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On Sharp Aperture-Weighted Estimates for Square Functions

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Abstract Let $S_{\alpha,\psi}(f)$ be the square function defined by means of the cone in $\mathbb{R}^{n+1}_+$ of aperture α, and a standard kernel ψ. Let $[w]_{A_p}$ denote the A_p characteristic of the weight w. We show that for any $1 < p < \infty$ and $\alpha \geq 1$,

$$\|S_{\alpha,\psi}\|_{L^p(w)} \lesssim \alpha^n [w]_{A_p}^{\max\left(\frac{1}{2},\frac{1}{p-1}\right)}.$$

For each fixed α the dependence on $[w]_{A_p}$ is sharp. Also, on all class A_p the result is sharp in α. Previously this estimate was proved in the case $\alpha = 1$ using the intrinsic square function. However, that approach does not allow to get the above estimate with sharp dependence on α. Hence we give a different proof suitable for all $\alpha \geq 1$ and avoiding the notion of the intrinsic square function.

Keywords Littlewood–Paley operators · Sharp weighted inequalities · Sharp aperture dependence

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1 Introduction

Let ψ be an integrable function, $\int_{\mathbb{R}^n} \psi = 0$, and, for some $\varepsilon > 0$,

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$$|\psi(x)| \le \frac{c}{(1+|x|)^{n+\varepsilon}} \quad \text{and} \quad \int_{\mathbb{R}^n} |\psi(x+h)-\psi(x)|dx \le c|h|^{\varepsilon}. \tag{1.1}$$

Let $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}_+$ and $\Gamma_\alpha(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < \alpha t\}$. Set $\psi_t(x) = t^{-n}\psi(x/t)$. Define the square function $S_{\alpha,\psi}(f)$ by

$$S_{\alpha,\psi}(f)(x) = \left(\int_{\Gamma_\alpha(x)} |f*\psi_t(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \quad (\alpha > 0).$$

We drop the subscript α if $\alpha = 1$.

Given a weight w, define its A_p characteristic by

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q w\,dx \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}}\,dx \right)^{p-1},$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

It was proved in [13] that for any $1 < p < \infty$,

$$\|S_\psi\|_{L^p(w)} \le c_{p,n,\psi}[w]_{A_p}^{\max\left(\frac{1}{2},\frac{1}{p-1}\right)}, \tag{1.2}$$

and this estimate is sharp in terms of $[w]_{A_p}$ (we also refer to [13] for a detailed history of closely related results).

Similarly one can show that

$$\|S_{\alpha,\psi}\|_{L^p(w)} \le c_{p,n,\psi}\gamma(\alpha)[w]_{A_p}^{\max\left(\frac{1}{2},\frac{1}{p-1}\right)} \quad (\alpha \ge 1,\, 1 < p < \infty); \tag{1.3}$$

however, the sharp dependence on α in this estimate cannot be determined by means of the approach from [13]. The aim of this paper is to find the sharp $\gamma(\alpha)$ in (1.3).

Let us explain first why the method from [13] gives a rough estimate for $\gamma(\alpha)$. The proof in [13] was based on the intrinsic square function $G_{\alpha,\beta}(f)$ by Wilson [19] defined as follows. For $0 < \beta \le 1$, let $\mathcal{C}_\beta$ be the family of functions supported in the unit ball with mean zero and such that for all x and x', $|\varphi(x)-\varphi(x')| \le |x-x'|^\beta$. If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $(y,t) \in \mathbb{R}^{n+1}_+$, we define $A_\beta(f)(y,t) = \sup_{\varphi\in\mathcal{C}_\beta} |f*\varphi_t(y)|$ and

$$G_{\alpha,\beta}(f)(x) = \left(\int_{\Gamma_\alpha(x)} \big(A_\beta(f)(y,t)\big)^2 \frac{dydt}{t^{n+1}} \right)^{1/2}.$$

Set $G_{1,\beta}(f) = G_\beta(f)$.

The intrinsic square function has several interesting features (established in [19]). First, though $G_\beta(f)$ is defined by means of kernels with uniform compact support, it pointwise dominates $S_\psi(f)$. Also there is a pointwise relation between $G_{\alpha,\beta}(f)$ with different apertures:

$$G_{\alpha,\beta}(f)(x) \le \alpha^{(3/2)n+\beta} G_\beta(f)(x) \quad (\alpha \ge 1). \tag{1.4}$$

Notice that for the usual square functions $S_{\alpha,\psi}(f)$ such a pointwise relation is not available.

In [13], (1.2) with $G_\beta(f)$ instead of $S_\psi(f)$ was obtained. Combining this with (1.4), we would obtain that one can take $\gamma(\alpha) = \alpha^{(3/2)n+\beta}$ in (1.3) assuming that $\psi \in \mathcal{C}_\beta$. For non-compactly supported ψ some additional ideas from [19] can be used that lead to even worse estimate on $\gamma(\alpha)$. Observe also that it is not clear to us whether (1.4) can be improved.

It is easy to see that the dependence $\gamma(\alpha) = \alpha^{(3/2)n+\beta}$ in (1.3) is far from the sharp one. For instance, it is obvious that the information on β should not appear in (1.3). All this indicates that the intrinsic square function approach is not suitable for our purposes in determining the sharp $\gamma(\alpha)$.

Suppose we seek for $\gamma(\alpha)$ in the form $\gamma(\alpha) = \alpha^r$. Then a simple observation shows that $r \ge n$ for any $1 < p < \infty$. Indeed, consider the Littlewood–Paley function $g^*_{\mu,\psi}(f)$ defined by

$$g^*_{\mu,\psi}(f)(x) = \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|} \right)^{\mu n} |f * \psi_t(y)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2}.$$

Using the standard estimate

$$g^*_{\mu,\psi}(f)(x) \le S_\psi(f)(x) + \sum_{k=0}^{\infty} 2^{-k\mu n/2} S_{2^{k+1},\psi}(f)(x),$$

we obtain that (1.3) for some $p = p_0$ and $\gamma(\alpha) = \alpha^{r_0}$ implies

$$\|g^*_{\mu,\psi}\|_{L^{p_0}(w)} \lesssim \left(\sum_{k=0}^{\infty} 2^{-k\mu n/2} 2^{k r_0} \right) [w]_{A_{p_0}}^{\max\left(\frac{1}{2}, \frac{1}{p_0-1}\right)}. \tag{1.5}$$

This means that if $\mu > 2r_0/n$, then $g^*_{\mu,\psi}$ is bounded on $L^{p_0}(w)$, $w \in A_{p_0}$. From this, by the Rubio de Francia extrapolation theorem, $g^*_{\mu,\psi}$ is bounded on the unweighted L^p for any $p > 1$, whenever $\mu > 2r_0/n$. But it is well known [8] that $g^*_{\mu,\psi}$ is not bounded on L^p if $1 < \mu < 2$ and $1 < p \le 2/\mu$. Hence, if $r_0 < n$, we would obtain a contradiction to the latter fact for p sufficiently close to 1.

