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On the Dynamics of WKB Wave Functions Whose Phase are Weak KAM Solutions of H–J Equation

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Abstract In the framework of toroidal Pseudodifferential operators on the flat torus $\mathbb{T}^n := (\mathbb{R}/2\pi\mathbb{Z})^n$ we begin by proving the closure under composition for the class of Weyl operators $\mathrm{Op}^w_\hbar(b)$ with symbols $b \in S^m(\mathbb{T}^n \times \mathbb{R}^n)$. Subsequently, we consider $\mathrm{Op}^w_\hbar(H)$ when $H = \frac{1}{2}|\eta|^2 + V(x)$ where $V \in C^\infty(\mathbb{T}^n)$ and we exhibit the toroidal version of the equation for the Wigner transform of the solution of the Schrödinger equation. Moreover, we prove the convergence (in a weak sense) of the Wigner transform of the solution of the Schrödinger equation to the solution of the Liouville equation on $\mathbb{T}^n \times \mathbb{R}^n$ written in the measure sense. These results are applied to the study of some WKB type wave functions in the Sobolev space $H^1(\mathbb{T}^n;\mathbb{C})$ with phase functions in the class of Lipschitz continuous weak KAM solutions (positive and negative type) of the Hamilton–Jacobi equation $\frac{1}{2}|P + \nabla_x v(P,x)|^2 + V(x) = \bar{H}(P)$ for $P \in \ell\mathbb{Z}^n$ with $\ell > 0$, and to the study of the backward and forward time propagation of the related Wigner measures supported on the graph of $P + \nabla_x v$.

Keywords Toroidal pseudodifferential operators · Wigner measures · Hamilton–Jacobi equation

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1 Introduction

In this paper we study WKB type wave functions on flat torus $\mathbb{T}^n := (\mathbb{R}/2\pi\mathbb{Z})^n$, namely functions of the form

$$\psi_\hbar(x) = a(x)e^{iS(x)/\hbar}, \ x \in \mathbb{T}^n, \ n \geq 1 \tag{1.1}$$

where $a = a_{\hbar,P}$ is a family of functions in $L^2(\mathbb{T}^n; \mathbb{R})$ and $S(x) = P \cdot x + v(x), \ P \in \ell\mathbb{Z}^n, \ell > 0, \hbar^{-1} \in \ell^{-1}\mathbb{N}$, the phase $v(x) = v(P, x)$ is a Lipschitz continuous weak KAM solution (of positive or negative type) of the stationary Hamilton–Jacobi equation

$$H(x, P + \nabla_x v(P, x)) = \bar{H}(P) \tag{1.2}$$

for Hamiltonian $H(x, \eta) := \frac{1}{2}|\eta|^2 + V(x)$, $V \in C^\infty(\mathbb{T}^n)$, see Sect. 2.2.1 for precise definitions.

It is well known that in the case where v is a smooth function (i.e. at least C^2), the wave function $\psi_\hbar$ is, under general conditions on the family $a = a_{\hbar,P}$, a Lagrangian distribution associated to the Lagrangian manifold $\Lambda_P := \{(x, \eta) \in \mathbb{T}^n \times \mathbb{R}^n, \ \eta = P + \nabla_x v(P, x)\}$. Therefore, it has an associated monokinetic Wigner measure taking the form

$$dw(x, \eta) = \delta(\eta - (P + \nabla_x v(P, x)))|a_0(x)|^2 dx. \tag{1.3}$$

Moreover, it remains of the same type under time propagation associated with the Schrödinger equation whose quantum Hamiltonian is the quantization of the function $H(x, \eta)$ (see Sect. 2.1 for details on the toroidal quantization) leading to a Wigner measure

$$dw_t(x, \eta) = \delta(\eta - (P + \nabla_x v(P, x)))|a_0^t(x)|^2 dx \tag{1.4}$$

where the density $|a_0^t(x)|^2$ satisfies a transport equation in such a way that dw_t is the pushforward of dw by the Hamiltonian flow of H.

The goal of this paper is to show what remains of this construction in the case where v is a solution of (1.2) with only a Lipschitz continuity property, a regularity which is far from being used in the framework of standard microlocal Analysis on this type of wave functions.

Note that propagation of monokinetic Wigner measures with low regularity momentum profiles and application to the classical limit of propagation of WKB type wave functions have been recently studied in [5]. The regularity assumption in [5] is much stronger than ours, but at the contrary the construction in [5] works for any profile with a given regularity as we need our phase function to be a solution of the Hamilton–Jacobi equation. Therefore, the two papers are complementary.

The precise definition of our WKB states, especially of the amplitude in (1.1), is given in Sect. 4.2, Definition 4.3 where a family of examples are given in the Remark 4.4 following the definition. We underline that these WKB states are different from the usual Bloch wave functions, as used for example in [32] where for $\hbar = 1$, the wave functions take the form $\psi(x) = e^{2\pi i P \cdot x}\phi(x)$ with $P \in \mathbb{R}^n$ and ϕ is $\mathbb{Z}^n$-periodic. The similarity with our setting is for the $2\pi\mathbb{Z}^n$—periodic term $a(x)e^{iv(x)/\hbar}$ whereas the difference is for our assumption on $P \in \ell\mathbb{Z}^n$ which makes our functions $\psi_\hbar$ periodic. For the more general approach called Bloch decomposition of wave functions we address the reader to [17] and the references therein.

Note moreover that WKB states on the torus with phase functions issued from weak KAM theory have been used in [10,11] where it has been studied L^2—energy quasimode estimates. In [27] a class of WKB states on the torus with regularized phase function have been defined in such a way that the associated Wigner measures are coinciding with the Legendre transform of the so-called Mather measures.

In the present paper we will work with true solutions of Hamilton–Jacobi equation for the phase and will use a kind of regularization for the amplitude, as no canonical function choice is linked to the latter out weak KAM theory.

Our first main result concerns the Wigner measure dw, as defined in Sect. 2.1.3, Definition 2.6, associated with our family of WKB states. It claims, Theorem 4.9, that dw is as expected monokinetic in the sense that it has the form

$$dw(x,\eta) = \delta(\eta - (P + \nabla_x v(P,x)))dm_P(x) \tag{1.5}$$

where the limit in the measure sense $dm_P(x) = \lim_{\hbar\to 0} |a_{\hbar,P}(x)|^2 dx$ exists thanks to Definition 4.3. In fact, we also assume that

$$dm_P \ll d\sigma_P := \pi_\star(dw_P) \tag{1.6}$$

where dw_P is the Legendre transform of a Mather P—minimal measure (see Sect. 2.2.2). This setting implies that any measure $dw(x,\eta)$ as in (1.5) is absolutely continuous to dw_P itself, as shown in Lemma 4.8. We underline that $d\sigma_P$ solves the continuity equation

$$0 = \int_{\mathbb{T}^n} \nabla_x f(x) \cdot (P + \nabla_x v(P,x))\, d\sigma_P(x) \quad \forall f \in C^\infty(\mathbb{T}^n), \tag{1.7}$$

and this can be interpreted as the consequence of an asymptotic free current density condition for wave functions $\psi_\hbar$ of type (1.1), as we show in Proposition 4.11. We recall that the usual construction of WKB wave functions works within the assumption of smoothness for the map $x \longmapsto v(P,x)$. In this case, the determination of an amplitude function $a_P(x)$ is related to the solution of the continuity equation (1.7) written in the strong sense for $\sigma_P(x) = a_P^2(x)$, namely $\mathrm{div}_x[(P + \nabla_x v(P,x))\sigma_P(x)] = 0$.

The assumption (1.6) on dm_P together with the monokinetic form of dw_P with support contained in the graph of a weak KAM solution of the Hamilton–Jacobi equation allow to study very much easily the time propagation of such measures, which remains of monokinetic type. This is in fact our second main result, which deals with the classical limit of the Wigner transform of the evolved WKB state. It is contained within Theorem 5.1 and Proposition 5.3 where the propagation

$$dw_t(x,\eta) = \delta(\eta - (P + \nabla_x v(P,x)))\mathbf{g}(t,P,x)d\sigma_P(x) \tag{1.8}$$

both forward and backward (they are different in our situation) in time is exhibited.

The paper is organized as follows: Sect. 2 is devoted to some preliminaries concerning the Weyl quantization on the torus (Sect. 2.1), as well as the weak KAM theory and Aubry–Mather theory (Sect. 2.2). Section 3 concerns the dynamics of the Wigner transform on the torus and Sect. 4 the classical limit of the Wigner transform, including the Sect. 4.2 where the monokinetic property of the Wigner measures of our WKB state is established. Its time propagation is studied in the final Sect. 5.

2 Preliminaries

2.1 The Weyl Quantization on the Torus

2.1.1 Settings

Let us consider the flat torus $\mathbb{T}^n := (\mathbb{R}/2\pi\mathbb{Z})^n$. The class of symbols $b \in S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, $0 \leq \delta, \rho \leq 1$, consisting of those functions in $C^\infty(\mathbb{T}^n \times \mathbb{R}^n; \mathbb{R})$ which are 2π-periodic in x (that is, in each variable x_j, $1 \leq j \leq n$) and for which for all $\alpha, \beta \in \mathbb{Z}^n_+$ there exists $C_{\alpha\beta} > 0$ such that $\forall (x, \eta) \in \mathbb{T}^n \times \mathbb{R}^n$

$$|\partial_x^\beta \partial_\eta^\alpha b(x, \eta)| \leq C_{\alpha\beta m} \langle \eta \rangle^{m-\rho|\alpha|+\delta|\beta|} \tag{2.1}$$

where $\langle \eta \rangle := (1 + |\eta|^2)^{1/2}$. In particular, the set $S^m_{1,0}(\mathbb{T}^n \times \mathbb{R}^n)$ is denoted by $S^m(\mathbb{T}^n \times \mathbb{R}^n)$.

The toroidal Pseudodifferential Operator associated to $b \in S^m(\mathbb{T}^n \times \mathbb{R}^n)$ reads

$$b(X, D)\psi(x) := (2\pi)^{-n} \sum_{\kappa \in \mathbb{Z}^n} \int_{\mathbb{T}^n} e^{i\langle x-y,\kappa\rangle} b(x, \kappa)\psi(y)dy, \quad \psi \in C^\infty(\mathbb{T}^n; \mathbb{C}), \tag{2.2}$$

see [29]. Here we have used Euclidean symbols, but we address the reader to Remark 2.1 about the link with so-called the toroidal symbols. In particular, notice that it is given a map $b(X, D) : C^\infty(\mathbb{T}^n) \longrightarrow \mathcal{D}'(\mathbb{T}^n)$. We recall that $u \in \mathcal{D}'(\mathbb{T}^n)$ are the linear maps $u : C^\infty(\mathbb{T}^n) \longrightarrow \mathbb{C}$ such that $\exists C > 0$ and $k \in \mathbb{N}$, for which $|u(\phi)| \leq C \sum_{|\alpha| \leq k} \|\partial_x^\alpha \phi\|_\infty \forall \phi \in C^\infty(\mathbb{T}^n)$, see for example Definition 2.1.1 of [19]. Given a symbol $b \in S^m(\mathbb{T}^n \times \mathbb{R}^n)$, the (toroidal) Weyl quantization reads

$$\mathrm{Op}^w_\hbar(b)\psi(x) := (2\pi)^{-n} \sum_{\kappa \in \mathbb{Z}^n} \int_{\mathbb{T}^n} e^{i\langle x-y,\kappa\rangle} b(y, \hbar\kappa/2)\psi(2y - x)dy, \quad \psi \in C^\infty(\mathbb{T}^n). \tag{2.3}$$

In particular, it holds that

$$\mathrm{Op}^w_\hbar(b)\psi(x) = (\sigma(X, D) \circ T_x \psi)(x) \tag{2.4}$$

where $T_x : C^\infty(\mathbb{T}^n) \to C^\infty(\mathbb{T}^n)$ defined as $(T_x\psi)(y) := \psi(2y - x)$ is linear, invertible and L^2-norm preserving, and σ is a suitable toroidal symbol related to b, i.e. $\sigma \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \triangle_\eta^\alpha D_y^{(\alpha)} b(y, \hbar\eta/2)\big|_{y=x}$ (see Theorem 4.2 in [29] or also Theorem 2.1 in [27]).

Starting from quantization in (2.3), we now introduce the Wigner transform $W_\hbar\psi$ by

$$W_\hbar\psi(x,\eta) := (2\pi)^{-n}\int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta\rangle}\psi(x-z)\psi^\star(x+z)dz, \quad \eta\in\frac{\hbar}{2}\mathbb{Z}^n, \tag{2.5}$$

which is well defined also for $\psi\in L^2(\mathbb{T}^n)$. For $b\in S^m(\mathbb{T}^n\times\mathbb{R}^n)$ the Wigner distribution reads

$$\langle\psi,\mathrm{Op}^w_\hbar(b)\psi\rangle = \sum_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n}\int_{\mathbb{T}^n} b(x,\eta)W_\hbar\psi(x,\eta)dx, \quad \psi\in C^\infty(\mathbb{T}^n). \tag{2.6}$$

For $b\in S^0(\mathbb{T}^n\times\mathbb{R}^n)$ and $\psi\in L^2(\mathbb{T}^n)$, the mean value $\langle\psi,\mathrm{Op}^w_\hbar(b)\psi\rangle_{L^2(\mathbb{T}^n)}$ is well defined thanks to the L^2 - boundedness estimate of $\mathrm{Op}^w_\hbar(b)$, see Theorem 2.3.

Remark 2.1 Before recalling the notion of toroidal symbols and toroidal amplitudes, we need first to recall the notion of partial difference operator $\triangle$. Given $f:\mathbb{Z}^n_\kappa\longrightarrow\mathbb{C}$, it is defined the

$$\triangle_{\kappa_j} f(\kappa) := f(\kappa+e_j)-f(\kappa) \tag{2.7}$$

where $e_j\in\mathbb{N}^n$, $(e_j)_j=1$ and $(e_j)_i=1$ if $i\neq j$. The composition provide $\triangle^\alpha_\kappa f(\kappa) := \triangle^{\alpha_1}_{\kappa_1} f(\kappa)\ldots\triangle^{\alpha_n}_{\kappa_n} f(\kappa)$ for any $\alpha\in\mathbb{N}^n_0$. We recall now that toroidal symbols $\widetilde{b}\in S^m_{\rho,\delta}(\mathbb{T}^n\times\mathbb{Z}^n)$, $m\in\mathbb{R}$, $0\le\delta,\rho\le 1$, are those functions which are smooth in x for all $\kappa\in\mathbb{Z}^n$, 2π-periodic in x and for which for all $\alpha,\beta\in\mathbb{Z}^n_+$ there exists $C_{\alpha\beta m}>0$ such that $\forall(x,\kappa)\in\mathbb{T}^n\times\mathbb{Z}^n$

$$|\partial^\beta_x\triangle^\alpha_\kappa\,\widetilde{b}(x,\kappa)|\le C_{\alpha\beta m}\langle\kappa\rangle^{m-\rho|\alpha|+\delta|\beta|} \tag{2.8}$$

where $\langle\kappa\rangle := (1+|\kappa|^2)^{1/2}$. As usually, $S^m(\mathbb{T}^n\times\mathbb{Z}^n)$ stands for $S^m_{1,0}(\mathbb{T}^n\times\mathbb{Z}^n)$. In the same way, it is defined the set of toroidal amplitudes $S^m_{\rho,\delta}(\mathbb{T}^n\times\mathbb{T}^n\times\mathbb{Z}^n)$.

The link between this class of symbols and the Euclidean ones $S^m_{\rho,\delta}(\mathbb{T}^n\times\mathbb{R}^n)$ is shown within Theorem 5.2 in [29]. Namely, for any $\widetilde{b}\in S^m_{\rho,\delta}(\mathbb{T}^n\times\mathbb{Z}^n)$ there exists $b\in S^m_{\rho,\delta}(\mathbb{T}^n\times\mathbb{R}^n)$ such that $\widetilde{b}=b|_{\mathbb{T}^n\times\mathbb{Z}^n}$, and conversely for any b there exists $\widetilde{b}$ such that this restriction holds true. Moreover, the extended symbol is unique modulo a function in $S^{-\infty}(\mathbb{T}^n\times\mathbb{R}^n)$.

Remark 2.2 In [18] it is considered the phase space Fourier representation,

$$b(x,\eta)=F(\widehat{b}) := (2\pi)^{-n}\int_{\mathbb{R}^n}\sum_{q\in\mathbb{Z}^n}\widehat{b}(q,p)e^{i(\langle p,\eta\rangle+\langle q,x\rangle)}dp, \qquad (q,p)\in\mathbb{Z}^n\times\mathbb{R}^n,$$
$$(x,\eta)\in\mathbb{T}^n\times\mathbb{R}^n, \tag{2.9}$$

(in the sense of distributions) and the operator $U_\hbar(q,p)\psi(x) := e^{i(q\cdot x+\hbar p\cdot q/2)}\psi(x+\hbar p)$ which is well defined on $L^2(\mathbb{T}^n)$ for any fixed $(q,p)\in\mathbb{Z}^n\times\mathbb{R}^n$. In this framework, the Weyl quantization of a symbol $b\in S^m(\mathbb{T}^n\times\mathbb{R}^n)$ is given by

$$\mathrm{Op}_\hbar^w(b)\psi(x) := (2\pi)^{-n}\int_{\mathbb{R}^n}\sum_{q\in\mathbb{Z}^n}\widehat{b}(q,p)U_\hbar(q,p)\psi(x)dp. \tag{2.10}$$

Consequently, the corresponding Wigner transform and Wigner distribution are

$$\widehat{W}_\hbar\psi(q,p) := \langle\psi, U_\hbar(q,p)\psi\rangle_{L^2}, \tag{2.11}$$

$$\langle\psi, \mathrm{Op}_\hbar^w(b)\psi\rangle := \int_{\mathbb{R}^n}\sum_{q\in\mathbb{Z}^n}\widehat{b}(q,p)\widehat{W}_\hbar\psi(q,p)dp. \tag{2.12}$$

In fact, as shown by Proposition 2.3 in [27], the Weyl quantizations as in (2.3) and (2.10) coincide.

2.1.2 Composition and Boundedness for Weyl Operators

In the following we recall a result on $L^2(\mathbb{T}^n)$-boundedness for a class of Weyl operators involved in our paper.

