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Ulyanov-type Inequalities Between Lorentz–Zygmund Spaces

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Abstract We establish inequalities of Ulyanov-type for moduli of smoothness relating the source Lorentz–Zygmund space $L^{p,r}(\log L)^{\alpha-\gamma}$, $\gamma > 0$, and the target space $L^{p^*,s}(\log L)^{\alpha}$ over $\mathbb{R}^n$ if $1 < p < p^* < \infty$ and over $\mathbb{T}^n$ if $1 < p \le p^* < \infty$. The stronger logarithmic integrability (corresponding to $L^{p^*,s}(\log L)^{\alpha}$) is balanced by an additional logarithmic smoothness.

Keywords Moduli of smoothness · K-functionals · Lorentz–Zygmund spaces

Mathematics Subject Classification Primary 41A17 · 46E30 · Secondary 42B15 · 46E35

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1 Introduction

Limiting problems in analysis require much more refined scales of function spaces than the classical Lebesgue spaces. For example, significant improvements of classical inequalities, like the Hausdorff–Young inequality for the Fourier transform, has been obtained by using Lorentz $L^{p,r}$-spaces (see, e.g., [24]). These are interpolation spaces constructed via Peetre's K-functional (see [4]) and represent a refinement of the Lebesgue L^p-scale. A different extension of the L^p-scale, given by the Zygmund spaces $L^p(\log L)^\alpha$, has come into play, e.g., in [17] in order to obtain estimates of the eigenvalues of certain degenerate elliptic differential operators. It is also well known that the Sobolev space $W_k^p(\mathbb{R}^n)$ in the limiting situation when $p = \frac{n}{k} > 1$ is not embedded into the space $L^\infty(\mathbb{R}^n)$ but into the space $L^{\infty,p}(\log L)^{-1}(\mathbb{R}^n)$ as shown by Brézis and Wainger [5] and Hansen [22]. Thus, Lorentz–Zygmund spaces $L^{p,r}(\log L)^\alpha$ appear naturally even in the context of classical Sobolev spaces.

A different approach deriving embedding theorems for $L^p(\mathbb{T})$-functions with a certain smoothness was used by Ul'yanov [42,43]. The essential tool has been the inequality

$$\omega_k(f,\delta)_{p^*} \lesssim \left(\int_0^\delta [t^{-\sigma}\omega_k(f,t)_p]^{p^*}\frac{dt}{t}\right)^{1/p^*}, \qquad \frac{1}{p^*} = \frac{1}{p} - \sigma,$$
$$0 < \sigma < 1/p, \quad k \in \mathbb{N}, \tag{1.1}$$

for functions $f \in L^p(\mathbb{T})$, $1 \le p < \infty$, nowadays known as Ulyanov's inequality. Here the k-th order modulus of smoothness $\omega_k(f,\delta)_p$ is defined in the standard way by

$$\omega_k(f,\delta)_p := \sup_{|h|\le\delta} \|\Delta_h^k f\|_p, \quad \Delta_h f(x) = f(x+h) - f(x), \quad \Delta_h^k = \Delta_h \Delta_h^{k-1}.$$

Throughout the paper we use the notation $A \lesssim B$, with $A, B \ge 0$, for the estimate $A \le c\,B$, where c is a positive constant, independent of the appropriate variables in A and B. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ (and say that A is equivalent to B). For two normed spaces X and Y, we will use the notation $Y \hookrightarrow X$ if $Y \subset X$ and $\|f\|_X \lesssim \|f\|_Y$ for all $f \in Y$.

On the one hand, Ulyanov's approach gives sharp embedding results for certain degrees of smoothness, e.g., if $\omega_k(f,\delta)_p = O(\delta^\alpha)$, $0 < \alpha < k$—cf. Remark 1.3 (d) below. On the other hand, (1.1) has the obvious shortcoming that its left-hand side cannot decrease faster than $O(\delta^{k-\sigma})$ even for $f \in C^\infty(\mathbb{T})$; however, for any $f \in C^\infty(\mathbb{T})$ one has $\omega_k(f,\delta)_r = O(\delta^k)$, $1 \le r \le \infty$. A replacement of $\omega_k(f,\delta)_p$ in (1.1) by $\omega_{k+1}(f,\delta)_p$ leads to the contradiction $\omega_k(f,\delta)_{p^*} = O(\delta^{k+1-\sigma})$, $0 < \sigma < 1/p$, for smooth $f \in C^\infty(\mathbb{T})$.

Thus one has to use moduli of smoothness $\omega_\kappa(f,\delta)_p$ of fractional order $\kappa > 0$ of a function $f \in L^p(\mathbb{R}^n)$ (or $f \in L^p(\mathbb{T}^n)$), $1 \le p < \infty$, given by (cf. [7, p. 788])

$$\omega_\kappa(f,\delta)_p := \sup_{|h|\le\delta} \left\|\Delta_h^\kappa f(x)\right\|_{L^p}, \qquad \Delta_h^\kappa f(x) = \sum_{\nu=0}^{\infty}(-1)^\nu \binom{\kappa}{\nu} f\left(x+\nu h\right). \tag{1.2}$$

Then a typical sharp Ulyanov-type inequality for $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, reads as follows ([32], [39])

$$\omega_\kappa(f,\delta)_{p^*} \lesssim \left(\int_0^\delta [t^{-\sigma}\omega_{\kappa+\sigma}(f,t)_p]^{p^*} \frac{dt}{t} \right)^{1/p^*}, \qquad \frac{1}{p^*} = \frac{1}{p} - \frac{\sigma}{n}, \quad 0 < \sigma < n/p. \tag{1.3}$$

The importance of Ulyanov-type inequalities results from its relation to problems in the theory of function spaces, approximation theory, and interpolation theory—see, e.g., [10], [21], [23], [25], [32], [39]. The interplay of Ulyanov inequalities and embedding theorems shows that it is quite natural to consider the moduli of smoothness in the framework of the Lorentz–Zygmund spaces.

To define the Lorentz–Zygmund spaces $L^{p,r}(\log L)^\alpha(\mathbb{R}^n)$, $1 \le p, r \le \infty$, $\alpha \in \mathbb{R}$, we introduce the logarithmic function $\ell(t) = (1 + |\log t|)$, $t > 0$. A measurable function f belongs to the space $L^{p,r;\alpha} \equiv L^{p,r}(\log L)^\alpha(\mathbb{R}^n)$ if

$$\|f\|_{p,r;\alpha} := \begin{cases} \left(\int_0^\infty [t^{1/p}\,\ell^\alpha(t) f^*(t)]^r \frac{dt}{t} \right)^{1/r} < \infty &, \ r < \infty \\ \sup_{t>0} t^{1/p}\,\ell^\alpha(t) f^*(t) < \infty &, \ r = \infty, \end{cases}$$

where f^* denotes the non-increasing rearrangement of f. Thus $L^p = L^{p,p:0}$ and $\|f\|_p = \|f\|_{p,p;0}$. In the case of the torus, the integration extends over the interval $(0, 1)$—see [3, p. 253]; the Lorentz–Zygmund spaces are rearrangement invariant Banach function spaces if $p > 1$. For all these concepts see, e.g., [2], [3, Chap. 4, p. 253]. Likewise (1.2), the fractional modulus of smoothness of a function $f \in L^{p,r}(\log L)^\alpha(X)$, $X = \mathbb{R}^n$ or $X = \mathbb{T}^n$, is defined by

$$\omega_\kappa(f,\delta)_{p,r;\,\alpha} := \sup_{|h|\le\delta} \left\| \Delta_h^\kappa f(x) \right\|_{L^{p,r}(\log L)^\alpha(X)}.$$

The main goal of the paper is to prove sharp Ulyanov inequalities for the Lorentz–Zygmund spaces $L^{p,r}(\log L)^\alpha$ over $\mathbb{R}^n$ or $\mathbb{T}^n$. Let us first formulate and comment our two main results on the Euclidean space $\mathbb{R}^n$ (Theorem 1.1) and on the torus $\mathbb{T}^n$ (Theorem 1.2).

Theorem 1.1 *Let* $\kappa > 0$, $1 < p < \infty$, $0 < \sigma < n/p$, *and* $\alpha \in \mathbb{R}$.

(i) If $\gamma \ge 0$ *and* $1 \le r \le s \le \infty$, *then*

$$\omega_\kappa(f,\delta)_{p^*,s;\,\alpha} \lesssim \left(\int_0^\delta [t^{-\sigma}\ell^\gamma(t)\,\omega_{\kappa+\sigma}(f,t)_{p,r;\,\alpha-\gamma}]^s \frac{dt}{t} \right)^{1/s},$$
$$\delta \to 0+, \quad \frac{1}{p^*} = \frac{1}{p} - \frac{\sigma}{n}, \tag{1.4}$$

for all $f \in L^{p,r}(\log L)^{\alpha-\gamma}(\mathbb{R}^n)$.

(ii) If $\gamma < 0$ *and* $1 \le r \le s \le \infty$, *then inequality* (1.4) *holds only if* $f = 0$.

(iii) If $\gamma \ge 0$ *and* $1 \le s < r < \infty$, *then inequality* (1.4) *is not true for all* $f \in L^{p,r}(\log L)^{\alpha-\gamma}(\mathbb{R}^n)$.

Theorem 1.1 (i) shows how the logarithmic component in smoothness on the right-hand side of (1.4) leads to an additional logarithmic integrability on its left-hand side.

In the next theorem, concerning results on the torus $\mathbb{T}^n$, we consider not only the sublimiting case $0 < \sigma < n/p$, or equivalently, $p < p^* < \infty$ (part (a)) but also the limiting case $\sigma = 0$, or equivalently, $p = p^*$ (part (b)).

Theorem 1.2 *(a) Let* $\kappa > 0$, $1 < p < \infty$, $0 < \sigma < n/p$, $1 \le r \le s \le \infty$, $\alpha \in \mathbb{R}$, *and* $\gamma \ge 0$. *Then*

$$\omega_\kappa(f,\delta)_{p^*,s;\alpha} \lesssim \left(\int_0^\delta [t^{-\sigma}\ell^\gamma(t)\,\omega_{\kappa+\sigma}(f,t)_{p,r;\alpha-\gamma}]^s \frac{dt}{t}\right)^{1/s},$$

$$\delta \to 0+, \quad \frac{1}{p^*} = \frac{1}{p} - \frac{\sigma}{n}, \tag{1.5}$$

for all $f \in L^{p,r}(\log L)^{\alpha-\gamma}(\mathbb{T}^n)$. *Inequality* (1.5) *holds for* $\gamma < 0$ *only if* f *is constant.*

(b) Let $\kappa > 0$, $1 < p < \infty$, *and* $\alpha \in \mathbb{R}$.

(i) If $1 < r \le s < \infty$ *then, for all* $f \in L^{p,r}(\log L)^{\alpha-\gamma}(\mathbb{T}^n)$ *when* $\delta \to 0+$,

$$\omega_\kappa(f,\delta)_{p,s;\alpha} \lesssim \left(\int_0^\delta [\ell^{\gamma-1/s}(t)\,\omega_\kappa(f,t)_{p,r;\alpha-\gamma}]^s \frac{dt}{t}\right)^{1/s}$$

$$+\ell^\gamma(\delta)\,\omega_\kappa(f,\delta)_{p,r;\alpha-\gamma}, \quad \gamma > 0. \tag{1.6}$$

(ii) If $1 \le s < r < \infty$ *then, for all* $f \in L^{p,r}(\log L)^{\alpha-\gamma}(\mathbb{T}^n)$ *when* $\delta \to 0+$,

$$\omega_\kappa(f,\delta)_{p,s;\alpha} \lesssim \left(\int_0^\delta [\ell^{\gamma-1/r}(t)\,\omega_\kappa(f,t)_{p,r;\alpha-\gamma}]^s \frac{dt}{t}\right)^{1/s}$$

$$+\ell^{\gamma+1/s-1/r}(\delta)\,\omega_\kappa(f,\delta)_{p,r;\alpha-\gamma}, \quad \gamma > 1/r - 1/s. \tag{1.7}$$

Remark 1.3 (a) In the case $1 < p < \infty$, $1 \le s < r < \infty$, $n = 1$, $\kappa = 1$, and $\alpha = \gamma = 0$ Theorem 1.2 (b) (ii) is contained in [31, p. 336].

