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A Note on Spaces of Absolutely Convergent Fourier Transforms

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Abstract Let $\mathcal{F}f$ be an abolutely convergent Fourier transform on the real line. We extend the following result of K. Karlander to $\mathbf{R}^n$ for $n \geq 1$: Any closed reflexive subspace $\{\mathcal{F}f\}$ of the space of continuous functions vanishing at infinity is of finite dimension.

Keywords Absolutely convergent Fourier transforms · Reflexivity · Weakly sequential completeness

Mathematics Subject Classification 42A38 · 42A55 · 42B10 · 42B35 · 46E15

1 Introduction

1.1. The notation used in this note is explained in Sect. 2. The space of absolutely convergent Fourier transforms $\left[\mathcal{F}L^1\right](\Xi)$ is dense in the space $\overline{\mathcal{C}_0}(\Xi)$ of continuous functions which vanish at infinity. Here Ξ is the dual group of a locally compact abelian group X. See Segal [17, Lemma 2, p. 158]. Hence we may claim that sufficiently many functions are absolutely convergent Fourier transforms. On the other hand, $\left[\mathcal{F}L^1\right](\Xi)$ either coincides with $\overline{\mathcal{C}_0}(\Xi)$ or is of the first category in that space. This follows from a theorem on bounded linear mappings (cf. Banach [2, Théorème 3, p. 38], [3, Theorem 3, p. 24]) applied to the *Fourier transformation*. $\left[\mathcal{F}L^1\right](\Xi)$ being of the first category manifests that few functions are absolutely convergent Fourier tranforms. A further indication that there are few such functions is the following result:

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Theorem 1.1.1 (K. Karlander [10]) *Let $\mathcal{Y}$ be a closed subspace of the space of continuous functions on the real line which vanish at infinity. Assume that all elements in $\mathcal{Y}$ are absolutely convergent Fourier transforms and that $\mathcal{Y}$ is reflexive. Then $\mathcal{Y}$ is of finite dimension.*

1.2. The purpose of this note is to extend Theorem 1.1.1 in as much as we replace $L^1(\mathbf{R})$ by $L^1(\mathbf{R}^n)$ for $n \geq 1$. See Theorem 3.1. To prove this theorem we extract two lemmata. See Sects. 4.1 and 4.4. The proof follows the ideas in [10] closely.

1.3. The space $\overline{\mathcal{C}_0}(\mathbf{R}^n)$ has closed reflexive subspaces of infinite dimension. (Cf. the theorem of Banach and Mazur [2, Théorème 9, p. 185], [3, Theorem 9, p. 112].) Hence it is meaningful to consider the problem discussed in Theorem 1.1.1.

Let $\mathcal{Y}$ be a closed subspace of $\overline{c_0}(\mathbf{Z}^n)$ of infinite dimension. Then $\mathcal{Y}$ is not reflexive. Cf. Sect. 2.3. Hence the problem discussed in Theorem 1.1.1 is not meaningful if $L^1(\mathbf{R})$ is replaced by $L^1(\mathbf{T}^n)$.

2 Notation and Preparation

2.1. Function Spaces. $c_0(M)$ is the set of all *functions* a on any set M such that a vanishes except at finitely many points. For the function e_m, an element in the canonical basis, the only exception is $e_m(m) = 1$. By $\overline{c_0}(M)$ we denote the completion of $c_0(M)$ under the uniform norm.

By $\overline{\mathcal{C}_0}(\mathbf{R}^n)$ we denote the completion of $\mathcal{C}_0(\mathbf{R}^n)$, the space of functions F continuous on $\mathbf{R}^n$ such that supp F is compact, under the uniform norm. By X we denote a locally compact abelian group with dual group Ξ. To obtain $\overline{\mathcal{C}_0}(\Xi)$ we apply the previous definition regarding completion. Spaces of continuous functions on certain compact sets will occur and their definitions and norms are obvious.

In most cases when the space $L^1(S, \Sigma, \mu)$ occurs S will be either $\mathbf{R}^n$ or $\mathbf{T}^n$ endowed with Lebesgue measure. We will then write $L^1(\mathbf{R}^n)$ and $L^1(\mathbf{T}^n)$ respectively. If μ is the counting measure then $L^1(S, \Sigma, \mu) = \ell^1(S)$.