Theorem 2.3 (see [18]) *Let* $\mathrm{Op}_\hbar^w(b)$ *as in (2.10) with* $b \in S^0_{0,0}(\mathbb{T}^n\times\mathbb{R}^n)$. *Let* $N = n/2+1$ *when n is even,* $N = (n+1)/2+1$ *when n is odd. Then, for* $\psi \in C^\infty(\mathbb{T}^n)$

$$\|\mathrm{Op}_\hbar^w(b)\psi\|_{L^2(\mathbb{T}^n)} \le \frac{2^{n+1}}{n+2}\,\frac{\pi^{(3n-1)/2}}{\Gamma((n+1)/2)}\sum_{|\alpha|\le 2N}\|\partial_x^\alpha b\|_{L^\infty(\mathbb{T}^n\times\mathbb{R}^n)}\,\|\psi\|_{L^2(\mathbb{T}^n)}. \tag{2.13}$$

By using standard arguments (such as Hahn–Banach Theorem, see for example [28]) the above class of operators can be extended as bounded linear operators on $L^2(\mathbb{T}^n)$.This is the toroidal counterpart of the well known Calderon–Vaillancourt Theorem for PDO on $\mathbb{R}^n$ (see for example [25]).

By applying some results in [29], we now prove the main composition properties of the toroidal Weyl operators (see also [18], for a similar result involving a smaller class of symbols).

Theorem 2.4 *Let* $\ell, m \in \mathbb{R}$, $a \in S^\ell(\mathbb{T}^n\times\mathbb{R}^n)$ *and* $b \in S^m(\mathbb{T}^n\times\mathbb{R}^n)$. *Then,*

$$\mathrm{Op}_\hbar^w(a)\circ\mathrm{Op}_\hbar^w(b) = \mathrm{Op}_\hbar^w(a\sharp b) \tag{2.14}$$

where $a\sharp b = a\cdot b + O(\hbar)$ *in* $S^{\ell+m}(\mathbb{T}^n\times\mathbb{R}^n)$. *Moreover,*

$$[\mathrm{Op}_\hbar^w(a), \mathrm{Op}_\hbar^w(b)] = \mathrm{Op}_\hbar^w(a\sharp b - b\sharp a) \tag{2.15}$$

where the Moyal bracket reads $\{a,b\}_{\mathrm{M}} := a\sharp b - b\sharp a = -i\hbar\{a,b\} + O(\hbar^2)$ *in* $S^{\ell+m-1}(\mathbb{T}^n\times\mathbb{R}^n)$ *and the Poisson bracket* $\{a,b\} := \nabla_\eta a\cdot\nabla_x b - \nabla_x a\cdot\nabla_\eta b$.

Proof To begin, we observe that $T_\omega\psi(y) := \psi(2y-\omega)$ can be written as

$$T_\omega\psi(y) = (2\pi)^{-n}\sum_{\kappa\in\mathbb{Z}^n}\int_{\mathbb{T}^n} e^{i\langle(2y-\omega)-z,\kappa\rangle}\psi(z)dz, \quad \forall\,\psi\in C^\infty(\mathbb{T}^n;\mathbb{C}), \tag{2.16}$$

and hence $\mathrm{Op}^w_\hbar(b)\psi(x) = (\sigma(X,D)\circ T_{\omega=x}\,\psi)(x)$ with $\sigma \sim \sum_{\alpha\geq 0}\frac{1}{\alpha!}\triangle^\alpha_\eta D^{(\alpha)}_y$ $b(y,\hbar\eta/2)\big|_{y=x}$.
By recalling (2.4) together with Theorem 8.4 of [29], it follows

$$\mathrm{Op}^w_\hbar(b)\psi(x) = (2\pi)^{-n}\sum_{\kappa\in\mathbb{Z}^n}\int_{\mathbb{T}^n} e^{i\langle x-z,\kappa\rangle}c(\hbar,x,z,\kappa)\psi(z)dz \tag{2.17}$$

with amplitude $c(\hbar,\cdot)\in C^\infty(\mathbb{T}^n\times\mathbb{T}^n\times\mathbb{R}^n)$ such that $|\partial^\alpha_x\partial^\gamma_z c(\hbar,x,z,\kappa)| \leq C_{\alpha\gamma}\langle\kappa\rangle^{\ell+m}$. Thus, $c(\hbar,\cdot)\in S^{\ell+m}(\mathbb{T}^n\times\mathbb{T}^n\times\mathbb{R}^n)$ and its restriction on the integer frequencies fulfills $c(\hbar,\cdot)\in S^{\ell+m}(\mathbb{T}^n\times\mathbb{T}^n\times\mathbb{Z}^n)$ as recalled in Remark 2.1. In particular, a direct look at the asymptotics gives $c = b(z,\hbar\kappa)+O(\hbar)$ in $S^m(\mathbb{T}^n\times\mathbb{T}^n\times\mathbb{Z}^n)$. Now, apply Theorem 4.2 of [29], so that there exists a unique toroidal symbol $\sigma(\hbar,\cdot)\in S^m(\mathbb{T}^n\times\mathbb{Z}^n)$ such that

$$\mathrm{Op}^w_\hbar(b)\psi(x) = (2\pi)^{-n}\sum_{\kappa\in\mathbb{Z}^n}\int_{\mathbb{T}^n} e^{i\langle x-y,\kappa\rangle}\sigma(\hbar,y,\kappa)\psi(y)dy \tag{2.18}$$

where moreover it turns out that $\sigma(\hbar,y,\kappa) = b(y,\hbar\kappa)+O(\hbar)$ in $S^m(\mathbb{T}^n\times\mathbb{Z}^n)$. By Theorem 4.3 of [29], it follows the existence of $\widehat{a\sharp b}(\hbar,\cdot)\in S^{\ell+m}(\mathbb{T}^n\times\mathbb{Z}^n)$ such that

$$\mathrm{Op}^w_\hbar(a)\circ\mathrm{Op}^w_\hbar(b)\psi(x) = (2\pi)^{-n}\sum_{\kappa\in\mathbb{Z}^n}\int_{\mathbb{T}^n} e^{i\langle x-y,\kappa\rangle}\widehat{a\sharp b}(\hbar,y,\kappa)\psi(y)dy \tag{2.19}$$

and $\widehat{a\sharp b}(\hbar,y,\kappa) = a\cdot b(y,\hbar\kappa)+O(\hbar)$ in $S^{\ell+m}(\mathbb{T}^n\times\mathbb{Z}^n)$. Now apply this operator on $T_x^{-1}\circ T_x\psi$, use again Theorems 8.4 and 4.2 of [29], in order to get

$$\mathrm{Op}^w_\hbar(a)\circ\mathrm{Op}^w_\hbar(b) = \mathrm{Op}^w_\hbar(\widetilde{a\sharp b}) \tag{2.20}$$

and in addition $\widetilde{a\sharp b}(\hbar,y,\kappa) = a\cdot b(y,\kappa)+O(\hbar)$ in $S^{\ell+m}(\mathbb{T}^n\times\mathbb{Z}^n)$. By Theorem 5.2 of [29] we get an Euclidean symbol $a\sharp b\in S^{\ell+m}(\mathbb{T}^n\times\mathbb{R}^n)$ which is an extension of $\widetilde{a\sharp b}$ modulo $S^{-\infty}(\mathbb{T}^n\times\mathbb{R}^n)$, and thus such that

$$\mathrm{Op}^w_\hbar(a)\circ\mathrm{Op}^w_\hbar(b) = \mathrm{Op}^w_\hbar(a\sharp b) \tag{2.21}$$

where $a\sharp b(\hbar,y,\kappa) = a\cdot b(y,\kappa)+O(\hbar)$ but now in $S^{\ell+m}(\mathbb{T}^n\times\mathbb{R}^n)$. By looking at the second order expansion of the symbols $a\sharp b$ and $b\sharp a$, it follows $a\sharp b - b\sharp a = -i\hbar\{a,b\}+O(\hbar^2)$ in $S^{\ell+m-1}(\mathbb{T}^n\times\mathbb{R}^n)$. □

2.1.3 Wigner Measures

To begin, let us recall that in the framework of the usual Weyl quantization on $\mathbb{R}^n$ it can be considered the following space of test functions (see for example [3,22])

$$\mathcal{A} := \{\varphi\in C_0(\mathbb{R}^n_x\times\mathbb{R}^n_\xi)\mid \|\varphi\|_{\mathcal{A}} := \int_{\mathbb{R}^n}\sup_{x\in\mathbb{R}^n}|\mathcal{F}_\xi\varphi(x,z)|\,dz < +\infty\} \tag{2.22}$$

where $C_0(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ denotes the set of continuous functions tending to zero at infinity, and $\mathcal{F}_\xi$ is the usual Fourier transform in the frequency variables, i.e. $\mathcal{F}_\xi\varphi(x,z) := \int_{\mathbb{R}^n} e^{-i\xi\cdot z}\varphi(x,\xi)d\xi$. In particular, $\mathcal{A}$ is a Banach space and it is a dense subset of $C_0(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$. Hence, its dual space $\mathcal{A}'$ contains $C_0'(\mathbb{R}^n_x \times \mathbb{R}^n_\xi) = \mathcal{M}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ the space of not necessarily nonnegative Radon measures on $\mathbb{R}^{2n}$ of finite mass. As shown in Proposition III.1 of [22], it holds the inequality

$$\left| \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} W_\hbar\psi_\hbar(x,\xi)\varphi(x,\xi)dxd\xi \right| \leq (2\pi)^{-n}\|\varphi\|_{\mathcal{A}} \cdot \|\psi_\hbar\|^2_{L^2}, \tag{2.23}$$

and hence for any family of wave functions such that $\|\psi_\hbar\|_{L^2(\mathbb{R}^n)} \leq C$ there exists a sequence $\hbar_j \longrightarrow 0^+$ as $j \longrightarrow +\infty$ such that $W_{\hbar_j}\psi_{\hbar_j}$ is converging in $\mathcal{A}'$ to some $W \in \mathcal{A}'$ (thanks to Banach–Alaoglu Theorem). Moreover, through the use of Husimi transform, it can be proved that in fact any such limit $W \in \mathcal{A}'$ fulfills also $W \in \mathcal{M}^+(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, i.e. positive Radon measure of finite mass.

We underline that there is an estimate analogous to (2.23) for our toroidal framework which takes the form

$$\left| \sum_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n} \int_{\mathbb{T}^n} W_\hbar\psi_\hbar(x,\eta)g(x,\eta)dx \right| \leq (2\pi)^{-n} \sup_{(x,\eta)\in\mathbb{T}^n\times\mathbb{R}^n} |g(x,\eta)| \cdot \|\psi_\hbar\|^2_{L^2} \tag{2.24}$$

for all continuous bounded functions $g : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$. Indeed, we observe that for states $\psi_\hbar \in L^2(\mathbb{T}^n)$, by writing the Fourier series $\psi_\hbar(x) = \sum_{\alpha\in\mathbb{Z}^n} \widehat{\psi}_{\hbar,\alpha}\, e^{i\langle x,\alpha\rangle}$ we have

(i) $\sum_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n} W_\hbar\psi_\hbar(x,\eta) = |\psi_\hbar(x)|^2$,

(ii) $(2\pi)^{-n}\int_{\mathbb{T}^n} W_\hbar\psi_\hbar(x,\eta)dx = \begin{cases} |\widehat{\psi}_{\hbar,\alpha}|^2 & \text{when } \eta = \hbar\alpha, \quad \alpha\in\mathbb{Z}^n, \\ 0 & \text{otherwise.} \end{cases}$

Hence, by property (ii) it follows the estimate (2.24).
In view of the above observations, we can now introduce the following

Definition 2.5 (*Test functions*) Let $C_0(\mathbb{T}^n_x \times \mathbb{R}^n_\eta)$ be the set of real valued continuous functions on $\mathbb{T}^n_x \times \mathbb{R}^n_\eta$ tending to zero at infinity in η-variables. We consider the subset of those $\phi \in C_0(\mathbb{T}^n_x \times \mathbb{R}^n_\eta)$ admitting the phase space Fourier representation $\phi = F(\widehat{\phi})$ as in (2.9) for some compactly supported $\widehat{\phi} : \mathbb{Z}^n \times \mathbb{R}^n \longrightarrow \mathbb{C}$. We define the set

$$A := \overline{\left\{\phi \in C_0(\mathbb{T}^n_x \times \mathbb{R}^n_\eta) \mid \operatorname{supp}(\widehat{\phi}) \text{ is compact}\right\}}^{L^\infty}. \tag{2.25}$$

Notice that A is a closed linear subset of $L^\infty(\mathbb{T}^n_x \times \mathbb{R}^n_\eta)$ hence it becomes a Banach space when equipped with the L^∞ - norm. We also underline that for any fixed $\phi \in C_0(\mathbb{T}^n_x \times \mathbb{R}^n_\eta)$ such that $\operatorname{supp}(\widehat{\phi})$ is compact, ϕ is also a C^∞ - function rapidly decreasing in η-variables. Hence, we can directly deal with the set of C^∞—functions vanishing at infinity in the η-variables $C_0^\infty(\mathbb{T}^n_x \times \mathbb{R}^n_\eta)$. Thus, we can write

$$A = \overline{\left\{ \phi \in C_0^\infty(\mathbb{T}^n_x \times \mathbb{R}^n_\eta) \mid \text{supp}(\widehat{\phi}) \text{ is compact} \right\}}^{L^\infty}. \tag{2.26}$$

Moreover, it can be easily seen that $A \subset C_b(\mathbb{T}^n \times \mathbb{R}^n)$.

We are now in the position to provide the

Definition 2.6 (*Wigner measures*) Let us fix $\{\psi_\hbar\}_{0<\hbar\leq 1} \in L^2(\mathbb{T}^n)$ with $\|\psi_\hbar\|_{L^2} \leq C \forall 0 < \hbar \leq 1$. We say that $dw \in \mathcal{M}(\mathbb{R}^n_x \times \mathbb{R}^n_\eta)$ is the Wigner measure of the sequence $\{\psi_\hbar\}_{0<\hbar\leq 1}$ if $\forall \phi \in A$

$$\sum_{\eta \in \frac{\hbar}{2}\mathbb{Z}^n} \int_{\mathbb{T}^n} \phi(x,\eta) W_\hbar \psi_\hbar(x,\eta) dx \longrightarrow \int_{\mathbb{T}^n \times \mathbb{R}^n} \phi(x,\eta) dw(x,\eta) \tag{2.27}$$

for some sequence $\hbar = \hbar_j \longrightarrow 0^+$ as $j \longrightarrow +\infty$.

Remark 2.7 The Wigner transform of $\psi_\hbar \in C^\infty(\mathbb{T}^n)$

$$W_\hbar \psi_\hbar(x,\eta) := (2\pi)^{-n} \int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta\rangle} \psi_\hbar(x-z)\psi_\hbar^\star(x+z)dz, \quad \eta \in \frac{\hbar}{2}\mathbb{Z}^n, \tag{2.28}$$

works on test functions as

$$\sum_{\eta \in \frac{\hbar}{2}\mathbb{Z}^n} \int_{\mathbb{T}^n} \phi(x,\eta) W_\hbar \psi_\hbar(x,\eta) dx = \sum_{\kappa \in \mathbb{Z}^n} \int_{\mathbb{T}^n} \phi\Big(x, \frac{2}{\hbar}\kappa\Big) W_\hbar \psi_\hbar\Big(x, \frac{2}{\hbar}\kappa\Big) dx, \tag{2.29}$$

$$W_\hbar \psi_\hbar\Big(x, \frac{2}{\hbar}\kappa\Big) = (2\pi)^{-n} \int_{\mathbb{T}^n} e^{i\langle z,\kappa\rangle} \psi_\hbar(x-z)\psi_\hbar^\star(x+z)dz, \quad \kappa \in \mathbb{Z}^n. \tag{2.30}$$

Thus, we notice the $2\pi\mathbb{Z}^n$-periodicity properties

$$W_\hbar \psi_\hbar\Big(x, \frac{2}{\hbar}(\kappa + 2\pi\alpha)\Big) = W_\hbar \psi_\hbar\Big(x, \frac{2}{\hbar}\kappa\Big) \quad \forall \alpha \in \mathbb{Z}^n, \tag{2.31}$$

$$W_\hbar \psi_\hbar\Big(x + 2\pi\alpha, \frac{2}{\hbar}\kappa\Big) = W_\hbar \psi_\hbar\Big(x, \frac{2}{\hbar}\kappa\Big) \quad \forall \alpha \in \mathbb{Z}^n. \tag{2.32}$$

From (2.28) we also easily obtain the estimate

$$\sup_{\eta \in \frac{\hbar}{2}\mathbb{Z}^n} \sup_{x \in \mathbb{T}^n} |W_\hbar \psi_\hbar(x,\eta)| \leq (2\pi)^{-n} \|\psi_\hbar\|^2_{L^2}. \tag{2.33}$$

Notice that if $\eta \notin \frac{\hbar}{2}\mathbb{Z}^n$ then (2.28) is not defined, since we are computing the integral over the torus and thus we need the $2\pi\mathbb{Z}^n$ periodicity with respect to x-variables of the function within the integral. For this reason, we cannot regard $W_\hbar\psi_\hbar(x,\eta)$ as a wellposed function belonging to $L^\infty(\mathbb{T}^n_x \times \mathbb{R}^n_\eta)$ even if we exhibited the estimate (2.33). This is one of the main differences with the Weyl quantization on $\mathbb{R}^n$ where the Wigner transform $W_\hbar\psi_\hbar(x,\xi)$, when $\psi_\hbar \in L^2(\mathbb{R}^n)$, is a well defined function in $L^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ for any $\hbar > 0$.

In the toroidal framework of this paper, under the general assumption $\|\psi_\hbar\|_{L^2} \le C$ with $C > 0$ independent of $\hbar$, we obtain semiclassical limits in A' (see Lemma 2.8) and for suitably defined wave functions (as for example the WKB ones shown in Sect. 4.2) we recover semiclassical limits by probability measures on $\mathbb{T}^n \times \mathbb{R}^n$.

