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L^p-Integrability, Dimensions of Supports of Fourier Transforms and Applications

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Abstract It is proved that there does not exist any non zero function in $L^p(\mathbb{R}^n)$ with $1 \leq p \leq 2n/\alpha$ if its Fourier transform is supported by a set of finite packing α-measure where $0 < \alpha < n$. It is shown that the assertion fails for $p > 2n/\alpha$. The result is applied to prove L^p Wiener Tauberian theorems for $\mathbb{R}^n$ and $M(2)$.

Keywords Supports of Fourier Transform · Hausdorff dimension · Packing measure · Salem sets · Ahlfors–David regular sets · Wiener Tauberian theorems

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1 Introduction

A classical result of Wiener [15] states that the translates of a function $f \in L^1(\mathbb{R}^n)$ span a dense subset of $L^1(\mathbb{R}^n)$ if and only if the Fourier transform, $\widehat{f}$ of f is not zero at any point on $\mathbb{R}^n$. That is, if $f \in L^1(\mathbb{R}^n)$ and

$$\widehat{f}(t) = \frac{1}{(\sqrt{2\pi})^n} \int\limits_{\mathbb{R}^n} f(x) e^{-ix\cdot t} dx,$$

then for ${}^x f(y) = f(-x+y)$, we have $\overline{span\ \{{}^x f : x \in \mathbb{R}^n\}} = L^1(\mathbb{R}^n)$ if and only if $\widehat{f}(t) \neq 0\ \forall\ t \in \mathbb{R}^n$. In fact, if $g \in L^\infty(\mathbb{R}^n)$ is such that $\int_{\mathbb{R}^n} {}^x f(y) g(y) dy =$

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$0 \; \forall \, x \in \mathbb{R}^n$, we get $\widetilde{f} * g = 0$ where $\widetilde{f}(t) = f(-t)$. Distribution theory tells us that $supp \; \widehat{\widehat{g}} \subseteq \{x \in \mathbb{R}^n : \widehat{f}(x) = 0\}$ (which is Wiener Tauberian theorem in disguise. See [12]). If $\widehat{f}$ is nowhere vanishing then it follows that $g \equiv 0$. This crucial step in the proof of Wiener's theorem leads us to the study of functions f in $L^p(\mathbb{R}^n)$ with $supp \; \widehat{f}$ in a thin set.

This question also arises in PDE. If u is a tempered solution of the equation $P(D)u = 0$, where $P(D)$ is a constant coefficient differential operator then the Fourier transform $\hat{u}$ is supported in the zero set of the polynomial P. Agmon and Hörmander [1] studied the asymptotic properties of u. Agranovsky and Narayanan in [2] proved that if $f \in L^p(\mathbb{R}^n)$ and $supp \; \widehat{f}$ is carried by a C^1-manifold of dimension $d < n$, then $f \equiv 0$ if $p \leq \frac{2n}{d}$. Notice that in these results and other ones of similar nature, only the range $p > 2$ is interesting. If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$ and $\widehat{f}$ is supported in a set of measure zero then f is identically zero. If $p > 2$, then $\widehat{f}$ is a tempered distribution and the support of f is a closed set which may be thin.

Our aim in the first part of this paper is to extend the above results where the integer dimension manifold set is replaced with finite packing measurable set (see (1.1)). We also mention that an older result of Beurling (see [3]) says that if $f \in L^p(\mathbb{R})$, $p > 2$ and $\hat{f}$ is supported by a set of Hausdorff dimension less than $2/p$, then the function is identically zero. We show that *if* $f \in L^p(\mathbb{R}^n)$ *and supp* $\widehat{f}$ *is contained in a set* E, *which has a finite packing* α*-measure* (for $0 < \alpha < n$), *then* $f \equiv 0$ *if* $p \leq \frac{2n}{\alpha}$. By considering the cartesian product of the Salem set in $\mathbb{R}$ (see [13] and also page 263 in [5]), we show that our result is sharp.

In the second part of the paper we use the above result to prove some L^p-Wiener Tauberian theorems. Wiener [15] characterized the cyclic vectors (with respect to translations) in $L^p(\mathbb{R})$, for $p = 1, 2$, in terms of the zero set of the Fourier transform. He conjectured that a similar characterization should be true for $1 < p < 2$ (See page 93 in [15]). Lev and Olevskii in [8] recently proved that for any $1 < p < 2$ one can find two functions in $L^1(\mathbb{R}) \cap C_0(\mathbb{R})$, such that one is cyclic in $L^p(\mathbb{R})$ and the other is not, but their Fourier transforms have the same (compact) set of zeros. This disproves Wiener's conjecture. As is well known, there are no complete answers to L^p-Weiner-Tauberian theorems when $p \neq 1, 2$. See pages 234–236 in [5] for initial results and [8] for more references.

Beurling [3] proved that if the Hausdorff dimension of the closed set where the Fourier transform of f vanishes, is α, for $0 \leq \alpha \leq 1$, then the space of finite linear combinations of translates of f is dense in $L^p(\mathbb{R})$ for $2/(2-\alpha) < p$. Now using our result, we prove a similar result (including the end points for the range) on $\mathbb{R}^n$ where sets of Hausdorff dimension are replaced with the sets of finite packing $\alpha-$measure. Herz studied some versions of L^p- Wiener Tauberian theorems and gave alternative sufficient conditions for the translates of $f \in L^1 \cap L^p(\mathbb{R}^n)$ to span $L^p(\mathbb{R}^n)$ (See [6]). With an additional hypothesis on the zero sets of Fourier transform of f, we improve his result. Rawat and Sitaram [11] initiated the study of L^p-versions of the Wiener Tauberian theorem under the action of motion group $M(n)$ on $\mathbb{R}^n$.

We shall show that some of the results proved in [11] can be improved using our result. Finally we take up L^p-Wiener Tauberian theorem on the Euclidean motion group $M(2)$.

In the remaining of this section we recall certain definitions from Fractal geometry (See [4] and [9]). In the second section we prove the above mentioned result on the L^p-integrability and dimension of the support of $\widehat{f}$ and its sharpness. Finally in the third section we look at applications of the results proved in Sect. 2 to L^p-Wiener Tauberian theorems on $\mathbb{R}^n$ and the Euclidean motion group $M(2)$.