(b) The two terms on the right-hand side of (1.7) are independent of each other: Consider the case $p = r$, $\kappa > 0$, $\alpha = \gamma > 0$. There exists a sufficiently regular f such that $\omega_\kappa(f,t)_p \approx \ell^{-1/s+1/r-\gamma}(t)\big(\ell(\ell(t))\big)^{-\beta}$, where $\beta > 1/s$ (see [30, Thm. 2] for $\kappa \in \mathbb{N}$ and [34, Thm. 2.5] for $\kappa > 0$). Then the first integral term is equivalent to $\big(\ell(\ell(\delta))\big)^{1/s-\beta}$, while the second behaves like $\big(\ell(\ell(\delta))\big)^{-\beta}$. Next, if $\omega_\kappa(f,t)_p \approx t^\kappa$, then the first term leads to $\ell^{\gamma-1/r}(\delta)\,\delta^\kappa$, while the second one to $\ell^{1/s-1/r+\gamma}(\delta)\,\delta^\kappa$. Analogously, the independence of the two terms on the right-hand side of (1.6) can be shown: Consider $\omega_\kappa(f,t)_p \approx \ell^{-\gamma}(t)\big(\ell(\ell(t))\big)^{-\beta}$, $\beta > 1/s$, and $\omega_\kappa(f,t)_p \approx t^\kappa$.

(c) In the case $s = r = p$ and $n = 1$ estimate (1.6) is an improvement of

$$\omega_k(f,\delta)_{p,p;\gamma} \lesssim \int_0^\delta \ell^\gamma(u)\,\omega_k(f,u)_p \,\frac{du}{u}, \qquad \delta \to 0+, \;\; k \in \mathbb{N}, \;\; \gamma > 0,$$
$$f \in L^p(\mathbb{T}), \quad 1 < p < \infty,$$

(see [41]) which follows as a specification of an abstract Ulyanov-type inequality for semigroups in Banach spaces. Indeed,

$$\left(\int_0^\delta [\ell^{\gamma-1/r}(u)\omega_k(f,u)_p]^p \frac{du}{u}\right)^{1/p} \lesssim \left(\sum_{j=-\infty}^{0} \int_{2^{j-1}\delta}^{2^j\delta} [\ell^\gamma(u)\omega_k(f,u)_p]^p \,\frac{du}{u}\right)^{1/p}$$
$$\lesssim \left(\sum_{j=-\infty}^{0} \ell^{\gamma p}(2^{j-1}\delta)\omega_k^p(f,2^j\delta)_p\right)^{1/p} \lesssim \sum_{j=-\infty}^{0} \ell^\gamma(2^{j-1}\delta)\omega_k(f,2^j\delta)_p \int_{2^{j-1}\delta}^{2^j\delta} \frac{du}{u},$$

by the monotonicity properties of the modulus of smoothness. Here the last term is approximately $\int_0^\delta \ell^\gamma(u)\,\omega_k(f,u)_p\,\frac{du}{u}$. Moreover,

$$\ell^\gamma(\delta)\omega_k(f,\delta)_p \approx \ell^\gamma(\delta)\frac{\omega_k(f,\delta)_p}{\delta^k}\int_0^\delta u^{k-1}\,du \lesssim \int_0^\delta \ell^\gamma(u)\omega_k(f,u)_p\,\frac{du}{u}$$

since $\ell^\gamma(\delta)$ is decreasing and $\omega_k(f,\delta)_p/\delta^k$ is almost decreasing on $(0,1)$.

(d) As mentioned above, Ulyanov-type inequalities are closely related to embedding theorems for smooth function spaces (Lipschitz, Nikolskii–Besov, etc.). In particular, sharp Ulyanov inequalities imply the following embedding for the generalized Lipschitz spaces (Nikolskii spaces). Define

$$\mathrm{Lip}\,\big(\omega(\cdot), l, X\big) := \Big\{ f \in X : \omega_l(f,\delta)_X \le C\omega(\delta), \quad \delta \to 0+ \Big\},$$

where $\omega(\cdot)$ is a is non-decreasing function on $[0,1]$ such that $\omega(\delta) \to 0$ as $\delta \to 0+$ and $\delta^{-l}\omega(\delta)$ is non-increasing (see [34]). Then Theorem 1.1 implies that

$$\mathrm{Lip}\,\big(\omega(\cdot), \kappa+\sigma, L^{p,r}(\log L)^{\alpha-\gamma}(\mathbb{R}^n)\big) \subset \mathrm{Lip}\,\big(\widetilde{\omega}(\cdot), \kappa, L^{p^*,s}(\log L)^{\alpha}(\mathbb{R}^n)\big) \tag{1.8}$$

provided that

$$\left(\int_0^\delta [t^{-\sigma}\ell^\gamma(t)\,\omega(t)]^s \frac{dt}{t}\right)^{1/s} = O\Big(\widetilde{\omega}(t)\Big). \tag{1.9}$$

Moreover, as it was shown in [32], this estimate is sharp for the Lebesgue spaces, that is, in order that embedding (1.8) holds it is necessary that condition (1.9) is valid. This result and similar ones that can be obtained from Theorems 1.1-1.2 extend several known embedding theorems (see [14], [21], [31], [42,43]). For the case $p = 1$ or $q = \infty$ see also [15, Rem. 3.7], [35], [36].

(e) We mention one simple consequence of Theorems 1.1-1.2 for embeddings of the Besov spaces $B^{(p,r;\beta),s}_{\sigma,\gamma}$. To this end, we define the *Besov-type space* $B^{(p,r;\beta),s}_{\sigma,\gamma}(X)$, $X = \mathbb{R}^n$ or $X = \mathbb{T}^n$, $\beta, \gamma \in \mathbb{R}$, $s > 0$, by

$$B^{(p,r;\beta),s}_{\sigma,\gamma} := \Bigg\{ f \in L^{p,r}(\log L)^\beta : |f|_{B^{(p,r;\beta),s}_{\sigma,\gamma}} := \left(\int_0^1 [u^{-\sigma}\ell^\gamma(u)\omega_\eta(f,u)_{p,r;\beta}]^s \frac{du}{u}\right)^{1/s} < \infty \Bigg\}, \tag{1.10}$$

equipped with the norm

$$\|f\|_{B^{(p,r;\beta),s}_{\sigma,\gamma}} := \|f\|_{L^{p,r}(\log L)^\beta} + |f|_{B^{(p,r;\beta),s}_{\sigma,\gamma}},$$

where $\eta > \sigma \geq 0$. In the case $\alpha = \gamma = 0$, $p^* = s = r$, Theorem 1.1 (i) and Theorem 1.2 (a) immediately imply

$$B^{(p,p^*;0),p^*}_{\sigma,\mu+1/p^*} \hookrightarrow B^{(p^*,p^*;0),p^*}_{0,\mu}.$$

The corresponding result in [10, Cor. 5.3] gives only $B^{(p,p;0),p^*}_{\sigma,\mu+1} \hookrightarrow B^{(p^*,p^*;0),p^*}_{0,\mu}$. The general case with proofs are considered in Sect. 2.4 below.

In what follows a modification of the K-functional plays an essential role since they can be identified with the occurring moduli of smoothness. To make this more precise, introduce the Riesz potential space

$$H^{p,r;\alpha}_\lambda(\mathbb{R}^n) := \{g \in L^{p,r}(\log L)^\alpha(\mathbb{R}^n) : \ |g|_{H^{p,r;\alpha}_\lambda} := \|(-\Delta)^{\lambda/2} g\|_{p,r;\alpha} < \infty\}, \quad \lambda > 0,$$

where $(-\Delta)^{\lambda/2}$ is to be understood in the standard way (cf. [4, p. 147]). As K-functional of the couple $(L^{p,r}(\log L)^\alpha(\mathbb{R}^n), H^{p,r;\alpha}_\lambda(\mathbb{R}^n))$, we will mainly use the expression

$$K(f,t;L^{p,r}(\log L)^\alpha, H^{p,r;\alpha}_\lambda) := \inf_{g\in H^{p,r;\alpha}_\lambda} \Big(\|f-g\|_{p,r;\alpha} + t|g|_{H^{p,r;\alpha}_\lambda}\Big).$$

The following lemma contains some characterizations of this K-functional; here we use the notation $\mathcal{F}$ for the Fourier transformation and $\mathcal{F}^{-1}$ for its inverse.

Lemma 1.4 *Let* $1 < p < \infty$, $1 \le r \le \infty$, $\alpha \in \mathbb{R}$, *and* $\lambda > 0$. *Define on* $L^{p,r}(\log L)^\alpha(\mathbb{R}^n)$ *the generalized Weierstrass means* W_t^λ *and de la Vallée-Poussin means* V_t *by*

$$W_t^\lambda f := \mathcal{F}^{-1}[e^{-(t|\xi|)^\lambda}] * f, \qquad V_t f := \mathcal{F}^{-1}[\chi(t|\xi|)] * f, \qquad t > 0,$$

where $\chi \in C^\infty[0,\infty)$ *is such that* $\chi(u) = 1$ *for* $0 \le u \le 1$ *and* $\chi(u) = 0$ *for* $u \ge 2$. *Then*

$$K(f, t^\lambda; L^{p,r}(\log L)^\alpha, H_\lambda^{p,r;\alpha}) \approx \|f - W_t^\lambda f\|_{p,r;\alpha}, \tag{1.11}$$

$$K(f, t^\lambda; L^{p,r}(\log L)^\alpha, H_\lambda^{p,r;\alpha}) \approx \|f - V_t f\|_{p,r;\alpha} + t^\lambda |V_t f|_{H_\lambda^{p,r;\alpha}}, \tag{1.12}$$

$$\omega_\lambda(f,t)_{p,r;\alpha} \approx K(f, t^\lambda; L^{p,r}(\log L)^\alpha, H_\lambda^{p,r;\alpha}). \tag{1.13}$$

On $L^p(\mathbb{R}^n)$, $1 \le p < \infty$, the first two characterizations are folklore and (1.13) for $1 < p < \infty$ has been proved by Wilmes [45]. For the sake of completeness, we give a proof of (1.11), (1.12), (1.13) in Sect. 2.1.

The paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.1. Since the Fourier multipliers with respect to $\mathbb{R}^n$ have periodic counterparts (cf. [33, Chap. VII]), the abstract arguments are independent of the underlying measure space, and the Wilmes' characterization also holds in the periodic situation [46], we obtain the sublimiting case $0 < \sigma < n/p$ of Theorem 1.2 as that of Theorem 1.1 (i); details are left to the reader. Finally, in Sect. 3 we treat the limiting case $\sigma = 0$ for Lorentz–Zygmund spaces over $\mathbb{T}^n$.

2 The Sublimiting Case $p < p^*$ for Lorentz–Zygmund Spaces Over $\mathbb{R}^n$

The proof of Theorem 1.1 (i) essentially runs as follows: Replace the modulus of smoothness on the left-hand side of (1.4) by an appropriate (modified) K-functional, estimate the latter by a K-functional with respect to $L^{p,r}(\log L)^{\alpha-\gamma}$-spaces, apply a Holmstedt-type formula (cf. [19, Thm. 3.1 (c)]) and go back to the associated modulus of smoothness on $L^{p,r}(\log L)^{\alpha-\gamma}$. For this purpose, we have to prove a series of results, e.g., an embedding of a homogeneous Besov-type space into some Lorentz–Zygmund space, etc. Throughout the proof of Theorem 1.1, it will be convenient to work with the norm

$$\|f\|^\sharp_{B_{\sigma,\gamma}^{(p,r;\beta),s}} := \|f\|_{L^{p,r}(\log L)^\beta} + |f|^\sharp_{B_{\sigma,\gamma}^{(p,r;\beta),s}},$$

where

$$|f|^\sharp_{B_{\sigma,\gamma}^{(p,r;\beta),s}} := \left(\int_0^\infty [u^{-\sigma} \ell^\gamma(u) \omega_\eta(f,u)_{p,r;\beta}]^s \frac{du}{u} \right)^{1/s},$$

being equivalent to $\|f\|_{B_{\sigma,\gamma}^{(p,r;\beta),s}}$—see (1.10).

2.1 Auxiliary Means

We start with the proof of (1.11), (1.12), and (1.13).

Proof of Lemma 1.4. First we analyze the proofs in $L^p(\mathbb{R}^n)$. We start with (1.11). By [38, Cor. 2.3]),

$$\|\mathcal{F}^{-1}[e^{-(t|\xi|)^\lambda}]\|_1 + \Big\|\mathcal{F}^{-1}\Big[\frac{1-e^{-(t|\xi|)^\lambda}}{(t|\xi|)^\lambda}\Big]\Big\|_1 + \Big\|\mathcal{F}^{-1}\Big[\frac{(t|\xi|)^\lambda e^{-(t|\xi|)^\lambda}}{1-e^{-(t|\xi|)^\lambda}}\Big]\Big\|_1 \lesssim 1, \quad t > 0. \tag{2.1}$$

Therefore, using Minkowski's inequality and the boundedness of the first two terms in (2.1), we get for any $g \in H^p_\lambda$,

$$\|f - W_t^\lambda f\|_p \le \|(f-g) - W_t^\lambda(f-g)\|_p + \|g - W_t^\lambda g\|_p \lesssim \|f-g\|_p + t^\lambda |g|_{H^p_\lambda}$$

since $g - W_t^\lambda g = \mathcal{F}^{-1}[(1-e^{-(t|\xi|)^\lambda})(t|\xi|)^{-\lambda}] * t^\lambda(-\Delta)^{\lambda/2} g$. Taking the infimum over all g, we arrive at the part $'' \gtrsim ''$ of the estimate in (1.11). Similarly, using the boundedness of the third term in (2.1), we obtain the converse estimate

$$K(f, t^\lambda; L^p, H^p_\lambda) \le \|f - W_t^\lambda f\|_p + t^\lambda |W_t^\lambda f|_{H^p_\lambda} \lesssim \|f - W_t^\lambda f\|_p\,,$$

completing the proof of (1.11).