2.2. The Fourier Transformation. By $e^{ix\xi}$ we denote the action of $\xi \in \Xi$ on $x \in X$. For $f \in L^1(X)$ the integral

$$[\mathcal{F}f](\xi) = \int_X \overline{e^{ix\xi}} f(x)\, dx$$

with respect to the Haar measure is absolutely convergent for all $\xi \in \Xi$. The function $\mathcal{F}f$ is the *Fourier transform* of f. The mapping $\mathcal{F}$, the *Fourier transformation,* is bounded, linear and injective $L^1(X) \longrightarrow \overline{\mathcal{C}_0}(\Xi)$ with range $\left[\mathcal{F}L^1\right](\Xi)$.

2.3. Let μ be a positive measure. The space $L^1(S, \Sigma, \mu)$ is weakly sequentially complete. For $S = [a, b]$ endowed with Lebesgue measure this result is a theorem of Steinhaus Cf. [18]. For the general case see e.g. Dunford and Schwartz [5, Theorem IV.8.6, p. 290]. In $\overline{c_0}(\mathbf{Z}_+)$ the sequence $(e_1 + e_2 + \cdots + e_m)_{m \in \mathbf{Z}_+}$ is weakly Cauchy but does not converge weakly. Hence $\overline{c_0}(\mathbf{Z}_+)$ is not weakly sequentially complete.

We will use the following result whose proof is straightforward: A closed subspace of a weakly sequentially complete Banach space is weakly sequentially complete.

The proof of the following standard result may be found e.g. in Lax [12, Theorem 15, p. 82]: A closed subspace of a reflexive Banach space is reflexive. In conjuction with Lindenstrauss and Tzafriri [14, Proposition 2.a.2, p. 53] this gives the following useful result: Neither $\ell^1(\mathbf{Z}^n)$ nor $\overline{c_0}(\mathbf{Z}_+)$ has reflexive subspaces of infinite dimension.

3 Main Theorem and Earlier Results

Theorem 3.1 *Let $\mathcal{Y}$ be a closed subspace of $\overline{\mathcal{C}_0}(\mathbf{R}^n)$ such that $\mathcal{Y}$ is a subset of $[\mathcal{F}L^1](\mathbf{R}^n)$. If $\mathcal{Y}$ is reflexive then it is of finite dimension.*

3.2. The result on *density* mentioned in Sect. 1.1 extends to the transformation of Fourier and Stieltjes. See Hewitt [9, Theorem 1.3, p. 664].

In addition to the *category* result in Sect. 1.1 we have the following: If $[\mathcal{F}L^1](\Xi)$ coincides with the codomain, then X is finite. See Segal [17, p. 157]. Cf. also Edwards [6], Rajagopalan [15], Friedberg [7], Graham [8] and Basit [4]. A similar result is valid for the transformation of Fourier and Stieltjes. See Hewitt [9, Theorem 1.2, p. 664]. Cf. also Edwards [6].

The result in [10] extended in Theorem 3.1 is of the following nature: On $\mathcal{Y}$, a closed subspace of the codomain of $\mathcal{F}$, we impose additional conditions. The conclusion is that $\mathcal{Y}$ is of *finite dimension*. It is hence relevant to relate our result to earlier results which are of a similar pattern.

Theorem 3.2.1 (Rajagopalan [15, Theorem 2, p. 87] and Sakai [16, Proposition 2, p. 661].) *Let $\mathcal{Y}$ be a weakly sequentially complete C^*-algebra. Then $\mathcal{Y}$ is of finite dimension.*

Theorem 3.2.2 (Cf. Albiac and Kalton [1, Corollary 2.3.8, p. 38].) *Let $\mathcal{Y}$ be a Banach space such that every weakly convergent sequence is convergent. If $\mathcal{Y}$ is reflexive then it is of finite dimension.*

3.3. In the survey paper by Liflyand et al. [13] *necessary conditions* and *sufficient conditions* are given for a function to be an absolutely convergent Fourier transform. It contains an extensive list of references and also a table of functions which are absolutely convergent Fourier transforms.

