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Applications of Fourier Analysis in Homogenization of Dirichlet Problem III: Polygonal Domains

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Abstract In this paper we prove convergence results for the homogenization of the Dirichlet problem for elliptic equations in divergence form with rapidly oscillating boundary data and non oscillating coefficients in convex polygonal domains. Our analysis is based on integral representation of solutions. Under a certain Diophantine condition on the boundary of the domain and smooth coefficients we prove pointwise, as well as L^p convergence results. For larger exponents p we prove that the L^p convergence rate is close to optimal. We also suggest several directions of possible generalization of the results in this paper.

Keywords Homogenization · Dirichlet problem · Polygonal domain · Fourier analysis

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1 Introduction and Main Results

Elliptic boundary value problems with rapidly oscillating boundary data as well as oscillating coefficients has been much in focus lately, due to its importance for higher order approximation in homogenization theory. Higher order approximation gives rise to the so-called boundary-layer phenomena, which roughly states that the solutions to elliptic problems with oscillating coefficients and boundary data should have concentration near the boundary of the domain with no periodic character. We refer the readers to [3] for some background, and examples of applications where oscillating data plays central role.

For a smooth and uniformly convex domains in $\mathbb{R}^d$, $(d \geq 2)$ in a recent work [11], Gérard-Varet, and Masmoudi proved convergence rate of order any $\alpha < (d-1)/(3d+5)$ in L^2 for solutions to elliptic system of divergence type, with periodically oscillating coefficients and boundary data. This is one of the few results where the speed for such type of homogenization problem is established. In the same setting, for homogenization of non-oscillating operators and oscillating boundary data, in dimensions greater than two we showed [2] a power convergence rate of order $1/p$ in L^p, for all $1 \leq p < \infty$, and proved that the rate $1/p$ can not be improved. Also, combining our methods from [2] with a recent results due to Kenig et al. [17], in a setting of the paper [11], i.e. for periodically oscillating operator and boundary data, for a certain class of operators we proved the same $1/p$ power convergence rate for homogenization in dimensions greater than two.

A wider range of treatments of the problem, but with no particular speed of convergence, can be found in recent works: [6,8–10,18,19].

In case, when the operator is fixed, and only the boundary data is oscillating, the convergence result was proved in [18] for some general class of domains. For elliptic systems of divergence type, the current authors found partial convergence rate for the pointwise convergence, and an optimal rate of the convergence in L^p norm in dimensions greater than three, when the domain in question is strictly convex and smooth; see [1,2]. In this paper, we continue our program of studying the problem of homogenization of the boundary data with fixed operator. Here we shall consider convex polygonal domains. We note that the homogenization of the Dirichlet problem for elliptic systems of divergence form set in convex polygonal domains with periodically oscillating operator, zero Dirichlet data, and fixed source term is studied in [12]. The main goal of [12] is to analyze higher order two-scale approximations to solutions, which is carried out under certain Diophantine condition on the normal vectors of the bounding hyperplanes of the domain, and restrictive regularity assumptions on initial terms in the two-scale expansion. We refer the reader to [12] for the details.

To fix the ideas, let D be a bounded convex polygonal domain in $\mathbb{R}^d$ $(d \geq 2)$, that is a convex domain bounded by some number of hyperplanes

$$D = \bigcap_{j=1}^{N} \{x \in \mathbb{R}^d : \nu_j \cdot x > c_j\}, \tag{1.1}$$

where $c_j \in \mathbb{R}$ and $\nu_j \in \mathbb{S}^{d-1}$. Denote by Γ the boundary of D. Let also $A(y) = (A^{ij}(y))$, $1 \leq i, j \leq d$, be an $\mathbb{R}^{d^2}$-valued function defined on $\mathbb{R}^d$, and g be a complex valued function defined on $\mathbb{T}^d$-the unit torus in $\mathbb{R}^d$. We study asymptotic behavior of solutions to the following problem:

$$\begin{cases} \mathcal{L}u_\varepsilon(x) = 0, & \text{in } D, \\ u_\varepsilon(x) = g(x/\varepsilon), & \text{on } \Gamma, \end{cases} \tag{1.2}$$

where $\varepsilon > 0$ is a small parameter, and using the summation convention of repeated indices the operator $\mathcal{L}$ is defined as

$$\mathcal{L}u := -\frac{\partial}{\partial x_i}\left[A^{ij}(x)\frac{\partial u}{\partial x_j}\right] = -\text{div}\,[A(x)\nabla u]\,.$$

For (1.2) we consider the corresponding homogenized problem

$$\begin{cases} \mathcal{L}u_0(x) = 0, & \text{in } D, \\ u_0(x) = \overline{g}, & \text{on } \Gamma, \end{cases} \tag{1.3}$$

where $\overline{g} = \int_{\mathbb{T}^d} g(y)dy$.