Now consider (1.12). Since $V_t f \in C^\infty \cap L^p$ for any $f \in L^p$, the part $'' \lesssim ''$ is trivial. To verify the converse inequality, we note that, by [38, Cor. 2.3],

$$\|\mathcal{F}^{-1}[\chi(t|\xi|)]\|_1 + \Big\|\mathcal{F}^{-1}\Big[\frac{1-\chi(t|\xi|)}{(t|\xi|)^\lambda}\Big]\Big\|_1 + \Big\|\mathcal{F}^{-1}\Big[\frac{(t|\xi|)^\lambda \chi(t|\xi|)}{1-e^{-(t|\xi|)^\lambda}}\Big]\Big\|_1 \lesssim 1, \qquad t > 0. \tag{2.2}$$

The first two estimates show that $\|f - V_t f\|_p \lesssim K(f, t^\lambda; L^p, H^p_\lambda)$. Together with (1.11), the estimate of the third term in (2.2) finally implies that $t^\lambda |V_t f|_{H^p_\lambda} \lesssim K(f, t^\lambda; L^p, H^p_\lambda)$.

Concerning the proof of (1.13) in $L^p(\mathbb{R}^n)$, in the proof given by Wilmes [45] there are only used pointwise identities, L^p-norm triangle inequalities, and the L^p-boundedness of linear operators generated by Fourier multipliers.

To extend (1.11)–(1.13) from $L^p(\mathbb{R}^n)$ to $L^{p,r}(\log L)^\alpha(\mathbb{R}^n)$, we note that the Lorentz–Zygmund spaces are interpolation spaces thus normable and triangle inequalities hold. Further, Corollary 3.15 from [16] states that any quasilinear bounded operator $T : L^p \to L^p$, $1 < p < \infty$, is also bounded on the interpolation spaces $L^{p,r}(\log L)^\alpha$, $\alpha \in \mathbb{R}$, $1 \le r \le \infty$. Thus, the L^p-norm estimates mentioned above are also valid for the $L^{p,r}(\log L)^\alpha$-norms. This concludes the proofs of (1.11)–(1.13). □

Next we consider a theorem on *fractional integration*, a slight variant of [29, Thm. 2.1], which is based on a modified Bessel potential operator. We define the Riesz potential operator with logarithmic component by

$$I^{\sigma,\gamma} f := k_{\sigma,\gamma} * f$$

where $k_{\sigma,\gamma}$ is the function satisfying

$$\mathcal{F}[k_{\sigma,\gamma}](\xi) = |\xi|^{-\sigma} \log^{-\gamma}(e + |\xi|^2), \quad 0 < \sigma < n, \quad \gamma > 0.$$

Analogously to [29], we obtain that

$$k_{\sigma,\gamma}(x) \lesssim |x|^{\sigma-n} \ell^{-\gamma}(|x|), \quad k^*_{\sigma,\gamma}(t) \le k^{**}_{\sigma,\gamma}(t) \lesssim t^{\sigma/n-1} \ell^{-\gamma}(t), \tag{2.3}$$

where $k^{**}_{\sigma,\gamma}(t) := t^{-1} \int_0^t k^*_{\sigma,\gamma}(u)\, du$ is the maximal function of k^* (cf. [3, p. 52]).

Lemma 2.1 *Let* $1 < p < \infty$, $1 \le r \le s \le \infty$, $\alpha \in \mathbb{R}$, $\gamma \ge 0$, $0 < \sigma < n/p$, *and* $1/p^* = 1/p - \sigma/n$. *Then*

$$\|I^{\sigma,\gamma} f\|_{p^*,s;\alpha} \lesssim \|f\|_{p,r;\alpha-\gamma} \quad \textit{for all } f \in L^{p,r}(\log L)^{\alpha-\gamma}.$$

The proof is analogous to that of [29, Thm. 2.1] since in [29] only estimates (2.3) were used to get the corresponding result for the Bessel-type potential operator.

The next lemma deals with a *Bernstein inequality* for logarithmic derivatives. Throughout the paper we put

$$B_R(0) := \{\xi \in \mathbb{R}^n : |\xi| \le R\}.$$

Lemma 2.2 *Let* $1 < p < \infty$, $1 \le r \le \infty$, $\alpha \in \mathbb{R}$, *and* $\gamma > 0$. *Then*

$$\|\mathcal{F}^{-1}[\log^{\gamma}(e + |\xi|^2)\widehat{g}\,]\|_{p,r;\alpha-\gamma} \lesssim \begin{cases} \ell^{\gamma}(R)\, \|g\|_{p,r;\alpha-\gamma}\,, & 1 \le R, \\ \|g\|_{p,r;\alpha-\gamma}, & 0 < R < 1, \end{cases}$$

for all $g \in S'$ *with* $\operatorname{supp} \widehat{g} \subset B_R(0)$.

Proof Let $\chi \in C^{\infty}[0, \infty)$ be as in Lemma 1.4. Again, in view of [16, Cor. 3.15], we only need to show that

$$\|\mathcal{F}^{-1}[\log^{\gamma}(e + |\xi|^2)\chi(|\xi|^2/R^2)\,]\|_1 \lesssim \ell^{\gamma}(R), \qquad R \ge 1,$$

which immediately follows by [38, Cor. 2.3]. □

A combination of these two lemmas gives the following embedding.

Lemma 2.3 *Let* $1 < p < \infty$, $1 \le r \le s \le \infty$, $\alpha \in \mathbb{R}$, $\gamma > 0$, $0 < \sigma < n/p$, *and* $1/p^* = 1/p - \sigma/n$. *Then*

$$\|I^{\sigma,0} g\|_{p^*,s;\alpha} \lesssim \begin{cases} \ell^{\gamma}(R)\, \|g\|_{p,r;\alpha-\gamma}, & 1 \le R, \\ \|g\|_{p,r;\alpha-\gamma}, & 0 < R < 1, \end{cases}$$

for all entire functions $g \in L^{p,r}(\log L)^{\alpha-\gamma}$ *with* $\operatorname{supp} \widehat{g} \subset B_R(0)$.

Proof Note that $I^{\sigma,0}g = I^{\sigma,\gamma-\gamma}g = I^{\sigma,\gamma}\mathcal{F}^{-1}[\log^\gamma(e+|\xi|^2)\widehat{g}]$ and, therefore, by Lemmas 2.1 and 2.2,

$$\|I^{\sigma,0}g\|_{p^*,s;\alpha} \lesssim \|\mathcal{F}^{-1}[\log^\gamma(e+|\xi|^2)\widehat{g}]\|_{p,r;\alpha-\gamma} \lesssim \ell^\gamma(R)\,\|g\|_{p,r;\alpha-\gamma}\,, \qquad R \ge 1.$$

□

The following variant of *Nikolskii's inequality* will turn out to be useful.

Lemma 2.4 *Let* $1 < p < p^* < \infty$, $1 \le r \le s \le \infty$, $\alpha \in \mathbb{R}$, *and* $\gamma \ge 0$. *Then*

$$\|g\|_{p^*,s;\alpha} \lesssim R^{n(1/p-1/p^*)}\ell^\gamma(R)\,\|g\|_{p,r;\alpha-\gamma}$$

for all $g \in L^{p,r}(\log L)^{\alpha-\gamma}$ *with* $\operatorname{supp}\widehat{g} \subset B_R(0)$, $R > 0$.

Proof Take χ from Lemma 1.4 and define $v_R(x) := \mathcal{F}^{-1}[\chi(|\xi|/R)](x)$. Then

$$|v_R(x)| \lesssim \frac{R^n}{(1+R|x|)^n}, \qquad v_R^*(t) \lesssim \frac{R^n}{(1+Rt^{1/n})^n}, \qquad v_R^{**}(t) \lesssim \min\left\{R^n, \frac{1}{t}\right\}.$$

By the assumption on the support of the Fourier transform of g, we have $v_R * g = g$. Therefore, by O'Neil's inequality,

$$g^*(t) = (v_R * g)^*(t) \lesssim t\, v_R^{**}(t) g^{**}(t) + \int_t^\infty v_R^*(u) g^*(u)\, du.$$

Hence,

$$\|g\|_{p^*,s;\alpha} \lesssim \left(\int_0^\infty \left[t^{1/p^*}\ell^\alpha(t)\min\left\{R^n, \frac{1}{t}\right\}\int_0^t g^*(u)\,du\right]^s \frac{dt}{t}\right)^{1/s}$$
$$+ R^n\left(\int_0^\infty \left[t^{1/p^*}\ell^\alpha(t)\int_t^\infty \frac{g^*(u)}{(1+Ru^{1/n})^n}\,du\right]^s \frac{dt}{t}\right)^{1/s} =: N_1 + N_2.$$

Observing that $t^\varepsilon \ell^\gamma(t)$, $\varepsilon > 0$, is almost increasing and $t^{-\varepsilon}\ell^\gamma(t)$ is almost decreasing, elementary estimates lead to

$$N_1 \le R^n\left(\int_0^{R^{-n}} \left[\{t^{1/p^*+1-1/p}\ell^\gamma(t)\}\, t^{1/p-1}\ell^{\alpha-\gamma}(t)\int_0^t g^*(u)\,du\right]^s \frac{dt}{t}\right)^{1/s}$$
$$+ \left(\int_{R^{-n}}^\infty \left[\{t^{1/p^*-1/p}\ell^\gamma(t)\}\, t^{1/p-1}\ell^{\alpha-\gamma}(t)\int_0^t g^*(u)\,du\right]^s \frac{dt}{t}\right)^{1/s}$$
$$\lesssim R^{n(1/p-1/p^*)}\ell^\gamma(R)\left(\int_0^\infty \left[t^{1/p-1}\ell^{\alpha-\gamma}(t)\int_0^t g^*(u)\,du\right]^s \frac{dt}{t}\right)^{1/s}.$$

Now apply a Hardy-type inequality [16, Lemma 3.1 (i)] to obtain (cf. the estimate [29, (2.5)])

$$N_1 \lesssim R^{n(1/p-1/p^*)} \ell^{\gamma}(R) \, \|g\|_{p,r;\alpha-\gamma} .$$

Similarly, handle the term N_2, use [16, Lemma 3.1 (ii)] (cf. the estimate [29, (2.6)]) to arrive at

$$N_2 \lesssim R^n \left(\int_0^\infty \left[t^{1/p^*+1-1/r} \ell^{\alpha}(t) \frac{g^*(t)}{(1+Rt^{1/n})^n} \right]^r dt \right)^{1/r} = R^n \left(\int_0^{R^{-n}} \dots + \int_{R^{-n}}^{\infty} \dots \right)^{1/r} .$$

Apply Minkowski's inequality, observe that

$$(1+Rt^{1/n})^n \approx \begin{cases} 1, & 0 < t < R^{-n}, \\ R^n t, & t \geq R^{-n}, \end{cases}$$

and use again the monotonicity properties of $t^{\pm\varepsilon} \ell^{\gamma}(t)$ to get $N_2 \lesssim R^{n(1/p-1/p^*)} \ell^{\gamma}(R) \, \|g\|_{p,r;\alpha-\gamma}$. □

Next we need an analog of Lemma 2.1 with Besov-type spaces (1.10) involved instead of Riesz-type potential spaces. Note that definition (1.10) is independent of $\eta > 0$ when $\sigma > 0$. This follows from the Marchaud inequality

$$\omega_{\sigma}(f, t)_{p,r;\beta} \lesssim t^{\sigma} \int_t^{\infty} u^{-\sigma-1} \omega_{\sigma+\kappa}(f, u)_{p,r;\beta} \, du \tag{2.4}$$

and a Hardy-type inequality [16, Lemma 3.1 (ii)]. To deduce (2.7), we refer to an abstract Marchaud inequality from [40]—see the next remark.