4 Two Lemmata for the Proof of the Main Result

Lemma 4.1 *Assume that $\mathcal{Y}$ is a closed subspace of $\overline{\mathcal{C}_0}(\mathbf{R}^n)$ such that $\mathcal{Y}$ is a subset of $[\mathcal{F}L^1](\mathbf{R}^n)$. If $\overline{B^n}$ denotes the closed unit ball of $\mathbf{R}^n$, then there are positive numbers α and $C_{4.1}$ independent of $\mathcal{F}f \in \mathcal{Y}$ such that*

$$\|\mathcal{F}f\|_{\overline{\mathcal{C}_0}(\mathbf{R}^n)} \leq C_{4.1} \|\mathcal{F}f\|_{\mathcal{C}(\alpha\overline{B^n})}. \tag{4.1}$$

Proof To simplify notation we write $\overline{\mathcal{C}_0}$, L^1 and c_0 instead of $\overline{\mathcal{C}_0}(\mathbf{R}^n)$, $L^1(\mathbf{R}^n)$ and $c_0(\mathbf{Z}_+)$ respectively. □

4.1.1. Let $\mathcal{X} = \mathcal{F}^{-1}\mathcal{Y}$. For all $f \in \mathcal{X}$ we have

$$\|\mathcal{F}f\|_{\overline{\mathcal{C}_0}} \le \|f\|_{L^1} \tag{4.2}$$

and $\mathcal{X}$ is a closed subspace of L^1. According to the open mapping theorem there is a number $C_{4.3}$ independent of $f \in \mathcal{X}$ such that

$$\|f\|_{L^1} \le C_{4.3}\,\|\mathcal{F}f\|_{\overline{\mathcal{C}_0}}. \tag{4.3}$$

4.1.2. Let ε_m be a positive number for each $m \in \mathbf{Z}_+$ such that $\varepsilon = \sum_1^\infty \varepsilon_m < 1$. For the purpose of deriving a contradiction we assume that for each choice of positive numbers α and $C_{4.4}$ there is an $\mathcal{F}f \in \mathcal{Y}$ such that

$$C_{4.4}\,\|\mathcal{F}f\|_{\mathcal{C}(\alpha\overline{B^n})} < \|\mathcal{F}f\|_{\overline{\mathcal{C}_0}}. \tag{4.4}$$

As basis for a recursion we choose for $\alpha = \alpha_1 > 0$ and $C_{4.4} = 1/\varepsilon_1$ a function $\mathcal{F}f_1 \in \mathcal{X}$ with $\|\mathcal{F}f_1\|_{\overline{\mathcal{C}_0}} = 1$ such that

$$\|\mathcal{F}f_1\|_{\mathcal{C}(\alpha_1\overline{B^n})} < \varepsilon_1.$$

If $\mathcal{F}f_l \in \mathcal{Y}$ with $\|\mathcal{F}f_l\|_{\overline{\mathcal{C}_0}} = 1$ for $l \in \{1,\dots,m\}$ as well as $\alpha_m > 0$ have been chosen we choose α_{m+1} so that

$$\sup\{|[\mathcal{F}f_l](\xi)| : l \in \{1,\dots,m\},\ |\xi| \ge \alpha_{m+1}\} \le \varepsilon_{m+1} \quad \text{and} \quad \alpha_{m+1} > \alpha_m.$$

By our assumption we can find for $\alpha = \alpha_{m+1}$ and $C_{4.4} = 1/\varepsilon_{m+1}$ a function $\mathcal{F}f_{m+1} \in \mathcal{Y}$ with $\|\mathcal{F}f_{m+1}\|_{\overline{\mathcal{C}_0}} = 1$ such that

$$\|\mathcal{F}f_{m+1}\|_{\mathcal{C}(\alpha_{m+1}\overline{B^n})} < \varepsilon_{m+1}.$$