Our main result shows that for any $1 < p < \infty$ one can take the optimal power growth $\gamma(\alpha) = \alpha^n$.

Theorem 1.1 *For any* $1 < p < \infty$ *and for all* $1 \le \alpha < \infty$,

$$\|S_{\alpha,\psi}\|_{L^p(w)} \le c_{p,n,\psi}\alpha^n [w]_{A_p}^{\max\left(\frac{1}{2},\frac{1}{p-1}\right)}.$$

By (1.5), we immediately obtain the following.

Corollary 1.2 *Let* $\mu > 2$. *Then for any* $1 < p < \infty$,

$$\|g^*_{\mu,\psi}(f)\|_{L^p(w)} \le c_{p,n,\mu,\psi}[w]_{A_p}^{\max\left(\frac{1}{2},\frac{1}{p-1}\right)}.$$

Observe that if $\mu = 2$, then $g^*_{2,\psi}$ is also bounded on $L^p(w)$ for $w \in A_p$ (see [17]). However, the sharp dependence on $[w]_{A_p}$ in the corresponding $L^p(w)$ inequality is unknown to us.

We emphasize that the growth $\gamma(\alpha) = \alpha^n$ is best possible in the weighted $L^p(w)$ estimate for $w \in A_p$. In the unweighted case a better dependence on α is known, namely, $\|S_{\alpha,\psi}\|_{L^p} \le c_{p,n,\psi}\alpha^{\frac{n}{\min(p,2)}}$, see [1, 18].

Some words about the proof of Theorem 1.1. As in [13], we use here the local mean oscillation decomposition. But in [13] we worked with the intrinsic square function, and due to the fact that this operator is defined by uniform compactly supported kernels, we arrived to the operator

$$\mathcal{A}(f)(x) = \Big(\sum_{j,k}(f_{\gamma Q_j^k})^2\chi_{Q_j^k}(x)\Big)^{1/2},$$

where Q_j^k is a sparse family (see Sect. 2.2 for the definition of this notion) and $\gamma > 1$ (here we use the standard notations $f_Q = \frac{1}{|Q|}\int_Q f$ and γQ is the γ-fold concentric dilate of Q). This operator can be handled sufficiently easily.

Here we work with the square function $S_{\alpha,\psi}(f)$ directly, more precisely we consider its smooth variant $\widetilde{S}_{\alpha,\psi}(f)$. Applying the local mean oscillation decomposition to $\widetilde{S}_{\alpha,\psi}(f)$, we obtain that $S_{\alpha,\psi}(f)$ is essentially pointwise bounded by $\alpha^n\mathcal{B}(f)$, where

$$\mathcal{B}(f)(x) = \sum_{m=0}^{\infty}\frac{1}{2^{m\delta}}\Big(\sum_{j,k}(f_{2^m Q_j^k})^2\chi_{Q_j^k}(x)\Big)^{1/2} \quad (\delta > 0).$$

Observe that this pointwise aperture estimate is interesting in its own right. In order to handle $\mathcal{B}$, we use a mixture of ideas from recent papers on a simple proof of the A_2 conjecture [14] and sharp weighted estimates for multilinear Calderón–Zygmund operators [5]. In particular, similarly to [14], we obtain the $X^{(2)}$-norm boundedness of $\mathcal{B}$ by $\mathcal{A}$ on an arbitrary Banach function space X.

The paper is organized as follows. The next section contains some preliminary information. In Sect. 3, we obtain the main estimate, namely, the local mean oscillation estimate of $\widetilde{S}_{\alpha,\psi}(f)$. The proof of Theorem 1.1 is contained in Sect. 4. Section 5 contains some concluding remarks concerning the sharp aperture-weighted weak type estimates for $S_{\alpha,\psi}(f)$.

2 Preliminaries

2.1 A Weak Type (1, 1) Estimate for Square Functions

It is well known that the operator $S_{\alpha,\psi}$ is of weak type $(1, 1)$. However, we could not find in the literature the sharp dependence on α in the corresponding inequality. Hence we give below an argument based on general square functions.

For a measurable function F on $\mathbb{R}^{n+1}_+$ define

$$S_\alpha(F)(x) = \left(\int_{\Gamma_\alpha(x)} |F(y,t)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}.$$

Lemma 2.1 *For any* $\alpha \geq 1$,

$$\|S_\alpha(F)\|_{L^{1,\infty}} \leq c_n \alpha^n \|S_1(F)\|_{L^{1,\infty}}. \tag{2.1}$$

Proof We will use the following estimate, which can be found in [18, p. 315]: if $\Omega \subset \mathbb{R}^n$ is an open set and

$$U = \{x \in \mathbb{R}^n : M\chi_\Omega(x) > 1/(2\alpha^n)\},$$

where M is the Hardy–Littlewood maximal operator, then

$$\int_{\mathbb{R}^n \setminus U} S_\alpha(F)(x)^2 dx \leq 2\alpha^n \int_{\mathbb{R}^n \setminus \Omega} S_1(F)(x)^2 dx$$

(observe that the definitions of $S_\alpha(F)$ here and in [18] differ by the factor $\alpha^{n/2}$.)