Let $\mathcal{H}_\alpha$ denote the Hausdorff α-dimensional outer measure. Let E be a non-empty bounded subset of $\mathbb{R}^n$. The ϵ**-covering number** of E, $N(E, \epsilon)$, is the smallest number of open balls of radius ϵ needed to cover E. The ϵ**-packing number** of E, $P(E, \epsilon)$, is the largest number of **disjoint** open balls of radius ϵ with centres in E. The ϵ**-packing** of E is any collection of disjoint balls $\{B_{r_k}(x_k)\}_k$ with centres $x_k \in E$ and radii satisfying $0 < r_k \le \epsilon/2$. Let $0 \le s < \infty$. For $0 < \epsilon < 1$ and $A \subset \mathbb{R}^n$, put

$$P^s_\epsilon(A) = sup\left\{\sum_k (2r_k)^s\right\}$$

where the supremum is taken over all permissible ϵ-packings, $\{B_{r_k}(x_k)\}_k$ of A. Then $P^s_\epsilon(A)$ is non-decreasing with respect to ϵ and we set the **packing pre measure**, P^s_0 as

$$P^s_0(A) = \lim_{\epsilon\downarrow 0} P^s_\epsilon(A).$$

We have $P^s_0(\emptyset) = 0$, P^s_0 is monotonic and finitely subadditive, but not countably subadditive. The **packing** $s-$ **measure** of A, $\mathcal{P}^s(A)$ is defined as

$$\mathcal{P}^s(A) = inf\left\{\sum_{i=1}^{\infty} P^s_0(A_i) : A = \cup_{i=1}^{\infty} A_i\right\}. \tag{1.1}$$

Then $\mathcal{P}^s$ is Borel regular (Theorem 3.11 in [4]). If ν is a measure, the α**-upper density of** ν at x, $\overline{D^\alpha}(\nu, x)$ is defined as

$$\overline{D^\alpha}(\nu, x) = \limsup_{r\to 0} (2r)^{-\alpha}\nu(B_r(x)),$$

where $B_r(x)$ is a ball of radius r with centre x. Similarly α**-lower density** of ν at x, $\underline{D^\alpha}(\nu, x)$ is defined using $liminf$. Let $\alpha < n$. A set $E \subset \mathbb{R}^n$ is said to be **Ahlfors–David regular** α**-set** if there exists a, b (both > 0) in $\mathbb{R}$ such that

$$0 < ar^\alpha \le \mathcal{H}_\alpha(E \cap B_r(x)) \le br^\alpha < \infty$$

for all $x \in E$ and $0 < r \le 1$. For all these definitions and similar ones we refer to [4] and [9].

2 Dimensions of Supports of Fourier Transforms

In this section we relate the dimension of the support of the Fourier transform of a function with its membership in L^p. In the following lemma, we recall some needed results (see pages 78–89 in [9]). For a non-empty subset A of $\mathbb{R}^n$, let $A(\epsilon) = \{x \in \mathbb{R}^n : d(x, A) < \epsilon\}$.

Lemma 2.1 *Fix $\epsilon > 0$. Let $|A(\epsilon)|$ denote the Lebesgue measure of $A(\epsilon)$, where A is a non-empty bounded subset of $\mathbb{R}^n$. Then,*

(1) $N(A, 2\epsilon) \le P(A, \epsilon) \le N(A, \epsilon/2)$,
(2) $\Omega_n P(A, \epsilon)\epsilon^n \le |A(\epsilon)| \le \Omega_n N(A, \epsilon)(2\epsilon)^n$, *where Ω_n denotes the volume of the unit ball in $\mathbb{R}^n$,*
(3) *For $0 \le s < \infty$, $P(A, \epsilon/2)\epsilon^s \le P^s_\epsilon(A)$,*
(4) *Let $B \subset \mathbb{R}^n$ be such that $\mathcal{H}_\alpha(B) < \infty$. Then*

$$2^{-\alpha} \le \overline{D^\alpha}(\mu, x) \le 1$$

for $\mathcal{H}_\alpha$ almost all $x \in B$, where $\mu = \mathcal{H}_\alpha|_B$.

The following lemma is crucial for us.

Lemma 2.2 *Let $0 \le \alpha < n$ and let $E \subset \mathbb{R}^n$ be such that $\mathcal{P}^\alpha(E) < \infty$. Let $S \subset E$ be bounded and $S(\epsilon) = \{x \in \mathbb{R}^n : d(x, S) < \epsilon\}$. Then*

$$\limsup_{\epsilon \to 0} |S(\epsilon)|\epsilon^{\alpha - n} < \infty,$$

where $|S(\epsilon)|$ denotes the Lebesgue measure of $S(\epsilon)$.

Proof Since we have $\mathcal{P}^\alpha(S) \le \mathcal{P}^\alpha(E) < \infty$, there exists a countable cover $\{\widetilde{A_i}\}$ of S such that $\sum P^\alpha_0(\widetilde{A_i}) < \infty$. Let $R > 0$ be such that $S \subset B_R(0)$. Then $\{A_i\}$ also covers S, where $A_i = \widetilde{A_i} \cap B_R(0)$ is bounded and $\sum P^\alpha_0(A_i) \le \sum P^\alpha_0(\widetilde{A_i}) < \infty$. By Lemma 2.1,

$$\begin{aligned} |A_i(\epsilon)| &\le \Omega_n (2\epsilon)^n N(A_i, \epsilon) \\ &\le \Omega_n (2\epsilon)^n P(A_i, \epsilon/2) \\ &\le \Omega_n 2^n \epsilon^{n-\alpha} P^\alpha_\epsilon(A_i). \end{aligned}$$

Hence $\epsilon^{\alpha-n}|A_i(\epsilon)| \le C_n P^\alpha_\epsilon(A_i)$ for some fixed constant C_n. We also have $|S(\epsilon)| \le \sum |A_i(\epsilon)|$. Hence, $\epsilon^{\alpha-n}|S(\epsilon)| \le C_n \sum P^\alpha_\epsilon(A_i)$. So,

$$\limsup_{\epsilon \to 0} \epsilon^{\alpha-n}|S(\epsilon)| \le C_n \sum P^\alpha_0(A_i) < \infty.$$

Hence we have $\epsilon^{\alpha-n}|S(\epsilon)|$ tending to a finite limit as $\epsilon \to 0$.