1.1 Standing Assumptions

We make the following assumptions:

(i) (Periodicity) The boundary data g is 1-periodic:

$$g(x+h) = g(x),\ \forall x \in \mathbb{R}^d,\ \forall h \in \mathbb{Z}^d.$$

(ii) (Ellipticity) There exists a constant $c > 0$ such that

$$c^{-1}\xi_i\xi_i \leq A^{ij}(x)\xi_i\xi_j \leq c\xi_i\xi_i,\ \forall x \in \mathbb{R}^d,\ \forall \xi \in \mathbb{R}^d.$$

(iii) (Convexity) D is convex and for any bounding hyperplane of D its normal vector is Diophantine in a sense of Definition 1.1 below.

(iv) For the convex polygonal domain D choose $\alpha_* > 0$ so that $\pi/(1+\alpha_*)$ be the maximal angle between any two adjacent faces of D.

(v) (Smoothness) The boundary value g and all elements of A are sufficiently smooth.

The following are the main results of the paper.

Theorem 1.1 (Pointwise convergence) *Retain the standing assumptions in Sect. 1.1, and if $\alpha_* > 1$ set $\beta = 1$, otherwise, let $0 < \beta < \alpha_*$ be any number. Then for each $\delta > 0$ small there exists a constant C depending on δ, β, D, $\mathcal{L}$, but independent of $\varepsilon > 0$, such that for all $x \in D$ one has*

$$|u_\varepsilon(x) - u_0(x)| \leq C\left(\frac{\varepsilon^\beta}{d(x)^{\beta+\delta}}\right)^{\frac{d-1}{d-1+\beta}},$$

where $d(x)$ denotes the distance of x to the boundary of D.

Using this we will have the following result.

Theorem 1.2 (L^p convergence) *Retain the standing assumptions in Section 1.1, and set* $\gamma = \frac{(d-1)\min\{1,\alpha_*\}}{d-1+\min\{1,\alpha_*\}}$. *Then for each* $1 \leq p < \infty$, *and* $\delta > 0$ *there exists a constant* C *depending on* p, D, $\mathcal{L}$, δ *but independent of* $\varepsilon > 0$ *such that*

$$||u_\varepsilon - u_0||_{L^p(D)} \leq C\varepsilon^{\min\{\gamma, \frac{1}{p}\}-\delta}.$$

Theorem 1.3 (Optimality) *Under the same conditions and notation of Theorem 1.1 for each* $1 \leq p < \infty$ *there exists a constant* C *depending on* p, D, $\mathcal{L}$, *but independent of* ε, *such that*

$$||u_\varepsilon - u_0||_{L^p(D)} \geq C\varepsilon^{\frac{1}{p}}||g - \overline{g}||_{L^\infty(\mathbb{T}^d)}.$$

Remark 1.4 Observe that Theorem 1.3 shows that for larger exponents p the L^p convergence rate is close to optimal independently of the structure of the domain. This fact for all $1 \leq p < \infty$ is due to concentration of solutions near the boundary of the domain. However for smaller values of p there are some limitations in the speed of convergence in Theorem 1.2 due to the largest angle of the polygon. We do not know if this convergence rate is optimal as well.

Remark 1.5 In Sect. 4 we suggest several directions of possible generalization of present results.

1.2 Preliminaries

We start with some auxiliary results. In the sequel we will denote by C an absolute constant which may vary from formula to formula. For $x \in \mathbb{R}^d$ and $r > 0$ we set by $B(x, r)$, or $B_r(x)$ an open ball of radius r centered at x. If ambiguity does not arise, for a vector $x \in \mathbb{R}^d$ we will write $|x|$ to denote its standard Euclidean norm.

Definition 1.1 A vector $\nu = (\nu_1, ..., \nu_d) \in \mathbb{R}^d$ is called Diophantine if there exists $0 < \tau(\nu) < \infty$ and $C > 0$ such that

$$|m \cdot \nu| > \frac{C}{|m|^{\tau(\nu)}},$$

for all $m = (m_1, ..., m_d) \in \mathbb{Z}^d \setminus \{0\}$, where $m \cdot \nu$ is the usual scalar product and $|m| = |m_1| + ... + |m_d|$. We denote the set of such vectors by $\Omega(\tau, C)$.

It is well known and easy to see that for any $\tau > d - 1$ the set $\bigcup_{C>0} \Omega(\tau, C)$ has full measure in each ball of $\mathbb{R}^d$. This shows that the Diophantine condition, as stated in (iii) of Standing Assumptions, is generic for all polygonal domains.