Let $\Omega_\xi = \{x : S_1(F)(x) > \xi\}$ and $U_\xi = \{x : M\chi_{\Omega_\xi}(x) > 1/2\alpha^n\}$. Using the weak type $(1, 1)$ estimate for M, Chebyshev's inequality, and the above estimate, we obtain

$$\begin{aligned}
&|\{x \in \mathbb{R}^n : S_\alpha(F)(x) > \xi\}| \\
&\leq |U_\xi| + |\{x \in \mathbb{R}^n \setminus U_\xi : S_\alpha(F)(x) > \xi\}| \\
&\leq c_n\alpha^n |\{x : S_1(F)(x) > \xi\}| + \frac{1}{\xi^2} \int_{\mathbb{R}^n \setminus U_\xi} S_\alpha(F)(x)^2 dx \\
&\leq c_n\alpha^n |\{x : S_1(F)(x) > \xi\}| + \frac{2\alpha^n}{\xi^2} \int_{\mathbb{R}^n \setminus \Omega_\xi} S_1(F)(x)^2 dx.
\end{aligned}$$

Further,

$$\int_{\mathbb{R}^n \setminus \Omega_\xi} S_1(F)(x)^2 dx \leq 2 \int_0^\xi \lambda |\{x : S_1(F)(x) > \lambda\}| d\lambda \leq 2\xi \|S_1(F)\|_{L^{1,\infty}}.$$

Combining this with the previous estimate gives

$$|\{x : S_\alpha(F)(x) > \xi\}| \le c_n \alpha^n |\{x : S_1(F)(x) > \xi\}| + \frac{4\alpha^n}{\xi}\|S_1(F)\|_{L^{1,\infty}},$$

which proves (2.1). □

Note that the sharp unweighted L^p estimates relating square functions of different apertures were obtained recently in [1].

By Lemma 2.1 and by the weak type (1, 1) estimate for $S_\psi(f)$ [9],

$$\|S_{\alpha,\psi}(f)\|_{L^{1,\infty}} \le c_{n,\psi}\alpha^n \|f\|_{L^1}. \tag{2.2}$$

2.2 Dyadic Grids and Sparse Families

Recall that the standard dyadic grid in $\mathbb{R}^n$ consists of the cubes

$$2^{-k}([0,1)^n + j), \quad k \in \mathbb{Z}, j \in \mathbb{Z}^n.$$

Denote the standard grid by $\mathcal{D}$.

By a *general dyadic grid* $\mathscr{D}$ we mean a collection of cubes with the following properties: (i) for any $Q \in \mathscr{D}$ its sidelength ℓ_Q is of the form 2^k, $k \in \mathbb{Z}$; (ii) $Q \cap R \in \{Q, R, \emptyset\}$ for any $Q, R \in \mathscr{D}$; (iii) the cubes of a fixed sidelength 2^k form a partition of $\mathbb{R}^n$.

Given a cube Q_0, denote by $\mathcal{D}(Q_0)$ the set of all dyadic cubes with respect to Q_0, that is, the cubes from $\mathcal{D}(Q_0)$ are formed by repeated subdivision of Q_0 and each of its descendants into 2^n congruent subcubes. Observe that if $Q_0 \in \mathscr{D}$, then each cube from $\mathcal{D}(Q_0)$ will also belong to $\mathscr{D}$.

We will use the following proposition from [10].

Proposition 2.2 *There are 2^n dyadic grids $\mathscr{D}_i$ such that for any cube $Q \subset \mathbb{R}^n$ there exists a cube $Q_i \in \mathscr{D}_i$ such that $Q \subset Q_i$ and $\ell_{Q_i} \le 6\ell_Q$.*

We say that $\{Q_j^k\}$ is a *sparse family* of cubes if: (i) the cubes Q_j^k are disjoint in j, with k fixed; (ii) if $\Omega_k = \cup_j Q_j^k$, then $\Omega_{k+1} \subset \Omega_k$; (iii) $|\Omega_{k+1} \cap Q_j^k| \le \frac{1}{2}|Q_j^k|$.

2.3 A "Local Mean Oscillation Decomposition"

The non-increasing rearrangement of a measurable function f on $\mathbb{R}^n$ is defined by

$$f^*(t) = \inf\{\alpha > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}| < t\} \quad (0 < t < \infty).$$

Given a measurable function f on $\mathbb{R}^n$ and a cube Q, the local mean oscillation of f on Q is defined by

$$\omega_\lambda(f; Q) = \inf_{c\in\mathbb{R}} \big((f-c)\chi_Q\big)^*\big(\lambda|Q|\big) \quad (0 < \lambda < 1).$$

By a median value of f over Q we mean a possibly nonunique, real number $m_f(Q)$ such that

$$\max\big(|\{x\in Q : f(x) > m_f(Q)\}|, |\{x\in Q : f(x) < m_f(Q)\}|\big) \le |Q|/2.$$

It is easy to see that the set of all median values of f is either one point or a closed interval. In the latter case we will assume for the definiteness that $m_f(Q)$ is the *maximal* median value. Observe that it follows from the definitions that

$$|m_f(Q)| \le (f\chi_Q)^*(|Q|/2). \tag{2.3}$$

Given a cube Q_0, the dyadic local sharp maximal function $m^{\#,d}_{\lambda;Q_0} f$ is defined by

$$m^{\#,d}_{\lambda;Q_0} f(x) = \sup_{x\in Q'\in\mathcal{D}(Q_0)} \omega_\lambda(f;Q').$$

The following theorem was proved in [15] (a very similar version can be found in [12]).

Theorem 2.3 *Let f be a measurable function on $\mathbb{R}^n$ and let Q_0 be a fixed cube. Then there exists a (possibly empty) sparse family of cubes $Q_j^k \in \mathcal{D}(Q_0)$ such that for a.e. $x\in Q_0$,*

$$|f(x) - m_f(Q_0)| \le 4m^{\#,d}_{\frac{1}{2^{n+2}};Q_0} f(x) + 2\sum_{k,j} \omega_{\frac{1}{2^{n+2}}}(f;Q_j^k)\chi_{Q_j^k}(x).$$

3 A Key Estimate

In this section we will obtain the main local mean oscillation estimate of $S_{\alpha,\psi}$. We consider a smooth version of $S_{\alpha,\psi}$ defined as follows. Let Φ be a Schwartz function such that

$$\chi_{B(0,1)}(x) \le \Phi(x) \le \chi_{B(0,2)}(x).$$

Define

$$\widetilde{S}_{\alpha,\psi}(f)(x) = \left(\iint_{\mathbb{R}^{n+1}_+} \Phi\Big(\frac{x-y}{t\alpha}\Big)|f*\psi_t(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2} \quad (\alpha > 0).$$