Theorem 2.3 *Let $f \in L^p(\mathbb{R}^n)$ be such that supp $\widehat{f}$ is contained in a set E where $\mathcal{P}^\alpha(E) < \infty$ for some $0 \le \alpha < n$. Then $f \equiv 0$, provided $p \le \frac{2n}{\alpha}$.*

Proof For the proof we closely follow the arguments in [1] (See page 174 of [7]). By convolving f with a compactly supported smooth function we can assume that $f \in L^p(\mathbb{R}^n)$ where $p = 2n/\alpha$. Choose an even function $\chi \in C_c^\infty(R^n)$ with support in unit ball and $\int_{\mathbb{R}^n} \chi(x)dx = 1$. Let $\chi_\epsilon(x) = \epsilon^{-n}\chi(x/\epsilon)$ and $u_\epsilon = u * \chi_\epsilon$ where $u = \widehat{f}$. Then by the Plancherel theorem,

$$\begin{aligned}\|u_\epsilon\|^2 &= \int\limits_{\mathbb{R}^n} |f(x)|^2 |\widehat{\chi}(\epsilon x)|^2 dx \\ &\leqslant C\epsilon^{\alpha-n} \sum_{j=-\infty}^{\infty} 2^{j(n-\alpha)} \sup_{2^j \leqslant |\epsilon x| \leqslant 2^{j+1}} |\widehat{\chi}_\epsilon(x)|^2 (2^{-j}\epsilon)^{n-\alpha} \int\limits_{2^j \leqslant |\epsilon x| \leqslant 2^{j+1}} |f(x)|^2 dx \\ &= C\epsilon^{\alpha-n} \sum_{j=-\infty}^{\infty} a_j b_j^\epsilon,\end{aligned}$$

where

$$a_j = 2^{j(n-\alpha)} \sup_{2^j \leqslant |x| \leqslant 2^{j+1}} |\widehat{\chi}(x)|^2,$$

and

$$b_j^\epsilon = (2^{-j}\epsilon)^{n-\alpha} \int\limits_{2^j \leqslant |\epsilon x| \leqslant 2^{j+1}} |f(x)|^2 dx.$$

Applying Holder's inequality,

$$|b_j^\epsilon| \leqslant C \left(\int\limits_{2^j\epsilon^{-1} \leqslant |x| \leqslant 2^{j+1}\epsilon^{-1}} |f(x)|^p dx \right)^{2/p},$$

which goes to zero as $\epsilon \to 0$, for any fixed j. Also we have $|b_j^\epsilon| \leqslant C\|f\|_p^2 < \infty$ for some constant C independent of ϵ and j. Since $\sum_j |a_j|$ is finite, by the dominated convergence theorem, we have $\sum_j a_j b_j^\epsilon \to 0$ as $\epsilon \to 0$.

Let $\psi \in C_c^\infty(\mathbb{R}^n)$. Let $S = supp\ \widehat{f} \cap supp\ \psi$. Then S is a bounded subset of E. By Lemma 2.2, we have $\epsilon^{\alpha-n}|S_\epsilon|$ tending to a finite limit as $\epsilon \to 0$. So,

$$
\begin{aligned}
| < u, \psi > |^2 &= \lim_{\epsilon \to 0} | < u_\epsilon, \psi > |^2 \\
&\leq \lim_{\epsilon \to 0} \|u_\epsilon\|_2^2 \int_{S_\epsilon} |\psi|^2 \\
&\leq c \|\psi\|_\infty^2 \lim_{\epsilon \to 0} \epsilon^{\alpha - n} |S_\epsilon| \sum_{j=-\infty}^{\infty} a_j b_j^\epsilon \\
&= 0
\end{aligned}
$$

Hence $f = 0$.

Remark 2.1 For an integer $0 \leq d < n$, any d-dimensional smooth manifold in $\mathbb{R}^n$ has both Hausdorff and Packing dimension as d. (See page 56 and 85 in [9]) Hence the above Theorem 2.3 extends Theorem 1 in [2].

Lemma 2.4 *Let $E \subset \mathbb{R}^n$ be such that $0 < \mathcal{H}_\alpha(E) < \infty$. Assume that there exists constants $0 < a < \infty$ and $r_\alpha > 0$ such that $ar^\alpha \leq \mathcal{H}_\alpha(E \cap B_r(x))$ for all $x \in E$ and for all $r < r_\alpha$. Then $\mathcal{P}^\alpha(E) < \infty$.*

Proof Consider $\mu_\alpha(A) = \mathcal{H}_\alpha(E \cap A)$ for all $A \subseteq \mathbb{R}^n$. Then μ_α is a finite Borel regular measure on $\mathbb{R}^n$. Since $ar^\alpha \leq \mathcal{H}_\alpha(E \cap B_r(x))$ for all $x \in E$ and for all $r < r_\alpha$, we have $a \leq \underline{D^\alpha}(\mu_\alpha, x)$ for all $x \in E$. Let $k = inf_{x \in E}\{\underline{D^\alpha}(\mu_\alpha, x)\}$. We have $0 < a \leq k$. By Lemma 2.1 (4), we have $k < \infty$. Also corresponding to each $E \subseteq \mathbb{R}^n$ is a Borel set $B \supseteq E$ such that $\mathcal{P}^\alpha(B) = \mathcal{P}^\alpha(E)$. (See Theorem 3.11(c) in [4]). Then, $\mu_\alpha(B) = \mathcal{H}_\alpha(E \cap B) < \infty$. By Theorem 3.16 in [4], we have $\mathcal{P}^\alpha(E) = \mathcal{P}^\alpha(B) \leq \mu_\alpha(B)/k < \infty$.

Remark 2.2 Ahlfors–David regular α-sets satisfy the hypothesis of the above lemma.