Lemma 2.8 *Let $\{\psi_\hbar(t)\}_{0<\hbar\le 1}$ a sequence in $C([-T,T];L^2(\mathbb{T}^n))$ such that $\|\psi_\hbar(t)\|_{L^2} \le C_T$ for all $t \in [-T,T]$ and $0<\hbar\le 1$. Then, there is a sequence $\hbar_j \longrightarrow 0^+$ as $j \longrightarrow +\infty$ such that $W_{\hbar_j}\psi_{\hbar_j} \rightharpoonup W$ in $L^\infty([-T,+T];A')$ with A as in Def 2.5.*

Proof Since we are assuming $\psi_\hbar \in C([-T,T];L^2(\mathbb{T}^n))$ with $\|\psi_\hbar(t)\|_{L^2} \le C_T$ then the estimate (2.24) implies that for $0<\hbar\le 1$, the family $W_\hbar\psi_\hbar$ is bounded in $L^\infty([-T,+T];A')$. However, $L^\infty([-T,+T];A')$ is the dual of the separable space $L^1([-T,+T];A)$ and hence the application of the Banach–Alaoglu Theorem provides the existence of a converging sequence $W_{\hbar_j}\psi_{\hbar_j} \rightharpoonup W$ in $L^\infty([-T,+T];A')$. □

We devote now our attention on the following (locally finite) Borel complex measure on $\mathbb{T}^n\times\mathbb{R}^n$. Let $\mathcal{X}_\Omega$ be the characteristic function of a Borel set $\Omega \subseteq \mathbb{T}^n\times\mathbb{R}^n$, we define

$$\mathbb{P}_\hbar(\Omega) := \sum_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n} \int_{\mathbb{T}^n} \mathcal{X}_\Omega(x,\eta) W_\hbar\psi_\hbar(x,\eta)dx. \tag{2.34}$$

which is a (complex valued) countably additive set function on the Borel sigma algebra of $\mathbb{T}^n\times\mathbb{R}^n$. In particular, we notice that if $\|\psi_\hbar\|_{L^2} = 1$ then $|\mathbb{P}_\hbar(\Omega)| \le 1$ for all $\Omega \subseteq \mathbb{T}^n\times\mathbb{R}^n$ and $|\mathbb{P}_\hbar(\mathbb{T}^n\times\mathbb{R}^n)| = 1$. As usual, we say that $\mathbb{P}_\hbar$ is weak (i.e. narrow) convergent to a Borel complex measure $\mathbb{P}$ if $\forall f \in C_b(\mathbb{T}^n\times\mathbb{R}^n)$ it holds

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} f(x,\eta)d\mathbb{P}_\hbar(x,\eta) \longrightarrow \int_{\mathbb{T}^n\times\mathbb{R}^n} f(x,\eta)d\mathbb{P}(x,\eta) \tag{2.35}$$

as $\hbar \longrightarrow 0^+$. In fact, since $f \in C_b(\mathbb{T}^n\times\mathbb{R}^n)$, it holds

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} f(x,\eta)d\mathbb{P}_\hbar(x,\eta) = \sum_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n} \int_{\mathbb{T}^n} f(x,\eta) W_\hbar\psi_\hbar(x,\eta)dx. \tag{2.36}$$

Definition 2.9 The family of (complex Borel) measures $\{\mathbb{P}_\hbar\}_{0<\hbar\le 1}$ on the probability space $\mathbb{T}^n\times\mathbb{R}^n$ (equipped with the Borel sigma algebra) is called *tight* if

$$\lim_{R\to+\infty} \sup_{0<\hbar\le 1} \int_{\mathbb{T}^n\times\{\mathbb{R}^n\setminus B_R\}} d\mathbb{P}_\hbar(x,\eta) = 0. \tag{2.37}$$

Thanks to a well-known Prokhorov's theorem, the set of measures $\{\mathbb{P}_\hbar\}_{0<\hbar\le 1}$ is relatively compact with respect to the weak topology if and only if is tight. Notice that the condition (2.37) reads equivalently as $\lim_{R\to+\infty}\sup_{0<\hbar\le 1}\mathbb{P}_\hbar(\mathbb{T}^n\times\{\mathbb{R}^n\setminus B_R\}) = 0$.

Remark 2.10 When $\mathbb{P}_\hbar = \mathbb{P}_\hbar^\pm$ is associated to the class of WKB wave functions $\varphi_\hbar^\pm$ described in Sect. 4.2, we will directly prove the weak convergence (with test functions in A) to some meaningful probability measures of monokinetic type (see Theorem 4.9). On the other hand, within Lemma 4.6 we will also prove that such measures $\mathbb{P}_\hbar^\pm$ fulfill the tightness condition (2.37), and in this way we can apply the next result on time propagation of tightness. This ensures the existence of the Wigner probability measure associated to the solution of the Schrödinger equation, and its coincidence with the solution of the underlying classical continuity equation, see Theorem 5.1 and Proposition 5.3.

Proposition 2.11 (Propagation of tightness) *Let* $H = \frac{1}{2}|\eta|^2 + V(x)$ *with* $V \in C^\infty(\mathbb{T}^n)$, $\psi_\hbar \in L^2(\mathbb{T}^n)$ *be such that* $\|\psi_\hbar\|_{L^2} \leq C$ *for all* $0 < \hbar \leq 1$. *Assume that* $\mathbb{P}_\hbar$ *as in (2.34) is tight. Define* $\psi_\hbar(t) := e^{-\frac{i}{\hbar}\mathrm{Op}_\hbar(H)t}\psi_\hbar$. *Then,* $\mathbb{P}_\hbar(t)$ *is tight for any* $t \in \mathbb{R}$.

Proof Let $Y \in C^\infty(\mathbb{R}^n_\eta; [0,1])$ be such that $Y(\eta) = 1$ on $|\eta| > 1$ and $Y(\eta) = 0$ on $|\eta| < 1/2$; for $R > 0$ define $Y_R(\eta) := Y(\eta/R)$. Then, $|\nabla_\eta Y| \leq C/R$ and $|\nabla^2_\eta Y| \leq C/R^2$ for some $C > 0$. In fact, we can regard $Y \in C_b^\infty(\mathbb{T}^n_x \times \mathbb{R}^n_\eta; [0,1])$. We now use the equation

$$\frac{d}{ds}\langle \psi_\hbar(s), \mathrm{Op}_\hbar(Y_R)\psi_\hbar(s)\rangle_{L^2} = \frac{i}{\hbar}\langle \psi_\hbar(s), [\mathrm{Op}_\hbar(Y_R), \mathrm{Op}_\hbar(H)]\psi_\hbar(s)\rangle_{L^2}. \tag{2.38}$$

Recalling Theorem 2.4, the commutator reads $[\mathrm{Op}_\hbar(Y_R), \mathrm{Op}_\hbar(H)] = \mathrm{Op}_\hbar(\{Y_R, H\}_M)$ where the Moyal bracket has the asymptotics $\{Y_R, H\}_M = -i\hbar\{Y_R, H\} + D_\hbar$ in $S^2(\mathbb{T}^n \times \mathbb{R}^n)$ and furthermore the remainder $D_\hbar \simeq O(\hbar^2)$ involves the second order derivatives of Y_R and H. But $|\partial_x^\alpha \partial_\eta^\beta H(z)| \leq c_1$ and $|\partial_x^\alpha \partial_\eta^\beta Y_R(z)| \leq c_2/R^2$ for $|\alpha + \beta| = 2$; hence $|D_\hbar| \simeq R^{-2}$ as $R \longrightarrow +\infty$ (uniformly on $\hbar$). Moreover $\{Y_R, H\}(z) = \partial_x Y_R \partial_\eta H - \partial_\eta Y_R \partial_x H = -\partial_\eta Y_R \partial_x H$ hence $|\{Y_R, H\}(z)| \leq c_3/R$. By recalling the L^2—boundedness of the Weyl operators with symbols in $S^0_{0,0}(\mathbb{T}^n \times \mathbb{R}^n)$ as shown in Theorem 2.3 and using the assumption $\|\psi_\hbar\|_{L^2} \leq C$, we deduce

$$\left|\frac{d}{ds}\langle \psi_\hbar(s), \mathrm{Op}_\hbar(Y_R)\psi_\hbar(s)\rangle_{L^2}\right| \leq K \cdot R^{-1} \tag{2.39}$$

for some $K > 0$ independent on $\hbar$ and t. Thus

$$\begin{aligned}\langle \psi_\hbar(t), \mathrm{Op}_\hbar(Y_R)\psi_\hbar(t)\rangle_{L^2} &= \langle \psi_\hbar(0), \mathrm{Op}_\hbar(Y_R)\psi_\hbar(0)\rangle_{L^2} \\ &\quad + \int_0^t \frac{d}{ds}\langle \psi_\hbar(s), \mathrm{Op}_\hbar(Y_R)\psi_\hbar(s)\rangle_{L^2} ds \end{aligned} \tag{2.40}$$

and

$$\begin{aligned}|\langle \psi_\hbar(t), \mathrm{Op}_\hbar(Y_R)\psi_\hbar(t)\rangle_{L^2}| &\leq |\langle \psi_\hbar(0), \mathrm{Op}_\hbar(Y_R)\psi_\hbar(0)\rangle_{L^2}| \\ &\quad + \left|\int_0^t \frac{d}{ds}\langle \psi_\hbar(s), \mathrm{Op}_\hbar(Y_R)\psi_\hbar(s)\rangle_{L^2} ds\right| \\ &\leq |\langle \psi_\hbar(0), \mathrm{Op}_\hbar(Y_R)\psi_\hbar(0)\rangle_{L^2}| + t\, K \cdot R^{-1} \end{aligned} \tag{2.41}$$

Notice that, from the property (ii) of $W_\hbar\psi_\hbar$, it follows

$$\langle\psi_\hbar, \mathrm{Op}_\hbar(Y_R)\psi_\hbar\rangle_{L^2} = \sum_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n}\int_{\mathbb{T}^n} W_\hbar\psi_\hbar(x,\eta)Y_R(\eta)dx \tag{2.42}$$

$$= \sum_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n} Y_R(\eta)\int_{\mathbb{T}^n} W_\hbar\psi_\hbar(x,\eta)dx = \sum_{\alpha\in\mathbb{Z}^n} Y_R(\hbar\alpha)|\widehat{\psi}_{\hbar,\alpha}|^2, \tag{2.43}$$

thus any term of the series is non negative. The same holds true for

$$\mathbb{P}_\hbar(\mathbb{T}^n\times U) = \sum_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n}\int_{\mathbb{T}^n} W_\hbar\psi_\hbar(x,\eta)\mathcal{X}_U(\eta)dx \tag{2.44}$$

$$= \sum_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n}\mathcal{X}_U(\eta)\int_{\mathbb{T}^n} W_\hbar\psi_\hbar(x,\eta)dx = \sum_{\alpha\in\mathbb{Z}^n}\mathcal{X}_U(\hbar\alpha)|\widehat{\psi}_{\hbar,\alpha}|^2, \tag{2.45}$$

where U is any Borel set in $\mathbb{R}^n$.

By defining $M_R := \mathbb{T}^n\times\{\mathbb{R}^n\backslash B_R\}$, and recalling that $Y_R(\eta) = 0$ for $|\eta| < R/2$ whereas $Y_R(\eta) = 1$ for $|\eta| > R$, we can write

$$\mathbb{P}_\hbar(t)(M_R) \leq \langle\psi_\hbar(t), \mathrm{Op}_\hbar(Y_R)\psi_\hbar(t)\rangle_{L^2} \tag{2.46}$$

$$\leq \langle\psi_\hbar(0), \mathrm{Op}_\hbar(Y_R)\psi_\hbar(0)\rangle_{L^2} + t\ K\cdot R^{-1} \tag{2.47}$$

$$\leq \mathbb{P}_\hbar(M_{R/2}) + t\ K\cdot R^{-1} \tag{2.48}$$

and hence (recalling the tightness assumption on $\mathbb{P}_\hbar$)

$$\lim_{R\to+\infty}\sup_{0<\hbar\leq 1}\mathbb{P}_\hbar(t)(M_R) = 0. \tag{2.49}$$

□

2.2 A Quick Review of Weak KAM Theory and Aubry–Mather theory

2.2.1 Weak Solutions of Hamilton–Jacobi Equation

As it is well known, the KAM theory investigates the persistence, under small perturbations, of some invariant tori of unperturbed integrable Hamiltonian systems. In the case where the unperturbed Hamiltonian depend only on the fiber variable of $T^*\mathbb{T}^n$, these tori are, for a perturbation small enough, the graphs of the gradients of functions that reduce in the unperturbed case to $x\mapsto P\cdot x,\ P\in\mathbb{R}^n$. It is therefore natural to look at unperturbed tori as gradients of functions of the form $x\mapsto P\cdot x + v(P,x)$. In the case where C^2 such functions exist the system is integrable and the weak KAM solutions fulfil this picture in the case of (much) less regularity.

More precisely the *weak KAM theory* deals with a class of Lipschitz continuous solutions of the Hamilton–Jacobi equation

$$H(x, P + \nabla_x v(P, x)) = \bar{H}(P), \quad P \in \mathbb{R}^n, \tag{2.50}$$

in the general assumption of Tonelli Hamiltonians $H \in C^\infty(\mathbb{T}^n \times \mathbb{R}^n; \mathbb{R})$, that is to say, for functions H such that $\eta \mapsto H(x, \eta)$ is strictly convex and uniformly superlinear in the fibers of the canonical projection $\pi : \mathbb{T}^n \times \mathbb{R}^n \longrightarrow \mathbb{T}^n$. The function $\bar{H}(P)$ is called the *effective Hamiltonian* and, as shown in [7] (see also [13]), it can be expressed by the inf-sup formula

$$\bar{H}(P) = \inf_{v \in C^\infty(\mathbb{T}^n;\mathbb{R})} \sup_{x \in \mathbb{T}^n} H(x, P + \nabla_x v(x)) \tag{2.51}$$

which is a convex function of $P \in \mathbb{R}^n$ (hence continuous). The Lax-Oleinik semigroup of negative and positive type is defined as

$$T_t^{\mp} u(x) := \inf_{\gamma} \left\{ u(\gamma(0)) \pm \int_0^t L(\gamma(s), \dot{\gamma}(s)) - P \cdot \dot{\gamma}(s)\, ds \right\},$$

where the infimum is taken over all absolutely continuous curves $\gamma : [0, t] \longrightarrow \mathbb{T}^n$ such that $\gamma(t) = x$. A function $v_- \in C^{0,1}(\mathbb{T}^n; \mathbb{R})$ is said to be a *weak KAM solution of negative type* for (2.50) if $\forall t \geq 0$

$$T_t^- v_- = v_- - t\, \bar{H}(P), \tag{2.52}$$

whereas it is said to be a *weak KAM solution of positive type* if $\forall t \geq 0$

$$T_t^+ v_+ = v_+ + t\, \bar{H}(P), \tag{2.53}$$

see Definition 4.7.6 in [14]. For any weak KAM solution it holds

$$\overline{\mathrm{Graph}(P + \nabla_x v_\pm(P, \cdot))} \subset \{(x, \eta) \in \mathbb{T}^n \times \mathbb{R}^n \mid H(x, \eta) = \bar{H}(P)\}. \tag{2.54}$$

Furthermore, the graphs are invariant under the backward (resp. forward) Hamiltonian flow, namely

$$\varphi_H^t\Big(\mathrm{Graph}(P + \nabla_x v_-(P, \cdot))\Big) \subseteq \mathrm{Graph}(P + \nabla_x v_-(P, \cdot)) \quad \forall t \leq 0 \tag{2.55}$$

and

$$\varphi_H^t\Big(\mathrm{Graph}(P + \nabla_x v_+(P, \cdot))\Big) \subseteq \mathrm{Graph}(P + \nabla_x v_+(P, \cdot)) \quad \forall t \geq 0 \tag{2.56}$$

see Theorems 4.9.2 and 4.9.3 in [14]. Moreover, it is proved that the maps $x \longmapsto (x, P + \nabla_x v_\pm(P, x))$ are continuous on $\mathrm{dom}(\nabla_x v_\pm) := \{x \in \mathbb{T}^n \mid \exists\, \nabla_x v_\pm(x)\}$. As showed within Theorem 7.6.2 of [14], all the Lipschitz continuous weak KAM solutions of negative type coincide with the so-called *viscosity solutions* in the sense of [8,20,21].

2.2.2 Mather Measures

The Aubry–Mather theory proves the existence of invariant and Action-minimizing measures as well as invariant and Action-minimizing sets in the phase space. Here we recall only those results which we are going to use in what follows, and for an exhaustive treatment we refer the reader to [23,26,30].

Recall that a compactly supported Borel probability measure $d\mu$ on the tangent bundle $T(\mathbb{T}^n) = \mathbb{T}^n \times \mathbb{R}^n$ is called *invariant* with respect to the Lagrangian flow $\phi^t : \mathbb{T}^n \times \mathbb{R}^n \longrightarrow \mathbb{T}^n \times \mathbb{R}^n$ related to a Lagrangian function $L(x, \xi)$ which is Legendre-related to a Tonelli Hamiltonian $H(x, \eta)$, if

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} f(\phi^t(x, \xi)) d\mu(x, \xi) = \int_{\mathbb{T}^n \times \mathbb{R}^n} f(x, \xi) d\mu(x, \xi)$$

for all $t \in \mathbb{R}$ and all $f \in C_0^\infty(\mathbb{T}^n \times \mathbb{R}^n; \mathbb{R})$. A Borel probability measure $d\mu$ is said to be *closed* if for every $g \in C^\infty(\mathbb{T}^n; \mathbb{R})$ one has

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \nabla_x g(x) \cdot \xi \; d\mu(x, \xi) = 0.$$

One says that an invariant compactly supported Borel probability measure $d\mu_P$ is a Mather P-minimal measure if for all $P \in \mathbb{R}^n$

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \big(L(x, \xi) - P \cdot \xi \big) \, d\mu_P(x, \xi) = \inf_{d\mu} \int_{\mathbb{T}^n \times \mathbb{R}^n} \big(L(x, \xi) - P \cdot \xi \big) \, d\mu(x, \xi),$$

where the infimum is taken over all invariant compactly supported Borel probability measures $d\mu$. Moreover, the minimizing value of the Action is related to the effective Hamiltonian as

$$-\bar{H}(P) = \int_{\mathbb{T}^n \times \mathbb{R}^n} \big(L(x, \xi) - P \cdot \xi \big) \, d\mu_P(x, \xi).$$

It has been also proved that the Mather measures of a Tonelli-Lagrangian are those which minimize the action in the class of all (compactly supported) closed measures (see for example [6]). As for the Mather set, it involves the supports of all Mather's measures, and is defined to be

$$\widetilde{\mathcal{M}}_P := \overline{\bigcup_{d\mu_P} \text{supp } d\mu_P}. \tag{2.57}$$

We recall that Mather proved in [26] that the set $\widetilde{\mathcal{M}}_P$ is not empty, compact and Lipschitz graphs above $\mathbb{T}^n$, namely the restriction of $\pi : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n$ to $\widetilde{\mathcal{M}}_P$ is an injective map and $\pi^{-1} : \pi(\widetilde{\mathcal{M}}_P) \to \widetilde{\mathcal{M}}_P$ is Lipschitz. For any fixed Mather measure $d\mu_P$, we denote by

$$dw_P := \mathcal{L}_\star(d\mu_P), \qquad d\sigma_P := \pi_\star(dw_P) = \pi_\star(d\mu_P), \tag{2.58}$$

the push forward by the Legendre transform $\mathcal{L}(x,\xi) = (x, \nabla_\xi L(x,\xi))$ and by the canonical projection $\pi(x,\eta) = x$ on $\mathbb{T}^n$.