It is easy to see that

$$S_{\alpha,\psi}(f)(x) \le \widetilde{S}_{\alpha,\psi}(f)(x) \le S_{2\alpha,\psi}(f)(x).$$

Hence, by (2.2),

$$\|\widetilde{S}_{\alpha,\psi}(f)\|_{L^{1,\infty}} \le c_{n,\psi}\alpha^n \|f\|_{L^1}. \tag{3.1}$$

Lemma 3.1 *For any cube* $Q \subset \mathbb{R}^n$,

$$\omega_\lambda(\widetilde{S}_{\alpha,\psi}(f)^2; Q) \le c_{n,\lambda,\psi}\alpha^{2n} \sum_{k=0}^{\infty} \frac{1}{2^{k\delta}} \left(\frac{1}{|2^k Q|} \int_{2^k Q} |f| \right)^2, \tag{3.2}$$

where $\delta = \varepsilon$ *from condition* (1.1) *if* $\varepsilon < 1$, *and* $\delta < 1$ *if* $\varepsilon = 1$.

Proof Given a cube Q, let $T(Q) = \{(y,t) : y \in Q, 0 < t < \ell_Q\}$, where ℓ_Q denotes the side length of Q. For $x \in Q$ we decompose $\widetilde{S}_{\alpha,\psi}(f)(x)^2$ into the sum of

$$I_1(f)(x) = \iint_{T(2Q)} \Phi\Big(\frac{x-y}{t\alpha}\Big)|f * \psi_t(y)|^2 \frac{dydt}{t^{n+1}}$$

and

$$I_2(f)(x) = \iint_{\mathbb{R}^{n+1}_+ \setminus T(2Q)} \Phi\Big(\frac{x-y}{t\alpha}\Big)|f * \psi_t(y)|^2 \frac{dydt}{t^{n+1}}.$$

Let us show first that

$$(I_1(f)\chi_Q)^*(\lambda|Q|) \le c_{n,\lambda,\psi}\alpha^{2n} \sum_{k=0}^{\infty} \frac{1}{2^{k\varepsilon}} \left(\frac{1}{|2^k Q|} \int_{2^k Q} |f| \right)^2. \tag{3.3}$$

Using that $(a+b)^2 \le 2(a^2+b^2)$, we get

$$I_1(f)(x) \le 2\big(I_1(f\chi_{4Q})(x) + I_1(f\chi_{\mathbb{R}^n \setminus 4Q})(x)\big).$$

Hence,

$$\begin{aligned}(I_1(f)\chi_Q)^*(\lambda|Q|) \le 2\big((I_1(f\chi_{4Q}))^*(\lambda|Q|/2) \\ +(I_1(f\chi_{\mathbb{R}^n \setminus 4Q})\chi_Q)^*(\lambda|Q|/2)\big).\end{aligned} \tag{3.4}$$

By (3.1),

$$\begin{aligned}(I_1(f\chi_{4Q}))^*(\lambda|Q|/2) &\le (\widetilde{S}_{\alpha,\psi}(f\chi_{4Q}))^*(\lambda|Q|/2)^2 \\ &\le c_{n,\lambda,\psi}\alpha^{2n} \left(\frac{1}{|4Q|} \int_{4Q} |f| \right)^2.\end{aligned} \tag{3.5}$$

Further, by (1.1), for $(y, t) \in T(2Q)$,

$$|(f\chi_{\mathbb{R}^n\setminus 4Q}) * \psi_t(y)| \leq c_\psi t^\varepsilon \int\limits_{\mathbb{R}^n\setminus 4Q} |f(\xi)| \frac{1}{(t+|y-\xi|)^{n+\varepsilon}} d\xi$$
$$\leq c_{n,\psi}(t/\ell_Q)^\varepsilon \sum_{k=0}^{\infty} \frac{1}{2^{k\varepsilon}} \frac{1}{|2^k Q|} \int\limits_{2^k Q} |f|.$$

Hence, using Chebyshev's inequality and that $\int_{\mathbb{R}^n} \Phi\left(\frac{x-y}{t\alpha}\right) dx \leq c_n(t\alpha)^n$, we have

$$\begin{aligned}
&(I_1(f\chi_{\mathbb{R}^n\setminus 4Q})\chi_Q)^*(\lambda|Q|/2)\\
&\leq \frac{2}{\lambda|Q|} \iint\limits_{T(2Q)} \Big(\int\limits_{\mathbb{R}^n} \Phi\Big(\frac{x-y}{t\alpha}\Big) dx\Big) |(f\chi_{\mathbb{R}^n\setminus 4Q}) * \psi_t(y)|^2 \frac{dydt}{t^{n+1}}\\
&\leq c_{n,\lambda,\psi}\alpha^n (1/\ell_Q)^{2\varepsilon} \left(\sum_{k=0}^{\infty} \frac{1}{2^{k\varepsilon}} \frac{1}{|2^k Q|} \int\limits_{2^k Q} |f|\right)^2 \int\limits_0^{2\ell_Q} t^{2\varepsilon-1} dt\\
&\leq c_{n,\lambda,\psi}\alpha^n \left(\sum_{k=0}^{\infty} \frac{1}{2^{k\varepsilon}} \frac{1}{|2^k Q|} \int\limits_{2^k Q} |f|\right)^2 .
\end{aligned}$$

By Hölder's inequality,

$$\left(\sum_{k=0}^{\infty} \frac{1}{2^{k\varepsilon}} \frac{1}{|2^k Q|} \int\limits_{2^k Q} |f|\right)^2 \leq \left(\sum_{k=0}^{\infty} \frac{1}{2^{k\varepsilon}}\right) \sum_{k=0}^{\infty} \frac{1}{2^{k\varepsilon}} \left(\frac{1}{|2^k Q|} \int\limits_{2^k Q} |f|\right)^2 .$$

Combining this with the previous estimate and with (3.5) and (3.4) proves (3.3).