Remark 2.5 Let $(X, \|\cdot\|)$ be a (complex) Banach space and $\{T(t)\}_{t\geq 0}$ be an equi-bounded $(\mathcal{C}_0)$-semigroup of linear operators from X into itself with infinitesimal generator A_T (cf. [4, § 6.7]), i.e.,

$$T(t_1+t_2) = T(t_1) + T(t_2) \text{ for all } t_1, \ t_2 \geq 0, \quad T(0) = I,$$
$$\|T(t)\| \leq C \text{ with a constant } C \text{ independent of } t \geq 0,$$
$$\lim_{t\to 0+} \|T(t)f - f\| = 0 \text{ for each } f \in X \quad ((\mathcal{C}_0)\text{-property}),$$
$$\lim_{t\to 0+} \left\| \frac{T(t)f - f}{t} - A_T f \right\| = 0 \text{ for all } f \in D(A_T) \quad (\text{domain of } A_T).$$

The operator A_T is closed, $D(A_T)$ is a Banach space under the graph norm $\|g\| + \|A_T g\|$, and the associated K-functional is given by

$$K(f, t; X, D(A_T)) := \inf_{g\in D(A_T)} \Big\{ \|f - g\| + t \, \|A_T g\| \Big\}.$$

If one defines the fractional power $(-A_T)^\mu,\ \mu>0$, of $(-A_T)$ by the strong limit

$$(-A_T)^\mu f := \text{s--}\lim_{t\to 0+}\frac{[I-T(t)]^\mu}{t^\mu}f,$$

then $(-A_T)^\mu$ is closed and [40, (1.12) and (1.5)] imply that

$$K(f,t^\mu;X,D((-A_T)^\mu)) \lesssim t^\mu \int_t^\infty u^{-\mu-1}K(f,u^{\mu+\kappa};X,D((-A_T)^{\mu+\kappa}))\,du$$
$$\text{for any } \kappa>0. \tag{2.5}$$

Now observe that for $X = L^{p,r}(\log L)^\beta(\mathbb{R}^n),\ 1<p<\infty$, the generalized Weierstrass means $\{\mathfrak{W}_t^\mu\}_{t>0}$,

$$\mathfrak{W}_t^\mu f := \begin{cases} \mathcal{F}^{-1}[e^{-t|\xi|^\mu}]*f\,, & t>0\\ f\,, & t=0,\end{cases} \tag{2.6}$$

differing from the Weierstrass means of Lemma 1.4 in the normalization of the parameter $t>0$, form (cf. [4, § 6.7] and [16, Cor. 3.15]) an equibounded $(\mathcal{C}_0)$-semigroup of linear operators of the required type and

$$D(A_{\mathfrak{W}^\mu}) = D((-A_{\mathfrak{W}^1})^\mu) = H_\mu^{p,r;\beta}.$$

Thus, (2.5) in combination with (1.13) gives the Marchaud inequality (2.7).

An important role in the proof of Theorem 1.1 is played by the following lemma, generalizing several known results—cf. [25] and [26].

Lemma 2.6 *Let* $1<p<\infty,\ 0<\sigma<n/p,\ 1\le r\le s\le\infty,\ \alpha\in\mathbb{R}$, *and* $\gamma\ge 0$. *Then*

$$\|f\|_{p^*,s;\alpha} \lesssim |f|^\sharp_{B^{(p,r;\alpha-\gamma),s}_{\sigma,\gamma}} \quad \textit{for all } f\in B^{(p,r;\alpha-\gamma),s}_{\sigma,\gamma}\,,\quad \frac{1}{p^*}=\frac{1}{p}-\frac{\sigma}{n}.$$

Proof Consider the partition of unity on $\mathbb{R}^n$,

$$\sum_{j=-\infty}^{\infty}\varphi_j(|\xi|)=1 \text{ for } \xi\neq 0,\qquad \varphi_j(t)=\varphi(2^{-j}t),\quad \varphi(t):=\chi(t)-\chi(2t),$$

with the cut-off function χ from Lemma 1.4. Set

$$f_j := \mathcal{F}^{-1}[\varphi_j(|\xi|)]*f,\quad j\in\mathbb{Z}. \tag{2.7}$$

Under the assumption that

$$\|f\|^s_{p^*,s;\alpha} \lesssim \sum_{j=-\infty}^{\infty} [\,\ell^\gamma(2^j)2^{j\sigma}\|f_j\|_{p,r;\alpha-\gamma}\,]^s \tag{2.8}$$

holds, we show that the assertion of Lemma 2.6 is true. To this end, we first note that

$$\begin{aligned}\|\mathcal{F}^{-1}[\varphi_j(|\xi|)] * f\|_{p,r;\alpha-\gamma} &\le \|f - V_{2^{-j}}f\|_{p,r;\alpha-\gamma} + \|f - V_{2^{1-j}}f\|_{p,r;\alpha-\gamma}\\ &\lesssim K(f, 2^{-j(\kappa+\sigma)}; L^{p,r;\alpha-\gamma}, H^{p,r;\alpha-\gamma}_{\kappa+\sigma})\end{aligned} \tag{2.9}$$

by Lemma 1.4. Therefore,

$$\begin{aligned}\|f\|^s_{p^*,s;\alpha} &\lesssim \sum_{j=-\infty}^{\infty} [\,\ell^\gamma(2^j)2^{j\sigma}K(f, 2^{-j(\kappa+\sigma)}; L^{p,r;\alpha-\gamma}, H^{p,r;\alpha-\gamma}_{\kappa+\sigma})\,]^s \int_{2^{j-1}}^{2^j} \frac{dt}{t}\\ &\approx \int_0^\infty [\,\ell^\gamma(t)t^{-\sigma}K(f, t^{\kappa+\sigma}; L^{p,r;\alpha-\gamma}, H^{p,r;\alpha-\gamma}_{\kappa+\sigma})\,]^s \frac{dt}{t}\,,\end{aligned}$$

and Lemma 2.6 is established in view of (1.13) and (1.10) provided that (2.8) is valid.

We prove (2.8) by an argument communicated to us by A. Seeger. Choose $\widetilde{\varphi} \in C^\infty(0,\infty)$ with $\operatorname{supp}\widetilde{\varphi} \subset (1/4, 4)$ and $\widetilde{\varphi} = 1$ on $\operatorname{supp}\varphi$; set $\widetilde{\varphi}_j = \widetilde{\varphi}(2^{-j}\cdot)$. Define $T_{\widetilde{\varphi}_j}f := \mathcal{F}^{-1}[\widetilde{\varphi}_j(|\xi|)] * f$. Then $T_{\widetilde{\varphi}_j}f_j = f_j$ for the f_j's from (2.7). Recall that $\operatorname{supp}\widehat{f_j} \subset \{\xi \in \mathbb{R}^n : 2^{j-1} \le |\xi| \le 2^{j+1}\}$ and, therefore, by the Nikolskii inequality from Lemma 2.4,

$$\|\ell^{-\gamma}(2^j)2^{-j\sigma}T_{\widetilde{\varphi}_j}f\|_{p^*,s;\alpha} \lesssim \|T_{\widetilde{\varphi}_j}f\|_{p,r;\alpha-\gamma} \lesssim \|f\|_{p,r;\alpha-\gamma}\,, \quad \sigma = n(1/p - 1/p^*), \tag{2.10}$$

for all $f \in L^{p,r;\alpha-\gamma}$. Now fix p, $1 < p < \infty$; choose p_0^*, p_1^* such that $p < p_0^* < p^* < p_1^* < \infty$. Set $\sigma_0 = n(1/p - 1/p_0^*)$, $\sigma_1 = n(1/p - 1/p_1^*)$, consequently $\sigma_0 \ne \sigma_1$. Then (2.10) also holds with this fixed p but with (p^*, σ) replaced by (p_i^*, σ_i), $i = 0, 1$. Hence, for an arbitrary sequence $(F_j)_{j\in\mathbb{Z}}$, $F_j \in L^{p,r;\alpha-\gamma}$, we have

$$\begin{aligned}\Big\|\sum_{j\in\mathbb{Z}} T_{\widetilde{\varphi}_j}F_j\Big\|_{p_i^*,s;\alpha} &\lesssim \sum_{j\in\mathbb{Z}} \|T_{\widetilde{\varphi}_j}F_j\|_{p_i^*,s;\alpha} \lesssim \sum_{j\in\mathbb{Z}} \ell^\gamma(2^j)\,2^{j\sigma_i}\,\|T_{\widetilde{\varphi}_j}F_j\|_{p,r;\alpha-\gamma}\\ &\lesssim \sum_{j\in\mathbb{Z}} \ell^\gamma(2^j)\,2^{j\sigma_i}\,\|F_j\|_{p,r;\alpha-\gamma}.\end{aligned} \tag{2.11}$$

Now apply an interpolation argument: Define the sequence space $\ell^q_\sigma(X)$, X a normed space, as the space of X-valued sequences $(F_j)_{j\in\mathbb{Z}}$ with

$$\|(F_j)_j\|_{\ell^q_\sigma} := \Big(\sum_{j\in\mathbb{Z}} [\,2^{j\sigma}\|\ell^\gamma(2^j)\,F_j\|_X\,]^q\Big)^{1/q} < \infty\,,$$

and a linear operator S by

$$S\big((F_j)_j\big) := \sum_{j\in\mathbb{Z}} T_{\widetilde{\varphi}_j} F_j .$$

Then (2.11) means that

$$S : \ell^1_{\sigma_i}(L^{p,r;\alpha-\gamma}) \to L^{p_i^*,s;\alpha}, \qquad i = 0, 1.$$

Since $\sigma_0 \neq \sigma_1$, we obtain, by [4, Thm. 5.6.1 (dotted version)] that

$$(\ell^1_{\sigma_0}(X), \ell^1_{\sigma_1}(X))_{\theta,q} = \ell^q_\sigma(X), \qquad \sigma = (1-\theta)\sigma_0 + \theta\sigma_1, \quad 0 < \theta < 1, \quad 1 \le q \le \infty. \tag{2.12}$$

Moreover, $\big(L^{p_0^*,s;\alpha}, L^{p_1^*,s;\alpha}\big)_{\theta,q} = L^{p^*,q;\alpha}$, where $1/p^* = (1-\theta)/p_0^* + \theta/p_1^*$ and $0 < \theta < 1$. Thus, the real interpolation implies that

$$S : \ell^1_\sigma(L^{p,r;\alpha-\gamma}) \to L^{p^*,q;\alpha}. \tag{2.13}$$

Choose $F_j = f_j$ with f_j from (2.7). Then

$$S\big((F_j)_j\big) = S\big((f_j)_j\big) = \sum_{j\in\mathbb{Z}} T_{\widetilde{\varphi}_j} f_j = \sum_{j\in\mathbb{Z}} f_j = f$$

and, by (2.13),

$$\|f\|_{p^*,q;\alpha} \lesssim \left(\sum_{j\in\mathbb{Z}} [\, 2^{j\sigma} \|\ell^\gamma(2^j)\, f_j\|_{p,r;\alpha-\gamma}]^q\right)^{1/q},$$

which gives (2.8) on putting $q = s$. □

As already announced, we want to apply an appropriate *Holmstedt formula* for the proof of Theorem 1.1 (i). To this purpose, we introduce slowly varying functions. A measurable function $b : (0,\infty) \to (0,\infty)$ is said to be *slowly varying* on $(0,\infty)$, notation $b \in SV(0,\infty)$ if, for each $\varepsilon > 0$, there is an increasing function g_ε and a decreasing $g_{-\varepsilon}$ such that $t^\varepsilon b(t) \approx g_\varepsilon(t)$ and $t^{-\varepsilon} b(t) \approx g_{-\varepsilon}(t)$ for all $t \in (0,\infty)$. Clearly, one has that $\ell^\gamma \in SV(0,\infty)$, $\gamma \in \mathbb{R}$. For the sake of simplicity, in the following we assume that the functions $t^{\pm\varepsilon} b(t)$ are already monotone. To describe the framework of the desired Holmstedt formula, let (X, Y) be a compatible couple of Banach spaces, where $Y \subset X$ has a seminorm $|\cdot|_Y$ such that $\|\cdot\|_Y := \|\cdot\|_X + |\cdot|_Y$ is a norm on Y. We will work with the (modified) K-functional

$$K(f, t; X, Y) := \inf_{g\in Y} (\|f - g\|_X + t|g|_Y)$$

and we will state a slight variant of the Holmstedt formula involving slowly varying functions given in [19, Thm. 3.1 (c)] without proof.

Lemma 2.7 *Let* $0 \le \theta \le 1$, $1 \le s \le \infty$, *and* $b \in SV(0,\infty)$. *Define the interpolation space* $(X,Y)_{\theta,s;b}$ *by*

$$(X,Y)_{\theta,s;b} := \left\{ f \in X : |f|_{\theta,s;b} = \left(\int_0^\infty [t^{-\theta} b(t) K(f,t;X,Y)]^s \frac{dt}{t} \right)^{1/s} < \infty \right\}.$$

If $0 < \theta < 1$, *then*

$$K(f, t^{1-\theta} b(t); (X,Y)_{\theta,s;b}, Y) \approx \left(\int_0^t [u^{-\theta} b(u) K(f,u;X,Y)]^s \frac{du}{u} \right)^{1/s}$$

for all $f \in X$ *and all* $t > 0$.