Next we show that Theorem 2.3 is sharp. First, let us recall a well known example due to Salem which shows that there exists a measure ν supported on a Cantor type set $K \subseteq \mathbb{R}$, of Hausdorff dimension β, $0 < \beta < 1$ with Fourier tranform $\widehat{\nu}$ belonging to $L^q(\mathbb{R})$ for all $q > 2/\beta$ (See [13] and page 263–271 in [5]). Let $M = K \times K \times \ldots \times K$ (n times) and $\mu = \nu \times \nu \times \ldots \times \nu$ (n times). Then μ is supported in M and $\widehat{\mu} \in L^q(\mathbb{R}^n)$ for $q > \frac{2}{\beta} = \frac{2n}{\alpha}$ where $\alpha = n\beta$. Closely following the proof in [5] (page 33) we show that not only the Hausdorff dimension of M is α, but M also satisfies the hypothesis of the above Lemma 2.4 and thus proving that the range in Theorem 2.3 is the best possible.

First, we briefly recall how the above set $K \subseteq \mathbb{R}$ is constructed. Choose a positive number η and an integer N so that $N\eta < 1$ and

$$N\eta^\beta = 1. \tag{2.1}$$

Choose N independent points a_i in the unit interval [0, 1] in such a way that $0 \leq a_1 < a_2 < \ldots < a_N \leq 1 - \eta$ and widely enough spaced so that the distance between two a_i is larger than η. The set K is constructed as the intersection of decreasing sequence of compact sets K_j, where K_j's are defined as follows:

Choose an increasing sequence of non-zero positive numbers η_j converging to η where

$$\eta\left(1-\frac{1}{(j+1)^2}\right) \leq \eta_j \leq \eta \tag{2.2}$$

for all j. The first set, K_1, is the union of N intervals of length η_1 of the form $[a_k, a_k+\eta_1]$. The second set K_2, has N^2 intervals of length $\eta_1\eta_2$ of the form $[a_i + a_j\eta_1, a_i + a_j\eta_1 + \eta_1\eta_2]$ and so on. Inductively, we obtain a sequence K_j of decreasing sets of length $\eta_1\eta_2\ldots\eta_j$. Then $K = \cap_j K_j$. It is known that the Hausdorff dimension of K is β. (see [13] and page 268 in [5])

Lemma 2.5 *Hausdorff dimension of* $M = K \times K \times \ldots \times K$ *(n times) equals* $\alpha = n\beta$ *and* $0 \leq \mathcal{P}^\alpha(M) < \infty$.

Proof Let $M_j = K_{j_1} \times K_{j_2} \times \ldots \times K_{j_n}$ for $j = (j_1, j_2, \ldots, j_n)$. Among the coverings of M which compete in the definition of $\mathcal{H}_\alpha(M)$, (the Hausdorff measure of M) are the coverings M_j themselves, consisting of $N^{j_1+j_2+\ldots+j_n}$ cubes of volume $\Pi_{i=1}^n(\eta_1\eta_2\ldots\eta_{j_i})$. Since $\eta_1 \leq \eta_2 \leq \ldots < \eta$, we obtain

$$\mathcal{H}_\alpha(M) \leq N^{j_1}(\eta_1^\beta\eta_2^\beta\ldots\eta_{j_1}^\beta)\ldots N^{j_n}(\eta_1^\beta\eta_2^\beta\ldots\eta_{j_n}^\beta) \leq N^{j_1}\eta^{j_1\beta}\ldots N^{j_n}\eta^{j_n\beta} = 1,$$

and see that the dimension of M is at most α.

To show that the dimension of M is exactly α, we show that $\mathcal{H}_\alpha(M)$ is not 0. First we prove that M satisfies the hypothesis of Lemma 2.4.

Let $0 < r < 1$ and $x \in M$, that is let $x = (x_1, x_2, \ldots, x_n)$ where $x_m \in K$ for all m. For every m, by construction of K, there exists a smallest integer t_m such that $K \cap (x_m - r, x_m + r)$ contains at least one interval I_{t_m} of length $\eta_1 \ldots \eta_m$. Thus

$$K \cap (x_m - r, x_m + r) \supseteq K \cap I_{t_1} \times \ldots \times K \cap I_{t_n}. \tag{2.3}$$

Since Hausdorff measure is translation invariant, we can assume $2r \leq \eta_1 \ldots \eta_{t_m-1}$. Since $\alpha = n\beta$,

$$(2r)^\alpha \leq \Pi_{m=1}^n(\eta_1^\beta \ldots \eta_{t_m-1}^\beta). \tag{2.4}$$

In computing the Hausdorff measure, it is enough to take the infimum of Σd_i^α over all coverings of $M \cap B_r(x)$ by countable families of (sufficiently small) open balls A_i, where the end points of the projection of A_i to m^{th} axis is in the complement of $K \cap (x_m - r, x_m + r)$. From the compactness, it is also clear that these coverings consist of only a finite number of disjoint, open cubes. Let $\{U_i\}$ be one such family of sufficiently small cubes that cover $M \cap B_r(x)$, where the end points of the projection of U_i to m^{th} axis is in the complement of $K \cap (x_m - r, x_m + r)$.

Let p_{i_m} be the smallest integer p such that m^{th} projection of U_i contains at least one interval of K_p and $P_i = (p_{i_1}, p_{i_2}, \ldots p_{i_n})$. Then, from (2.3), $t_m \leq p_{i_m}$. Let

$$p_{i_m} = t_m + s_{i_m} \tag{2.5}$$

Let m^{th} projection of U_i contain $k_i^{(m)}$ number of constituent intervals of K_{p_m}. Then U_i contain $k_i = \Pi_{m=1}^n k_i^{(m)}$ number of cubes of $M_{P_i} = K_{p_{i_1}} \times \ldots \times K_{p_{i_n}}$. Let d_i