2.2.3 Aubry Set

About the definition of the Aubry set $\widetilde{\mathcal{A}}_P$ (in the tangent bundle of $\mathbb{T}^n$) involving regular P-minimizers we refer to [14]; we recall here that its Legendre transform can be given by

$$\mathcal{A}_P^* = \bigcap_{v \in S_P^{\mp}} \{(x, P + \nabla_x v(P,x)) \mid x \in \mathbb{T}^n \text{ s.t. } \exists\, \nabla_x v(P,x)\} \tag{2.59}$$

where the intersection is taken over all Lipschitz continuous weak KAM solutions $S_P^{\mp}$ of negative (resp. positive) type of the Hamilton–Jacobi equation (2.50). This set is invariant under the Hamiltonian flow and

$$\mathcal{M}_P^\star := \mathcal{L}(\widetilde{\mathcal{M}}_P) \subseteq \mathcal{A}_P^*. \tag{2.60}$$

The set $\mathcal{A}_P^\star$ is compact, the restriction of $\pi : \mathbb{T}^n \times \mathbb{R}^n \longrightarrow \mathbb{T}^n$ to $\mathcal{A}_P^\star$ is an injective map and moreover $\pi^{-1} : \pi(\mathcal{A}_P^\star) \longrightarrow \mathcal{A}_P^\star$ is a Lipschitz map (see [14,30]).

3 The Dynamics of the Wigner Transform on the Torus

3.1 The Schrödinger Equation on the Torus

Let us consider the classical Hamiltonian $H = \frac{1}{2}|\eta|^2 + V(x)$, with $V \in C^\infty(\mathbb{T}^n;\mathbb{R})$. Thus we have $H \in S^2(\mathbb{T}^n \times \mathbb{R}^n)$, namely the symbol class described in (2.1) with $m = 2$. We now consider the Schrödinger equation:

$$\begin{aligned} i\hbar\partial_t \psi_\hbar(t,x) &= \mathrm{Op}_\hbar^w(H)\psi_\hbar(t,x) \\ \psi_\hbar(0,x) &= \varphi_\hbar(x) \end{aligned} \tag{3.1}$$

where $\mathrm{Op}_\hbar^w(H)$ is the Weyl quantization of H as in (2.3). As for the initial datum, we can require $\varphi_\hbar \in W^{2,2}(\mathbb{T}^n;\mathbb{C})$ and $\|\varphi_\hbar\|_{L^2} \leq C \forall 0 < \hbar \leq 1$. The one parameter group of unitary operators $e^{-\frac{i}{\hbar}\mathrm{Op}_\hbar^w(H)t}$ can be defined on the whole $L^2(\mathbb{T}^n;\mathbb{C})$. In fact, this is because the Schrödinger operator $\hat{H}_\hbar := -\frac{1}{2}\hbar^2\Delta_x + V(x)$ is coinciding with $\mathrm{Op}_\hbar^w(H)$. This is the content of the Lemma 6.1 shown in the Appendix.

3.2 The Equation for the Wigner Transform

In this section we provide a result on the equation for the Wigner transform of the solution of the Schrödinger equation written on the torus. The well known arguments within the framework of the Weyl quantization on $\mathbb{R}^n$ (see for example [3,4,17,22]) must be adapted for the Weyl quantization on $\mathbb{T}^n$.

The first result reads as follows

Proposition 3.1 *Let $\psi_\hbar$ be the solution of (3.1), $t > 0$ and $f \in C_c^\infty((0,t) \times \mathbb{T}^n \times \mathbb{R}^n; \mathbb{R})$ such that $\forall s \in (0,t)$ it holds $f(s,\cdot) \in A$ as in Definition 2.5. Then,*

$$\int_0^t \sum_{\eta \in \frac{\hbar}{2}\mathbb{Z}^n} \int_{\mathbb{T}^n} \Big[\Big(\partial_s f + \eta \cdot \nabla_x f\Big)(s,x,\eta) W_\hbar \psi_\hbar(s,x,\eta) + f(s,x,\eta)\mathcal{E}_\hbar \psi_\hbar(s,x,\eta)\Big] dx ds = 0 \tag{3.2}$$

where

$$\mathcal{E}_\hbar \psi_\hbar(s,x,\eta) := \frac{i}{(2\pi)^n \hbar} \int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta\rangle} \{V(x+z) - V(x-z)\} \times \psi_\hbar(s,x-z)\overline{\psi}_\hbar(s,x+z) dz. \tag{3.3}$$

Proof We interpret all the subsequent partial derivatives in the distributional sense of A'. To begin,

$$\partial_t W_\hbar \psi = (2\pi)^{-n} \int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta\rangle} \partial_t \psi_\hbar(t,x-z)\overline{\psi}_\hbar(t,x+z)dz + (2\pi)^{-n} \int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta\rangle} \psi_\hbar(t,x-z)\partial_t \overline{\psi}_\hbar(t,x+z)dz. \tag{3.4}$$

Since $\psi_\hbar$ solves the Schrödinger equation, it follows

$$\partial_t \psi_\hbar(t,x-z)\overline{\psi}_\hbar(t,x+z) + \psi_\hbar(t,x-z)\partial_t \overline{\psi}_\hbar(t,x+z) \tag{3.5}$$

$$= \frac{i\hbar}{2}[(\Delta_x \psi_\hbar(t,x-z))\overline{\psi}_\hbar(t,x+z) - \psi_\hbar(t,x-z)\Delta_x \overline{\psi}_\hbar(t,x+z)] \tag{3.6}$$

$$+ i\hbar^{-1}[V(x+z) - V(x-z)]\psi_\hbar(t,x-z)\overline{\psi}_\hbar(t,x+z).$$

Now recall the simple equality $(\Delta_x f)g - f\Delta_x g = \mathrm{div}_x[(\nabla_x f)g - f\nabla_x g]$, so that

$$(\Delta_x \psi_\hbar(t,x-z))\overline{\psi}_\hbar(t,x+z) - \psi_\hbar(t,x-z)\Delta_x \overline{\psi}_\hbar(t,x+z) = -\mathrm{div}_x \nabla_z [\psi_\hbar(t,x-z)\overline{\psi}_\hbar(t,x+z)]. \tag{3.7}$$

Then, insert (3.7) in (3.6), so that

$$\partial_t \psi_\hbar(t,x-z)\overline{\psi}_\hbar(t,x+z) + \psi_\hbar(t,x-z)\partial_t \overline{\psi}_\hbar(t,x+z) \tag{3.8}$$

$$= -\frac{i\hbar}{2}\,\mathrm{div}_x \nabla_z [\psi_\hbar(t, x - z)\overline{\psi}_\hbar(t, x + z)] \tag{3.9}$$
$$+\frac{i}{\hbar}\,[V(x + z) - V(x - z)]\psi_\hbar(t, x - z)\overline{\psi}_\hbar(t, x + z).$$

Moreover, an easy computation involving integration by parts shows

$$\eta \cdot \nabla_x W_\hbar\psi_\hbar = \frac{i\hbar}{2}\,(2\pi)^{-n}\int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta\rangle}\mathrm{div}_z \nabla_x [\psi_\hbar(t, x - z)\overline{\psi}_\hbar(t, x + z)]dz. \tag{3.10}$$

Thanks to the equality $\mathrm{div}_z \nabla_x [\psi_\hbar(t, x - z)\overline{\psi}_\hbar(t, x + z)] = \mathrm{div}_x \nabla_z [\psi_\hbar(t, x - z)\overline{\psi}_\hbar(t, x + z)]$ and by (3.9)–(3.10) we directly obtain the statement. □

Lemma 3.2 *Let* $\epsilon > 0$ *and* $g(\epsilon, \cdot) : \mathbb{T}^n \longrightarrow \mathbb{R}^+$ *defined as*

$$g(\epsilon, y) := \frac{1}{(2\pi)^n}\sum_{\kappa_0\in\mathbb{Z}^n} e^{-\epsilon|\kappa_0|^2}e^{-i\langle y,\kappa_0\rangle} = \frac{1}{(2\pi)^n}\sum_{\xi\in\mathbb{Z}^n}\left(\frac{\pi}{\epsilon}\right)^{\frac{n}{2}} e^{-|\xi-y|^2(4\epsilon)^{-1}}. \tag{3.11}$$

Then, $\forall\psi \in C^\infty(\mathbb{T}^n; \mathbb{C})$

$$\lim_{\epsilon\to 0^+}\int_{\mathbb{T}^n} g(\epsilon, y - y_0)\psi(y_0)dy_0 = \psi(y). \tag{3.12}$$

Proof Let $G(\kappa_0, \epsilon, y) := e^{-\epsilon|\kappa_0|^2}e^{-i\langle y,\kappa_0\rangle}$, then $\widehat{G}(\xi, \epsilon, y) := \int_{\mathbb{R}^n} e^{-i\langle\xi,\kappa_0\rangle}G(\kappa_0, \epsilon, y)d\kappa_0$ reads

$$\widehat{G}(\xi, \epsilon, y) = \left(\frac{\pi}{\epsilon}\right)^{\frac{n}{2}} e^{-|\xi-y|^2(4\epsilon)^{-1}}$$

By applying the Poisson's summation formula (see for example [9]),

$$g(\epsilon, y) = \frac{1}{(2\pi)^n}\sum_{\xi\in\mathbb{Z}^n}\left(\frac{\pi}{\epsilon}\right)^{\frac{n}{2}} e^{-|\xi-y|^2(4\epsilon)^{-1}} = \frac{1}{(2\pi)^n}\sum_{\xi\in\mathbb{Z}^n}\left(\frac{\pi}{\epsilon}\right)^{\frac{n}{2}} e^{-|2\pi\xi-2\pi y|^2(16\pi^2\epsilon)^{-1}}. \tag{3.13}$$

Now recall the identification $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$, fix the periodicity domain $y_0 \in Q_n := [0, 2\pi]^n$, so that

$$\lim_{\epsilon\to 0^+}\int_{Q_n} g(\epsilon, y - y_0)\psi(y_0)dy_0 \tag{3.14}$$
$$= \lim_{\epsilon\to 0^+}\left(\frac{1}{4\pi\epsilon}\right)^{\frac{n}{2}}\sum_{\xi\in\mathbb{Z}^n}\int_{\mathbb{R}^n} e^{-|2\pi\xi-2\pi(y-y_0)|^2(16\pi^2\epsilon)^{-1}}\psi(y_0)\mathcal{X}_{Q_n}(y_0)dy_0 \tag{3.15}$$
$$= \lim_{\epsilon\to 0^+}\left(\frac{1}{4\pi\epsilon}\right)^{\frac{n}{2}}\int_{\mathbb{R}^n} e^{-|y-y_0|^2(4\epsilon)^{-1}}\psi(y_0)\mathcal{X}_{Q_n}(y_0)dy_0 = \psi(y). \tag{3.16}$$

□

In the following, we provide the evolution equation for the Wigner transform $W_\hbar\psi_\hbar$ of the solution of the Schrödinger's equation on the torus,

$$\partial_t W_\hbar\psi_\hbar + \eta\cdot\nabla_x W_\hbar\psi_\hbar + \mathcal{K}_\hbar \star_\eta W_\hbar\psi_\hbar = 0 \tag{3.17}$$

written in the distributional sense. More precisely, $\forall f \in C_c^\infty((0,t)\times\mathbb{T}^n\times\mathbb{R}^n;\mathbb{R})$ such that $f(s,\cdot)\in A\forall s\in(0,t)$ as in Def 2.5 it holds

$$\int_0^t \sum_{\eta\in\frac{\hbar}{2}\mathbb{Z}^n}\int_{\mathbb{T}^n}\Big[\Big(\partial_s f + \eta\cdot\nabla_x f\Big)(s,x,\eta)W_\hbar\psi_\hbar(s,x,\eta)$$
$$+f(s,x,\eta)\,\mathcal{K}_\hbar\star_\eta W_\hbar\psi_\hbar(s,x,\eta)\Big]dxds = 0 \tag{3.18}$$

where for $\eta\in\frac{\hbar}{2}\mathbb{Z}^n$

$$\mathcal{K}_\hbar(s,x,\eta) := \frac{i}{(2\pi)^n\hbar}\int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta\rangle}\{V(x+z)-V(x-z)\}dz, \tag{3.19}$$

$$\mathcal{K}_\hbar\star_\eta W_\hbar\psi_\hbar(s,x,\eta) := \sum_{\kappa_0\in\mathbb{Z}^n}\mathcal{K}_\hbar\Big(s,x,\eta-\frac{\hbar}{2}\kappa_0\Big)W_\hbar\psi_\hbar\Big(s,x,\frac{\hbar}{2}\kappa_0\Big). \tag{3.20}$$

Theorem 3.3 *Let $\psi_\hbar$ be the solution of (3.1). Then, it holds*

$$\partial_t W_\hbar\psi_\hbar + \eta\cdot\nabla_x W_\hbar\psi_\hbar + \mathcal{K}_\hbar\star_\eta W_\hbar\psi_\hbar = 0 \tag{3.21}$$

in the distributional sense as in (3.18).

Proof We exhibit a short proof based on the previous result, namely we simply show that convolution (3.20) is well defined and coincides with the remainder term (3.3). Since $V\in C^\infty(\mathbb{T}^n;\mathbb{R})$, the related Fourier components $V_\omega := (2\pi)^{-n}\int_{\mathbb{T}^n} e^{i\omega z}V(z)dz$, $\omega\in\mathbb{Z}^n$, fulfill $|V_\omega|\le c_j\langle\omega\rangle^j\forall j\in\mathbb{N}$ and some $c_j>0$. An easy computation shows that

$$\mathcal{K}_\hbar\Big(s,x,\frac{\hbar}{2}\kappa\Big) = \frac{i}{(2\pi)^n\hbar}(e^{-i\kappa\cdot x}V_\kappa - e^{+i\kappa\cdot x}V_\kappa^\star),\quad \kappa\in\mathbb{Z}^n. \tag{3.22}$$

Moreover, $\|W_\hbar\psi_\hbar(s,\cdot)\|_\infty \le (2\pi)^{-n}C^2\forall s\in\mathbb{R}$. Thus, the series in (3.20) is absolutely convergent, and we can write down the regularization (useful in the subsequent computations):

$$\mathcal{K}_\hbar\star_\eta W_\hbar\psi_\hbar = \lim_{\epsilon\to0^+}\sum_{\kappa_0\in\mathbb{Z}^n} e^{-\epsilon|\kappa_0|^2}\mathcal{K}_\hbar\Big(s,x,\eta-\frac{\hbar}{2}\kappa_0\Big)W_\hbar\psi_\hbar\Big(s,x,\frac{\hbar}{2}\kappa_0\Big). \tag{3.23}$$

We look at the regularization:

$$\sum_{\kappa_0\in\mathbb{Z}^n} e^{-\epsilon|\kappa_0|^2}\mathcal{K}_\hbar\Big(s,x,\eta-\frac{\hbar}{2}\kappa_0\Big)W_\hbar\psi_\hbar\Big(s,x,\frac{\hbar}{2}\kappa_0\Big) \tag{3.24}$$

$$= \sum_{\kappa_0\in\mathbb{Z}^n} e^{-\epsilon|\kappa_0|^2}\frac{i}{(2\pi)^n\hbar}\int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta-\frac{\hbar}{2}\kappa_0\rangle}\{V(x+z)-V(x-z)\}dz \tag{3.25}$$

$$\times\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle\tilde{z},\frac{\hbar}{2}\kappa_0\rangle}\psi_\hbar(s,x-\tilde{z})\psi^\star_\hbar(s,x+\tilde{z})d\tilde{z}$$

$$= \frac{i}{(2\pi)^n\hbar}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta\rangle}\Big[\frac{1}{(2\pi)^n}\sum_{\kappa_0\in\mathbb{Z}^n} e^{-\epsilon|\kappa_0|^2}e^{-i\langle z-\tilde{z},\kappa_0\rangle}\Big] \tag{3.26}$$

$$\times\{V(x+z)-V(x-z)\}\psi_\hbar(s,x-\tilde{z})\psi^\star_\hbar(s,x+\tilde{z})dz d\tilde{z}$$

However, for any fixed $\epsilon > 0$, the function

$$g(\epsilon, z-\tilde{z}) := \frac{1}{(2\pi)^n}\sum_{\kappa_0\in\mathbb{Z}^n} e^{-\epsilon|\kappa_0|^2}e^{-i\langle z-\tilde{z},\kappa_0\rangle} \tag{3.27}$$

defines a tempered distribution on $C^\infty(\mathbb{T}^n;\mathbb{C})$ converging to $\delta(z-\tilde{z})$ as $\epsilon\to 0^+$ (see Lemma 3.2).

To conclude,

$$\mathcal{K}_\hbar \star_\eta W_\hbar\psi_\hbar = \lim_{\epsilon\to 0^+}\frac{i}{(2\pi)^n\hbar}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta\rangle}g(\epsilon,z-\tilde{z})\{V(x+z)-V(x-z)\}$$
$$\psi_\hbar(s,x-\tilde{z})\psi^\star_\hbar(s,x+\tilde{z})dzd\tilde{z}$$
$$= \frac{i}{(2\pi)^n\hbar}\int_{\mathbb{T}^n}\lim_{\epsilon\to 0^+}\int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\eta\rangle}g(\epsilon,z-\tilde{z})\{V(x+z)-V(x-z)\}$$
$$\psi_\hbar(s,x-\tilde{z})\psi^\star_\hbar(s,x+\tilde{z})dzd\tilde{z}$$
$$= \frac{i}{(2\pi)^n\hbar}\int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle\tilde{z},\eta\rangle}\{V(x+\tilde{z})-V(x-\tilde{z})\}\psi_\hbar(s,x-\tilde{z})\psi^\star_\hbar(s,x+\tilde{z})d\tilde{z} =: \mathcal{E}_\hbar\psi_\hbar. \tag{3.28}$$

□

4 Semiclassical Limits of Wigner Transforms on the Torus

4.1 The Liouville Equation

This section is devoted to the Liouville equation written in the measure sense on $\mathbb{T}^n\times\mathbb{R}^n$ solved by the semiclassical asymptotics of the toroidal Wigner transform.