2.2 Proof of Theorem 1.1 (i).

Using the characterization (1.13), we can reduce the problem to estimates between K-functionals. Thus,

$$\omega_\kappa(f,t)_{p^*,s;\alpha} \approx K(f, t^\kappa; L^{p^*,s;\alpha}, H_\kappa^{p^*,s;\alpha}) \le \|f-g\|_{p^*,s;\alpha} + t^\kappa \|(-\Delta)^{\kappa/2} g\|_{p^*,s;\alpha} \quad \text{for all } g \in H_\kappa^{p^*,s;\alpha},$$

in particular, in view of Lemma 2.1, for all $g \in H_{\kappa+\sigma}^{p,r;\alpha-\gamma}$. Consider $g_t = V_t g$, the de la Vallée-Poussin means of $g \in H_{\kappa+\sigma}^{p,r;\alpha-\gamma}$ from Lemma 1.4. Note that

$$|g_t|_{H_{\kappa+\sigma}^{p,r;\alpha-\gamma}} \lesssim |g|_{H_{\kappa+\sigma}^{p,r;\alpha-\gamma}}$$

since $\|\mathcal{F}^{-1}[\chi(t|\xi|)]\|_1 = O(1)$ by [38, Cor. 2.3]. As $\operatorname{supp}\widehat{g_t} \subset B_{2/t}(0)$, Lemmas 2.6 and 2.3 imply that

$$\omega_\kappa(f,t)_{p^*,s;\alpha} \lesssim |f-g_t|^\sharp_{B_{\sigma,\gamma}^{(p,r;\alpha-\gamma),s}} + \ell^\gamma(t)\, t^\kappa \|(-\Delta)^{(\kappa+\sigma)/2} g_t\|_{p,r;\alpha-\gamma}. \tag{2.14}$$

We want to apply the Holmstedt formula from Lemma 2.7. To this end, we have to get rid of the de la Vallée-Poussin means, i.e., we have to estimate g_t by g. Clearly,

$$\omega_\kappa(f,t)_{p^*,s;\alpha} \lesssim |f-g|^\sharp_{B_{\sigma,\gamma}^{(p,r;\alpha-\gamma),s}} + |g-g_t|^\sharp_{B_{\sigma,\gamma}^{(p,r;\alpha-\gamma),s}} + \ell^\gamma(t)\, t^\kappa |g_t|_{H_{\kappa+\sigma}^{p,r;\alpha-\gamma}}$$

and, by the above argument, $|g_t|_{H_{\kappa+\sigma}^{p,r;\alpha-\gamma}} \lesssim |g|_{H_{\kappa+\sigma}^{p,r;\alpha-\gamma}}$. Observe that

$$|g-g_t|^\sharp_{B_{\sigma,\gamma}^{(p,r;\alpha-\gamma),s}} \approx \left(\left(\int_0^t + \int_t^\infty \right) [u^{-\sigma} \ell^\gamma(u) K(g-g_t, u^{\kappa+\sigma}; L^{p,r;\alpha-\gamma}, H_{\kappa+\sigma}^{p,r;\alpha-\gamma})]^s \frac{du}{u} \right)^{1/s}.$$

Since $K(g-g_t,u^{\kappa+\sigma})\le u^{\kappa+\sigma}|g-g_t|_{H^{p,r;\alpha-\gamma}_{\kappa+\sigma}}$ and $K(g-g_t,u^{\kappa+\sigma})\le \|g-g_t\|_{p,r;\alpha-\gamma}$, we see that

$$|g-g_t|^{\sharp}_{B^{(p,r;\alpha-\gamma),s}_{\sigma,\gamma}}\lesssim \ell^{\gamma}(t)\,t^{\kappa}|g_t|_{H^{p,r;\alpha-\gamma}_{\kappa+\sigma}}+\ell^{\gamma}(t)\,t^{-\sigma}\|g-g_t\|_{p,r;\alpha-\gamma}.$$

The estimate

$$\|g-g_t\|_{p,r;\alpha-\gamma}\lesssim K(g,t^{\kappa+\sigma};L^{p,r;\alpha-\gamma},H^{p,r;\alpha-\gamma}_{\kappa+\sigma})\lesssim t^{\kappa+\sigma}|g|_{H^{p,r;\alpha-\gamma}_{\kappa+\sigma}}$$

follows from Lemma 1.4, the definition of the K-functional and the fact that $g\in H^{p,r;\alpha-\gamma}_{\kappa+\sigma}$. Thus, (2.14) holds for all $g\in H^{p,r;\alpha-\gamma}_{\kappa+\sigma}$, which implies that

$$\omega_{\kappa}(f,t)_{p^*,s;\alpha}\lesssim \inf_{g\in H^{p,r;\alpha-\gamma}_{\kappa+\sigma}}\left(|f-g|^{\sharp}_{B^{(p,r;\alpha-\gamma),s}_{\sigma,\gamma}}+\ell^{\gamma}(t)\,t^{\kappa}|g|_{H^{p,r;\alpha-\gamma}_{\kappa+\sigma}}\right). \qquad (2.15)$$

Now $B^{(p,r;\alpha-\gamma),s}_{\sigma,\gamma}=(L^{p,r;\alpha-\gamma},H^{p,r;\alpha-\gamma}_{\kappa+\sigma})_{\theta,s;\ell^{\gamma}}$, $\theta=\sigma/(\kappa+\sigma)$, which directly follows from the definition of the interpolation space given in Lemma 2.7 and the characterization (1.13) of the involved K-functional. If we change the variable t^{κ} to $t^{1-\theta}$ and set $\rho(t)=t^{1-\theta}\ell^{\gamma}(t)$, we can interpret the right-hand side of (2.15) as $K\big(f,\rho(t);(L^{p,r;\alpha-\gamma},H^{p,r;\alpha-\gamma}_{\kappa+\sigma})_{\theta,s;\ell^{\gamma}},H^{p,r;\alpha-\gamma}_{\kappa+\sigma}\big)$. By Lemma 2.7, the latter can be reformulated as follows

$$\begin{aligned}&K\big(f,\rho(t);(L^{p,r;\alpha-\gamma},H^{p,r;\alpha-\gamma}_{\kappa+\sigma})_{\theta,s;\ell^{\gamma}},H^{p,r;\alpha-\gamma}_{\kappa+\sigma}\big)\\ &\quad\approx\left(\int_0^t[u^{-\theta}\ell^{\gamma}(u)K(f,u;L^{p,r;\alpha-\gamma},H^{p,r;\alpha-\gamma}_{\kappa+\sigma})]^s\frac{du}{u}\right)^{1/s}.\end{aligned}$$

Hence, using the change of variables and (1.13), we arrive at

$$\begin{aligned}\omega_{\kappa}(f,t)_{p^*,s;\alpha}&\lesssim\left(\int_0^t[u^{-\sigma}\ell^{\gamma}(u)K(f,u^{\kappa+\sigma};L^{p,r;\alpha-\gamma},H^{p,r;\alpha-\gamma}_{\kappa+\sigma})]^s\frac{du}{u}\right)^{1/s}\\ &\approx\left(\int_0^t[u^{-\sigma}\ell^{\gamma}(u)\,\omega_{\kappa+\sigma}(f,u)_{p,r;\alpha-\gamma}]^s\frac{du}{u}\right)^{1/s}.\end{aligned}$$

□

2.3 Proof of Theorem 1.1 (ii, iii)

First let $\gamma<0$ and $1\le r\le s\le\infty$. Then it follows from (1.13) that $f\in H^{p,r;\,\alpha-\gamma}_{\kappa+\sigma}$ implies that $\omega_{\kappa+\sigma}(f,t)_{p,r;\alpha-\gamma}=O(t^{\kappa+\sigma}),\ t\to 0+$. Together with (1.4) and the assumption $\gamma < 0$, this gives $\omega_\kappa(f,\delta)_{p^*,s;\,\alpha} = o(\delta^\kappa)$, $\delta \to 0+$. Therefore, by (1.13), it remains to show that

$$K(f,\delta^\kappa; L^{p^*,s}(\log L)^\alpha, H_\kappa^{p^*,s;\alpha}) = o(\delta^\kappa) \text{ as } \delta \to 0+ \qquad \Longrightarrow \qquad f = 0.$$

From the proof of Lemma 1.4 it is clear that

$$\delta^\kappa \|(-\Delta)^{\kappa/2} W_\delta^\kappa f\|_{p^*,s;\alpha} \lesssim \|f - W_\delta^\kappa f\|_{p^*,s;\alpha} \lesssim K(f,\delta^\kappa; L^{p^*,s}(\log L)^\alpha, H_\kappa^{p^*,s;\alpha}).$$

Thus, by the Fatou property of the Lorentz–Zygmund spaces and the hypothesis, we have

$$\|(-\Delta)^{\kappa/2} f\|_{p^*,s;\alpha} \le \liminf_{\delta\to 0+} \|(-\Delta)^{\kappa/2} W_\delta^\kappa f\|_{p^*,s;\alpha} = 0.$$

But $(-\Delta)^{\kappa/2} f = 0$ yields $f = 0$ since $f \in L^{p^*,s}(\log L)^\alpha$. □

(iii) The case $\gamma \ge 0$ and $s < r$ will be proved by contradiction. Assuming that the Ulyanov-type inequality (1.4) holds, we prove that a fractional integration theorem follows, which is false in Lorentz–Zygmund spaces. To this end, consider the set of entire functions of exponential type

$$E_{p,r;\alpha-\gamma;R} := \left\{ P_R \in L^{p,r}(\log L)^{\alpha-\gamma}(\mathbb{R}^n) : \operatorname{supp} \widehat{P_R} \subset B_R(0), \quad R > 0 \right\},$$

where $\alpha \in \mathbb{R}$ and $\gamma \ge 0$. Then $\bigcup_{R>0} E_{p,r;\alpha-\gamma;R}$ is dense in $L^{p,r}(\log L)^{\alpha-\gamma}$ since, by (1.12),

$$\|f - V_t f\|_{p,r;\alpha-\gamma} \lesssim K(f, t^\lambda; L^{p,r}(\log L)^{\alpha-\gamma}, H_\lambda^{p,r;\,\alpha-\gamma}).$$

Moreover, the following Riesz-type inequality holds for $1 < p^* < \infty$, $1 \le s \le \infty$, $\kappa > 0$,

$$|P_{1/\delta}|_{H_\kappa^{p^*,s;\alpha}} \lesssim \delta^{-\kappa} \omega_\kappa(P_{1/\delta},\delta)_{p^*,s;\alpha}\,, \quad P_{1/\delta} \in E_{p,r;\alpha-\gamma;1/\delta}. \tag{2.16}$$

Indeed, this is proved in [45] for $p^* = s$ and the argument used in the proof of Lemma 1.4 shows that (2.16) is true. Formula (1.13) and the definition of the K-functional imply that

$$\begin{aligned}\omega_{\kappa+\sigma}(P_{1/\delta},t)_{p,r;\alpha-\gamma} &\approx K(P_{1/\delta}, t^{\kappa+\sigma}; L^{p,r}(\log L)^{\alpha-\gamma}, H_{\kappa+\sigma}^{p,r;\alpha-\gamma}) \\ &\lesssim t^{\kappa+\sigma} |P_{1/\delta}|_{H_{\kappa+\sigma}^{p,r;\alpha-\gamma}}.\end{aligned} \tag{2.17}$$

Estimates (2.16) and (2.17) applied to (1.4) lead to

$$|P_{1/\delta}|_{H_\kappa^{p^*,s;\alpha}} \lesssim \delta^{-\kappa} \left(\int_0^\delta [t^{-\sigma} t^{\kappa+\sigma} |P_{1/\delta}|_{H_{\kappa+\sigma}^{p,r;\alpha-\gamma}}]^s \frac{dt}{t} \right)^{1/s} \approx |P_{1/\delta}|_{H_{\kappa+\sigma}^{p,r;\alpha-\gamma}}.$$

Since the estimates involved are independent of $\delta > 0$, we get

$$\|I^{\sigma,\gamma} f\|_{p^*,s;\alpha} \lesssim \|f\|_{p,r;\alpha-\gamma}\,, \qquad f \in L^{p,r}(\log L)^{\alpha-\gamma}, \tag{2.18}$$

where $1 < p < \infty$, $0 < \sigma < n/p$, and $\frac{1}{p^*} = \frac{1}{p} - \frac{\sigma}{n}$. The following example provides a contradiction to (2.18). Let

$$f_\beta(x) = \begin{cases} |x|^{-n/p}(\log \frac{1}{|x|})^{-\beta}, & |x| < 1/2; \\ 0, & |x| \geq 1/2. \end{cases}$$

