(\frac{t}{2\pi n}\right)\sin(t)\frac{dt}{2\pi n} \\ &= \frac{2i}{2\pi n}\sum_{k=0}^{n-1}\int_{k\pi}^{(k+1)\pi} f\left(\frac{t}{2\pi n}\right)\sin(t)dt. \end{aligned}$$

Now,

$$\begin{aligned} &\int_{k\pi}^{(k+1)\pi} f\left(\frac{t}{2\pi n}\right)\sin(t)dt = \int_0^{\pi} f\left(\frac{x+k\pi}{2\pi n}\right)\sin(x+k\pi)dx \\ &= (-1)^k\int_0^{\pi} f\left(\frac{x}{2\pi n} + \frac{k}{2n}\right)\sin(x)dx = (-1)^k\int_0^{\pi} a_k(x)\sin(x)dx, \end{aligned}$$

where, for $k = 0, 1, \ldots, n-1$,

$$a_k(x) = f\left(\frac{x}{2\pi n} + \frac{k}{2n}\right), x \in [0, \pi].$$

(notice that $x \in [0, \pi] \iff \frac{x}{2\pi n} + \frac{k}{2n} \in [\frac{k}{2n}, \frac{k+1}{2n}]$, and then, since f is non decreasing, the equality $a_k(x) = a_{k+1}(x)$ holds for almost all $x \in [0, \pi]$ if and only if f is almost everywhere constant on $[\frac{k}{2n}, \frac{k+2}{2n}]$.)

Since

$$\frac{x}{2\pi n} + \frac{k}{2n} \leq \frac{1}{2} \iff x + k\pi \leq n\pi \iff x \leq (n-k)\pi,$$

and because f is non decreasing

$$\left(0 < x < \frac{1}{2} \Rightarrow \frac{1}{2} - x < \frac{1}{2} < \frac{1}{2} + x \Rightarrow f\left(\frac{1}{2} - x\right) \leq f\left(\frac{1}{2} + x\right) = -f\left(\frac{1}{2} - x\right) \Rightarrow 2f\left(\frac{1}{2} - x\right) \leq 0.\right)$$

we have

$$a_k(x) \leq 0, a_k(x) \leq a_{k+1}(x).$$

for almost all $x \in [0, \frac{1}{2}]$.

For $p = 0, 1, \ldots, n-1$, set

$$S_p(x) = \sum_{k=0}^{p} (-1)^k a_k(x).$$

Then

$$I_n = \frac{i}{\pi n} \int_0^{\pi} \sum_{k=0}^{n-1} (-1)^k a_k(x) \sin(x) dx = \frac{i}{\pi n} \int_0^{\pi} S_{n-1}(x) \sin(x) dx. \tag{8}$$

Then $S_p(x) \leq 0$. In fact the following inequalities

$$\begin{aligned} S_0(x) &= a_0(x) \leq 0, \\ S_1(x) &= a_0(x) - a_1(x) \leq 0, \\ S_{2q}(x) &= S_{2q-2}(x) + (a_{2q}(x) - a_{2q-1}(x)) \geq S_{2q-2}(x), \\ S_{2q+1}(x) &= S_{2q-1}(x) + (a_{2q}(x) - a_{2q+1}(x)) \leq S_{2q-1}(x); \\ S_{2q+1}(x) &= S_{2q}(x) - a_{2q+1}(x) \geq S_{2q}(x), \end{aligned}$$

prove, by induction, that $S_p(x) \leq 0$, for all $p \leq n-1$, and also

$$S_0(x) \leq S_{2q-2}(x) \leq S_{2q}(x) \leq S_{2r+1}(x) \leq S_{2r-1}(x) \leq S_1(x) \leq 0.$$

It follows

$$\int_0^{\pi} S_{n-1}(x) \sin(x) dx \leq \int_0^{\pi} S_1(x) \sin(x) dx \leq 0.$$

Then, by (8), $I_n = 0$ if and only if $S_{n-1} = 0$ almost everywhere on $[0, \pi]$.