Theorem 4.1 *Let* $\psi_\hbar(t) := e^{-\frac{i}{\hbar}\mathrm{Op}^w_\hbar(H)t}\varphi_\hbar$ *where* $\varphi_\hbar\in L^2(\mathbb{T}^n;\mathbb{C})$ *and* $\|\varphi_\hbar\|_{L^2}\le C$. *Let* $\{dw_t\}_{t\in[-T,T]}$ *be a limit of* $W_\hbar\psi_\hbar(t)$ *in* $L^\infty([-T,+T];\mathcal{A}')$ *along a sequence of values of* $\hbar\to 0$. *Then,*

$$\partial_t w_t + \eta \cdot \nabla_x w_t - \nabla_x V(x) \cdot \nabla_\eta w_t = 0 \tag{4.1}$$

in the distributional sense.

Proof To begin, we prove that

$$\frac{d}{dt} \int_{\mathbb{T}^n \times \mathbb{R}^n} \phi(x,\eta) dw_t(x,\eta) = \int_{\mathbb{T}^n \times \mathbb{R}^n} \{H, \phi\}(x,\eta) dw_t(x,\eta) \tag{4.2}$$

for any $\phi \in A$, see (2.26). To this aim, we observe that the Schrödinger equation implies

$$\frac{d}{dt} \langle \psi_\hbar(t), \mathrm{Op}_\hbar^w(\phi)\psi_\hbar(t)\rangle_{L^2} = -(i\hbar)^{-1} \langle \psi_\hbar(t), [\mathrm{Op}_\hbar^w(H), \mathrm{Op}_\hbar^w(\phi)]\psi_\hbar(t)\rangle_{L^2} \tag{4.3}$$

where $H := \frac{1}{2}|\eta|^2 + V(x)$. Hence, for $t \geq 0$,

$$\begin{aligned} &\langle \psi_\hbar(t), \mathrm{Op}_\hbar^w(\phi)\psi_\hbar(t)\rangle_{L^2} - \langle \varphi_\hbar, \mathrm{Op}_\hbar^w(\phi)\varphi_\hbar\rangle_{L^2} \\ &\quad = -\int_0^t (i\hbar)^{-1} \langle \psi_\hbar(s), [\mathrm{Op}_\hbar^w(H), \mathrm{Op}_\hbar^w(\phi)]\psi_\hbar(s)\rangle_{L^2} ds \end{aligned} \tag{4.4}$$

where $\psi_\hbar(t = 0) =: \varphi_\hbar \in L^2(\mathbb{T}^n; \mathbb{C})$ with $\|\varphi_\hbar\|_{L^2} \leq C \forall 0 < \hbar \leq 1$. Moreover, thanks to Theorem 2.4, the Weyl symbol of the commutator (namely the Moyal bracket of symbols H and ϕ) reads

$$\{H, \phi\}_{\mathrm{M}} = -i\hbar\{H, \phi\} + r \tag{4.5}$$

where r has order $O(\hbar^2)$ when estimated in $S^{2+m}(\mathbb{T}^n \times \mathbb{R}^n)$ for any $m \in \mathbb{R}$, and thus also in $S^0(\mathbb{T}^n \times \mathbb{R}^n)$,

$$|\partial_x^\beta \partial_\eta^\alpha r(x,\eta)| \leq C_{\alpha\beta}\, \hbar^2 \langle \eta \rangle^{-|\alpha|}. \tag{4.6}$$

The related remainder operator $\mathrm{Op}_\hbar^w(r)$ is thus L^2-bounded, with (time independent) norm estimate thanks to Theorem 2.3 with order $O(\hbar^2)$. This directly gives

$$\lim_{\hbar \to 0^+} \hbar^{-1} \left| \int_0^t \langle \psi_\hbar(s), \mathrm{Op}_\hbar^w(r)\psi_\hbar(s)\rangle_{L^2} ds \right| \leq \lim_{\hbar \to 0^+} t\,\hbar^{-1} \|\mathrm{Op}_\hbar^w(r)\|_{L^2 \to L^2} = 0, \tag{4.7}$$

since $\|\psi_\hbar(s)\|_{L^2} = \|\psi_\hbar(s = 0)\|_{L^2} = \|\varphi_\hbar\|_{L^2} \leq C$. The first term in (4.4) reads

$$\langle \psi_\hbar(t), \mathrm{Op}_\hbar^w(\phi)\psi_\hbar(t)\rangle_{L^2} = \sum_{\eta \in \frac{\hbar}{2}\mathbb{Z}^n} \int_{\mathbb{T}^n} \phi(x,\eta) W_\hbar \psi_\hbar(t,x,\eta) dx. \tag{4.8}$$

Let $w_t(x,\eta)$ be a family of Radon measures of finite mass on $\mathbb{T}^n \times \mathbb{R}^n$ for any $t \in [-T, T]$ which is a limit of $W_\hbar \psi_\hbar$ in $L^\infty([-T, +T]; A')$ along a sequence of values of $\hbar_j \to 0$. The related semiclassical limit of (4.8) reads

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \phi(x,\eta) dw_t(x,\eta). \tag{4.9}$$

If we now look at

$$\sum_{\eta \in \frac{\hbar}{2}\mathbb{Z}^n} \int_{\mathbb{T}^n} \{H,\phi\}(x,\eta) W_\hbar \psi_\hbar(t,x,\eta) dx \tag{4.10}$$

we recall that ϕ is rapidly decreasing in η-variables and the phase space transform $\widehat{\phi}$ has compact support, hence also $\{H,\phi\} \in A$. As a consequence, we can extract a subsequence $\hbar_{j(\alpha)} \to 0$ so that the semiclassical limit of the righthand side of (4.4) reads

$$\int_0^t \int_{\mathbb{T}^n \times \mathbb{R}^n} \{H,\phi\}(x,\eta) dw_s(x,\eta) ds. \tag{4.11}$$

We therefore deduce

$$\begin{aligned} &\int_{\mathbb{T}^n \times \mathbb{R}^n} \phi(x,\eta) dw_t(x,\eta) - \int_{\mathbb{T}^n \times \mathbb{R}^n} \phi(x,\eta) dw_0(x,\eta) \\ &= \int_0^t \int_{\mathbb{T}^n \times \mathbb{R}^n} \{H,\phi\}(x,\eta) dw_s(x,\eta) ds \end{aligned} \tag{4.12}$$

and observe that the righthand side is differentiable for any $t \in \mathbb{R}$ (and thanks to the equivalence, the lefthand side too). We now take the time derivative of both sides and get equation (4.2). On the other hand, since H is smooth, it is easily seen that equation (4.2) has a unique solution in $C_{weak}([-T,+T]; \mathcal{M}^+(\mathbb{T}^n \times \mathbb{R}^n))$, i.e. the topology on $\mathcal{M}^+(\mathbb{T}^n \times \mathbb{R}^n)$ is given by the Lévy-Prokhorov metric which metrizes the weak convergence w.r.t. continuous and bounded test functions, and this solution is given by the push forward of the initial data $(\varphi_H^t)_\star(dw_0)$ involving the Hamiltonian flow. However, this is also the unique solution of the Liouville equation written in the following weak sense

$$\begin{aligned} \int_0^t \int_{\mathbb{T}^n \times \mathbb{R}^n} [\partial_s f(s,x,\eta) + \{H,f\}(s,x,\eta)]\, dw_s(x,\eta) ds = 0 \\ \forall f \in C_c^\infty((0,t) \times \mathbb{T}^n \times \mathbb{R}^n; \mathbb{R}), \end{aligned} \tag{4.13}$$

as shown within Sect. 8.1 in [1]. In view of Remark 4.2, our limits $\{dw_t\}_{t \in [-T,T]}$ are in fact continuous path of nonnegative Radon measures, and hence coinciding with the continuous solution $(\varphi_H^t)_\star(dw_0)$ of the Liouville equation. □

Remark 4.2 About the above result, we recall Lemma 3.2 of [3], and we focus the attention on the additional continuous regularity of the limits $\{dw_t\}_{t \in [-T,T]}$ of $W_\hbar \psi_\hbar(t)$ in $L^\infty([-T,+T]; A')$ passing through sequences as $\hbar \to 0$. In fact, it can be easily proved that for our test functions $\phi \in A$, the related functions

$$\Phi_{\hbar,\phi}(t) := \sum_{\eta \in \frac{\hbar}{2}\mathbb{Z}^n} \int_{\mathbb{T}^n} \phi(x,\eta) W_\hbar \psi_\hbar(t,x,\eta) dx \tag{4.14}$$

are differentiable and fulfill

$$\sup_{-T\le t\le T}\left|\frac{d}{dt}\Phi_{\hbar,\phi}(t)\right|\le C_{\phi,T}. \tag{4.15}$$

This can be proved thanks to the phase space representation (2.12) for (4.14), recalling that the phase space transform $\widehat{\phi}$ is supposed to be compactly supported and by the equation for the Wigner transform (3.18) with test functions $f=\mathcal{X}(t)\phi(x)$. Then, by following the same arguments in Lemma 3.2 of [3], it follows that $dw\in C_{weak}([-T,+T];\mathcal{M}^+(\mathbb{T}^n\times\mathbb{R}^n))$ and that for any $-T\le t\le T$ it holds the weak limit $W_\hbar\psi_\hbar(t)\rightharpoonup dw_t$ with test functions $\phi\in A\subset C_b(\mathbb{T}^n\times\mathbb{R}^n)$, namely as in Definition 2.6.

4.2 WKB Wave Functions of Positive and Negative Type

We begin this section introducing a class of WKB-type wave functions in $H^1(\mathbb{T}^n;\mathbb{C})$ associated with weak KAM solutions of the stationary Hamilton–Jacobi equation.

Definition 4.3 Let $P\in\ell\,\mathbb{Z}^n$ for some $\ell>0$ and $\hbar^{-1}\in\ell^{-1}\mathbb{N}$. Let $v_\pm(P,\cdot)\in C^{0,1}(\mathbb{T}^n;\mathbb{R})$ be weak KAM solutions of the H–J equation (2.50) (in the sense of [14], see subsection 2.2.1). Select $a^\pm_{\hbar,P}\in H^1(\mathbb{T}^n;\mathbb{R}^+)$ such that

$$\mathrm{dom}(a^\pm_{\hbar,P})\subseteq\mathrm{dom}(\nabla_x v_\pm(P,\cdot)):=\{x\in\mathbb{T}^n\mid\exists\,\nabla_x v_\pm(P,x)\} \tag{4.16}$$

$\|a^\pm_{\hbar,P}\|_{L^2}=1$ and $\hbar\,\|a^\pm_{\hbar,P}\|_{H^1}\longrightarrow 0$ as $\hbar\longrightarrow 0^+$. We suppose that the following weak limit upon passing through a subsequence $\hbar_j\longrightarrow 0^+$

$$dm^\pm_P(x):=\lim_{\hbar_j\longrightarrow 0^+}|a^\pm_{\hbar_j,P}(x)|^2dx \tag{4.17}$$

fulfills $dm^\pm_P\ll d\sigma_P:=\pi_\star(dw_P)$ where dw_P is the Legendre transform of a Mather P-minimal measure. The WKB wave functions of negative type are defined by

$$\varphi^-_\hbar(x):=a^-_{\hbar,P}(x)\,e^{\frac{i}{\hbar}[P\cdot x+v_-(P,x)]} \tag{4.18}$$

and the WKB wave functions of positive type

$$\varphi^+_\hbar(x):=a^+_{\hbar,P}(x)\,e^{\frac{i}{\hbar}[P\cdot x+v_+(P,x)]}. \tag{4.19}$$

Let us point out that though the definitions (4.18), (4.19) seems to recall Bloch wave expansions, the parameter P is reduced to values belonging to $\hbar\mathbb{Z}^n$ and therefore $\varphi^\pm_\hbar$ are truly periodic functions.

Remark 4.4 (Example) About the previous definition, we exhibit an explicit construction for $a^{\pm}_{\hbar,P}$. In fact, consider $\rho \in C^{\infty}_0(\mathbb{R}^n)$ such that $0 \le \rho$, supp $\rho \subset Q_n := [0, 2\pi]^n$ and $\int \rho(x)dx = 1$. For a fixed $\alpha > 0$ let

$$\Phi_{\alpha,\hbar}(x) := \hbar^{-n\alpha} \sum_{k\in\mathbb{Z}^n} \rho\Big(\frac{x-2\pi k}{\hbar^{\alpha}}\Big). \tag{4.20}$$

Then $\int_{\mathbb{T}^n} \Phi_{\alpha,\hbar}(x)dx = 1$, and if $f \in L^1(\mathbb{T}^n)$ we have, by the periodicity,

$$\Phi_{\alpha,\hbar} \star f(x) = \int_{\mathbb{T}^n} \Phi_{\alpha,\hbar}(x-y)f(y)dy = \int_{Q_n} \rho(z)f(x-\hbar^{\alpha}z)dz$$

For a fixed (P-dependent) Borel positive measure $dm^{\pm}_P$ on $\mathbb{T}^n$ with supp$(dm^{\pm}_P) \subseteq$ dom$(\nabla_x v_{\pm}(P,\cdot))$, an amplitude function can be given by

$$a^{\pm}_{\hbar,P}(x) := \Big\{ \int_{\mathbb{T}^n} \frac{1}{c_0}\Big(\hbar^{\epsilon} + \Phi_{\gamma,\hbar}(x-y)\Big)dm^{\pm}_P(y)\Big\}^{1/2}\Big|_{\mathrm{dom}(\nabla v_{\pm})}, \tag{4.21}$$

where $\epsilon, \gamma > 0$ with $0 < \epsilon+\gamma(n+1) < 1$, $c_0 = c_0(\hbar) = \|\hbar^{\epsilon}+\rho\|_{L^1(Q_n)} = 1+O(\hbar^{\epsilon})$. Notice that $a > \hbar^{\epsilon/2}c_0^{-1/2}$ and $x \mapsto a^{\pm}_{\hbar,P}(x)$ is 2π-periodic (in each variable). This means that it is a well-defined positive function on the torus. The function (4.21) fulfills (see Proposition 4.6 in [27])

(i) $\int_{\mathbb{T}^n} |a^{\pm}_{\hbar,P}(x)|^2 dx = 1$
(ii) $\hbar^2 \int_{\mathbb{T}^n} |\nabla_x a^{\pm}_{\hbar,P}(x)|^2 dx \le \|\nabla_x\rho\|^2_{L^{\infty}} \hbar^{2(1-\epsilon-(n+1)\gamma)}$
(iii) $\lim_{\hbar\to 0+} \int_{\mathbb{T}^n} f(x)|a^{\pm}_{\hbar,P}(x)|^2 dx = \int_{\mathbb{T}^n} f(x)dm^{\pm}_P(x)$,$\forall$ bounded Borel measurable $f : \mathbb{T}^n \longrightarrow \mathbb{R}$ whose discontinuity set has zero $dm^{\pm}_P$-measure.

Before to conclude this construction, we need to remind (2.58)–(2.60) which ensure that the supports of the projected (on $\mathbb{T}^n$) Mather measures $d\sigma_P$ are all contained in the domains dom$(\nabla_x v_{\pm}(P,\cdot))$.

In the following, we provide two useful Lemma involving our class of WKB functions.

Lemma 4.5 *Let $\varphi^{\pm}_{\hbar}$ be as in Definition 4.3. Then, $\varphi^{\pm}_{\hbar} \in H^1(\mathbb{T}^n;\mathbb{C})$.*

Proof The L^2-norm simply reads $\|\varphi^{\pm}_{\hbar}\|_{L^2} = \|a^{\pm}_{\hbar,P}\|_{L^2} < +\infty$, whereas

$$\|\nabla_x\varphi^{\pm}_{\hbar}\|_{L^2} \le \frac{1}{\hbar}\|(P+\nabla_x v_{\pm})a^{\pm}_{\hbar,P}\|_{L^2} + \|\nabla_x a^{\pm}_{\hbar,P}\|_{L^2}$$

Recalling (2.54) and the setting of $a_{\hbar,P}$, for any fixed $0 < \hbar \le 1$ it follows that

$$\|\nabla_x\varphi^{\pm}_{\hbar}\|_{L^2} \le \frac{1}{\hbar}\|P+\nabla_x v_{\pm}\|_{L^{\infty}} + \|\nabla_x a^{\pm}_{\hbar,P}\|_{L^2} < +\infty.$$

□

Lemma 4.6 *Let $\varphi_\hbar^\pm$ be as in Definition 4.3. Let $\mathbb{P}_\hbar^\pm$ be as in (2.34) associated to $\varphi_\hbar^\pm$. Then, the family of measures $\{\mathbb{P}_\hbar^\pm\}_{0\leq\hbar\leq 1}$ is tight.*

Proof Let $M_R := \mathbb{T}^n \times \{\mathbb{R}^n \backslash B_R\}$ and $U_R := \mathbb{R}^n \backslash B_R$. Thanks to (2.45)

$$\mathbb{P}_\hbar^\pm(\mathbb{T}^n \times U_R) = \sum_{\alpha\in\mathbb{Z}^n} \mathcal{X}_{U_R}(\hbar\alpha)|\widehat{\phi}_{\hbar,\alpha}^\pm|^2 \tag{4.22}$$

where the Fourier components read

$$\widehat{\phi}_{\hbar,\alpha}^\pm := (2\pi)^{-n}\int_{\mathbb{T}^n} e^{-i\alpha\cdot x}\varphi_\hbar^\pm(x)dx = (2\pi)^{-n}\int_{\mathbb{T}^n} e^{-i\alpha\cdot x} a_{\hbar,P}^\pm(x)\, e^{\frac{i}{\hbar}[P\cdot x + v_\pm(P,x)]}dx \tag{4.23}$$

$$= (2\pi)^{-n}\int_{\mathbb{T}^n} a_{\hbar,P}^\pm(x)\, e^{\frac{i}{\hbar}v_\pm(P,x)} e^{\frac{i}{\hbar}(-\hbar\alpha+P)\cdot x}dx \tag{4.24}$$

and $P \in \ell\mathbb{Z}^n$ for some fixed $\ell > 0$; moreover we underline that the series (4.22) is computed over $|\hbar\alpha| > R$ (or equivalently $|\alpha| > R\hbar^{-1}$). In the case $R > |P|$, it holds the equality

$$\widehat{\phi}_{\hbar,\alpha}^\pm = \frac{(-i\hbar)}{|-\hbar\alpha+P|^2}(-\hbar\alpha+P)\cdot(2\pi)^{-n}\int_{\mathbb{T}^n} a_{\hbar,P}^\pm(x)\, e^{\frac{i}{\hbar}v_\pm(P,x)}\nabla_x e^{\frac{i}{\hbar}(-\hbar\alpha+P)\cdot x}dx. \tag{4.25}$$