denote the diameter of U_i. Then

$$d_i^n \geq k_i \Pi_{m=1}^n (\eta_1 \eta_2 \ldots \eta_{p_{i_m}}). \tag{2.6}$$

Let j_m's be large such that $\cup U_i$ contains $M_j \cap M \cap B_r(x)$ where $M_j = K_{j_1} \times \ldots \times K_{j_n}$ and $M_j \subset M_{P_i}$, for all i. Then U_i contains $k_i N^{(j_1 - p_{i_1} + \ldots + j_n - p_{i_n})}$ cubes of M_j. By (2.5), U_i contains $k_i N^{(j_1 - t_1 - s_{i_1} + \ldots + j_n - t_n - s_{i_n})}$ cubes of M_j. However by (2.3),

$$M \cap B_r(x) \cap M_j \subseteq M_t \subset M \cap B_r(x),$$

where $M_t = (K \cap I_{t_1}) \times \ldots \times (K \cap I_{t_n})$. So the number of cubes of M_j covered by $\cup U_i$ is at least $N^{j_1 - t_1 + \ldots + j_n - t_n}$. Since $\sum_i k_i N^{(j_1 - t_1 - s_{i_1} + \ldots + j_n - t_n - s_{i_n})}$ is the total number of cubes of M_j covered by $\cup U_i$,

$$\sum_i k_i N^{(j_1 - t_1 - s_{i_1} + \ldots + j_n - t_n - s_{i_n})} \geq N^{j_1 - t_1 + \ldots + j_n - t_n}. \tag{2.7}$$

The Eq. (2.6) implies that

$$\begin{aligned} d_i^\alpha &\geq (k_i \Pi_{m=1}^n (\eta_1 \eta_2 \ldots \eta_{p_{i_m}}))^\beta \\ &\geq (2r)^\alpha (k_i \Pi_{m=1}^n (\eta_{t_m} \eta_{t_m+1} \ldots \eta_{p_{i_m}}))^\beta \text{(from (2.0.4))} \\ &\geq (2r)^\alpha \left(k_i \Pi_{m=1}^n \eta_{t_m} \eta^{p_{i_m} - t_m} \left[\left(1 - \frac{1}{(t_m+1)^2}\right) \ldots \left(1 - \frac{1}{p_{i_m}^2}\right) \right] \right)^\beta \text{ from(2.0.2)} \end{aligned}$$

Since η_m is an increasing sequence and by (2.2), $\eta_{t_1} \eta_{t_2} \ldots \eta_{t_n} \geq (\frac{3}{4}\eta)^n$. Fix $C = (\frac{3}{4}\eta)^n$. Thus

$$\begin{aligned} d_i^\alpha &\geq C(2r)^\alpha \left(k_i \eta^{(p_{i_1} + \ldots + p_{i_n} - (t_1 + \ldots t_n))} \Pi_{m=1}^n \left[\left(1 - \frac{1}{t_m + 1}\right) \left(1 + \frac{1}{p_{i_m}}\right) \right] \right)^\beta \\ &\geq C(2r)^\alpha \left(k_i \eta^{(p_{i_1} + \ldots + p_{i_n} - (t_1 + \ldots t_n))} \Pi_{m=1}^n \left[\frac{1}{2} \left(1 + \frac{1}{p_{i_m}}\right) \right] \right)^\beta \\ &> C r^\alpha k_i^\beta \eta^{(p_{i_1} + \ldots + p_{i_n} - (t_1 + \ldots t_n))\beta}, \end{aligned}$$

From (2.1), we have

$$N^{(j_1 + \ldots j_n) - (p_{i_1} + \ldots p_{i_n})} \eta^{(j_1 + \ldots j_n - t_1 - \ldots t_n)\beta} = \eta^{(p_{i_1} + \ldots + p_{i_n} - (t_1 + \ldots t_n))\beta}.$$

Thus

$$d_i^\alpha \geq C r^\alpha k_i^\beta N^{(j_1 + \ldots j_n) - (p_{i_1} + \ldots p_{i_n})} \eta^{(j_1 + \ldots j_n - t_1 - \ldots t_n)\beta}. \tag{2.8}$$

Also, there exists a constant $C_{N,n}$ $(= 2^n (N-1)^n)$, such that $1 \leq k_i \leq C_{N,n}$ because of the choice of p_{i_k}. Let $L = (C_{N,n})^{\beta - 1}$. Since $0 < \beta < 1$,

$$k_i^\beta > L k_i \tag{2.9}$$

From (2.5) and (2.9), summing over i in (2.8), we have

$$\begin{aligned}\Sigma_i d_i^{\alpha} &\geq CLr^{\alpha}\eta^{(j_1+\ldots j_n-t_1-\ldots-t_n)\beta}\Sigma_i k_i N^{(j_1+\ldots j_n)-(t_1+\ldots t_n+s_{i_1}+\ldots s_{i_n})}\\ &\geq CLr^{\alpha}\eta^{(j_1+\ldots j_n-t_1-\ldots-t_n)\beta}N^{j_1+\ldots j_n-t_1-\ldots-t_n}\ \text{(from (2.0.7))}\\ &= CLr^{\alpha}\ \ \text{(from (2.0.1))}\\ &> 0\end{aligned}$$

Thus $\mathcal{H}_\alpha(M\cap B_r(x))\geq CLr^{\alpha}$ for all $x\in M$ and $0<r<1$. Similarly we prove that $\mathcal{H}_\alpha(M)>0$. By Lemma 2.4, $\mathcal{P}^{\alpha}(M)<\infty$.

Remark 2.3 As remarked by one of the referees of this paper, the set constructed in Lemma 2.5 is fractal even if α is an integer. In [2], the authors proved the sharpness of Theorem 2 (in [2]) for any integer $\alpha\geq n/2$ by constructing a smooth manifold $M\subset\mathbb{R}^n$ and μ supported on M such that the Fourier transform $f=\hat{\mu}\in L^p(\mathbb{R}^n)$ for all $p>2n/\alpha$. It would be interesting to see if this can be done for all integers α between 0 and n.