Let $x, x_0 \in Q$, and let us estimate now $|I_2(f)(x) - I_2(f)(x_0)|$. We have

$$\begin{aligned}
&|I_2(f)(x) - I_2(f)(x_0)|\\
&\leq \sum_{k=1}^{\infty} \iint\limits_{T(2^{k+1}Q)\setminus T(2^k Q)} \left|\Phi\Big(\frac{x-y}{t\alpha}\Big) - \Phi\Big(\frac{x_0-y}{t\alpha}\Big)\right| |f * \psi_t(y)|^2 \frac{dydt}{t^{n+1}}.
\end{aligned}$$

Suppose $(y, t) \in T(2^{k+1}Q) \setminus T(2^k Q)$. If $y \in 2^k Q$, then $t \geq 2^k \ell_Q$. On the other hand, if $y \in 2^{k+1}Q \setminus 2^k Q$, then for any $x \in Q, |y - x| \geq \frac{2^k-1}{2}\ell_Q$. Hence, if $t < \frac{2^k-1}{4\alpha}\ell_Q$, then $|y - x|/\alpha t > 2$ and $|y - x_0|/\alpha t > 2$, and therefore,

$$\Phi\Big(\frac{x-y}{t\alpha}\Big) - \Phi\Big(\frac{x_0-y}{t\alpha}\Big) = 0.$$

Assume that $t \geq \frac{2^k-1}{4\alpha}\ell_Q$. This easily implies $t \geq 2^{k-3}\ell_Q/\alpha$. Thus, using that

$$\left|\Phi\Big(\frac{x-y}{t\alpha}\Big) - \Phi\Big(\frac{x_0-y}{t\alpha}\Big)\right| \leq \frac{\sqrt{n}\ell_Q}{\alpha t}\|\nabla\Phi\|_{L^\infty},$$

we get

$$\begin{aligned}&\left|\Phi\Big(\frac{x-y}{t\alpha}\Big) - \Phi\Big(\frac{x_0-y}{t\alpha}\Big)\right|\chi_{\{T(2^{k+1}Q)\setminus T(2^kQ)\}}(y,t)\\ &\leq c_n\frac{\ell_Q}{\alpha t}\chi_{\{(y,t):y\in 2^{k+1}Q,2^{k-3}\ell_Q/\alpha\leq t\leq 2^{k+1}\ell_Q\}}(y,t).\end{aligned}$$

Hence,

$$\begin{aligned}&\iint\limits_{T(2^{k+1}Q)\setminus T(2^kQ)}\left|\Phi\Big(\frac{x-y}{t\alpha}\Big) - \Phi\Big(\frac{x_0-y}{t\alpha}\Big)\right||f*\psi_t(y)|^2\frac{dydt}{t^{n+1}}\\ &\leq c_n\frac{\ell_Q}{\alpha}\int\limits_{2^{k-3}\ell_Q/\alpha}^{2^{k+1}\ell_Q}\int\limits_{2^{k+1}Q}|f*\psi_t(y)|^2\frac{dydt}{t^{n+2}} \leq c_n(J_1+J_2),\end{aligned}$$

where

$$J_1 = \frac{\ell_Q}{\alpha}\int\limits_{2^{k-3}\ell_Q/\alpha}^{2^{k+1}\ell_Q}\int\limits_{2^{k+1}Q}|(f\chi_{2^{k+2}Q})*\psi_t(y)|^2\frac{dydt}{t^{n+2}}$$

and

$$J_2 = \frac{\ell_Q}{\alpha}\int\limits_{2^{k-3}\ell_Q/\alpha}^{2^{k+1}\ell_Q}\int\limits_{2^{k+1}Q}|(f\chi_{\mathbb{R}^n\setminus 2^{k+2}Q})*\psi_t(y)|^2\frac{dydt}{t^{n+2}}.$$

Let us first estimate J_1. Using Minkowski's integral inequality, we obtain

$$J_1 \leq \frac{\ell_Q}{\alpha}\left(\int\limits_{2^{k+2}Q}|f(\xi)|\Big(\int\limits_{2^{k-3}\ell_Q/\alpha}^{2^{k+1}\ell_Q}\int\limits_{2^{k+1}Q}\psi_t(y-\xi)^2\frac{dydt}{t^{n+2}}\Big)^{1/2}d\xi\right)^2.$$

Since

$$\int\limits_{2^{k+1}Q}\psi_t(y-\xi)^2dy \leq \frac{\|\psi\|_{L^\infty}}{t^n}\|\psi_t\|_{L^1} = \frac{\|\psi\|_{L^\infty}\|\psi\|_{L^1}}{t^n},$$

we get

$$J_1 \le c_\psi \frac{\ell_Q}{\alpha}\Big(\int_{2^{k+2}Q} |f(\xi)|d\xi\Big)^2 \int_{2^{k-3}\ell_Q/\alpha}^{\infty} \frac{dt}{t^{2n+2}}$$
$$\le c_{n,\psi}\alpha^{2n}2^{-k}\Big(\frac{1}{|2^{k+2}Q|}\int_{2^{k+2}Q} |f(\xi)|d\xi\Big)^2.$$

We turn to the estimate of J_2. By (1.1), for $(y,t) \in T(2^{k+1}Q)$,

$$|(f\chi_{\mathbb{R}^n\setminus 2^{k+2}Q}) * \psi_t(y)| \le c_\psi t^\varepsilon \int_{\mathbb{R}^n\setminus 2^{k+2}Q} |f(\xi)|\frac{1}{(t+|y-\xi|)^{n+\varepsilon}}d\xi$$
$$\le c_{n,\psi}(t/\ell_Q)^\varepsilon \sum_{i=k}^{\infty}\frac{1}{2^{i\varepsilon}}\frac{1}{|2^iQ|}\int_{2^iQ}|f|.$$

Therefore,

$$J_2 \le c_{n,\psi}\frac{\ell_Q}{\alpha}\Big(\sum_{i=k}^{\infty}\frac{1}{2^{i\varepsilon}}\frac{1}{|2^iQ|}\int_{2^iQ}|f|\Big)^2\frac{1}{\ell_Q^{2\varepsilon}}\int_{2^{k-3}\ell_Q/\alpha}^{2^{k+1}\ell_Q}\int_{2^{k+1}Q}\frac{dydt}{t^{n+2-2\varepsilon}}$$
$$\le c_{n,\psi}\alpha^{n-2\varepsilon}2^{(2\varepsilon-1)k}\Big(\sum_{i=k}^{\infty}\frac{1}{2^{i\varepsilon}}\frac{1}{|2^iQ|}\int_{2^iQ}|f|\Big)^2.$$