We have thus constructed the set $\Phi = \{f_m \in \mathcal{X} : m \in \mathbf{Z}_+\} \subset L^1$ such that

$$\sup\{|[\mathcal{F}f_l](\xi)| : l \in \{1,\dots,m\},\ |\xi| \ge \alpha_{m+1}\} \le \varepsilon_{m+1}, \tag{4.5}$$

$$\|\mathcal{F}f_m\|_{\overline{\mathcal{C}_0}} = 1 \tag{4.6}$$

and

$$\|\mathcal{F}f_{m+1}\|_{\mathcal{C}(\alpha_{m+1}\overline{B^n})} < \varepsilon_{m+1} \tag{4.7}$$

for each $m \in \mathbf{Z}_+$ as well as the increasing sequence $(\alpha_m)_{m\in\mathbf{Z}_+}$ of positive numbers.

4.1.3. For each $m \in \mathbf{Z}_+$ we choose $b_m \in \mathbf{R}^n$ such that $|[\mathcal{F}f_m](b_m)| = \|\mathcal{F}f_m\|_{\overline{\mathcal{C}_0}} = 1$. Then $\alpha_m < |b_m| < \alpha_{m+1}$. Given any $a \in c_0$ let $N \in \mathbf{Z}_+$ and $k \in \mathbf{Z}_+$ be such that

$a(N+r)=0$ for all $r \in \mathbf{Z}_+$ and $|a(k)\,[\mathcal{F} f_k]\,(b_k)| = \|a\|_{\overline{c_0}}$. Write

$$\|a\|_{\overline{c_0}} = \left| \sum_{m=1}^{N} a(m)\,[\mathcal{F} f_m]\,(b_k) - {\sum_{m=1}^{N}}' a(m)\,[\mathcal{F} f_m]\,(b_k) \right|,$$

where we have omitted the term $a(k)\,[\mathcal{F} f_k]\,(b_k)$ from the second sum. We apply the triangle inequality to get

$$\|a\|_{\overline{c_0}} \le \left\| \sum_{1}^{N} a(m)\mathcal{F} f_m \right\|_{\overline{\mathcal{C}_0}} + |a(k)| \left(\sum_{1}^{k-1} |[\mathcal{F} f_m]\,(b_k)| + \sum_{k+1}^{N} |[\mathcal{F} f_m]\,(b_k)| \right).$$

In the parenthesis we use (4.5) and (4.7) respectively for the first and second group of terms respectively. (If $k=1$ or $k=N$ then there is only one group of terms.) We get

$$\|a\|_{\overline{c_0}} < \left\| \sum_{1}^{N} a(m)\mathcal{F} f_m \right\|_{\overline{\mathcal{C}_0}} + \|a\|_{\overline{c_0}}\,\varepsilon.$$

We have proved that there is a number $C_{4.8}$ independent of $a \in c_0$ such that

$$\|a\|_{\overline{c_0}} \le C_{4.8} \left\| \sum_{m \in \mathbf{Z}_+} a(m)\mathcal{F} f_m \right\|_{\overline{\mathcal{C}_0}} . \tag{4.8}$$

4.1.4. There is a vector $b \in \mathbf{R}^n$ such that

$$\left\| \sum_{1}^{N} a(m)\mathcal{F} f_m \right\|_{\overline{\mathcal{C}_0}} = \left| \sum_{1}^{N} a(m)\,[\mathcal{F} f_m]\,(b) \right| \le \|a\|_{\overline{c_0}} \sum_{1}^{N} |[\mathcal{F} f_m]\,(b)|\,.$$

Furthermore, there is a unique positive integer k such that $\alpha_k \le |b| < \alpha_{k+1}$. Hence we write

$$\left\| \sum_{1}^{N} a(m)\mathcal{F} f_m \right\|_{\overline{\mathcal{C}_0}} \le \|a\|_{\overline{c_0}} \left[\left(\sum_{1}^{k-1} + \sum_{k+1}^{N} \right) |[\mathcal{F} f_m]\,(b)| + |[\mathcal{F} f_k]\,(b)| \right].$$

In the parenthesis we again use (4.5) and (4.7) respectively for the first and second group of terms respectively. We get

$$\left\| \sum_{1}^{N} a(m)\mathcal{F} f_m \right\|_{\overline{\mathcal{C}_0}} < \|a\|_{\overline{c_0}}\,(\varepsilon + 1)\,.$$