The integration by parts gives

$$\begin{aligned}\widehat{\phi}_{\hbar,\alpha}^\pm =& \frac{(i\hbar)}{|-\hbar\alpha+P|^2}(-\hbar\alpha+P)\cdot(2\pi)^{-n}\int_{\mathbb{T}^n} \nabla_x a_{\hbar,P}^\pm(x)\, e^{\frac{i}{\hbar}v_\pm(P,x)} e^{\frac{i}{\hbar}(-\hbar\alpha+P)\cdot x}dx \\ &-\frac{1}{|-\hbar\alpha+P|^2}(-\hbar\alpha+P)\cdot(2\pi)^{-n} \\ &\int_{\mathbb{T}^n} a_{\hbar,P}^\pm(x)(\nabla_x v_\pm(P,x)) e^{\frac{i}{\hbar}v_\pm(P,x)} e^{\frac{i}{\hbar}(-\hbar\alpha+P)\cdot x}dx\end{aligned} \tag{4.26}$$

We are now in the position to provide an estimate for $|\widehat{\phi}_{\hbar,\alpha}^\pm|$, indeed some easy computations together with the application of Cauchy–Schwarz inequality give

$$|\widehat{\phi}_{\hbar,\alpha}^\pm| \leq \frac{(2\pi)^{-n/2}}{|-\hbar\alpha+P|}\Big(\|\hbar\nabla_x a_{\hbar,P}^\pm\|_{L^2} + (2\pi)^{-n/2}\|\nabla_x v_\pm(P,\cdot)\|_{L^\infty}\Big) \tag{4.27}$$

Recalling (2.54) we have $\|\nabla_x v_\pm(P,\cdot)\|_{L^\infty} < +\infty$ for any fixed $P \in \ell\mathbb{Z}^n$. We also remind that $\|\hbar\nabla_x a_{\hbar,P}^\pm\|_{L^2} \to 0$ as $\hbar \to 0^+$. To conclude, by defining

$$C_{n,P} := (2\pi)^{-n}\Big(\sup_{0<\hbar\leq 1}(\|\hbar\nabla_x a_{\hbar,P}^\pm\|_{L^2}) + (2\pi)^{-n/2}\|\nabla_x v_\pm(P,\cdot)\|_{L^\infty}\Big)^2 \tag{4.28}$$

it follows (when $R > |P|$)

$$|\mathbb{P}_\hbar^\pm(\mathbb{T}^n \times U_R)| \leq \sum_{\alpha\in\mathbb{Z}^n, |\hbar\alpha|>R} \frac{C_{n,P}}{|-\hbar\alpha + P|^2} \leq \int_{\mathbb{R}^n/B_R(0)} \frac{C_{n,P}}{|-y+P|^2} dy \quad (4.29)$$

The last ($\hbar$-independent) upper bound implies that

$$\lim_{R\to+\infty} \sup_{0<\hbar\leq 1} |\mathbb{P}_\hbar^\pm(\mathbb{T}^n \times U_R)| = 0. \quad (4.30)$$

We next exhibit a property of the involved monokinetic measures.

Proposition 4.7 *Let $dm_P^\pm$ as in (4.17) and $v_-(P,\cdot) \in C^{0,1}(\mathbb{T}^n;\mathbb{R})$ be a weak KAM solution of negative type for the H–J equation (2.50). Define the lifted Borel measure on $\mathbb{T}^n \times \mathbb{R}^n$ by*

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta) d\widetilde{m}_P^\pm(x,\eta) := \int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x, P + \nabla_x v_-(P,x)) dm_P^\pm(x), \quad \forall\, \phi \in A. \quad (4.31)$$

Then, $d\widetilde{m}_P^\pm$ does not depend on the choice of $v_-(P,\cdot)$, namely

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta) d\widetilde{m}_P^\pm(x,\eta) = \int_{\mathbb{T}^n} \phi(x, P + \nabla_x v'_-(P,x)) dm_P^\pm(x) \quad (4.32)$$

for any other weak KAM of negative type $v'_-(P,x)$. Moreover, for any weak KAM of positive type $v_+(P,x)$ it holds

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta) d\widetilde{m}_P^\pm(x,\eta) = \int_{\mathbb{T}^n} \phi(x, P + \nabla_x v_+(P,x)) dm_P^\pm(x) \quad (4.33)$$

Finally, there exists a Borel measurable function $g^\pm(P,\cdot) : \mathbb{T}^n \to \mathbb{R}^+$ such that

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x) dm_P^\pm(x) = \int_{\mathbb{T}^n} \phi(x)\, g^\pm(P,x) d\sigma_P(x). \quad (4.34)$$

Proof For any $v_\pm(P,\cdot) \in C^{0,1}(\mathbb{T}^n;\mathbb{R})$ which is a weak KAM solution of Hamilton–Jacobi equation (2.50), the map $x \mapsto \nabla_x v_\pm(P,x)$ is continuous and uniformly bounded on its domain of definition $\mathrm{dom}(\nabla_x v_\pm(P,\cdot)) \subseteq \mathbb{T}^n$. Moreover, since we assumed $dm_P^\pm \ll d\sigma_P$ then $\mathrm{supp}(dm_P^\pm) \subseteq \mathrm{supp}(d\sigma_P)$. By recalling that $\mathrm{supp}(d\sigma_P) \subseteq \pi(\mathcal{M}_P^\star) \subseteq \pi(\mathcal{A}_P^\star)$ and thanks to the localization the Aubry set $\mathcal{A}_P^\star$ shown in Sect. 2.2.3, it follows

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta) d\widetilde{m}_P^\pm(x,\eta) = \int_{\mathbb{T}^n} \phi(x, P + \nabla_x v_\pm(P,x)) dm_P^\pm(x) \quad (4.35)$$

for any $v_\pm(P,\cdot) \in C^{0,1}(\mathbb{T}^n;\mathbb{R})$ weak KAM solutions of Hamilton–Jacobi equation. Finally, the assumption on the absolute continuity of $dm_P^\pm$ with respect to $d\sigma_P$ together with the well known Radon-Nikodym derivative provides the existence of a Borel measurable $g^{\pm}(P,x)$ satisfying (4.34). □

Lemma 4.8 *Let*

$$d\widetilde{m}_P^{\pm}(x,\eta) := \delta(\eta - P - \nabla_x v_{\pm}(P,x))dm_P^{\pm}(x) \tag{4.36}$$

be as in Proposition 4.7. Then, $d\widetilde{m}_P^{\pm}$ is absolutely continuous to dw_P, i.e. the Legendre transform of a Mather P-minimal measure. In particular, there exists a Borel measurable function $g^{\pm}(P,\cdot) : \mathbb{T}^n \to \mathbb{R}^+$ such that

$$d\widetilde{m}_P^{\pm}(x,\eta) = g^{\pm}(P,x)dw_P(x,\eta) \tag{4.37}$$

where $dw_P(x,\eta) = \delta(\eta - P - \nabla_x v_{\pm}(P,x))d\sigma_P(x)$.

Proof By the assumption within Definition 4.3, it holds $dm_P \ll \pi_\star(dw_P) =: d\sigma_P$ where dw_P is Legendre transform of a Mather P-minimal measure $d\mu_P$ as in (2.58). Equivalently, we can take $d\sigma_P := \pi_\star(d\mu_P)$. Thus, there exists a Borel measurable function $g^{\pm}(P,\cdot) : \mathbb{T}^n \to \mathbb{R}^+$ such that

$$d\widetilde{m}_P^{\pm}(x,\eta) = g^{\pm}(P,x)\delta(\eta - P - \nabla_x v_{\pm}(P,x))d\sigma_P(x). \tag{4.38}$$

In fact, it holds the equality $\delta(\eta - P - \nabla_x v_{\pm}(P,x))d\sigma_P(x) = dw_P(x,\eta)$ thanks to the inclusion

$$\mathrm{supp}(dw_P) \subseteq \mathcal{A}_P^{\star} \subseteq \mathrm{Graph}(P + \nabla_x v_{\pm}(P,\cdot)),$$

see Lemma 3.1 shown in [15]. The (4.37) follows directly.

We are now ready to provide the result involving the semiclassical limits of the Wigner transform for the above class of WKB-type wave functions.

Theorem 4.9 *Let $P \in \ell\,\mathbb{Z}^n$ for some $\ell > 0$, $\hbar^{-1} \in \ell^{-1}\mathbb{N}$, $v_{\pm}$ be weak KAM solutions of H–J equation (2.50) and $\varphi_\hbar^{\pm}$ be the associated WKB wave functions as in Definition 4.3, $dm_P^{\pm}$ as in Definition 4.3. Then,*

$$\lim_{\hbar\to 0^+} W_\hbar\varphi_\hbar^{\pm}(x,\eta) = \delta(\eta - P - \nabla_x v_{\pm}(P,x))dm_P^{\pm}(x) =: d\widetilde{m}_P^{\pm}(x,\eta) \tag{4.39}$$

in A' for test functions A as in Definition 2.5, and passing through a subsequence.

Proof The Wigner transform in the variables $(q,p) \in \mathbb{Z}^n \times \mathbb{R}^n$:

$$\begin{aligned}\widehat{W}_\hbar\varphi_\hbar^{\pm}(q,p) &:= \int_{\mathbb{T}^n} \varphi_\hbar^{\pm}(y)^\star e^{i(q\cdot y+\hbar p\cdot q/2)}\varphi_\hbar^{\pm}(y+\hbar p)dy \\ &= \int_{\mathbb{T}^n} e^{i[\hbar p\cdot q/2+P\cdot p]}\, e^{iq\cdot y} e^{\frac{i}{\hbar}[v_{\pm}(P,y+\hbar p)-v_{\pm}(P,y)]} a_{\hbar,P}^{\pm}(y)a_{\hbar,P}^{\pm}(y+\hbar p)dy.\end{aligned} \tag{4.40}$$

By the H^1-regularity of $a^{\pm}_{\hbar,P}$, it holds $a^{\pm}_{\hbar,P}(y+\hbar p) = a^{\pm}_{\hbar,P}(y) + \hbar \int_0^1 p \cdot \nabla_x a^{\pm}_{\hbar,P}(y + \lambda\hbar p) d\lambda$ and

$$\|a^{\pm}_{\hbar,P}(\diamond + \hbar p) - a^{\pm}_{\hbar,P}(\diamond)\|_{L^2} \leq |p|\, \hbar\, \|a^{\pm}_{\hbar,P}\|_{H^1} \tag{4.41}$$

Thus,

$$\begin{aligned}\widehat{W}_{\hbar}\varphi^{\pm}_{\hbar}(q,p) = &\int_{\mathbb{T}^n} e^{i[\hbar p\cdot q/2 + P\cdot p]}\, e^{iq\cdot y} e^{\frac{i}{\hbar}[v_{\pm}(P,y+\hbar p) - v_{\pm}(P,y)]} a^{\pm}_{\hbar,P}(y)^2 dy\\ &+ R_{\hbar}(q,p)\end{aligned} \tag{4.42}$$

where

$$\begin{aligned}R_{\hbar}(q,p) := &\int_{\mathbb{T}^n} e^{i[\hbar p\cdot q/2 + P\cdot p]}\, e^{iq\cdot y} e^{\frac{i}{\hbar}[v_{\pm}(P,y+\hbar p) - v_{\pm}(P,y)]} a^{\pm}_{\hbar,P}(y)\\ &\times [a^{\pm}_{\hbar,P}(y+\hbar p) - a^{\pm}_{\hbar,P}(y)] dy\end{aligned}$$

and $\forall (q,p) \in \mathbb{Z}^n \times \mathbb{R}^n$

$$\begin{aligned}|R_{\hbar}(q,p)| &\leq \mathrm{vol}(\mathbb{T}^n)\|a_{\hbar,P}\|_{L^2}\|a^{\pm}_{\hbar,P}(\diamond + \hbar p) - a^{\pm}_{\hbar,P}(\diamond)\|_{L^2}\\ &\leq (2\pi)^n\, |p|\, \hbar\, \|a^{\pm}_{\hbar,P}\|_{H^1}.\end{aligned} \tag{4.43}$$

For any $\phi \in A$ the related $\mathrm{supp}(\phi)$ is compact, and hence

$$\sum_{q\in\mathbb{Z}^n} \int_{\mathbb{R}^n} \widehat{\phi}(q,p)\widehat{W}_{\hbar}\varphi^{\pm}_{\hbar}(q,p)(q,p) dp \tag{4.44}$$

$$\begin{aligned}&= \sum_{q\in\mathbb{Z}^n} \int_{\mathbb{R}^n} \widehat{\phi}(q,p) \int_{\mathbb{T}^n} e^{i[\hbar p\cdot q/2 + P\cdot p]}\, e^{iq\cdot y} e^{\frac{i}{\hbar}[v_{\pm}(P,y+\hbar p) - v_{\pm}(P,y)]} a^{\pm}_{\hbar,P}(y)^2 dy dp\\ &\quad + \sum_{q\in\mathbb{Z}^n} \int_{\mathbb{R}^n} \widehat{\phi}(q,p) R_{\hbar}(q,p) dp.\end{aligned} \tag{4.45}$$

An easy computation shows that

$$\Big| \sum_{q\in\mathbb{Z}^n} \int_{\mathbb{R}^n} \widehat{\phi}(q,p) R_{\hbar}(q,p) dp \Big| \leq \sum_{q\in\mathbb{Z}^n} \int_{\mathbb{R}^n} |\widehat{\phi}(q,p)| (2\pi)^n\, |p|\, \hbar\, \|a^{\pm}_{\hbar,P}\|_{H^1} dp$$

and hence, since $\mathrm{supp}(\widehat{\phi})$ is compact and $\hbar \|a^{\pm}_{\hbar,P}\|_{H^1} \longrightarrow 0$ as $\hbar \longrightarrow 0^+$ (see Remark 4.4) it follows

$$(2\pi)^n \sum_{q\in\mathbb{Z}^n} \int_{\mathbb{R}^n} |\widehat{\phi}(q,p)|\, |p| dp\, \hbar\, \|a^{\pm}_{\hbar,P}\|_{H^1} \longrightarrow 0^+ \quad \text{as} \quad \hbar \longrightarrow 0^+. \tag{4.46}$$

In view of (4.46) and the compactness of $\mathrm{supp}(\widehat{\phi})$, the (4.44) reads

$$\sum_{q\in\mathbb{Z}^n}\int_{\mathbb{R}^n}\widehat{\phi}(q,p)\lim_{\hbar\to 0^+}\int_{\mathbb{T}^n} e^{i[\hbar p\cdot q/2+P\cdot p]}\ e^{iq\cdot y}e^{\frac{i}{\hbar}[v_\pm(P,y+\hbar p)-v_\pm(P,y)]}|a^{\pm}_{\hbar,P}(y)|^2 dydp. \tag{4.47}$$

By looking at the integral

$$\int_{\mathbb{T}^n} e^{i[\hbar p\cdot q/2+P\cdot p]}\ e^{iq\cdot y}e^{\frac{i}{\hbar}[v_\pm(P,y+\hbar p)-v_\pm(P,y)]}|a^{\pm}_{\hbar,P}(y)|^2 dy \tag{4.48}$$

we observe that $e^{i(\hbar p\cdot q/2)}e^{\frac{i}{\hbar}[v_\pm(P,y+\hbar p)-v_\pm(P,y)]}$ is a family of uniformly bounded continuous functions on $\mathbb{T}^n$ such that

$$\lim_{\hbar\to 0^+} e^{i(\hbar p\cdot q/2)}e^{\frac{i}{\hbar}[v_\pm(P,y+\hbar p)-v_\pm(P,y)]} = e^{ip\cdot\nabla_x v_\pm(P,y)} \tag{4.49}$$

$\forall (q,p)\in\mathrm{supp}(\widehat{\phi})$ and $\forall y\in\mathrm{dom}(\nabla_x v_\pm(P,\cdot))$, since any map $x\longmapsto\nabla_x v_\pm(P,x)$ is continuous on $\mathrm{dom}(\nabla_x v_\pm(P,\cdot))$ (as we recall in Sect. 2.2.1). By the inclusions

$$\mathrm{supp}(dm^{\pm}_P)\subseteq\mathrm{supp}(d\sigma_P)\subseteq\mathrm{dom}(\nabla_x v_\pm(P,\cdot)) \tag{4.50}$$

we deduce that (4.49) is (possibly) not fulfilled only for a set of zero $dm^{\pm}_P$ measure.

Hence, we can apply Lemma 6.3 for the classical limit of the integral (4.48) to obtain

$$\int_{\mathbb{T}^n} e^{iP\cdot p}\ e^{iq\cdot y}e^{ip\cdot\nabla_x v_\pm(P,y)}dm^{\pm}_P(y). \tag{4.51}$$

We deduce that (4.47) reads

$$\sum_{q\in\mathbb{Z}^n}\int_{\mathbb{R}^n}\widehat{\phi}(q,p)\Big(\int_{\mathbb{T}^n} e^{iP\cdot p}\ e^{iq\cdot y}e^{ip\cdot\nabla_x v_\pm(P,y)}dm^{\pm}_P(y)\Big)dp. \tag{4.52}$$

$$=\int_{\mathbb{T}^n}\sum_{q\in\mathbb{Z}^n}\int_{\mathbb{R}^n}\widehat{\phi}(q,p)e^{iP\cdot p}\ e^{iq\cdot y}e^{ip\cdot\nabla_x v_\pm(P,y)}dp\ dm^{\pm}_P(y) \tag{4.53}$$

where we used again the compactness of $\mathrm{supp}(\widehat{\phi})$. Through the inverse phase-space Fourier transform the above expression becomes

$$\int_{\mathbb{T}^n}\phi(y,P+\nabla_x v_\pm(P,y))\,dm^{\pm}_P(y). \tag{4.54}$$

□

Remark 4.10 Let $P\in\ell\,\mathbb{Z}^n$ for some $\ell>0$ and $\varphi^{\pm}_\hbar$ as in Definition 4.3. Define the current

$$J^{\pm}_\hbar(x):=\hbar\,\mathrm{Im}((\varphi^{\pm}_\hbar)^\star\nabla_x\varphi^{\pm}_\hbar(x))=(P+\nabla_x v_\pm(P,x))|a^{\pm}_{\hbar,P}(x)|^2 \tag{4.55}$$

The (formal) free current equation $\mathrm{div}_x J_\hbar^\pm(x) = 0$ becomes well-posed in the weak sense:

$$\int_{\mathbb{T}^n} \nabla_x f(x) \cdot J_\hbar^\pm(x)\, dx = 0 \qquad \forall f \in C^\infty(\mathbb{T}^n;\mathbb{R}). \tag{4.56}$$

In particular, we recall the inclusion (2.54) which implies, together with the assumptions on $a_{\hbar,P}^\pm$, the estimate $\sup_{0<\hbar\le 1} \|J_\hbar^\pm\|_{L^1} \le \|P + \nabla_x v_\pm(P,\cdot)\|_{L^\infty} < +\infty$. However, the low regularity $v_\pm(P,\cdot) \in C^{0,1}(\mathbb{T}^n;\mathbb{R}^n)$ does not guarantee the existence of some amplitude function satisfying this equation, hence we have to write the asymptotic condition

$$\Big| \int_{\mathbb{T}^n} \nabla_x f(x) \cdot J_{\hbar_j}^\pm(x)\, dx \Big| \longrightarrow 0, \quad \forall f \in C^\infty(\mathbb{T}^n;\mathbb{R}) \tag{4.57}$$

for a sequence $\{\hbar_j^{-1}\}_{j\in\mathbb{N}} \in \ell^{-1}\mathbb{N}$ with $\hbar_j \longrightarrow 0^+$ as $j \longrightarrow +\infty$.

