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A Note on Recent Papers by Grafakos and Teschl, and Estrada

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Abstract We indicate how recent results of Grafakos and Teschl (J Fourier Anal Appl 19:167–179, 2013), and Estrada (J Fourier Anal Appl 20:301–320, 2014), relating the Fourier transform of a radial function in $\mathbb{R}^n$ and the Fourier transform of the same function in $\mathbb{R}^{n+2}$ and $\mathbb{R}^{n+1}$, respectively, are located within known results on transplantation for Hankel transforms.

Keywords Radial function · Fourier transform · Hankel transform

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Working with Fourier transforms of radial functions in $\mathbb{R}^n$ indispensably leads to Hankel transforms. Indeed, it is well known that the Fourier transform of a radial function in $\mathbb{R}^n$, $n \geq 1$, reduces directly to the *modified Hankel transform* H_μ of order $\mu = n/2 - 1$. Moreover, the radial part of the standard Laplacian in $\mathbb{R}^n$ is the Bessel operator $L_\mu = \frac{d^2}{dx^2} + \frac{2\mu+1}{x}\frac{d}{dx}$, $\mu = n/2 - 1$, which is the natural 'Laplacian' in harmonic analysis associated with H_μ. More precisely, if f is a radial function on $\mathbb{R}^n$, $f(x) = f_0(|x|)$, then $\mathcal{F}_n f(x) = H_\mu f_0(|x|)$ and $\Delta_n f(x) = (L_\mu f_0)(|x|)$. Here $\mathcal{F}_n$

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denotes the Fourier transform on $\mathbb{R}^n$,

$$\mathcal{F}_n f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(y) \exp(-i\langle x, y\rangle)\, dy, \qquad x \in \mathbb{R}^n,$$

$\Delta_n = \sum_1^n \partial_j^2$ is the Laplacian in $\mathbb{R}^n$, and H_μ denotes the modified Hankel transform of order $\mu \geq -1/2$, the integral transform given by

$$H_\mu g(x) = \int_0^\infty g(y) \frac{J_\mu(xy)}{(xy)^\mu}\, dm_\mu(y), \qquad dm_\mu(y) = y^{2\mu+1} dy, \quad x > 0,$$

defined for appropriate functions on $\mathbb{R}_+ = (0, \infty)$; J_μ denotes here the Bessel function of the first kind of order μ. The Hankel transform possesses the well known properties: $(H_\mu \circ H_\mu)g = g$, and $\|H_\mu g\|_{L^2(\mathbb{R}_+, m_\mu)} = \|g\|_{L^2(\mathbb{R}_+, m_\mu)}$, both identities on appropriate subclass of functions, say, for $g \in \mathcal{S}(\mathbb{R}_+)$, the space of restrictions to $\mathbb{R}_+$ of even Schwartz functions on $\mathbb{R}$. It is known that H_μ is a continuous bijection of $\mathcal{S}(\mathbb{R}_+)$, see e.g. [3], and extends to an isometric isomorphism on $L^2(\mathbb{R}_+, m_\mu)$.

Now, given a function $g \in \mathcal{S}(\mathbb{R}_+)$, denote $g_\mu := H_\mu g$. Then, for $\mu, \nu \geq -1/2$,

$$g_\mu = T_\nu^\mu g_\nu,$$

where $T_\nu^\mu := H_\mu \circ H_\nu$ is the *transplantation operator*, see [5, p. 56], which is well defined on $\mathcal{S}(\mathbb{R}_+)$. (A comment: this transplantation operator does not fit into the framework of [4].) Its exact form is known: for $\nu > \mu \geq -1/2$,

$$T_\nu^\mu g(x) = c_{\nu,\mu} \int_x^\infty (y^2 - x^2)^{\nu-\mu-1} y g(y)\, dy, \qquad x > 0,$$

see [5, (3.9)], where $c_{\nu,\mu}$ is a constant independent on g (the proof is based on an integral formula expressing J_μ through J_ν). In particular,

$$g_\mu(x) = T_{\mu+1}^\mu g_{\mu+1}(x) = c_{\mu+1,\,\mu} \int_x^\infty y g_{\mu+1}(y)\, dy, \qquad x > 0,$$

which immediately shows that $\frac{d}{dx} g_\mu(x) = -c_{\mu+1,\,\mu}\, x g_{\mu+1}(x)$, and this, when specified to $\mu = n/2 - 1$, is the formula [2, Theorem 1.1 (1)].

Similarly,

$$g_\mu(x) = T^\mu_{\mu+1/2} g_{\mu+1/2}(x) = c_{\mu+1/2,\,\mu} \int_x^\infty (y^2 - x^2)^{-1/2} y g_{\mu+1/2}(y)\,dy$$
$$= c_{\mu+1/2,\,\mu} \int_0^\infty g_{\mu+1/2}\big(\sqrt{s^2 + x^2}\big)\,ds,$$

and this, when specified to $\mu = n/2 - 1$, is [1, (1.3)], one of the main results of [1]. Iterating this formula k times and then integrating in polar coordinates brings

$$g_\mu(x) = C_{\mu,\,k} \int_0^\infty \dots \int_0^\infty g_{\mu+k/2}\Big(\sqrt{s_1^2 + \dots + s_k^2 + x^2}\Big)\,ds_1 \dots ds_k$$
$$= C_{\mu,\,k}\, 2^{-k} \omega_{k-1} \int_0^\infty g_{\mu+k/2}\big(\sqrt{s^2 + x^2}\big) s^{k-1}\,ds,$$

which, when specified to $\mu = n/2 - 1$, is [1, (1.4)]; here $C_{\mu,\,k} = \prod_{j=1}^k c_{\mu+j/2,\,\mu+(j-1)/2}$ and ω_{k-1} is the surface area of the unit sphere S^{k-1} in $\mathbb{R}^k$.

We remark that the exact form of T^μ_ν is also known when $\mu > \nu \ge -1/2$:

$$T^\mu_\nu g(x) = d_{\nu,\,\mu} \frac{1}{x^{2\mu}} \int_0^x (x^2 - y^2)^{\mu-\nu-1} L_\nu^{\mu-\nu} g(y)\,dm_\mu(y), \qquad x > 0,$$

where L_ν^δ denotes the δ-fractional power of L_ν (or rather its natural self-adjoint extension) given spectrally on $\mathrm{Dom}(L_\nu^\delta) = \{f \in L^2 \colon (\cdot)^{2\delta} H_\nu f \in L^2\}$ by $H_\nu(L_\nu^\delta f)(y) = y^{2\delta} H_\nu f(y)$, see [5, p. 58]. In particular,

$$g_{\nu+1}(x) = T^{\nu+1}_\nu g_\nu(x) = d_{\nu,\,\nu+1} \frac{1}{x^{2\nu+2}} \int_0^x L_\nu g_\nu(y)\,dm_\mu(y), \qquad x > 0.$$

Finally, we mention that to keep this note compact we did not specify the exact values of the constants involved. Nevertheless, an inspection shows that the constants appearing in the relevant formulas above are consistent (after taking into account different normalizations of Fourier transforms) with those in [1, (1.2), (1.3), (1.4)]. Perhaps the simplest way to see this consistency is to employ a concrete g, say $g(x) = \exp(-x^2)$ (it is known that $g_\mu(x) = 2^{-\mu-1}\exp(-x^2/4)$). Also, to be concise we assumed $g \in \mathcal{S}(\mathbb{R}_+)$, but simple density arguments allow to enlarge applicability of the relevant formulas to more general classes of functions.

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