We have proved that there is a number $C_{4.9}$ independent of $a \in c_0$ such that

$$\left\| \sum_{m \in \mathbf{Z}_+} a(m) \mathcal{F} f_m \right\|_{\overline{\mathcal{C}_0}} \leq C_{4.9} \|a\|_{\overline{c_0}}. \tag{4.9}$$

4.1.5. For all $a \in c_0$ we have

$$\|a\|_{\overline{c_0}} \leq C_{4.8} \left\| \sum_{m \in \mathbf{Z}_+} a(m) f_m \right\|_{L^1} \tag{4.10}$$

according to (4.8) and (4.2) respectively. On the other hand, for all $a \in c_0$ we have

$$\left\| \sum_{m \in \mathbf{Z}_+} a(m) f_m \right\|_{L^1} \leq C_{4.3} C_{4.9} \|a\|_{\overline{c_0}} \tag{4.11}$$

according to (4.3) and (4.9) respectively.

4.1.6. Let $\mathcal{X}_1$ be the closed linear span of Φ in L^1. According to (4.10) and (4.11) respectively the mapping $e_m \mapsto f_m$ can be extended to an isomorphism $\overline{c_0} \longrightarrow \mathcal{X}_1$. But $\mathcal{X}_1$ is as a closed subspace of a weakly sequentially complete space itself weakly sequentially complete. This contradicts the aforementioned isomorphy. (Cf. Sect. 2.3.)

4.2. In the preceding proof we use only boundedness and linearity properties of the Fourier transformation $\mathcal{F}$. Symmetry properties of that linear mapping are not needed for the argument.

Proposition 4.2.1 *Let $\mathcal{W}$ be a weakly sequentially complete Banach space and let the mapping T: $\mathcal{W} \longrightarrow \overline{\mathcal{C}_0}(\mathbf{R}^n)$ be bounded, linear and injective. Assume that $\mathcal{Y}$ is a closed subspace of $\overline{\mathcal{C}_0}(\mathbf{R}^n)$ such that $\mathcal{Y}$ is a subset of $T(\mathcal{W})$. Then there are positive numbers α and C independent of $F \in \mathcal{Y}$ such that*

$$\|F\|_{\overline{\mathcal{C}_0}(\mathbf{R}^n)} \leq C \|F\|_{\mathcal{C}(\alpha \overline{B^n})}.$$

4.3. Notation. For each $c = (c_1, \ldots, c_n) \in \mathbf{Z}^n$ let

$$Q_c = \{x \in \mathbf{R}^n : c_k \leq x_k \leq c_k + 1,\ k \in \{1, 2, \ldots, n\}\}.$$

As c ranges over $\mathbf{Z}^n$ we obtain the collection $\mathcal{Q}$ of sets Q_c whose union is $\mathbf{R}^n$. The intersection of a pair of terms in this union has Lebesgue measure 0. If β is a positive number we replace c_k and $c_k + 1$ by βc_k and $\beta(c_k + 1)$ respectively so as to obtain βQ_c. For $\beta \neq 1$ the union of βQ_c has the same disjointness property as for $\beta = 1$. It is clear that

$$\sup\left\{\left|x - x'\right| : x, x' \in \beta Q,\ Q \in \mathcal{Q}\right\} \leq \beta\sqrt{n}. \tag{4.12}$$

Lemma 4.4 *Let $\mathcal{X}$ be a reflexive subspace of $L^1(\mathbf{R}^n)$ of infinite dimension. Then for each choice of positive numbers β and $C_{4.14}$ there is an $f \in \mathcal{X}$ such that*

$$\sum_{Q \in \mathcal{Q}} \left| \int_{\beta Q} f \right| < C_{4.14} \, \|f\|_{L^1(\mathbf{R}^n)} \, . \tag{4.13}$$

Proof Assume that there is a choice of positive numbers β and $C_{4.14}$ independent of $f \in \mathcal{X}$ such that the opposite of (4.13) holds. Since we also have

$$\sum_{Q \in \mathcal{Q}} \left| \int_{\beta Q} f \right| \leq \|f\|_{L^1(\mathbf{R}^n)}$$

the mapping $T : \mathcal{X} \longrightarrow \ell^1(\mathbf{Z}^n)$ given by

$$[Tf](c) = \int_{\beta Q_c} f \, , \quad c \in \mathbf{Z}^n$$

is an isomorphism between the reflexive space $\mathcal{X}$ and a subspace of $\ell^1(\mathbf{Z}^n)$. This is impossible. (Cf. Sect. 2.3.)