The above observations become meaningful in view of the following result.

Proposition 4.11 *Let $P \in \ell\mathbb{Z}^n$ for some $\ell > 0$, $v_\pm(P,\cdot) \in C^{0,1}(\mathbb{T}^n;\mathbb{R})$ be a weak KAM solution for (2.50). Then, there exist $a_{\hbar,P}^\pm$ as in Remark 4.4 such that the (unique) weak-$\star$ limit $dm_P(x) := \lim_{j\to+\infty} |a_{\hbar_j,P}^\pm(x)|^2 dx$ equal $d\sigma_P := \pi_\star(dw_P)$ where dw_P is the Legendre transform of a Mather P-minimal measure and*

$$\Big| \int_{\mathbb{T}^n} \nabla_x f(x) \cdot J_{\hbar_j}^\pm(x) dx \Big| \longrightarrow 0 \;\text{ as } j \longrightarrow +\infty \;\; \forall f \in C^\infty(\mathbb{T}^n;\mathbb{R}). \tag{4.58}$$

Proof Let $d\sigma_P := \pi_\star(dw_P) = d\mu_P$ with dw_P as in (2.58). Then, $d\sigma_P$ is a Borel probability measure $\mathbb{T}^n$ with

$$\mathrm{supp}(d\sigma_P) \subseteq \pi_\star(\mathcal{M}_P^\star) \subseteq \pi_\star(\mathcal{A}_P^\star) \subseteq \mathrm{dom}(\nabla_x v_\pm(P,\cdot)). \tag{4.59}$$

Moreover, it holds

$$\int_{\mathbb{T}^n} \nabla_x f(x) \cdot (P + \nabla_x v_\pm(P,x))\, d\sigma_P(x) = 0 \;\; \forall f \in C^\infty(\mathbb{T}^n;\mathbb{R}). \tag{4.60}$$

Indeed, $dw_P := \mathcal{L}_\star(d\mu_P)$ and $d\mu_P$ is invariant under Lagrangian flow, hence closed, which means that

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \nabla_x f(x) \cdot \xi \, d\mu_P(x,\xi) = 0 \;\; \forall f \in C^\infty(\mathbb{T}^n;\mathbb{R}).$$

Here the Lagrangian reads $L(x,\xi) = \frac{1}{2}|\xi|^2 + V(x)$ and thus the Legendre transform $\mathcal{L}(x,\xi) = (x,\xi)$, which gives

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \nabla_x f(x) \cdot \eta \, dw_P(x,\eta) = 0 \;\; \forall f \in C^\infty(\mathbb{T}^n;\mathbb{R}).$$

By Lemma 3.1 in [15], we have necessary $\mathrm{supp}(dw_P) \subseteq \mathcal{A}_P^\star \subseteq \mathrm{Graph}(P + \nabla_x v_\pm(P,\cdot))$. Thus, we can restrict $dw_P|_{\mathrm{Graph}(P+\nabla_x v_\pm(P,\cdot))}$ since $\mathrm{Graph}(P + \nabla_x v_\pm (P,\cdot))$ are Borel measurable subsets of $\mathbb{T}^n \times \mathbb{R}^n$ containing the support of this measure. Hence

$$\int_{\mathrm{Graph}(P+\nabla_x v_\pm(P,\cdot))} \nabla_x f(x)\cdot \eta \, dw_P(x,\eta) = 0 \quad \forall f \in C^\infty(\mathbb{T}^n;\mathbb{R}).$$

The canonical projection $\pi : \mathrm{Graph}(P + \nabla_x v_\pm(P,\cdot)) \to \mathbb{T}^n$ is a Borel measurable map, because of $\overline{\mathrm{Graph}(P + \nabla_x v_\pm(P,\cdot))} = \mathbb{T}^n$. We can apply the change of variables and get (4.60).

Now, define the Borel probability measure $dm_P(x) := d\sigma_P(x)$ on $\mathbb{T}^n$. Recalling Remark 4.4, there exists $a^\pm_{\hbar,P} \in H^1(\mathbb{T}^n;\mathbb{R}^+)$ such that $\lim_{\hbar_j\to 0^+} |a^\pm_{\hbar_j,P}|^2 dx = dm_P$ in the weak-$\star$ convergence of Borel measures on $\mathbb{T}^n$. Notice that now we do not write dm_P as $dm^\pm_P$ since in fact the inclusion (4.59) holds.

Thus, we look at

$$\int_{\mathbb{T}^n} \nabla_x f(x)\cdot J^\pm_\hbar(x)dx = \int_{\mathbb{T}^n} \nabla_x f(x)\cdot (P + \nabla_x v_\pm(P,x))\, |a^\pm_{\hbar,P}(x)|^2 dx. \tag{4.61}$$

and observe that the function

$$x \longmapsto \nabla_x f(x)\cdot (P + \nabla_x v_\pm(P,x))$$

is a bounded Borel measurable function, and $x \mapsto \nabla_x v_\pm(P,x)$ is continuous on its domain of definition. Hence, the set of $x \in \mathbb{T}^n$ such that $\exists \{x_k\}_{k\in\mathbb{N}} \subset \mathbb{T}^n$, $\lim_{k\to+\infty} x_k = x$ and

$$\lim_{k\to+\infty} \nabla_x f(x_k)\cdot (P + \nabla_x v_\pm(P,x_k)) \neq \nabla_x f(x)\cdot (P + \nabla_x v_\pm(P,x))$$

is a set of zero dm_P-measure. We now apply Lemma 6.3 to get

$$\lim_{j\to+\infty} \int_{\mathbb{T}^n} \nabla_x f(x)\cdot (P + \nabla_x v_\pm(P,x))\, |a^\pm_{\hbar_j,P}(x)|^2 dx \tag{4.62}$$

$$= \int_{\mathbb{T}^n} \nabla_x f(x)\cdot (P + \nabla_x v_\pm(P,x))\, dm_P(x) = 0 \tag{4.63}$$

where the last equality is given by the above setting $dm_P(x) := d\sigma_P(x)$ and thanks to (4.60). □

5 Propagation of Wigner Measures on Weak KAM Tori

5.1 The Forward and Backward Propagation

The main result of the section reads as

Theorem 5.1 *Let $\varphi_\hbar^\pm$ be as in Definition 4.3 and $\psi_\hbar(t) := e^{-\frac{i}{\hbar}Op_\hbar^w(H)t}\varphi_\hbar$. Let $d\widetilde{m}_P^\pm(t)$ be a limit of $W_\hbar\psi_\hbar(t)$ in $L^\infty([-T,+T];A')$, and $d\widetilde{m}_P^\pm$, $g_\pm(P,x)$ be as in Proposition 4.7. Then, $d\widetilde{m}_P^\pm(t) = (\varphi_H^t)_\star(d\widetilde{m}_P^\pm) \in \mathcal{M}^{1+}(\mathbb{T}^n\times\mathbb{R}^n)$. Moreover, $\forall\phi\in A$ and $\forall t\geq 0$*

$$\int_{\mathbb{T}^n\times\mathbb{R}^n}\phi(x,\eta)\,d\widetilde{m}_P^+(t,x,\eta) = \int_{\mathbb{T}^n}\phi(x,P+\nabla_x v_+(P,x))\,\mathbf{g}_+(t,P,x)d\sigma_P(x) \tag{5.1}$$

$$\mathbf{g}_+(t,P,x) := g_+(P,\pi\circ\varphi_H^{-t}(x,P+\nabla_x v_-(P,x))) \tag{5.2}$$

Whereas $\forall t\leq 0$

$$\int_{\mathbb{T}^n\times\mathbb{R}^n}\phi(x,\eta)\,d\widetilde{m}_P^-(t,x,\eta) = \int_{\mathbb{T}^n}\phi(x,P+\nabla_x v_-(P,x))\,\mathbf{g}_-(t,P,x)d\sigma_P(x) \tag{5.3}$$

$$\mathbf{g}_-(t,P,x) := g_-(P,\pi\circ\varphi_H^{-t}(x,P+\nabla_x v_+(P,x))) \tag{5.4}$$

Proof By Theorem 4.1 and Remark 4.2, any distributional limit dw of the Wigner transform $W_\hbar\psi_\hbar(t)$ in $L^\infty([-T,+T];A')$ solves the Liouville equation and $dw\in C_{weak}([-T,+T];\mathcal{M}^+(\mathbb{T}^n\times\mathbb{R}^n))$. Hence, thanks to the uniqueness for the continuous solutions of this continuity equation, it holds $dw_t = (\varphi_H^t)_\star(d\mathrm{w}(0))$. On the other hand, for our initial data $\varphi_\hbar^\pm$ we proved, within Theorem 4.9, that the Wigner transform $W_\hbar\varphi_\hbar^\pm$ is weak converging (for test functions in A) to the monokinetic probability measures $d\widetilde{m}_P^\pm\in\mathcal{M}^{1+}(\mathbb{T}^n\times\mathbb{R}^n)$. Moreover, recalling Lemma 4.6, the complex measures $\mathbb{P}_\hbar^\pm$ are tight and hence their time evolution $\mathbb{P}_\hbar^\pm(t)$ is tight as well (see Proposition 2.11). This implies that there exist semiclassical limits of $\mathbb{P}_\hbar^\pm(t)$ in the sense of (2.35), namely there exist weak limits of $W_\hbar\psi_\hbar(t)$ with respect to test functions in $C_b(\mathbb{T}^n\times\mathbb{R}^n)\supset A$ to some Borel measures for any fixed t. In fact, this means that it must be that $dw_t = (\varphi_H^t)_\star(d\widetilde{m}_P^\pm)\in\mathcal{M}^{1+}(\mathbb{T}^n\times\mathbb{R}^n)$. From now on, we write $d\widetilde{m}_P^\pm(t) := (\varphi_H^t)_\star(d\widetilde{m}_P^\pm)$.
Next, we underline that $\forall\phi,\psi\in A$

$$\int_{\mathbb{T}^n\times\mathbb{R}^n}\phi(x,\eta)\,d\widetilde{m}_P^\pm(t,x,\eta) = \int_{\mathbb{T}^n\times\mathbb{R}^n}\phi\circ\varphi_H^t(x,\eta)\,d\widetilde{m}_P^\pm(x,\eta) \tag{5.5}$$

$$\int_{\mathbb{T}^n\times\mathbb{R}^n}\psi(x,\eta)\,d\widetilde{m}_P^\pm(x,\eta) = \int_{\mathbb{T}^n}\psi(x,P+\nabla_x v_\pm(P,x))\,g_\pm(P,x)d\sigma_P(x). \tag{5.6}$$

Hence

$$\int_{\mathbb{T}^n\times\mathbb{R}^n}\phi(x,\eta)\,d\widetilde{m}_P^\pm(t,x,\eta) = \int_{\mathbb{T}^n}\phi\circ\varphi_H^t(x,P+\nabla_x v_\pm(P,x))\,g_\pm(P,x)d\sigma_P(x). \tag{5.7}$$

We now recall that $d\sigma_P := \pi_\star(dw_P)$ where dw_P is the Legendre transform of a Mather P-minimal measure, which takes the monokinetic form

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta)\, dw_P(x,\eta) = \int_{\mathbb{T}^n} \phi(x, P+\nabla_x v_\pm(P,x))\, d\sigma_P(x) \tag{5.8}$$

and dw_P is invariant under the Hamiltonian flow. This is a consequence of Lemma 3.1 in [15], which gives $\mathrm{supp}(dw_P) \subseteq \mathcal{A}_P^\star$ and thanks to the inclusion $\mathcal{A}_P^\star \subseteq \mathrm{Graph}(P + \nabla_x v_\pm(P,\cdot))$.

Hence, we can rewrite

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta)\, d\widetilde{m}_P^\pm(t,x,\eta) = \int_{\mathbb{T}^n\times\mathbb{R}^n} \phi\circ\varphi_H^t(x,\eta)\; g_\pm(P,\pi(x,\eta))dw_P(x,\eta). \tag{5.9}$$

By the generalized change of variables,

$$\begin{aligned}&\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta)\, d\widetilde{m}_P^\pm(t,x,\eta)\\ &\quad= \int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta)\; g_\pm(P,\pi\circ\varphi_H^{-t}(x,\eta))(\varphi_H^{-t})_\star dw_P(x,\eta)\end{aligned} \tag{5.10}$$

and thanks to the invariance of dw_P,

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta)\, d\widetilde{m}_P^\pm(t,x,\eta) = \int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta)\; g_\pm(P,\pi\circ\varphi_H^{-t}(x,\eta))dw_P(x,\eta). \tag{5.11}$$

By (5.8)

$$\begin{aligned}&\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\eta)\, d\widetilde{m}_P^\pm(t,x,\eta)\\ &\quad= \int_{\mathbb{T}^n} \phi(x, P+\nabla_x v_\pm(P,x))\; g(P,\pi\circ\varphi_H^{-t}(x, P+\nabla_x v_\pm(P,x)))\, d\sigma_P(x).\end{aligned} \tag{5.12}$$

Thus, we can define

$$\mathbf{g}_+(t,P,x) := g_+(P,\pi\circ\varphi_H^{-t}(x, P+\nabla_x v_-(P,x))) \quad \text{for } t\geq 0 \tag{5.13}$$

and

$$\mathbf{g}_-(t,P,x) := g_-(P,\pi\circ\varphi_H^{-t}(x, P+\nabla_x v_+(P,x))) \quad \text{for } t\leq 0. \tag{5.14}$$

□

Remark 5.2 We notice that the supports of the measures $d\widetilde{m}_P^\pm(t)$ are contained, for any $t\in\mathbb{R}$, in the Mather set $\mathcal{M}_P^\star \subseteq \mathcal{A}_P^\star$ in the phase space which is invariant under the Hamiltonian flow as well as $\mathcal{A}^\star_P$. Hence, these are also contained in any set $\mathrm{Graph}(P+\nabla_x v_\pm(P,\cdot))$ and this means that we could write several possible equivalent Borel measurable density functions $\mathbf{g}_\pm(t,P,x)$. However, within the next result we underline that the functions $\mathbf{g}_+$ (linked to the vector field $P+\nabla_x v_+$) solve a time-forward continuity equation whereas $\mathbf{g}_-$ (linked to the vector field $P+\nabla_x v_-$) solve a time-backward equation.

Proposition 5.3 *Let $\mathbf{g}_\pm$ and $d\sigma_P$ as in Theorem 5.1. Then, for $t\ge 0$ and $\forall f\in C_c^\infty((0,t)\times\mathbb{T}^n;\mathbb{R})$*

$$\int_0^t\int_{\mathbb{T}^n}[\partial_s f(s,x)+\nabla_x f(s,x)\cdot(P+\nabla_x v_+(P,x))]\,\mathbf{g}_+(s,P,x)d\sigma_P(x)ds=0 \tag{5.15}$$

whereas for $t\le 0$ and $\forall f\in C_c^\infty((t,0)\times\mathbb{T}^n;\mathbb{R})$

$$\int_t^0\int_{\mathbb{T}^n}[\partial_s f(s,x)+\nabla_x f(s,x)\cdot(P+\nabla_x v_-(P,x))]\,\mathbf{g}_-(s,P,x)d\sigma_P(x)ds=0 \tag{5.16}$$

Proof We recall $\varphi^t_H|_{\mathcal{A}^\star_P}:\mathcal{A}^\star_P\to\mathcal{A}^\star_P$ is a one parameter group of homeomorphisms on the closed invariant graph $\mathcal{A}^\star_P$ on $\mathbb{T}^n$, hence

$$\begin{aligned}\mathbf{g}_+\,d\sigma_P&=\pi_\star d\widetilde{m}_P(t)=\pi_\star(\varphi^t_H)_\star d\widetilde{m}_P(0)=\pi_\star\Big(\varphi^t_H|_{\mathcal{A}^\star_P}\Big)_\star d\widetilde{m}_P(0)\\&=\Big(\pi(\varphi^t_H|_{\mathcal{A}^\star_P})\Big)_\star d\widetilde{m}_P(0)\end{aligned} \tag{5.17}$$

The map $\pi(\varphi^t_H|_{\mathcal{A}^\star_P}):\pi(\mathcal{A}^\star_P)\to\pi(\mathcal{A}^\star_P)$ is a one parameter group of homeomorphisms associated with the vector field

$$\mathrm{b}_\pm(x):=\frac{d}{dt}\pi(\varphi^t_H(x,P+\nabla_x v_\pm(P,x)))\Big|_{t=0}=\nabla_\eta H(x,P+\nabla_x v_\pm(P,x)) \tag{5.18}$$

defined for any $x\in\pi(\mathcal{A}^\star_P)$ but also in the bigger sets $\mathrm{dom}(\nabla_x v_\pm(P,\cdot))$ defined a.e. $x\in\mathbb{T}^n$. Here $H(x,\eta)=\frac{1}{2}|\eta|^2+V(x)$ and thus $\nabla_\eta H(x,\eta)=\eta$. About the regularity, we have $\mathrm{b}_\pm\in L^\infty(\mathbb{T}^n;\mathbb{R}^n)$. Write down the ODE

$$\dot\gamma=\mathrm{b}_\pm(\gamma) \tag{5.19}$$

with $\gamma(0)=x\in\mathrm{dom}(\nabla_x v_\pm(P,\cdot))$ but remind the inclusions (see Sect. 2.2.3)

$$\varphi^t_H\Big(\mathrm{Graph}(P+\nabla_x v_+(P,\cdot))\Big)\subseteq\mathrm{Graph}(P+\nabla_x v_+(P,\cdot))\quad\forall t\ge 0 \tag{5.20}$$

$$\varphi^t_H\Big(\mathrm{Graph}(P+\nabla_x v_-(P,\cdot))\Big)\subseteq\mathrm{Graph}(P+\nabla_x v_-(P,\cdot))\quad\forall t\le 0. \tag{5.21}$$

Thus, even if we have the low regularity $\mathrm{b}_\pm\in L^\infty(\mathbb{T}^n;\mathbb{R}^n)$ and not (in general) in the larger $W^{1,\infty}(\mathbb{T}^n;\mathbb{R}^n)$, the equation (5.19) is well posed and solved for $t\ge 0$

and $\gamma(0) = x \in \operatorname{dom}(\nabla_x v_+(P,\cdot))$, or in the case $t \leq 0$ and $\gamma(0) = x \in \operatorname{dom}(\nabla_x v_-(P,\cdot))$. We are now in the position to apply the same proof of Proposition 2.1 in [2] and get the statement.