Then

$$f_\beta^*(t) \approx \begin{cases} t^{-1/p}(\log \frac{1}{t})^{-\beta}, & t < |B_{1/2}(0)| \\ 0, & \text{otherwise} \end{cases}$$

and $\|f_\beta^*(t)\|_{p,r;\alpha-\gamma} \approx \left(\int_0^{|B_{1/2}(0)|}[\,\ell^{\alpha-\gamma-\beta}(u)\,]^r \frac{du}{u}\right)^{1/r}$. The latter converges if $\beta > \alpha - \gamma + 1/r$ and then $f_\beta \in L^{p,r}(\log L)^{\alpha-\gamma}$. On the other hand, for $0 < |x| \leq 1/2$,

$$\begin{aligned} I^{\sigma,\gamma} f_\beta(x) &\approx \int\limits_{|y|<1/2} \frac{1}{|x-y|^{n-\sigma}\ell^\gamma(|x-y|)}|y|^{-n/p}\log^{-\beta}(1/|y|)\,dy \\ &\gtrsim \int\limits_{|x|/4<|y|<|x|/2} \frac{1}{|x|^{n-\sigma}\ell^\gamma(1/|x|)}|x|^{-n/p}\log^{-\beta}(1/|x|)\,dy \\ &\approx |x|^{\sigma-n/p}\log^{-\gamma-\beta}(1/|x|). \end{aligned}$$

Hence,

$$(I^{\sigma,\gamma} f_\beta)^*(t) \gtrsim t^{\sigma/n-1/p}\log^{-\gamma-\beta}(1/t), \qquad 0 < t < |B_{1/2}(0)|$$

and $\|I^{\sigma,\gamma} f_\beta\|_{p^*,s;\alpha} \gtrsim \left(\int_0^{|B_{1/2}(0)|}[\,\ell^{\alpha-\gamma-\beta}(t)\,]^s \frac{dt}{t}\right)^{1/s}$, which diverges if $\beta < \alpha - \gamma + 1/s$. Since $r > s$, there exists β such that $\alpha - \gamma + 1/r < \beta < \alpha - \gamma + 1/s$ and hence (2.18) does not hold for $s < r$ and $\sigma = n(1/p - 1/p^*)$. □

2.4 Embedding Results for Besov Spaces

Let us give two (nonlimiting) embeddings of the $B_{\sigma,\gamma}^{(p,r;\beta),s}$-spaces that follow from Theorems 1.1 (i) and 1.2 (a).

Corollary 2.8 *Let* $1 < p < \infty$, $0 < \sigma = n(\frac{1}{p} - \frac{1}{p^*}) < n/p$, $\alpha \in \mathbb{R}$, $\gamma \geq 0$ *and* $1 \leq r \leq s \leq \infty$. *Then, for* $X = \mathbb{R}^n$ *or* $X = \mathbb{T}^n$, *we have* (*see the definition of the Besov spaces in* (1.10))

$$B_{\lambda+\sigma,\mu+\gamma}^{(p,r;\alpha-\gamma),\xi}(X) \hookrightarrow B_{\lambda,\mu}^{(p^*,s;\alpha),\xi}(X), \qquad \lambda, \xi > 0, \quad \mu \in \mathbb{R}, \tag{2.19}$$

and

$$B^{(p,r;\alpha-\gamma),\xi}_{\sigma,\mu+\gamma+\max\{1/\xi,1/s\}}(X) \hookrightarrow B^{(p^*,s;\alpha),\xi}_{0,\mu}(X), \qquad \mu,\xi > 0. \tag{2.20}$$

In the case $\alpha = \gamma = 0$, $p^* = s$, $p = r$, (2.19) coincides with the embedding [10, Cor. 5.3 (i)]. On the other hand, in this case (2.20) improves the embedding [10, Cor. 5.3 (ii)] since $B^{(p,p;0),\xi}_{\sigma,\mu+\max\{1/\xi,1/s\}} \hookrightarrow B^{(p^*,p^*;0),\xi}_{0,\mu}$ is sharper than $B^{(p,p;0),\xi}_{\sigma,\mu+1} \hookrightarrow B^{(p^*,p^*;0),\xi}_{0,\mu}$ when $\max\{1/\xi, 1/s\} < 1$.

Proof We only prove the limiting case (2.20). By Theorems 1.1 (i) and 1.2 (a) we have

$$\begin{aligned} |f|^{\xi}_{B^{(p^*,s;\alpha),\xi}_{0,\mu}} &= \int_0^1 [\,\ell^{\mu}(u)\,\omega_{\kappa}(f,u)_{p^*,s;\alpha}]^{\xi}\,\frac{du}{u} \\ &\lesssim \int_0^1 \ell^{\mu\xi}(u)\left(\int_0^u [\,t^{-\sigma}\,\ell^{\gamma}(t)\,\omega_{\kappa+\sigma}(f,t)_{p,r;\,\alpha-\gamma}]^{s}\,\frac{dt}{t}\right)^{\frac{\xi}{s}}\frac{du}{u}. \end{aligned}$$

Using Hardy's inequality (see, e.g., [2])

$$\int_0^1 \ell^{A}(t)\left(\int_0^t f(u)\frac{du}{u}\right)^{B}\frac{dt}{t} \lesssim \int_0^1 \ell^{A+B}(t)f^{B}(t)\frac{dt}{t}, \quad f \ge 0, \quad B \ge 1, \quad A > -1,$$

we get, in the case $\xi \ge s$,

$$|f|^{\xi}_{B^{(p^*,s;\alpha),\xi}_{0,\mu}} \le \int_0^1 [\,u^{-\sigma}\,\ell^{\mu+\gamma+1/s}(u)\,\omega_{\kappa+\sigma}(f,u)_{p,r;\alpha-\gamma}]^{\xi}\,\frac{du}{u} = |f|^{\xi}_{B^{(p,r;\alpha-\gamma),\xi}_{\sigma,\mu+\gamma+1/s}}.$$

Let us now consider the case $\xi < s$. Using Hardy's inequality (see [6,37])

$$\int_0^1 \ell^{A}(t)\left(\int_0^t f(u)\frac{du}{u}\right)^{B}\frac{dt}{t} \lesssim \int_0^1 \ell^{A+1}(t)f^{B}(t)\frac{dt}{t}, \qquad 0 < B < 1, \quad A > -1,$$

which is true for any non-negative function f satisfying $f(u) \le Cf(t)$ for $u/2 \le t \le u$, $0 \le u \le 1$, and monotonicity properties of the modulus of smoothness, we get

$$|f|^{\xi}_{B^{(p^*,s;\alpha),\xi}_{0,\mu}} \le \int_0^1 [\,u^{-\sigma}\,\ell^{\mu+\gamma+1/\xi}(u)\,\omega_{\kappa+\sigma}(f,u)_{p,r;\alpha-\gamma}]^{\xi}\,\frac{du}{u} = |f|^{\xi}_{B^{(p,r;\alpha-\gamma),\xi}_{\sigma,\mu+\gamma+1/\xi}}.$$

□

A similar approach via Ulyanov's inequalities can also be applied to obtain embedding theorems for general Calderón-type spaces

$$\Lambda^l(G,E)=\Big\{f\in G:\ \ \|f\|_G+\|\omega_l(f,\cdot)_G\|_E<\infty\Big\},$$

introduced by Calderón [8]. Note that the classical Besov spaces $B_\alpha^{p,q}$ are a particular case of the Calderón spaces.

3 The Limiting Case $\mathbf{p=p^*}$ for Lorentz–Zygmund Spaces Over $\mathbb{T}^{\mathbf{n}}$

In this section we discuss the limiting case $\sigma=0$, i.e., when $p=p^*$. When trying to follow the effective approach of Sect. 2, we encounter the difficulty that we cannot carry out the monotonicity arguments used in the proof of Lemma 2.4 on the whole half-line, but only on the interval $(0,1)$ or $(1,\infty)$ separately. There are two possibilities how to overcome this obstacle. One is to use the concept of broken indices for the log-function - see [18]. The other, which we make use of, is to restrict ourselves to the n-dimensional torus $\mathbb{T}^n$. In the following we use the standard Fourier series setting (cf. [33, Chap. VII]),

$$f(x)\sim\sum_{m\in\mathbb{Z}^n}\widehat{f}_m e^{2\pi imx},\qquad \widehat{f}_m=\int_{\mathbb{T}^n} f(x)e^{-2\pi imx}\,dx,\qquad f\in L^1(\mathbb{T}^n),$$

and denote by $\mathcal{T}_N$ the set of all trigonometric polynomials of degree N, i.e.,

$$\mathcal{T}_N:=\left\{T_N=\sum_{|m|\le N}c_m e^{2\pi imx}\ :\ c_m\in\mathbb{C},\ m\in\mathbb{Z}^n\right\},\qquad N\in\mathbb{N}_0.$$

Since in this section there will be no ambiguity, we use the notation of the previous sections though the underlying measure space is $\mathbb{T}^n$. Thus we write

$$L^{p,r;\alpha}:=\{f\in L^1(\mathbb{T}^n):\ \|f\|_{p,r;\alpha}:=\left(\int_0^1[t^{1/p}\ell^\alpha(t)f^*(t)]^r\frac{dt}{t}\right)^{1/r}<\infty\},$$

for the periodic Riesz-potential space $H_\lambda^{p,r;\alpha}(\mathbb{T}^n)$ of order $\lambda>0$

$$\begin{aligned}H_\lambda^{p,r;\alpha}&:=\{g\in L^{p,r;\alpha}:\ |g|_{H_\lambda^{p,r;\alpha}}\\&:=\|(-\Delta)^{\lambda/2}g\|_{p,r;\alpha}<\infty\},\quad(-\Delta)^{\lambda/2}g\sim\sum_{m\in\mathbb{Z}^n}|m|^\lambda\,\widehat{g}_m\,e^{2\pi imx},\end{aligned}$$

and for the associated K-functional $K(f, t^\lambda; L^{p,r;\alpha}(\mathbb{T}^n), H_\lambda^{p,r;\alpha}(\mathbb{T}^n))$

$$K(f, t^\lambda; L^{p,r;\alpha}, H_\lambda^{p,r;\alpha}) := \inf_{g \in H_\lambda^{p,r;\alpha}} (\|f - g\|_{p,r;\alpha} + t^\lambda |g|_{H_\lambda^{p,r;\alpha}}), \quad \lambda > 0.$$

On account of the Poisson-summation formula (see [44, p. 37]) we note that the periodic analogs of (1.11) and (1.12) hold; the periodic analog of (1.13) is due to Wilmes [46]. Hence the following variant of Lemma 1.4 is true.

Lemma 3.1 *Let* $1 < p < \infty$, $1 \le r \le \infty$, $\alpha \in \mathbb{R}$, *and* $\lambda > 0$. *Define the generalized Weierstrass means* W_t^λ *and de la Vallée-Poussin means* V_t *on the space* $L^{p,r}(\log L)^\alpha(\mathbb{T}^n)$ *by*

$$W_t^\lambda f := \sum_{m \in \mathbb{Z}^n} e^{-(t|m|)^\lambda} \widehat{f}_m e^{2\pi i m x}, \qquad V_t f := \sum_{|m| \le 2/t} \chi(t|m|) \widehat{f}_m e^{2\pi i m x}, \qquad t > 0,$$

where $\chi \in C^\infty[0, \infty)$ *is from* Lemma 1.4. *Then*

$$K(f, t^\lambda; L^{p,r}(\log L)^\alpha, H_\lambda^{p,r;\alpha}) \approx \|f - W_t^\lambda f\|_{p,r;\alpha}, \tag{3.1}$$

$$K(f, t^\lambda; L^{p,r}(\log L)^\alpha, H_\lambda^{p,r;\alpha}) \approx \|f - V_t f\|_{p,r;\alpha} + t^\lambda |V_t f|_{H_\lambda^{p,r;\alpha}}, \tag{3.2}$$

$$\omega_\lambda(f, t)_{p,r;\alpha} \approx K(f, t^\lambda; L^{p,r}(\log L)^\alpha, H_\lambda^{p,r;\alpha}). \tag{3.3}$$

3.1 Auxiliary Results

We start with deriving analogs of Lemmas 2.1, 2.2, and 2.4 in the limiting case $p = p^*$. These results will be used in the proof of Theorem 1.2 (b). Define a fractional integration $\tilde{I}^{0,\gamma}$ of logarithmic order $\gamma > 0$ via $\tilde{I}^{0,\gamma} f := \tilde{k}_{0,\gamma} * f$, where the Fourier series and the growth behavior (at the origin) of $\tilde{k}_{0,\gamma}$—see [44, Thm. 7 (ii)]—are given by

$$\tilde{k}_{0,\gamma}(x) \sim \sum_{m \in \mathbb{Z}^n} \frac{e^{2\pi i m x}}{\log^\gamma(e + |m|^2)}, \qquad |\tilde{k}_{0,\gamma}(x)| \lesssim \frac{1}{|x|^n} \log^{-\gamma-1} \frac{1}{|x|}, \quad x \to 0. \tag{3.4}$$

As the next result is a slight variant of [29, Thm. 2.4], we state it without proof.