3 Applications to Wiener Tauberian Theorems

3.1 L^p Wiener Tauberian Theorems on $\mathbb{R}^n$

In this section, we improve the results on L^p versions of Wiener Tauberian type theorems on $\mathbb{R}^n$ obtained in [11]. Consider the motion group $M(n)=\mathbb{R}^n\ltimes SO(n)$ with the group law

$$(x_1,k_1)(x_2,k_2)=(x_1+k_1x_2,k_1k_2).$$

For a function h on $\mathbb{R}^n$ and an arbitrary $g=(y,k)\in M(n)$, let ${}^g h$ be the function ${}^g h(x)=h(kx+y),\ x\in\mathbb{R}^n$. Let $\widehat{h}$ denote the Euclidean Fourier transform of the function h. For $h\in L^1\cap L^p(\mathbb{R}^n),\ 1\leq p\leq\infty$, let $S=\{r>0:\widehat{h}\equiv 0 \text{ on } C_r\}$, where C_r is the sphere of radius $r>0$ centered at origin in $\mathbb{R}^n$. Let $Y=Span\{{}^g h: g\in M(n)\}$. Then the main result from [11] is

Theorem 3.1 (1) *If $p=1$, then Y is dense in $L^1(\mathbb{R}^n)$ if and only if S is empty and $\widehat{h}(0)\neq 0$.*
(2) *If $1<p<\frac{2n}{n+1}$, then Y is dense in $L^p(\mathbb{R}^n)$ if and only if S is empty.*
(3) *If $\frac{2n}{n+1}\leq p<2$, and every point of S is an isolated point, then Y is dense in $L^p(\mathbb{R}^n)$.*
(4) *If $2\leq p\leq\frac{2n}{n-1}$, and S is of zero measure in $\mathbb{R}^+$, then Y is dense in $L^p(\mathbb{R}^n)$.*
(5) *If $\frac{2n}{n-1}<p<\infty$, then Y is dense in $L^p(\mathbb{R}^n)$ if and only if S is nowhere dense.*

We prove that the part (3) of the above theorem can be improved:

Theorem 3.2 *Let $f\in L^1(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$ and let $S=\{r>0:\widehat{f}\equiv 0 \text{ on } C_r\}$ be such that $\mathcal{P}^{\beta}(S)<\infty$, for some $0\leq\beta<1$. If $\frac{2n}{n+1-\beta}\leq p\leq 2$, then $Y=Span\{{}^g f: g\in M(n)\}$ is dense in $L^p(\mathbb{R}^n)$.*

Proof Fix $\epsilon < 1$. Suppose Y is not dense in $L^p(\mathbb{R}^n)$. Let $h \in L^q(\mathbb{R}^n)$ annihilate all the elements in Y, where $\frac{1}{p} + \frac{1}{q} = 1$. We can assume h to be smooth, bounded and radial (see the arguments in [11]). It follows that $h * f \equiv 0$. Then $supp\ \widehat{h}$ is contained in the zero set of $\widehat{f}$. Let α be such that $2 \le q = \frac{2n}{\alpha} \le \frac{2n}{n-1+\beta}$. Choose an even function $\chi \in C_c^\infty(\mathbb{R}^n)$ with support in the unit ball and $\int_{\mathbb{R}^n} \chi(x)dx = 1$. Let $\chi_\epsilon(x) = \epsilon^{-n}\chi(x/\epsilon)$ and $u_\epsilon = u * \chi_\epsilon$ where $u = \widehat{h}$. Since $2 \le q$, as in Theorem 2.3,

$$\|u_\epsilon\|^2 \leqslant C\epsilon^{\alpha-n} \sum_{j=-\infty}^{\infty} a_j b_j^\epsilon,$$

$$\text{where } a_j = 2^{j(n-\alpha)} \sup_{2^j \leqslant |x| \leqslant 2^{j+1}} |\widehat{\chi}(x)|^2$$

$$\text{and } b_j^\epsilon = (2^{-j}\epsilon)^{n-\alpha} \int_{2^j \leqslant |\epsilon x| \leqslant 2^{j+1}} |h(x)|^2 dx.$$

and $\sum_{j=-\infty}^{\infty} a_j b_j^\epsilon \to 0$ as $\epsilon \to 0$.

Let $\psi \in C_c^\infty(\mathbb{R}^n)$. Let $M = supp\ \widehat{h} \cap supp\ \psi$ and let $R_\psi > 0$ be such that M is contained in a ball of radius R_ψ. For $x \in M$, $\|x\| \in S$ and $\|x\| \le R_\psi$. Let $S_\psi = \{r \in S : r \le R_\psi\}$. Then S_ψ is a bounded subset of S. With similar arguments in Lemma 2.2, we prove that $\lim_{\epsilon\to 0} \epsilon^{\beta-1} \int_{M_\epsilon} |\psi(x)|^2 dx < \infty$:

Since $\mathcal{P}^\beta(S_\psi) \le \mathcal{P}^\beta(S) < \infty$, let $\{A_i\}$ be a cover of S_ψ such that $\sum_i P_0^\beta(A_i) < \infty$. Then $P_0^\beta(A_i \cap S_\psi) < \infty$. For $S_\psi^i = A_i \cap S_\psi$, let $P(S_\psi^i, \epsilon)$ be the maximum number of disjoint balls with centers $\{r_j\}$ in S_ψ^i, of radius ϵ and $N(S_\psi^i, \epsilon)$ be the ϵ-covering number of S_ϵ^i. Then

$$S_\psi^i \subseteq \cup_{j=1}^{N(S_\psi^i,\epsilon)} (r_j - \epsilon/2, r_j + \epsilon/2) \text{ and}$$

$$S_\psi(\epsilon) \subset \cup_i S_\psi^i(\epsilon) \subseteq \cup_i \cup_{j=1}^{N(S_\psi^i,\epsilon)} (r_j - \epsilon, r_j + \epsilon).$$