Combining the estimates for J_1 and J_2, we obtain

$$|I_2(f)(x) - I_2(f)(x_0)| \le c_{n,\psi}\alpha^{2n}\sum_{k=1}^{\infty}\frac{1}{2^k}\Big(\frac{1}{|2^kQ|}\int_{2^kQ}|f(\xi)|d\xi\Big)^2$$
$$+c_{n,\psi}\alpha^{n-2\varepsilon}\sum_{k=1}^{\infty}\frac{2^{2\varepsilon k}}{2^k}\Big(\sum_{i=k}^{\infty}\frac{1}{2^{i\varepsilon}}\frac{1}{|2^iQ|}\int_{2^iQ}|f|\Big)^2.$$

By Hölder's inequality,

$$\sum_{k=1}^{\infty}\frac{2^{2\varepsilon k}}{2^k}\left(\sum_{i=k}^{\infty}\frac{1}{2^{i\varepsilon}}\frac{1}{|2^iQ|}\int_{2^iQ}|f|\right)^2$$

$$\leq c_\varepsilon \sum_{k=1}^{\infty} \frac{2^{\varepsilon k}}{2^k} \sum_{i=k}^{\infty} \frac{1}{2^{i\varepsilon}} \left(\frac{1}{|2^i Q|} \int_{2^i Q} |f| \right)^2$$

$$\leq c_\varepsilon \sum_{k=1}^{\infty} \gamma(k,\varepsilon) \left(\frac{1}{|2^k Q|} \int_{2^k Q} |f| \right)^2,$$

where

$$\gamma(k,\varepsilon) = \begin{cases} \frac{1}{2^{\varepsilon k}}, & \varepsilon < 1 \\ \frac{k}{2^k}, & \varepsilon = 1. \end{cases}$$

Therefore,

$$|I_2(f)(x) - I_2(f)(x_0)| \leq c_{n,\psi} \alpha^{2n} \sum_{k=1}^{\infty} \gamma(k,\varepsilon) \left(\frac{1}{|2^k Q|} \int_{2^k Q} |f| \right)^2.$$

From this and from (3.3),

$$\omega_\lambda(\widetilde{S}_{\alpha,\psi}(f)^2; Q) \leq (I_1(f)\chi_Q)^*(\lambda|Q|) + \|I_2(f) - I_2(f)(x_0)\|_{L^\infty(Q)}$$
$$\leq c_{n,\lambda,\psi} \alpha^{2n} \sum_{k=0}^{\infty} \gamma(k,\varepsilon) \left(\frac{1}{|2^k Q|} \int_{2^k Q} |f| \right)^2,$$

which completes the proof. □

4 Proof of Theorem 1.1

4.1 Several Auxiliary Operators

Throughout this subsection we assume that $f, g \geq 0$. Given a sparse family $\mathcal{S} = \{Q_j^k\} \subset \mathscr{D}$, define

$$\mathcal{T}_{2,m}^{\mathcal{S}} f(x) = \left(\sum_{j,k} (f_{2^m Q_j^k})^2 \chi_{Q_j^k}(x) \right)^{1/2}.$$

The following result was proved in [4].

Lemma 4.1 *For any* $1 < p < \infty$,

$$\|\mathcal{T}_{2,0}^{\mathcal{S}}\|_{L^p(w)} \le c_{n,p}[w]_{A_p}^{\max(\frac{1}{2},\frac{1}{p-1})}.$$

Given a sparse family $\mathcal{S} = \{Q_j^k\} \subset \mathcal{D}$, define

$$\mathcal{M}_m^{\mathcal{S}}(f,g)(x) = \sum_{j,k}(f_{2^m Q_j^k})\left(\frac{1}{|2^m Q_j^k|}\int_{Q_j^k} g\right)\chi_{2^m Q_j^k}(x).$$

Applying Proposition 2.2, we decompose the family of cubes $\{Q_j^k\}$ into 2^n disjoint families F_i such that for any $Q_j^k \in F_i$ there exists a cube $P_{j,k}^{m,i} \in \mathscr{D}_i$ such that $2^m Q_j^k \subset P_{j,k}^{m,i}$ and $\ell_{P_{j,k}^{m,i}} \le 6\ell_{2^m Q_j^k}$. Hence,

$$\mathcal{M}_m^{\mathcal{S}}(f,g)(x) \le 6^{2n}\sum_{i=1}^{2^n}\mathcal{M}_{i,m}^{\mathcal{S}}(f,g)(x), \tag{4.1}$$

where

$$\mathcal{M}_{i,m}^{\mathcal{S}}(f,g)(x) = \sum_{j,k}(f_{P_{j,k}^{m,i}})\left(\frac{1}{|P_{j,k}^{m,i}|}\int_{Q_j^k} g\right)\chi_{P_{j,k}^{m,i}}(x).$$

The following statement was obtained in [5].

Lemma 4.2 *Suppose that the sum defining* $\mathcal{M}_{i,m}^{\mathcal{S}}(f,g)$ *contains finitely many terms. Then there are at most* 2^n *cubes* $Q_\nu \in \mathscr{D}_i$ *covering the support of* $\mathcal{M}_{i,m}^{\mathcal{S}}(f,g)$ *so that for every* Q_ν *there are two sparse families* $\mathcal{S}_{i,1}$ *and* $\mathcal{S}_{i,2}$ *from* $\mathscr{D}_i$ *having the property that for a.e.* $x \in Q_\nu$,

$$\mathcal{M}_{i,m}^{\mathcal{S}}(f,g)(x) \le c_n(m+1)\sum_{\kappa=1}^{2}\sum_{Q_j^k\in\mathcal{S}_{i,\kappa}} f_{Q_j^k}g_{Q_j^k}\chi_{Q_j^k}(x).$$

Observe that the proof of Lemma 4.2 is based on Theorem 2.3 along with [14, Lemma 3.2]. Formally Lemma 4.2 follows from [5, Lemma 4.2] taking there $m = 2$ (which corresponds to a bilinear case) and $l = m$, and from the subsequent argument in [5, Sect. 4.2].