Lemma 3.2 *Let* $1 < p < \infty$, $1 \le r \le s \le \infty$, $\alpha \in \mathbb{R}$ *and* $\gamma > 0$. *Then*

$$\|\tilde{I}^{0,\gamma} f\|_{p,s;\alpha} \lesssim \|f\|_{p,r;\alpha-\gamma} \quad \textit{for all} \quad f \in L^{p,r}(\log L)^{\alpha-\gamma}(\mathbb{T}^n).$$

By the Poisson-summation formula (see [44, p. 37]), it is clear that the proof of Lemma 2.2 also works in the periodic situation. Hence, we obtain the following lemma.

Lemma 3.3 *Let* $1 < p < \infty$, $1 \le r \le \infty$, $\alpha \in \mathbb{R}$ *and* $\gamma > 0$. *Then the Bernstein-type inequality*

$$\Big\| \sum_{|m|\le N} \log^\gamma(e+|m|^2)\, c_k e^{2\pi imx} \Big\|_{p,r;\alpha-\gamma} \lesssim \ell^\gamma(N) \Big\| \sum_{|m|\le N} c_m e^{2\pi imx} \Big\|_{p,r;\alpha-\gamma}$$

holds for all trigonometric polynomials of degree N.

A combination of these two lemmas yields a Nikolskii-type inequality for the limiting case.

Lemma 3.4 *Let* $1 < p < \infty$, $1 \le r \le s \le \infty$, $\alpha \in \mathbb{R}$, *and* $\gamma > 0$. *Then*

$$\Big\| \sum_{|m|\le N} c_m e^{2\pi imx} \Big\|_{p,s;\alpha} \lesssim \ell^\gamma(N) \Big\| \sum_{|m|\le N} c_m e^{2\pi imx} \Big\|_{p,r;\alpha-\gamma}$$

for all trigonometric polynomials in $\mathcal{T}_N$, $N \in \mathbb{N}$.

Proof By Lemma 3.2,

$$\begin{aligned}\Big\| \sum_{|m|\le N} c_m e^{2\pi imx} \Big\|_{p,s;\alpha} &= \Big\| \sum_{|m|\le N} \frac{\log^\gamma(e+|m|^2)}{\log^\gamma(e+|m|^2)} c_m e^{2\pi imx} \Big\|_{p,s;\alpha} \\ &\lesssim \Big\| \sum_{|m|\le N} \log^\gamma(e+|m|^2)\, c_m e^{2\pi imx} \Big\|_{p,r;\alpha-\gamma}\end{aligned}$$

and an application of Lemma 3.3 gives the assertion. □

To formulate an analog of Lemma 2.6 in our limiting case, we recall definition (1.10) of the Besov-type space involving only the logarithmic smoothness ℓ^γ, $\gamma > 0$,

$$\begin{aligned}&B_{0,\gamma}^{(p,r;\beta),s}(\mathbb{T}^n) \\ &:= \left\{ f \in L^{p,r;\beta} \,:\, |f|_{B_{0,\gamma}^{(p,r;\beta),s}} := \left(\int_0^1 [\,\ell^\gamma(u)\, \omega_\kappa(f,u)_{p,r;\beta}]^s \frac{du}{u} \right)^{1/s} < \infty \right\},\end{aligned}$$

where $\kappa > 0$. The notation $B_{0,\gamma}^{(p,r;\beta),s}$ is justified by the fact that the definition is independent of $\kappa > 0$. To verify this, we make use of the notion of the best approximation. Here $E_N(f)_{p,r;\beta}$ denotes the error of approximation of $f \in L^{p,r;\beta}$ by elements from $\mathcal{T}_N$, given by

$$E_N(f)_{p,r;\beta} := \inf\{\|f - T_N\|_{p,r;\beta} \,:\, T_N \in \mathcal{T}_N\}$$

and we call $T_N^{p,r;\beta}(f) \in \mathcal{T}_N$ the best approximant of $f \in L^{p,r;\beta}$ by $\mathcal{T}_N$. Next we observe that, for any $\kappa > 0$,

$$E_j(f)_{p,r;\,\alpha-\gamma} \lesssim \omega_\kappa(f,1/j)_{p,r;\alpha-\gamma} \lesssim \frac{1}{(j+1)^\kappa}\sum_{i=0}^{j}(i+1)^{\kappa-1}E_i(f)_{p,r;\alpha-\gamma}.\quad j\in\mathbb{N}_0. \tag{3.5}$$

Here the first estimate is the Jackson inequality, which can be easily derived from the classical Jackson's theorem for the integer order moduli of smoothness (see [12, Thm. 2.1]):

$$E_j(f)_{p,r;\,\alpha-\gamma} \lesssim \omega_{[\kappa]+1}(f,1/j)_{p,r;\alpha-\gamma} \lesssim \omega_\kappa(f,1/j)_{p,r;\alpha-\gamma}.$$

The second estimate in (3.5) is the weak inverse inequality, which is known (see [11, Thm. 2.3]) for the case $\kappa\in\mathbb{N}$. We can prove it, for any $\kappa>0$, as follows. By (3.2),

$$\omega_\kappa(f,1/2^m)_{p,r;\alpha-\gamma} \approx \|f-V_{2^{-m}}f\|_{p,r;\alpha-\gamma} + 2^{-m\kappa}|V_{2^{-m}}f|_{H_\kappa^{p,r;\,\alpha-\gamma}}.$$

Now we use the fact that the de la Vallée-Poussin sum satisfies $\|V_t f\|_{p,r;\alpha-\gamma}\le C\|f\|_{p,r;\alpha-\gamma}$ and $V_{1/N}T_N=T_N$, $T_N\in\mathcal{T}_N$. Therefore, one has (see also [9, Sect. 4])

$$\|f-V_{2^{-m}}f\|_{p,r;\alpha-\gamma} \lesssim E_{2^m}(f)_{p,r;\alpha-\gamma}. \tag{3.6}$$

We now need the Bernstein inequality in $L^{p,r;\,\alpha-\gamma}(\mathbb{T}^n)$,

$$\begin{aligned}|T_N|_{H_\kappa^{p,r;\,\alpha-\gamma}} = |V_{1/N}T_N|_{H_\kappa^{p,r;\,\alpha-\gamma}} &\lesssim N^\kappa K(T_N,t^\kappa;L^{p,r}(\log L)^\alpha, H_\kappa^{p,r;\,\alpha})\\ &\lesssim N^\kappa\|T_N\|_{p,r;\alpha-\gamma}\,,\end{aligned}$$

which follows from (3.2). This estimate and (3.6) yield

$$\begin{aligned}|V_{2^{-m}}f|_{H_\kappa^{p,r;\,\alpha-\gamma}} &= |\sum_{l=1}^{m}(V_{2^{-l}}f-V_{2^{-l+1}}f)+V_1f|_{H_\kappa^{p,r;\,\alpha-\gamma}}\\ &\lesssim \sum_{l=1}^{m}2^{l\kappa}\|V_{2^{-l}}f-V_{2^{-l+1}}f\|_{p,r;\alpha-\gamma}+\|V_1f\|_{p,r;\alpha-\gamma}\\ &\lesssim \sum_{l=0}^{m-1}2^{l\kappa}E_{2^l}(f)_{p,r;\alpha-\gamma}+E_0(f)_{p,r;\alpha-\gamma}.\end{aligned}$$

Thus, we get

$$\omega_\kappa(f,1/2^m)_{p,r;\alpha-\gamma} \lesssim 2^{-m\kappa}\Big(E_0(f)_{p,r;\alpha-\gamma}+\sum_{l=0}^{m-1}2^{l\kappa}E_{2^l}(f)_{p,r;\alpha-\gamma}\Big),$$

which is equivalent to the last estimate in (3.5). Using monotonicity properties of the modulus of smoothness, we get

$$\left(\int_0^1 [\,\ell^\gamma(u)\,\omega_\kappa(f,u)_{p,r;\beta}]^s \frac{du}{u}\right)^{1/s} \approx \left(\sum_{\nu=1}^{\infty}[\,\ell^\gamma(1/\nu)\,\omega_\kappa(f,1/\nu)_{p,r;\beta}]^s \frac{1}{\nu}\right)^{1/s}.$$

This estimate, (3.5), and Hardy's inequality imply that, for any $\gamma,\ s>0$,

$$\left(\int_0^1 [\,\ell^\gamma(u)\,\omega_\kappa(f,u)_{p,r;\beta}]^s \frac{du}{u}\right)^{1/s} \approx \left(\sum_{\nu=1}^{\infty}[\,\ell^\gamma(1/\nu)\,E_{\nu-1}(f)_{p,r;\beta}]^s \frac{1}{\nu}\right)^{1/s} \tag{3.7}$$

Note that in the case $0<s<1$ we use the following Hardy-type inequality for monotone sequences $\{\varepsilon_i\}$ (cf. [6]): $\sum_{\nu=1}^{\infty}\nu^{-1}\Big[\ell^\gamma(\nu)\nu^{-\kappa}\sum_{i=0}^{\nu}(i+1)^{\kappa-1}\varepsilon_i\Big]^s \lesssim \sum_{\nu=1}^{\infty}\nu^{-1}[\,\ell^\gamma(\nu)\,\varepsilon_{\nu-1}]^s$.
Finally, (3.7) immediately implies that the definition of $B_{0,\gamma}^{(p,r;\beta),s}$ is independent of $\kappa>0$.

Lemma 3.5 *If* $1<p,r<\infty$, $1\le s<\infty$, $\alpha\in\mathbb{R}$, *and* $\beta>-1/s$, *then*

$$B_{0,\beta}^{(p,r;\alpha),s}(\mathbb{T}^n)\hookrightarrow L^{p,s;\beta+\alpha+1/\max\{s,r\}}(\mathbb{T}^n).$$

In particular, we have

$$B_{0,\gamma-1/s}^{(p,r;\alpha-\gamma),s}(\mathbb{T}^n)\hookrightarrow L^{p,s;\alpha}(\mathbb{T}^n)\,,\qquad \gamma>0,\quad 1<r\le s<\infty, \tag{3.8}$$

and

$$B_{0,\gamma-1/r}^{(p,r;\alpha-\gamma),s}(\mathbb{T}^n)\hookrightarrow L^{p,s;\alpha}(\mathbb{T}^n)\,,\qquad \gamma>1/r-1/s,\quad 1\le s<r<\infty. \tag{3.9}$$

Proof By [28, Thm. 4.6],

$$L^{p,r;\alpha}(\mathbb{T}^n)\hookrightarrow L^{p_1}(\mathbb{T}^n)\quad\text{for any } p_1\in[1,p). \tag{3.10}$$

If $n\ge 2$, choose p_1 such that

$$\max\left\{1,\ \frac{np}{n+p}\right\}<p_1<\min\{p,n\}. \tag{3.11}$$

Together with the (generalized) Sobolev embedding theorem (cf., e.g., [16, Thm. 4.8 and Thm. 4.2]), embedding (3.10) implies that

$$W^1L^{p,r;\alpha}(\mathbb{T}^n)\hookrightarrow W^1L^{p_1}(\mathbb{T}^n)\hookrightarrow L^{p_1^*}(\mathbb{T}^n),\qquad \frac{1}{p_1^*}=\frac{1}{p_1}-\frac{1}{n}.$$

If $n = 1$, then, cf. [1, Lemma 5.8, p. 100], $W^1L^1(\mathbb{T}^n) \hookrightarrow C(\mathbb{T}^n)$, which, together with (3.10) shows that the embedding

$$W^1L^{p,r;\alpha}(\mathbb{T}^n) \hookrightarrow W^1L^{p_1}(\mathbb{T}^n) \hookrightarrow L^{p_1^*}(\mathbb{T}^n)$$

remains true with any $p_1^* \in [1,\infty]$, and hence with p_1^* satisfying $1 < p < p_1^* < \infty$. Combining the embedding $W^1L^{p,r;\alpha}(\mathbb{T}^n) \hookrightarrow L^{p_1^*}(\mathbb{T}^n)$ with the trivial embedding $L^{p,r;\alpha}(\mathbb{T}^n) \hookrightarrow L^{p,r;\alpha}(\mathbb{T}^n)$ and using a limiting interpolation, we arrive at

$$X := \Big(L^{p,r;\alpha}(\mathbb{T}^n), W^1L^{p,r;\alpha}(\mathbb{T}^n)\Big)_{0,s;\beta} \hookrightarrow \Big(L^{p,r;\alpha}(\mathbb{T}^n), L^{p_1^*}(\mathbb{T}^n)\Big)_{0,s;\beta} =: Y$$

for any $s \in [1,\infty]$ and $\beta \in \mathbb{R}$. Since (cf. [20, (1.6)])

$$K(f,t; L^{p,r;\alpha}(\mathbb{T}^n), W^1L^{p,r;\alpha}(\mathbb{T}^n)) \approx \min\{1,t\}\|f\|_{p,r;\alpha} + \omega_1(f,t)_{p,r;\alpha}$$

for all $f \in L^{p,r;\alpha}(\mathbb{T}^n) + W^1L^{p,r;\alpha}(\mathbb{T}^n)$ and every $t > 0$, one can show that $X = B^{(p,r;\alpha),s}_{0,\beta}(\mathbb{T}^n)$. Note that

$$\|f\|_X := \|f\|_{p,r;\alpha} + \left(\int_0^1 [\ell^\beta(t)\,\omega_1(f,t)_{p,r;\alpha}]^s \,\frac{dt}{t}\right)^{1/s}.$$

Moreover, if $1 \le s < \infty$ and $\beta > -1/s$, then, using [18, Thm. 5.9$^+$, Thm. 4.7$^+$ (ii), p. 952], we obtain that $Y \hookrightarrow L^{p,s;\beta+\alpha+1/\max\{s,r\}}(\mathbb{T}^n)$ and the result follows.