5 Proof of Theorem 3.1

5.1. For each $f \in L^1 = L^1(\mathbf{R}^n)$, for each $c \in \mathbf{Z}^n$ and for each $\beta > 0$ we have with the notation from Sect. 4.3

$$\left| \int_{\beta Q_c} e^{-ix\xi} f(x) \, dx \right| \leq \int_{\beta Q_c} \left| e^{-ix\xi} - e^{-iq_c\xi} \right| |f(x)| \, dx \, + \\ + \left| e^{-iq_c\xi} \int_{\beta Q_c} f \right| \leq \int_{\beta Q_c} |x - q_c| \, |\xi| \, |f(x)| \, dx + \left| \int_{\beta Q_c} f \right|$$

for any $q_c \in Q_c$. Summing with respect to c, taking supremum with respect to ξ and invoking (4.12) gives

$$\|\mathcal{F}f\|_{\mathcal{C}(\alpha\overline{B^n})} \leq \sup_{|\xi| \leq \alpha} \sum_{c \in \mathbf{Z}^n} \left| \int_{\beta Q_c} e^{-ix\xi} f(x) \, dx \right| \leq \alpha\beta\sqrt{n} \, \|f\|_{L^1} + \sum_{c \in \mathbf{Z}^n} \left| \int_{\beta Q_c} f \right| . \tag{5.1}$$

5.2. Assume that $\mathcal{Y}$ fulfills the assumptions of the theorem. As in the proof of Lemma 4.1 (cf. Sect. 4.1.1) the space $\mathcal{X} = \mathcal{F}^{-1}\mathcal{Y}$ is according to the open mapping theorem isomorphic to $\mathcal{Y}$. We have

$$\|f\|_{L^1} \leq C_{4.3} \, \|\mathcal{F}f\|_{\overline{\mathcal{C}_0}(\mathbf{R}^n)} \leq C_{4.3} C_{4.1} \, \|\mathcal{F}f\|_{\mathcal{C}(\alpha\overline{B^n})} \tag{5.2}$$

where we have used Lemma 4.1 in the second inequality. Collecting the estimates (5.1) and (5.2) gives that there is a number $C_{5.3}$ independent of f such that

$$\|f\|_{L^1} \le C_{5.3}\left[\alpha\beta\sqrt{n}\,\|f\|_{L^1} + \sum_{c\in\mathbf{Z}^n}\left|\int_{\beta Q_c} f\right|\right]. \tag{5.3}$$

5.3. By assumption, $\mathcal{Y}$ is reflexive. Hence $\mathcal{X}$ is reflexive. Assume that $\mathcal{X}$ is of infinite dimension. Then the assumptions of Lemma 4.4 are fulfilled, and hence for

$$\beta < \frac{1}{2\,C_{5.3}\,\alpha\sqrt{n}} \quad \text{and} \quad C_{4.14} = \frac{1}{2C_{5.3}}$$

there is an $f \in \mathcal{X}$ such that

$$\|f\|_{L^1} \le C_{5.3}\left[\alpha\beta\sqrt{n}\,\|f\|_{L^1} + \sum_{c\in\mathbf{Z}^n}\left|\int_{\beta Q_c} f\right|\right] < \frac{1}{2}\|f\|_{L^1} + \frac{1}{2}\|f\|_{L^1} = \|f\|_{L^1}.$$

6 A Closed Non-Reflexive Subspace

6.1. The following theorem is a corollary of Katznelson [11, Sect. 1.4, p. 137]. (Cf. also Zygmund [19, Vol. I, Theorem VI.6.1, p. 247].)