Let X be a Banach function space, and let X' denote the associate space (see [2, Ch. 1]). Given a Banach function space X, denote by $X^{(2)}$ the space endowed with the norm

$$\|f\|_{X^{(2)}} = \||f|^2\|_X^{1/2}.$$

It is well known [16, Ch. 1] that $X^{(2)}$ is also a Banach space.

Lemma 4.3 *For any Banach function space X,*

$$\sup_{\mathcal{S}\in\mathcal{D}} \|\mathcal{T}_{2,m}^{\mathcal{S}} f\|_{X^{(2)}} \le c_n m^{1/2} \max_{1\le i\le 2^n} \sup_{\mathcal{S}\in\mathscr{D}_i} \|\mathcal{T}_{2,0}^{\mathcal{S}} f\|_{X^{(2)}}.$$

Proof By the standard argument, it suffices to prove the estimate for a finite partial sum $\widetilde{\mathcal{T}}_{2,m}^{\mathcal{S}} f$ from the series defining $\mathcal{T}_{2,m}^{\mathcal{S}} f$. Fix $\mathcal{S}\in\mathcal{D}$. By duality, there exists $g\ge 0$ with $\|g\|_{X'}=1$ such that

$$\begin{aligned} \|\widetilde{\mathcal{T}}_{2,m}^{\mathcal{S}} f\|_{X^{(2)}}^2 &= \int_{\mathbb{R}^n} (\widetilde{\mathcal{T}}_{2,m}^{\mathcal{S}} f)^2 g\,dx = \sum_{j,k} (f_{2^m Q_j^k})^2 \int_{Q_j^k} g \\ &= \int_{\mathbb{R}^n} \mathscr{M}_m^{\mathcal{S}}(f,g) f\,dx, \end{aligned} \tag{4.2}$$

where the sum defining $\mathscr{M}_m^{\mathcal{S}}(f,g)$ contains finitely many terms. By Lemma 4.2 and by Hölder's inequality,

$$\begin{aligned} \int_{Q_\nu} \mathscr{M}_{i,m}^{\mathcal{S}}(f,g) f\,dx &\le c_n m \sum_{\kappa=1}^{2} \sum_{Q_j^k\in\mathcal{S}_{i,\kappa}} (f_{Q_j^k})^2 \int_{Q_j^k} g \\ &\le c_n m \sum_{\kappa=1}^{2} \int_{\mathbb{R}^n} (\mathcal{T}_{2,0}^{\mathcal{S}_{i,\kappa}} f)^2 g\,dx \\ &\le 2c_n m \sup_{\mathcal{S}\in\mathscr{D}_i} \|\mathcal{T}_{2,0}^{\mathcal{S}} f\|_{X^{(2)}}^2. \end{aligned}$$

Summing up over Q_ν and using (4.1), we obtain

$$\int_{\mathbb{R}^n} \mathscr{M}_m^{\mathcal{S}}(f,g) f\,dx \le c_n m \max_{1\le i\le 2^n} \sup_{\mathcal{S}\in\mathscr{D}_i} \|\mathcal{T}_{2,0}^{\mathcal{S}} f\|_{X^{(2)}}^2.$$

This along with (4.2) completes the proof.

4.2 Proof of Theorem 1.1

Let $Q \in \mathcal{D}$. By Lemma 3.1, for all $x \in Q$,

$$m^{\#,d}_{\frac{1}{2^{n+2}};Q}\big((\widetilde{S}_{\alpha,\psi}(f)^2)\big)(x) \le c_{n,\psi}\alpha^{2n} Mf(x)^2.$$

Hence, applying Theorem 2.3 to $\widetilde{S}_{\alpha,\psi}(f)^2$, we get that there exists a sparse family $\mathcal{S} = \{Q_j^k\} \subset \mathcal{D}(Q)$ such that for a.e. $x \in Q$,

$$|\widetilde{S}_{\alpha,\psi}(f)(x)^2 - m_Q(\widetilde{S}_{\alpha,\psi}(f)^2)| \le c_{n,\psi}\alpha^{2n}\Big(Mf(x)^2 + \sum_{m=0}^{\infty}\frac{1}{2^{m\delta}}\big(\mathcal{T}^{\mathcal{S}}_{2,m}f(x)\big)^2\Big).$$

Hence,

$$|\widetilde{S}_{\alpha,\psi}(f)^2 - m_Q(\widetilde{S}_{\alpha,\psi}(f)^2)|^{1/2} \le c_{n,\psi}\alpha^n\big(Mf(x) + \mathcal{T}(f)(x)\big), \tag{4.3}$$

where

$$\mathcal{T}(f)(x) = \sum_{m=0}^{\infty}\frac{1}{2^{m\delta/2}}\mathcal{T}^{\mathcal{S}}_{2,m}f(x).$$

Assuming, for instance, that $f \in L^1$, and using (2.3) and (3.1), we get

$$\lim_{|Q|\to\infty} m_Q(\widetilde{S}_{\alpha,\psi}(f)^2) = 0.$$

Therefore, letting Q tend to anyone of 2^n quadrants and using Fatou's lemma, by (4.3) we obtain

$$\|\widetilde{S}_{\alpha,\psi}(f)\|_{L^p(w)} \le c_{n,\psi}\alpha^n\big(\|Mf\|_{L^p(w)} + \|\mathcal{T}(f)\|_{L^p(w)}\big). \tag{4.4}$$

Combining Lemma 4.1 and Lemma 4.3 with $X = L^{3/2}(w)$ yields

$$\begin{aligned}\|\mathcal{T}(f)\|_{L^3(w)} &\le \sum_{m=0}^{\infty}\frac{1}{2^{m\delta/2}}\|\mathcal{T}^{\mathcal{S}}_{2,m}f\|_{L^3(w)}\\ &\le c_n\sum_{m=0}^{\infty}\frac{m^{1/2}}{2^{m\delta/2}}\max_{1\le i\le 2^n}\sup_{\mathcal{S}\in\mathscr{D}_i}\|\mathcal{T}^{\mathcal{S}}_{2,0}f\|_{L^3(w)}\\ &\le c_{n,\delta}[w]_{A_3}^{1/2}\|f\|_{L^3(w)}.\end{aligned}$$

Hence, by the sharp version of the Rubio de Francia extrapolation theorem (see [6] or [7]),

$$\|\mathcal{T}(f)\|_{L^p(w)} \le c_{n,p,\delta}[w]_{A_p}^{\max(\frac{1}{2},\frac{1}{p-1})}\|f\|_{L^p(w)} \quad (1 < p < \infty). \tag{4.5}$$

Thus, applying this result along with Buckley's estimate $\|M\|_{L^p(w)} \le c_{n,p}[w]_{A_p}^{\frac{1}{p-1}}$ (see [3]) and (4.4), we get

$$\|S_{\alpha,\psi}\|_{L^p(w)} \le \|\widetilde{S}_{\alpha,\psi}\|_{L^p(w)} \le c_{n,p,\psi}\alpha^n[w]_{A_p}^{\max(\frac{1}{2},\frac{1}{p-1})},$$

and therefore, the proof is complete.