Replace α by $\alpha-\gamma$, take $\beta = \gamma - 1/s$ and $\beta = \gamma - 1/r$, to obtain embeddings (3.10) and (3.9), respectively. □

3.2 Proof of Theorem 1.2 (b)

Unlike the proof of part (a), here we will use the technique based on estimates of the best approximations rather than a Holmstedt-type formula.

Proof By (3.3) and (3.2),

$$\omega_\kappa(f,1/N)_{p,s;\alpha} \lesssim \|f - V_{1/N}f\|_{p,s;\alpha} + N^{-\kappa}|V_{1/N}f|_{H_\kappa^{p,s;\alpha}} =: I + II. \quad (3.12)$$

(i) Let us first handle the case $r \le s$. Lemma 3.4 together with (3.2) and (3.3) gives

$$II \lesssim (\log N)^\gamma \omega_\kappa(f,1/N)_{p,r;\alpha-\gamma}. \quad (3.13)$$

Concerning I, we first observe that under our restriction on the parameters, by (3.10),

$$\|f\|_{p,s;\alpha} \lesssim \|f\|_{p,r;\alpha-\gamma} + \Big(\int_0^1 [\ell^{\gamma-1/s}(t)\,\omega_1(f,t)_{p,r;\alpha-\gamma}]^s \frac{dt}{t}\Big)^{1/s}$$

$$\lesssim \|f\|_{p,r;\,\alpha-\gamma} + \Big(\sum_{j=1}^{\infty}[\ell^{\gamma-1/s}(j)\,E_j(f)_{p,r;\alpha-\gamma}]^s \frac{1}{j}\Big)^{1/s} \tag{3.14}$$

for all $f \in L^{p,r}(\log L)^{\alpha-\gamma}(\mathbb{T}^n)$, where the latter inequality follows by (3.7). For arbitrary $g \in L^{p,r;\alpha-\gamma}(\mathbb{T}^n)$ set $f := g - T_N^{p,r;\alpha-\gamma}(g)$. This implies that

$$E_j(f)_{p,r;\alpha-\gamma} \le \|f\|_{p,r;\alpha-\gamma} = \|g - T_N^{p,r;\alpha-\gamma}(g)\|_{p,r;\alpha-\gamma} = E_N(g)_{p,r;\alpha-\gamma}\,, \quad 0 \le j \le N,$$

and

$$E_j(f)_{p,r;\alpha-\gamma} = E_j(g)_{p,r;\alpha-\gamma}\,, \qquad j \ge N.$$

Rewrite (3.14) for the above function $f = g - T_N^{p,r;\alpha-\gamma}(g)$ to get

$$\begin{aligned}
E_N(g)_{p,s;\,\alpha} &\lesssim \|g - T_N^{p,r;\alpha-\gamma}(g)\|_{p,s;\,\alpha} \\
&\lesssim E_N(g)_{p,r;\,\alpha-\gamma} + \Big[\Big(\sum_{j=1}^{N} + \sum_{j=N+1}^{\infty}\Big)[\ell^{\gamma-1/s}(j)\,E_j(f)_{p,r;\,\alpha-\gamma}]^s \frac{1}{j}\Big]^{1/s} \\
&\lesssim E_N(g)_{p,r;\,\alpha-\gamma} + E_N(g)_{p,r;\,\alpha-\gamma}\Big(\sum_{j=1}^{N}[\ell^{\gamma-1/s}(j)]^s \frac{1}{j}\Big)^{1/s} \\
&\quad + \Big(\sum_{j=N+1}^{\infty}[\ell^{\gamma-1/s}(j)\,E_j(g)_{p,r;\,\alpha-\gamma}]^s \frac{1}{j}\Big)^{1/s} \\
&\lesssim \ell^{\gamma}(N)\,E_N(g)_{p,r;\,\alpha-\gamma} \\
&\quad + \Big(\sum_{j=N+1}^{\infty}[\ell^{\gamma-1/s}(j)\,E_j(g)_{p,r;\,\alpha-\gamma}]^s \frac{1}{j}\Big)^{1/s}.
\end{aligned}$$

Observe that, by [12, Thm. 2.1],

$$E_j(g)_{p,r;\,\alpha-\gamma} \lesssim \omega_{[\kappa]+1}(g, 1/j)_{p,r;\,\alpha-\gamma} \lesssim \omega_\kappa(g, 1/j)_{p,r;\,\alpha-\gamma}\,,$$

and that, by (3.6), $\|g - V_{1/N}g\|_{p,s;\alpha} \lesssim E_N(g)_{p,s;\alpha}$ to get the desired estimate for I. Together with (3.13), this establishes (1.6).

(ii) Let us now consider the case $1 \le s < r$. Concerning I, we first observe that under our restriction on the parameters, by (3.9),

$$\|f\|_{p,s;\alpha} \lesssim \|f\|_{p,r;\alpha-\gamma} + \Big(\int_0^1 [\ell^{\gamma-1/r}(t)\, \omega_1(f,t)_{p,r;\alpha-\gamma}]^s \frac{dt}{t} \Big)^{1/s}.$$

Now follow straightforward the proof in (i) to obtain

$$E_N(g)_{p,s;\,\alpha} \lesssim \ell^{\gamma+1/s-1/r}(N)\, E_N(g)_{p,r;\,\alpha-\gamma} + \Big(\sum_{j=N+1}^{\infty} [\ell^{\gamma-1/r}(j)\, E_j(g)_{p,r;\,\alpha-\gamma}]^s \frac{1}{j} \Big)^{1/s}$$

for any $g \in L^{p,r;\alpha-\gamma}(\mathbb{T}^n)$. This implies that

$$I \lesssim \ell^{\gamma+1/s-1/r}(\delta)\, \omega_\kappa(f,\delta)_{p,r;\,\alpha-\gamma} + \Big(\int_0^\delta [\ell^{\gamma-1/r}(t)\, \omega_\kappa(f,t)_{p,r;\,\alpha-\gamma}]^s \frac{dt}{t} \Big)^{1/s}. \tag{3.15}$$

With regard to II, we need the following variant of Nikolskii's inequality for trigonometric polynomials $T_N \in \mathcal{T}_N(\mathbb{T}^n)$ which states that

$$\|T_N\|_{p,s;\alpha} \lesssim (\log N)^{\gamma+1/s-1/r} \|T_N\|_{p,r;\alpha-\gamma}\,, \qquad \gamma > 1/r - 1/s, \quad s < r, \tag{3.16}$$

and which will be proved below. Suppose (3.16) is true. Then

$$II \lesssim N^{-\kappa} (\log N)^{\gamma+1/s-1/r} |V_{1/N} f|_{H_\kappa^{p,r;\,\alpha-\gamma}} \lesssim (\log N)^{\gamma+1/s-1/r} \omega_\kappa(f, 1/N)_{p,r;\alpha-\gamma}\,,$$

by (3.2) and (3.3). In view of (3.15), this proves assertion (1.7).

To prove (3.16), we need the following Remez inequality (see [13] and also [27])

$$T_N^*(0) \le C(n) T_N^*(N^{-n}), \qquad N \in \mathbb{N}, \tag{3.17}$$

where T_N^* is the non-increasing rearrangement of T_N. Then

$$\|T_N\|_{p,s;\alpha}^s \lesssim \int_0^{N^{-n}} [\,t^{1/p} \ell^\alpha(t) T_N^*(t)\,]^s \frac{dt}{t} + \int_{N^{-n}}^1 [\,t^{1/p} \ell^\alpha(t) T_N^*(t)\,]^s \frac{dt}{t} =: I_1 + I_2.$$

By (3.17),

$$
\begin{aligned}
I_1 &\lesssim T_N^*(N^{-n})^s \int_0^{N^{-n}} [\, t^{1/p} \ell^{\alpha}(t)\,]^s \frac{dt}{t} \lesssim T_N^*(N^{-n})^s N^{-ns/p} \ell^{\alpha s}(N) \\
&\lesssim T_N^*(N^{-n})^s \ell^{\gamma s}(N) \left(\int_0^{N^{-n}} [\, t^{1/p} \ell^{\alpha-\gamma}(t)\,]^r \frac{dt}{t} \right)^{s/r} \\
&\lesssim \ell^{\gamma s}(N) \left(\int_0^{N^{-n}} [\, t^{1/p} \ell^{\alpha-\gamma}(t) T_N^*(t)\,]^r \frac{dt}{t} \right)^{s/r} \\
&\lesssim \ell^{\gamma s}(N) \| T_N \|_{p,r;\alpha-\gamma}^s .
\end{aligned}
$$

Finally, by Hölder's inequality,

$$
\begin{aligned}
I_2 &\lesssim \left(\int_{N^{-n}}^{1} [\, t^{1/p} \ell^{\alpha-\gamma}(t) T_N^*(t)\,]^r \frac{dt}{t} \right)^{s/r} \left(\int_{N^{-n}}^{1} \ell^{\gamma sr/(r-s)}(t) \frac{dt}{t} \right)^{(r-s)/r} \\
&\lesssim \ell^{\gamma s+1-s/r}(N) \left(\int_{N^{-n}}^{1} [\, t^{1/p} \ell^{\alpha-\gamma}(t) T_N^*(t)\,]^r \frac{dt}{t} \right)^{s/r} \\
&\lesssim \ell^{\gamma s+1-s/r}(N) \| T_N \|_{p,r;\alpha-\gamma}^s .
\end{aligned}
$$

Note that the power of the $\ell(N)$-factor is positive since $\gamma > 1/r - 1/s$. □

3.3 Embedding Results for Besov Spaces

We finish the paper with two limiting embeddings of the $B_{\sigma,\gamma}^{(p,r;\beta),s}$-spaces that follow from Theorem 1.2 (b).

Corollary 3.6 *Assume* $1 < p < \infty$, $\alpha \in \mathbb{R}$, *and either* $1 < r \le s < \infty$ *or* $1 \le s < r < \infty$. *Let* $\gamma + \max\{\frac{1}{s} - \frac{1}{r}, 0\} > 0$. *Then*

$$
B_{\lambda,\mu+\gamma+\max\{\frac{1}{s}-\frac{1}{r},0\}}^{(p,r;\alpha-\gamma),\xi}(\mathbb{T}^n) \hookrightarrow B_{\lambda,\mu}^{(p,s;\alpha),\xi}(\mathbb{T}^n), \qquad \lambda, \xi > 0, \quad \mu \in \mathbb{R}, \tag{3.18}
$$

and

$$
B_{0,\mu+\gamma+\max\{\frac{1}{\xi}-\frac{1}{s},0\}+\max\{\frac{1}{s}-\frac{1}{r},0\}}^{(p,r;\alpha-\gamma),\xi}(\mathbb{T}^n) \hookrightarrow B_{0,\mu}^{(p,s;\alpha),\xi}(\mathbb{T}^n), \qquad \xi > 0, \quad \mu\xi > -1. \tag{3.19}
$$

The proofs of (3.18) and (3.19) follow the same line as the proof of Corollary 2.8 using the Ulyanov inequalities from Theorem 1.2 (b) and suitable Hardy's inequalities.

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