If $x \in M(\epsilon)$, then $\|x\| \in S_\psi(\epsilon)$. We have,

$$\begin{aligned}\int_{M(\epsilon)} |\psi(x)|^2 dx &\le \int_{r\in S_\psi(\epsilon)} \int |\psi(r\omega)|^2 d\omega r^{n-1} dr \\ &\le (R_\psi + \epsilon)^{n-1} \int_{r\in S_\psi(\epsilon)} \int |\psi(r\omega)|^2 d\omega dr \\ &\le (R_\psi + 1)^{n-1} \|\psi\|_\infty^2 \Omega_n \sum_i \sum_{j=1}^{N(S_\psi^i,\epsilon)} \int_{r_j-\epsilon}^{r_j+\epsilon} dr\end{aligned}$$

$$= C_1 \sum_i N(S^i_\psi, \epsilon)(2\epsilon)$$
$$\leq 2C_1\epsilon \sum_i P(S^i_\psi, \epsilon/2) \quad \text{(by lemma 2.1)}$$

where $C_1 = (R_\psi + 1)^{n-1}\|\psi\|^2_\infty \Omega_n$ is a constant independent of ϵ and Ω_n is the volume of the unit sphere in $\mathbb{R}^n$. Thus,

$$\lim_{\epsilon\to 0} \epsilon^{\beta-1} \int\limits_{M(\epsilon)} |\psi(x)|^2 dx \leq 2C_1 \sum_i \lim_{\epsilon\to 0} \epsilon^\beta P(S^i_\psi, \epsilon/2) \leq 2C_1 \sum_i P^\beta_0(A_i) < \infty.$$

Hence,

$$\begin{aligned}
| < u, \psi > |^2 &= \lim_{\epsilon\to 0} | < u_\epsilon, \psi > |^2 \\
&\leq \lim_{\epsilon\to 0} \|u_\epsilon\|^2_2 \int\limits_{M_\epsilon} |\psi|^2 \\
&\leq C \lim_{\epsilon\to 0} \epsilon^{\alpha-n} \sum_{j=-\infty}^{\infty} a_j b^\epsilon_j \int\limits_{M_\epsilon} |\psi(x)|^2 dx \\
&\leq C \lim_{\epsilon\to 0} \epsilon^{\alpha-n-\beta+1} \epsilon^{\beta-1} \sum_{j=-\infty}^{\infty} a_j b^\epsilon_j \int\limits_{M_\epsilon} |\psi(x)|^2 dx \\
&\leq CC_1 \lim_{\epsilon\to 0} \sum_{j=-\infty}^{\infty} a_j b^\epsilon_j \\
&= 0,
\end{aligned}$$

since $2 \leq \frac{2n}{\alpha} \leq \frac{2n}{n-1+\beta}$, that is $0 \leq \alpha - n - \beta + 1$. Hence $h = 0$.

Remark 3.1 Suppose every point of S is an isolated point. Convolving f with an arbitrary Schwartz class function whose Fourier transform is compactly supported, we may assume that S is finite. The case $\beta = 0$ in the above theorem then implies part (3) of Theorem 3.1.

Now let f be an integrable function in $L^1 \cap L^p(\mathbb{R})$ and let F denote the closed set where the Fourier transform of f vanishes. In [3], A. Beurling proved that if for some p in $(1, 2)$, the space of finite linear combinations of translates of f is not dense in $L^p(\mathbb{R})$, then the Hausdorff dimension of F is at least $2 - (2/p)$ (see also page 312 in [5]). In other words, if the Hausdorff dimension of F is α, for $0 \leq \alpha \leq 1$, then the space of finite linear combinations of translates of f is dense in $L^p(\mathbb{R})$ for $2/(2-\alpha) < p < \infty$. Now using Theorem 2.3, we prove a similar result (including the end points for the range) on $\mathbb{R}^n$ where Hausdorff dimension is replaced with the packing dimension.

Theorem 3.3 *Let $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for $\frac{2n}{2n-\alpha} \leq p < \infty$ and let the zero set of $\widehat{f} \subseteq E$, where $\mathcal{P}^\alpha(E) < \infty$ for some $0 \leq \alpha < n$. Then $X = span\{^x f : x \in \mathbb{R}^n\}$ is dense in $L^p(\mathbb{R}^n)$.*

Proof Suppose X is not dense in $L^p(\mathbb{R}^n)$. Then there exists a non trivial, smooth and radial $h \in L^q(\mathbb{R}^n)$ such that $h * f_1 \equiv 0$ for all $f_1 \in X$(see the arguments in [11]). Clearly the zero set of $X(\subset L^1(\mathbb{R}^n))$, $\bigcap_{u \in X} \{s \in \mathbb{R}^n : \widehat{u}(s) = 0\}$ is equal to the zero set of $\widehat{f}$, $Z(\widehat{f})$. Hence $supp\ \widehat{h} \subseteq Z(\widehat{f})$. Since $\frac{2n}{2n-\alpha} \leq p < \infty$, we have $1 < q \leq \frac{2n}{\alpha}$. By Theorem 2.3, $h = 0$. Thus X is dense in $L^p(\mathbb{R}^n)$.

In [6], C. S Herz studied the versions of L^p- Wiener Tauberian theorems. From Theorem 1 and Theorem 4 of [6], we note that for $f \in L^1 \cap L^p(\mathbb{R}^n)$, $p < \infty$ the alternative sufficient conditions for the translates of f to span L^p are,

(1) $|K(\epsilon)| = o(\epsilon^{n(1-2/q)})$ for each compact subset K of E.
(2) $\dim E = \alpha < 2n/q$, with the proviso, if $n > 2$, that $q \leq 2n/(n-2)$.

where E denotes the zero set of $\widehat{f}$ and $\frac{1}{p} + \frac{1}{q} = 1$. With an additional hypothesis on E, using Theorem 3.3, we can improve the result in [6]:

Proposition 3.4 *For $f \in L^1 \cap L^p(\mathbb{R}^n)$, $1 \leq p < \infty$ a sufficient condition that the translates of f span L^p is : the zero set of $\widehat{f}$ has finite packing α- measure for $\alpha \leq 2n/q$ where $\frac{1}{p} + \frac{1}{q} = 1$.*

3.2 L^p Wiener Tauberian Theorem on $M(2)$

In this section, we look at one sided and two sided analogues of Wiener Tauberian Theorems on $M(2)$ and improve a few results from [10].