If $n = 2q + 1$, the equality

$$\begin{aligned} S_{n-1}(x) = S_{2q}(x) &= (a_0(x) - a_1(x)) + (a_2(x) - a_3(x)) + \ldots + (a_{2q-2}(x) \\ &\quad -(a_{2q-1}(x)) + a_{2q}(x) \end{aligned}$$

implies then the equalities

$$a_{2k} = a_{2k+1}(x), k = 0, 1, .., q-1,$$
$$a_{2q}(x) = 0.$$

Since f is increasing, the equality $a_{2k} = a_{2k+1}$ almost everywhere means that f is constant almost everywhere on the interval $[\frac{2k}{2(2q+1)}, \frac{2k+2}{2(2q+1)}] = [\frac{k}{2q+1}, \frac{k+1}{2q+1}] = [\frac{k}{n}, \frac{k+1}{n}]$. The last interval is for $k = q-1 = \frac{n-1}{2} - 1 = \frac{n-3}{2} = [\frac{n}{2}] - 1$, that is $[\frac{q-1}{2q+1}, \frac{q}{2q+1}]$. If we suppose that f is continuous at the points $\frac{k}{2q+1}, k = 1, \ldots, q$, these equalities implie

$$a_0(x) = a_1(x) = \ldots = a_{2q}(x) = a_{n-1}(x) = 0$$

which gives

$$f\left(\frac{x}{2\pi n}\right) = f\left(\frac{x}{2\pi n} + \frac{1}{2} - \frac{1}{2n}\right)$$

for almost all x, which means that the restriction of f to $[0, \frac{1}{2n}]$ is almost everywhere equal to its restriction to $[\frac{1}{2} - \frac{1}{2n}, \frac{1}{2}]$. which, since f is increasing, implies that f will be almost everywhere constant on $[0, \frac{1}{2}]$.

If $n = 2q$, then the equalities

$$S_{n-1} = S_{2q-1} = (a_0(x) - a_1(x)) + (a_2(x) - a_3(x)) + \ldots + (a_{2q-2}(x) - a_{2q-1}(x))$$

implie that $I_n = 0$ if and only if

$$a_0 = a_1, a_2 = a_3, \ldots, a_{2q-2} = a_{2q-1} = a_{n-1}$$

almost everywhere, which means that f is constant on the interval $[\frac{2k}{2n}, \frac{2k+2}{2n}] = [\frac{k}{n}, \frac{k+1}{n}]$, for $k = 0, 1, \ldots, q-1 = \frac{n}{2} - 1 = [\frac{n}{2}] - 1$. In this case, under the continuity of f,at the points $\{\frac{2k-1}{4q} : k = 1, \ldots, q\}$,

we see also that f will be constant almost everywhere on $[0, \frac{1}{2}]$.

We conclude that, if f is continuous at the points $\frac{k}{2q+1} : k = 1, \ldots, q$ and at the points $\frac{2k-1}{4q}, k = 1, \ldots, q$, for all $q \geq 1$, then $I_n = 0$ for some $n \neq 0$, implies that there is $c \geq 0$ such that $f = -c1_{[0,\frac{1}{2}]} + c1_{]\frac{1}{2},1]}$ almost everywhere. □

Corollary 2 *Let f be a real monotone Lebesgue integrable function on the interval $[0, 1]$, with the property $f(\frac{1}{2} - x) = -f(\frac{1}{2} + x)$, for almost all $-\frac{1}{2} < x < \frac{1}{2}$, and suppose that f has a discontinuity at an irrational point $\alpha \in]0, \frac{1}{2}[$. Then $\mathrm{Im}(\int_0^1 f(x)e^{i2\pi nx}dx) \neq 0, \forall n \neq 0$.*

Theorem 2 *Let f be non constant monotone (nondecreasing, or nonincreasing) Lebesgue integrable function on* [0, 1]. *Suppose that f is continuous or that for some irrational $\alpha \in [0, 1]$, f is not continuous at α. Then*

(i)

$$\int_0^1 f(x)e^{i2\pi nx}dx \neq 0, \forall n \neq 0.$$

(ii)

$$\int_0^1 h(x)e^{i2\pi nx}dx \neq 0, \forall n \neq 0,$$

where $h = f + g$, for any even function $g \in L^1(m)$. ($g(1-x) = g(x)$ for almost all $x \in [0, 1]$.)