Theorem 6.1.1 *Let $a \in c_0(\mathbf{Z})$ and consider for a fixed number $q > 1$*

$$g(x) = \sum_{m\in\mathbf{Z}} a(m)e^{ix\xi_m}, \quad x \in \mathbf{T} = [-\pi, \pi[\tag{6.1}$$

with $\xi_{-m} = -\xi_m$, $\xi_1 > 0$, $\xi_{m+1} > q\xi_m$ and $\xi_m \in \mathbf{Z}$ for all $m \in \mathbf{Z}_+$. Let $\mathcal{G}$ be the set of all functions g appearing in (6.1) and let T be the linear mapping from $\mathcal{G}$ to $\ell^1(\mathbf{Z})$ given by $Tg = a$. Then T has a unique extension, which we also denote by T, to the closure of $\mathcal{G}$ in $\mathcal{C}(\mathbf{T})$ such that

$$\|Tg\|_{\ell^1(\mathbf{Z})} \le C\,\|g\|_{\mathcal{C}(\mathbf{T})}$$

where is C independent of g.

6.2. Within the class of spaces which are closed subspaces $\mathcal{Y}$ of $\overline{\mathcal{C}_0}(\mathbf{R}^n)$ and subsets of $\left[\mathcal{F}L^1\right](\mathbf{R}^n)$ we have proved that reflexivity implies finite dimensionality. The opposite implication is trivial. Following Karlander's idea in [10, p. 312] we will now for $\mathbf{R}^n$ provide an example of a closed non-reflexive subspace $\mathcal{Y}$.

6.3. Let H be any *positive* function in $\left[L^1 \cap \overline{\mathcal{C}_0}\right](\mathbf{R}^n)$ with $\|H\|_{\overline{\mathcal{C}_0}(\mathbf{R}^n)} = 1$. For $a \in \ell^1(\mathbf{Z})$ we define

$$[Ta](\xi) = H(\xi)\sum_{k\in\mathbf{Z}} a(k)e^{i2^{|k|}\xi_1}, \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n$$

and we let $\mathcal{Y}$ be the image of $\ell^1(\mathbf{Z})$ under the mapping T. Then Ta is the Fourier transform of an $L^1(\mathbf{R}^n)$-function. Furthermore we have $\|Ta\|_{\overline{\mathcal{C}_0}(\mathbf{R}^n)} \le \|a\|_{\ell^1(\mathbf{Z})}$ and so $T : \ell^1(\mathbf{Z}) \longrightarrow \mathcal{Y}$ is a bounded linear bijection. If we can show that $\mathcal{Y}$ is closed then $\mathcal{Y}$ and $\ell^1(\mathbf{Z})$ are isomorphic by the open mapping theorem. In particular, $\mathcal{Y}$ is not reflexive.

6.4. To show that $\mathcal{Y}$ is closed we assume that F_m is in $\mathcal{Y}$ for each $m \in \mathbf{Z}_+$ and that F_m converges to F in $\overline{\mathcal{C}_0}(\mathbf{R}^n)$ as $m \to \infty$. If $G_m = F_m/H$ then G_m converges to a continuos function G uniformly on each compact set as $m \to \infty$. But

$$G_m(\xi) = \sum_{k\in\mathbf{Z}} a_m(k)e^{i2^{|k|}\xi_1}$$

for some $a_m \in \ell^1(\mathbf{Z})$. We now invoke Theorem 6.1.1 to conclude that there is a function $a \in \ell^1(\mathbf{Z})$ such that a_m converges to a in $\ell^1(\mathbf{Z})$ as $m \to \infty$. For fixed $\xi \in \mathbf{R}^n$ we have

$$F(\xi) = \lim_{m\to\infty} H(\xi)\sum_{k\in\mathbf{Z}} a_m(k)e^{i2^{|k|}\xi_1} = H(\xi)\sum_{k\in\mathbf{Z}} a(k)e^{i2^{|k|}\xi_1}.$$

We have proved that $F \in \mathcal{Y}$ and hence $\mathcal{Y}$ is closed.

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