The group $M(2)$ is the semi-direct product of $\mathbb{C}$ with the special orthogonal group $K = SO(2)$. The group law in $G = M(2)$ is given by

$$(z, e^{i\alpha})(w, e^{i\beta}) = (z + e^{i\alpha} w, e^{i(\alpha+\beta)}).$$

The Haar measure on G is given by $dg = dz d\alpha$ where dz is the Lebesgue measure on $\mathbb{C}$ and $d\alpha$ is the normalized Haar measure on S^1. For each $\lambda > 0$, we have a unitary irreducible representation of G realized on $H = L^2(K) = L^2([0, 2\pi], dt)$, given by

$$[\pi_\lambda(z, e^{it})u](s) = e^{i\lambda < z, e^{is} >} u(s-t),$$

for $(z, e^{it}) \in G$ and $u \in H$. Here $< z, w > = \text{Re} z.\bar{w}$. It is known that these are all the infinite dimensional, non equivalent unitary irreducible representations of G. Apart from the above family, we have another family $\{\chi_n, n \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of integers, of one dimensional unitary representations of G, given by $\chi_n(z, e^{i\alpha}) = e^{in\alpha}$. Then the unitary dual $\widehat{G}$, of G is the collection $\{\pi_\lambda, \lambda > 0\} \cup \{\chi_n : n \in \mathbb{Z}\}$ (see page 165, [14]).

For $f \in L^1(G)$, define the "group theoretic" Fourier transform of f as follows:

$$\pi_\lambda(f) = \int_G f(g)\pi_\lambda(g)dg, \ \lambda > 0$$

and

$$\chi_n(f) = \int_G f(z, e^{i\alpha})e^{-in\alpha}dzd\alpha, \ n \in \mathbb{Z}.$$

From the Plancherel theorem for G (see page 183, [14]) we have for $f \in L^2(G)$,

$$\|f\|_2^2 = \int_0^\infty \|\pi_\lambda(f)\|_{HS}^2 \lambda d\lambda,$$

where $\|.\|_{HS}$ denotes the Hilbert–Schmidt norm.

For $g_1, g_2 \in G$, the two sided translate, ${}^{g_1}f^{g_2}$ of f is the function defined by ${}^{g_1}f^{g_2}(g) = f(g_1^{-1}gg_2)$. For $f \in L^1(G) \cap L^p(G)$, let $S = \{a > 0 : \pi_a(f) = 0\}$, $X = Span\ \{{}^{g_1}f^{g_2} : g_1, g_2 \in G\}$, $S' = \{\lambda > 0 :$ Range of $\pi_\lambda(f)$ is not dense$\}$ and V_f be the closed subspace spanned by the right translates of f in $L^p(G)$.

Theorem 3.5 *Let $f \in L^1(G) \cap L^p(G)$.*

(1) For $\frac{4}{3-\alpha} \leq p < 2$, if $S = \{a > 0 : \pi_a(f) = 0\}$ is such that $\mathcal{P}^\alpha(S) < \infty$ for $0 \leq \alpha < 1$, then $X = span\{{}^{g_1}f^{g_2} : g_1, g_2 \in M(2)\}$ is dense in $L^p(M(2))$.
(2) If f is radial in the $\mathbb{R}^2$ variable and $\mathcal{P}^\alpha(S') < \infty$ for some $0 \leq \alpha < 1$, then $V_f = L^p(M(2))$ provided $\frac{4}{3-\alpha} \leq p \leq 2$.

Proof To prove part (1), we proceed as in the proof of Theorem 2.1 in [10]. It is enough to prove $L^p(G/K) \subseteq \overline{X}$.

For given $a, \epsilon > 0$, there exists constants $c_1, c_2, \ldots, c_m$, $w \in H$ and elements $x_1, x_2, \ldots x_m \in G$ such that $\|\sum_{j=1}^m c_j \pi_a(x_j)v_0 - w\| < \epsilon$, where v_0 is K-fixed vector $v_o \equiv 1 \in H$. Define $F_a = \sum_{j=1}^m c_j f^{x_j^{-1}}$. Then $\pi_a(F_a)v_0 \neq 0$. Let

$$F_a^\#(x) = \int_K F_a(xk)dk, \ x \in G.$$

Then whenever $\pi_a(f) \neq 0$, as in the proof of Theorem 2.1 in [10] we have a right K-invariant function $F_a^\#$ which can be considered as a function on $\mathbb{R}^2$, that is $F_a^\# \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ such that its Euclidean Fourier transform is not identically zero on the sphere $C_a = \{x \in \mathbb{R}^2 : \|x\| = a\}$.

Define $S_1 = \cap_{a \in S^c}\{r > 0 : \widehat{F_a^\#} \equiv 0 \text{ on } C_r\}$. Then $S_1 \subset S$. We have $\overline{Span\{{}^g F_a^\# : g \in G, a \in S_1^c\}} \subseteq \overline{Span\{{}^{g_1}f^{g_2} : g_1, g_2 \in G\}}$. Also using Theorem 3.2,

$\overline{Span\{{}^{g}F_{a}^{\#}: g \in G, a \in S_{1}^{c}\}} = L^{p}(G/K)$. Thus $L^{p}(G/K) \subseteq \overline{Span\{{}^{g_1}f^{g_2}: g_1, g_2 \in G\}} = \overline{X}$.

To prove part(2), we proceed as in the proof of (c) of Theorem 3.2 in [10]. Let $\phi(z)e^{im_0\alpha} \in L^q \cap L^\infty(M(2))$ kill all the functions in V_f where $\frac{1}{p}+\frac{1}{q}=1$. Then f being radial in the $\mathbb{R}^2$-variable we are led to the convolution equation $f_m *_{\mathbb{R}^2} \phi_m = 0$ where ϕ_m is defined by

$$\phi_m(z) = \int_0^{2\pi} \phi(e^{i\alpha}z)e^{i(m_0+m)\alpha}\,d\alpha.$$

and f_m is defined by

$$f_m(z) = \int_{S^1} f(z, e^{i\alpha})e^{-im\alpha}\,d\alpha$$

Taking Fourier transform we obtain that $supp\ \widehat{\phi_m}$ is contained in $\{z \in \mathbb{R}^2 : \|z\| \in S\}$. Proceeding as in the proof of Theorem 3.2, we have $< \phi_m, \psi >= 0$ for all $\psi \in C_c^\infty(\mathbb{R}^2)$ and m. Thus $\phi_m \equiv 0$ for all m.

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