Proof Suppose that f is non decreasing. Let $g(x) = f(x) - f(1-x)$, for $x \in [0, 1]$. Then g is also non decreasing, for in fact, let $x \leq y$. Then $1 - y \leq 1 - x$, and thus

$$g(x) - g(y) = f(x) - f(1-x) - f(y) + f(1-y) = (f(x) - f(y)) \\ +(f(1-y) - f(1-x)) \leq 0.$$

Also, note that, if we set $t = \frac{1}{2} + x$, then $\frac{1}{2} - x = 1 - t$, and $x \in [\frac{-1}{2}, \frac{1}{2}] \iff t \in [0, 1]$, so that

$$f\left(\frac{1}{2} - x\right) = -f\left(\frac{1}{2} + x\right), \forall x \in \left[\frac{-1}{2}, \frac{1}{2}\right] \iff f(1-t) = -f(t), \forall t \in [0, 1].$$

Then from

$$g(1-x) = f(1-x) - (f(1-1+x) = f(1-x) - f(x) = -g(x),$$

follows

$$g\left(\frac{1}{2} - x\right) = -g\left(\frac{1}{2} + x\right), \forall x \in \left[\frac{-1}{2}, \frac{1}{2}\right].$$

Now, if f is continuous, then clearly g is continuous. If f is not continuous at α, then g is not continuous at α also. In fact, if for a monotone function h, we set $h(x-0) = \lim_{t\to x, t<x} h(t)$ and $h(x+0) = \lim_{t\to x, t>x} h(t)$, then

$$\begin{aligned} g(\alpha - 0) - g(\alpha + 0) &= f(\alpha - 0) - f(1 - \alpha + 0) - f(\alpha + 0) + f(1 - \alpha - 0) \\ &= (f(\alpha - 0) - f(\alpha + 0)) + (f(1 - \alpha - 0) - f(1 - \alpha + 0)) \\ &\leq f(\alpha - 0) - f(\alpha + 0) < 0. \end{aligned}$$

It follows then from Lemma 3, that $\int_0^1 g(x)e^{i2\pi nx}dx \neq 0$, for $n \neq 0$. Now

$$\int_0^1 g(x)e^{i2\pi nx}dx = \int_0^1 f(x)e^{i2\pi nx}dx - \int_0^1 f(1-x)e^{i2\pi nx}dx,$$

and, with the replacement $1 - x = t$,

$$\int_0^1 f(1-x)e^{i2\pi nx}dx = \int_1^0 f(t)e^{i2\pi n(1-t)}(-dt) = \int_0^1 f(t)e^{-i2\pi nt}dt,$$

so that

$$\begin{aligned}\int_0^1 g(x)e^{i2\pi nx}dx &= \int_0^1 f(x)e^{i2\pi nx}dx - \int_0^1 f(x)e^{-i2\pi nx}dx \\ &= 2i \times \mathrm{Im}\left(\int_0^1 f(x)e^{i2\pi nx}dx\right),\end{aligned}$$

from which we conclude that $\mathrm{Im}(\int_0^1 f(x)e^{i2\pi nx}dx) \neq 0$, and this ends the proof of (i).

(ii) follows from (i), since $\mathrm{Im} \int g(x)e^{i2\pi nx}dx = 0$, and $\mathrm{Im} \int f(x)e^{i2\pi nx}dx \neq 0$. □

Acknowledgments I thank Jean-Paul Thouvenot for bringing to my attention the reference [15]. I thank him also for fruitful discussions.

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