About the explicit representation of the density $\mathbf{g}_+$ for $t \geq 0$, which can be seen as the Radon-Nikodym derivative of $\pi_\star d\widetilde{m}_P(t)$ with respect to $d\sigma_P$,

$$\int_{\mathbb{T}^n} \phi(x)\mathbf{g}_+(t,P,x)d\sigma_P(x) = \int_{\mathbb{T}^n} \phi(\pi(\varphi_H^t|_{\mathcal{A}_P^\star})(x))g_+(P,x)d\sigma_P(x) \tag{5.22}$$

$$= \int_{\mathbb{T}^n} \phi(x)g_+(P,\pi(\varphi_H^{-t}|_{\mathcal{A}_P^\star})(x))d\sigma_P(x) \tag{5.23}$$

since $d\sigma_P$ is invariant under $\pi(\varphi_H^{-t}|_{\mathcal{A}_P^\star})$. We are now looking at the Hamiltonian flow for negative times, and we recall the inclusions $\operatorname{supp}(d\sigma_P) \subseteq \mathcal{M}_P^\star \subseteq \mathcal{A}_P^\star \subseteq \operatorname{Graph}(P + \nabla_x v_\pm(P,\cdot))$, thus we can choose

$$\mathbf{g}_+(t,P,x) = g_+(P, \pi\circ\varphi_H^{-t}(x, P+\nabla_x v_-(P,x))) \quad \text{for } t \geq 0 \tag{5.24}$$

as we have chosen in (5.14). The same arguments for negative times provide

$$\mathbf{g}_-(t,P,x) = g_-(P, \pi\circ\varphi_H^{-t}(x, P+\nabla_x v_+(P,x))) \quad \text{for } t \leq 0. \tag{5.25}$$

as in (5.14). □

Remark 5.4 Let $\psi_\hbar^\pm(s,x) := e^{-\frac{i}{\hbar}\operatorname{Op}_\hbar^w(H)s}\varphi_\hbar^\pm(x)$, define the position density $\rho_\hbar^\pm(s,x) := |\psi_\hbar^\pm(s,x)|^2$ and the current density $J_\hbar^\pm(s,x) := \hbar\operatorname{Im}((\psi_\hbar^\pm)^\star\nabla_x\psi_\hbar^\pm(s,x))$. The (formal) conservation law reads

$$\partial_t\rho_\hbar^\pm(t,x) + \operatorname{div}_x J_\hbar^\pm(t,x) = 0. \tag{5.26}$$

In the next result we exhibit the well-posed setting.

Proposition 5.5 *Let $\psi_\hbar^\pm(s,x) := e^{-\frac{i}{\hbar}\operatorname{Op}_\hbar^w(H)s}\varphi_\hbar^\pm(x)$, $\rho_\hbar^\pm(s,x) := |\psi_\hbar^\pm(s,x)|^2$. Let $\varphi_{\hbar,\varepsilon}^\pm \in C^\infty(\mathbb{T}^n;\mathbb{C})$ such that $\|\varphi_{\hbar,\varepsilon}^\pm - \varphi_\hbar^\pm\|_{H^1} \to 0$ as $\varepsilon \to 0^+$. Define $J_{\hbar,\varepsilon}^\pm(s,x) := \hbar\operatorname{Im}((\psi_{\hbar,\varepsilon}^\pm)^\star\nabla_x\psi_{\hbar,\varepsilon}^\pm(s,x))$ and take a distributional limit $J_\hbar^\pm := \lim_{\varepsilon\to0^+} J_{\hbar,\varepsilon}^\pm$ in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$. Then,*

$$\int_0^t\int_{\mathbb{T}^n} \partial_s f(s,x)\rho_\hbar^\pm(s,x) + \nabla_x f(s,x)\cdot J_\hbar^\pm(s,x)\,dxds = 0 \quad \forall f \in C_c^\infty((0,t)\times\mathbb{T}^n;\mathbb{R}). \tag{5.27}$$

Proof This equation is well posed. Indeed,

$$E[\psi_{\hbar,\varepsilon}^\pm(s)] := \int_{\mathbb{T}^n} \frac{\hbar^2}{2}|\nabla_x\psi_{\hbar,\varepsilon}^\pm(s,x)|^2 + V(x)|\psi_{\hbar,\varepsilon}(s,x)|^2dx \tag{5.28}$$

$$= \int_{\mathbb{T}^n} \frac{\hbar^2}{2}|\nabla_x\psi_{\hbar,\varepsilon}^\pm(0,x)|^2 + V(x)|\psi_{\hbar,0}(s,x)|^2dx \tag{5.29}$$

$$\begin{aligned} &\to \int_{\mathbb{T}^n} \frac{\hbar^2}{2}|\nabla_x \varphi_\hbar^\pm(x)|^2 + V(x)|\varphi_\hbar^\pm(x)|^2 dx \quad \text{as} \quad \varepsilon \to 0^+ &(5.30)\\ &= \int_{\mathbb{T}^n} \Big(\frac{1}{2}|P + \nabla_x v_\pm(P,x)|^2 + V(x)\Big)|a_{\hbar,P}^\pm(x)|^2 dx \\ &\quad + \int_{\mathbb{T}^n} \frac{\hbar^2}{2}|\nabla_x a_{\hbar,P}^\pm(x)|^2 dx \\ &= \bar{H}(P) + \int_{\mathbb{T}^n} \frac{\hbar^2}{2}|\nabla_x a_{\hbar,P}^\pm(x)|^2 dx < +\infty \quad \forall\, 0 < \hbar < 1 &(5.31) \end{aligned}$$

since $\hbar\|\nabla_x a_{\hbar,P}^\pm\|_{L^2} \to 0$ thanks to the setting of $a_{\hbar,P}^\pm$.

Hence $\|J_{\hbar,\varepsilon}^\pm(s,\cdot)\|_{L^1} \le \|\psi_{\hbar,\varepsilon}(s,\cdot)\|_{L^2}\|\hbar\nabla_x\psi_{\hbar,\varepsilon}(s,\cdot)\|_{L^2} \le c\,\|\hbar\nabla_x\psi_{\hbar,\varepsilon}^\pm(s,\cdot)\|_{L^2} < +\infty$ uniformly in $(\varepsilon, s) \in (0,1] \times [0,t]$. We can take a distributional limit $J_\hbar^\pm := \lim_{\varepsilon\to 0^+} J_{\hbar,\varepsilon}^\pm$ in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$ and this gives

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{T}^n} \nabla_x f(s,x)\cdot J_{\hbar,\varepsilon}^\pm(s,x)\, dx = \int_{\mathbb{T}^n} \nabla_x f(s,x)\cdot J_\hbar^\pm(s,x)\, dx \quad \forall s \in (0,t)$$

Since $\rho_{\hbar,\varepsilon}^\pm$ is weak-$\star$ converging to the unique $\rho_\hbar^\pm \in L^1((0,T)\times\mathbb{T}^n;\mathbb{R}^+)$, we deduce that Eq. (5.27) is solved by $(\rho_{\hbar,\varepsilon}^\pm, J_{\hbar,\varepsilon}^\pm)$ in the distributional and in the strong sense, as well as being fulfilled by $(\rho_\hbar^\pm, J_\hbar^\pm)$ in the distributional sense. □

The last result of the section reads

Corollary 5.6 *Fix $P \in \mathbb{R}^n$. Suppose that $v_+(P,\cdot) = v_-(P,\cdot) \in C^2(\mathbb{T}^n;\mathbb{R})$ and $g(P,\cdot) \in W^{1,\infty}(\mathbb{T}^n;\mathbb{R}^+)$. Then, $\mathbf{g}_\pm$ as in Theorem 5.1 fulfill $\mathbf{g}_+ = \mathbf{g}_- \in L^1((0,T); W^{1,\infty}(\mathbb{T}^n;\mathbb{R}^+))$ and solves the transport equation*

$$\partial_t \mathbf{g}_\pm(t,P,x) + (P + \nabla_x v_\pm(P,x))\cdot\nabla_x \mathbf{g}_\pm(t,P,x) = 0 \quad \text{for} \quad t \in \mathbb{R} \tag{5.32}$$

with initial datum $\mathbf{g}_\pm(0,P,x) := g(P,x)$.

Proof The regularity $v_\pm(P,\cdot) \in C^2(\mathbb{T}^n;\mathbb{R})$ implies the C^1-regularity of the vector field $P + \nabla_x v_\pm(P,\cdot)$ on $\mathbb{T}^n$. By standard transport PDE arguments (see for example [1]) it follows the statement. □

Appendix

Lemma 6.1 *Let $\hat{H}_\hbar := -\frac{1}{2}\hbar^2\Delta_x + V(x)$, $H := \frac{1}{2}|\eta|^2 + V(x)$ and $\mathrm{Op}_\hbar^w(H)$ as in (2.3). Then,*

$$\mathrm{Op}_\hbar^w(H)\psi = \hat{H}_\hbar\psi, \quad \forall\,\psi \in C^\infty(\mathbb{T}^n;\mathbb{C}). \tag{6.1}$$

Proof To begin, we recall that for $b \in S^2(\mathbb{T}^n \times \mathbb{R}^n)$

$$\mathrm{Op}_\hbar^w(b)\psi(x) = (\sigma(X,D)\circ T_x\,\psi)(x). \tag{6.2}$$

where $(T_x\psi)(y) := \psi(2y - x)$ and $\sigma \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \triangle_\eta^\alpha D_y^{(\alpha)} b(y, \hbar\eta/2)\big|_{y=x}$ see Sect. 2.1. Moreover, it is easily proved that when $b = \frac{1}{2}|\eta|^2 + V(x)$

$$\sigma(X, D)\psi = \hat{H}_\hbar \psi \tag{6.3}$$

for $\psi \in C^\infty(\mathbb{T}^n; \mathbb{C})$ and that $(\sigma(X, D) \circ T_x\psi)(x) = \hat{H}_\hbar \psi(x)$. □

Remark 6.2 The operator $\hat{H}_\hbar : H^2(\mathbb{T}^n; \mathbb{C}) \to L^2(\mathbb{T}^n; \mathbb{C})$ is linear, selfadjoint and continuous. Hence, by standard results of evolution equations in Banach spaces, the solution of the Schrödinger equation (3.1) fulfills $\psi_\hbar \in C^0(\mathbb{R}; H^2(\mathbb{T}^n; \mathbb{C})) \cap C^1(\mathbb{R}; L^2(\mathbb{T}^n; \mathbb{C}))$. The one parameter group of unitary operators $e^{-\frac{i}{\hbar}\mathrm{Op}_\hbar^w(H)t}$ can be defined on $L^2(\mathbb{T}^n; \mathbb{C})$ and $e^{-\frac{i}{\hbar}\mathrm{Op}_\hbar^w(H)t}\varphi \in C^0(\mathbb{R}; L^2(\mathbb{T}^n; \mathbb{C}))$ (see for example [28]).

The following result is shown in [31].

Lemma 6.3 *Let X be a metric space. Let $d\mu_j$ $j \in \mathbb{N}$ and $d\mu$ Borel probability measures on X such that $d\mu_j \overset{w-\star}{\longrightarrow} d\mu$ as $j \to +\infty$. Let $f_k, f : X \longrightarrow \mathbb{R}$ ($k \in \mathbb{N}$) be Borel measurable functions such that*

$$\lim_{\lambda \to +\infty} \sup_{k \in \mathbb{N}} \int_{\{x \in X;\ |f_k(x)| > \lambda\}} |f_k(x)|\, d\mu_k(x) = 0. \tag{6.4}$$

Let

$$E := \Big\{ x \in X;\ \exists \{x_k\}_{k \in \mathbb{N}} \subset X,\ \lim_{k \to +\infty} x_k = x,\ \lim_{k \to +\infty} f_k(x_k) \neq f(x) \Big\}. \tag{6.5}$$

If $\mu(E) = 0$ then

$$\lim_{j \to +\infty} \int_X f_j(x) d\mu_j(x) = \int_X f(x) d\mu(x).$$

References

1. Ambrosio, L., Gigli, N., Savaré, G.: Gradient Flows in Metric Spaces and in the Space of Probability Measures, vol. 2. Birkhäuser Verlag, Basel (2008)
2. Ambrosio, L.: Lecture Notes in Mathematics. Transport equation and Cauchy problem for non-smooth vector fields. Calculus of variations and nonlinear partial differential equations, pp. 1–41. Springer, Berlin (2008)
3. Ambrosio, L., Friesecke, G., Giannoulis, J.: Passage from quantum to classical molecular dynamics in the presence of Coulomb interactions. Commun. Partial. Differ. Equ. **35**, 1490–1515 (2010)
4. Ambrosio, L., Figalli, A., Friesecke, G., Giannoulis, J., Paul, T.: Semiclassical limit of quantum dynamics with rough potentials and well posedness of transport equations with measure initial data. Commun. Pure Appl. Math. **64**(9), 1199–1242 (2011)
5. Bardos, C., Golse, F., Markowich, P., Paul, T.: Hamiltonian Evolution of Monokinetic Measures with Rough Momentum Profile (2012). http://www.math.polytechnique.fr/~paul/
6. Bernard, P.: On the number of Mather measures of Lagrangian systems. Arch. Ration. Mech. Anal. **197**(3), 1011 (2010)

7. Contreras, G., Iturriaga, R., Paternain, G.P., Paternain, M.: Lagrangian graphs, minimizing measures and Mane critical values. Geom. Funct. Anal. **5**, 788–809 (1998)
8. Crandall, M.G., Lions, P.L.: Viscosity solutions of Hamilton–Jacobi equations. Trans. Am. Math. Soc. **277**, 1–42 (1983)
9. Duistermaat, J.J., Kolk, J.A.C.: Distributions, Theory and Applications. Birkhäser, Boston (2010)
10. Evans, L.C.: Effective Hamiltonians and quantum states, Seminaire: Équations aux Dérivées Partielles, 2000–2001, Exp. No. XXII, 13 pp, École Polytech., Palaiseau (2001)
11. Evans, L.C.: Towards a quantum analog of weak KAM theory. Commun. Math. Phys. **244**(2), 311–334 (2004)
12. Evans, L.C.: Further PDE methods for weak KAM theory. Calc. Var. Partial Differ. Equ. **35**(4), 435–462 (2009)
13. Evans, L.C., Gomes, D.: Effective Hamiltonians and averaging for Hamiltonian dynamics I. Arch. Ration. Mech. Anal. **157**, 1–33 (2001)
14. Fathi, A.: Weak KAM Theorem in Lagrangian Dynamics, Pisa Version 16 (2005)
15. Fathi, A., Giuliani, A.: Uniqueness of invariant Lagrangian graphs in a homology or a cohomology class. Ann. Sc. Norm. Sup. Pisa Cl. Sci. **8**, 659–680 (2009)
16. Folland, G.: Harmonic Analysis in Phase Space, Annals of Mathematics Studies, p. 122. Princeton University Press, Princeton, NJ (1989)
17. Gérard, P., Markowich, P., Mauser, N.J., Poupaud, F.: Homogenization limits and Wigner transforms. Commun. Pure Appl. Math. **50**(4), 323–379 (1997)
18. Graffi, S., Paul, T.: Convergence of a quantum normal form and an exact quantization formula. J. Funct. Anal. **262**(7), 3340–3393 (2012)
19. Hörmander, L.: The Analysis of Linear Partial Differential Operators, vol. 2. Springer, Berlin (1983)
20. Lions, P.L.: Viscosity solutions of fully nonlinear second order equations and optimal stochastic control infinite dimensions. Acta Mathematica **161**(1), 243–278 (1988)
21. Lions, P.-L.: Lecture Notes in Mathematics. Viscosity solutions of fully nonlinear second order equations and optimal stochastic control in infinite dimensions. II. Optimal control of Zakai's equation. Stochastic partial differential equations and applications, II (Trento, 1988), pp. 147–170. Springer, Berlin (1989)
22. Lions, P.-L., Paul, T.: Sur les mesures de Wigner. (French) [On Wigner measures]. Rev. Mat. Iberoamericana **9**(3), 553 (1993)
23. Manè, R.: On the minimizing measures of Lagrangian dynamical systems. Nonlinearity **5**(3), 623–638 (1992)
24. Markowich, P., Paul, T., Sparber, C.: On the dynamics of Bohmian measures. Arch. Ration. Mech. Anal. **205**(3), 1031–1054 (2012)
25. Martinez, A.: An Introduction to Semiclassical and Microlocal Analysis. Springer, New York (2002)
26. Mather, J.N.: Action minimizing invariant measures for positive definite Lagrangian systems. Math. Z. **207**, 169–207 (1991)
27. Parmeggiani, A., Zanelli, L.: Wigner measures supported on weak KAM tori. Journal d'Analyse Mathematiques (in press)
28. Reed, M., Simon, B.: Methods of Modern Mathematical Physics 1, Functional Analysis, Revised and Enlarged version. Academic Press, New York
29. Ruzhansky, M., Turunen, V.: Quantization of pseudo-differential operators on the torus. J. Fourier Anal. Appl. **16**(6), 943–982 (2010)
30. Sorrentino, A.: Lecture notes on Mather's theory for Lagrangian systems. To be published by Princeton University Press. arXiv:1011.0590
31. Yang, X.: Integral convergence related to weak convergence of measures. Appl. Math. Sci. **5**, 2775–2779 (2011)
32. Wilcox, C.H.: Theory of Bloch waves. J. d'Analyse Math. **33**, 146–167 (1978)