5 Concluding Remarks

In a recent work [11], the following weak type estimate was obtained for $G_\beta(f)$ (and hence for $S_\psi(f)$): if $1 < p < 3$, then

$$\|G_\beta(f)\|_{L^{p,\infty}(w)} \lesssim [w]_{A_p}^{\max(\frac{1}{2},\frac{1}{p})}\Phi_p([w]_{A_p})\|f\|_{L^p(w)},$$

where $\Phi_p(t) = 1$ if $1 < p < 2$ and $\Phi_p(t) = 1 + \log t$ if $p \ge 2$. The proof was based on the local mean oscillation decomposition technique along with the estimate

$$\|\mathcal{T}_{2,0}^{\mathcal{S}} f\|_{L^{p,\infty}(w)} \lesssim [w]_{A_p}^{\max(\frac{1}{2},\frac{1}{p})}\Phi_p([w]_{A_p})\|f\|_{L^p(w)}. \tag{5.1}$$

Since the space $L^{p,\infty}(w)$ is normable if $p > 1$ (see, e.g., [2, p. 220]), combining Lemma 4.3 with $X = L^{1+\varepsilon,\infty}(w)$, $\varepsilon > 0$, and (5.1) yields for $2 < p < 3$ that

$$\|\mathcal{T} f\|_{L^{p,\infty}(w)} \lesssim [w]_{A_p}^{\max(\frac{1}{2},\frac{1}{p})}\Phi_p([w]_{A_p})\|f\|_{L^p(w)}. \tag{5.2}$$

Hence, exactly as above, by (4.3) (and by the weak type estimate for M proved in [3]), we obtain

$$\|S_{\alpha,\psi}(f)\|_{L^{p,\infty}(w)} \lesssim \alpha^n[w]_{A_p}^{\max(\frac{1}{2},\frac{1}{p})}\Phi_p([w]_{A_p})\|f\|_{L^p(w)} \quad (2 < p < 3).$$

We emphasize that our approach does not allow to extend this estimate to $1 < p \le 2$. This is clearly related to the same problem with (5.2). The limitation $2 < p < 3$ in (5.2) is due to Lemma 4.3 where the condition that X is a Banach function space was essential in the proof. This raises a natural question whether Lemma 4.3 holds under the condition that X is a quasi-Banach space. Observe that the same question can be asked regarding a recent estimate relating X-norms of Calderón–Zygmund and dyadic positive operators [15].

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References

1. Auscher, P.: Change of angle in tent spaces. C. R. Math. Acad. Sci. Paris **349**(5–6), 297–301 (2011)

2. Bennett, C., Sharpley, R.: Interpolation of Operators. Academic Press, New York (1988)
3. Buckley, S.M.: Estimates for operator norms on weighted spaces and reverse Jensen inequalities. Trans. Am. Math. Soc. **340**(1), 253–272 (1993)
4. Cruz-Uribe, D., Martell, J.M., Pérez, C.: Sharp weighted estimates for classical operators. Adv. Math. **229**(1), 408–441 (2012)
5. Damián, W., Lerner, A.K., Pérez, C.: Sharp weighted bounds for multilinear maximal functions and Calderón–Zygmund operators. http://arxiv.org/abs/1211.5115
6. Dragičević, O., Grafakos, L., Pereyra, M.C., Petermichl, S.: Extrapolation and sharp norm estimates for classical operators on weighted Lebesgue spaces. Publ. Math. **49**(1), 73–91 (2005)
7. Duoandikoetxea, J.: Extrapolation of weights revisited: new proofs and sharp bounds. J. Funct. Anal. **260**, 1886–1901 (2011)
8. Fefferman, C.: Inequalities for strongly singular convolution operators. Acta Math. **124**, 9–36 (1970)
9. García-Cuerva, J., Rubio de Francia, J.L.: Weighted Norm Inequalities and Related Topics. North-Holland Publishing Co., Amsterdam (1985)
10. Hytönen, T., Pérez, C.: Sharp weighted bounds involving A_∞. J. Anal. PDE **6**(4), 777–818 (2013)
11. Lacey, M.T., Scurry, J.: Weighted weak type estimates for square functions. http://arxiv.org/abs/1211.4219
12. Lerner, A.K.: A pointwise estimate for the local sharp maximal function with applications to singular integrals. Bull. Lond. Math. Soc. **42**(5), 843–856 (2010)
13. Lerner, A.K.: Sharp weighted norm inequalities for Littlewood–Paley operators and singular integrals. Adv. Math. **226**, 3912–3926 (2011)
14. Lerner, A.K.: A simple proof of the A_2 conjecture. Int. Math. Res. Not. **14**, 3159–3170 (2013)
15. Lerner, A.K.: On an estimate of Calderón–Zygmund operators by dyadic positive operators. J. Anal. Math. **121**(1), 141–161 (2013)
16. Lindenstrauss, J., Tzafriri, L.: Classical Banach Spaces II. Springer-Verlag, Berlin (1979)
17. Muckenhoupt, B., Wheeden, R.L.: Norm inequalities for the Littlewood–Paley functiong_λ^*. Trans. Am. Math. Soc. **191**, 95–111 (1974)
18. Torchinsky, A.: Real-Variable Methods in Harmonic Analysis. Academic Press, New York (1986)
19. Wilson, J.M.: The intrinsic square function. Rev. Mat. Iberoam. **23**, 771–791 (2007)