Lemma 1.6 *Let* $m \in \mathbb{Z}^d$ *be non zero, and assume that* $m_k \neq 0$, *for some* $1 \leq k \leq d$. *For a vector* $\nu = (\nu_1, \nu_2, ..., \nu_d) \in \Omega(\tau, c_0)$ *consider* $\Pi = \{x \in \mathbb{R}^d : \nu \cdot x = c,\ x_j \in [a_j, b_j],\ j = 1, 2, ..., d,\ j \neq k\}$, *and for* $\lambda > 1$ *set*

$$\mathcal{I}_\lambda := \int_\Pi e^{2\pi i \lambda m\cdot y} d\sigma(y).$$

Then for all $\lambda > 1$ *one has*

$$|\mathcal{I}_\lambda| \le C\lambda^{-(d-1)}||m||^{(d-1)\tau},$$

where the constant C *depends on* ν *and dimension* d *only.*

Proof Without loss of generality we will assume that $k = d$, that is $m_d \neq 0$. Since ν is Diophantine, all its components are non zero. In the domain of integration we have $\nu_1 y_1 + \ldots + \nu_{d-1} y_{d-1} + \nu_d y_d = c$, hence

$$y_d = \frac{c}{\nu_d} - \frac{1}{\nu_d}(\nu_1 y_1 + \nu_2 y_2 + \ldots + \nu_{d-1} y_{d-1}),$$

and substituting this in the integral we obtain

$$\mathcal{I}_\lambda = C \prod_{j=1}^{d-1} \int_{a_j}^{b_j} \exp\left[2\pi i \lambda \left(m_j - m_d \frac{\nu_j}{\nu_d}\right) y_j\right] dy_j. \tag{1.4}$$

From the Diophantine condition and the fact that $m_d \neq 0$ we have

$$|m_j - m_d \frac{\nu_j}{\nu_d}| = \frac{1}{|\nu_d|}|m_j \nu_d - m_d \nu_j| \ge \frac{C_\nu}{|\nu_d|} \frac{1}{(|m_j| + |m_d|)^\tau}, \tag{1.5}$$

for all $j = 1, 2, \ldots, d-1$. We now compute each of the integrals in (1.4), and applying (1.5) we get the desired estimate, finishing the proof. □

We now introduce some notation that will be used in the sequel. Let D be given as in (1.1). We say that $\Pi \subset \partial D$ is a ($(d-1)$-dimensional) *face* of the polygon D if for some $1 \le k \le N$ one has

$$\Pi = \{x \in \mathbb{R}^d : \nu_k \cdot x = c_k\} \cap \bigcap_{j=1, j\neq k}^{N} \{x \in \mathbb{R}^d : \nu_j \cdot x > c_j\}, \tag{1.6}$$

i.e., Π is just one of the flat portions of ∂D. For a given face Π, and a number $\rho > 0$ consider a strip of width ρ near the $(d-2)$-dimensional boundary of Π, and denote it by

$$\Pi_\rho = \{y \in \Pi : \mathrm{dist}(y, \partial\Pi) \le \rho\}. \tag{1.7}$$

For $1 \le k \le d$ set π_k to be the projection operator in the k-th direction, namely

$$\pi_k(x) = (x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_d), \text{ where } x \in \mathbb{R}^d.$$

We also set $\mathcal{H}^j$ for the j-dimensional Hausdorff measure, for $0 \le j \le d$.

Lemma 1.7 *Let D be a polygon as defined in (1.1), and $\Pi \subset \{x \in \mathbb{R}^d : \nu \cdot x = c\}$ be a face of D. Fix $1 \le k \le d$, then for any small number $\rho > 0$ there exist a finite number of measurable sets $\Gamma_j \subset \Pi$, $j = 1, 2, ..., M$ with disjoint $d-1$-dimensional interiors, and a measurable set $E \subset \Pi$ such that*

(i) $E \subset \Pi_{c_0\rho}$, *for some constant c_0 depending on Π and dimension d, but independent of ρ,*
(ii) $\Pi \backslash E = \bigcup_{j=1}^M \Gamma_j$, *and $\pi_k(\Gamma_j)$ is a $(d-1)$-dimensional cube of side length ρ with vertices in the lattice $\pi_k(\rho\mathbb{Z}^d)$, for $j = 1, 2, ..., M$.*
(iii) for $j = 1, 2, ..., M$ one has $\mathcal{H}^{d-1}(\Gamma_j) \approx \rho^{d-1}$, and $\mathrm{diam}(\Gamma_j) \approx \rho$, where constants in the equivalence depend on Π and dimension d, but are independent of ρ.

Proof We first construct the projections of the required sets in the projection of Π, and then lift it up to Π. To have a control on the lifted sets we need some control on the projection π_k. For any $x, y \in \Pi$ one has

$$\frac{|\nu_k|}{||\nu||}||x-y|| \le ||\pi_k(x) - \pi_k(y)|| \le ||x-y||, \tag{1.8}$$

where $\nu = (\nu_1, ..., \nu_d)$ is the unit outward normal vector of Π. The second inequality is obvious, for the first one observe that if $x \in \Pi$ then $x_k = \frac{c}{\nu_k} - \frac{1}{\nu_k}\sum_{i \ne k} \nu_i x_i$, from which we get

$$\begin{aligned} ||x-y||^2 &= \sum_{i\ne k}(x_i - y_i)^2 + \frac{1}{\nu_k^2}\left(\sum_{i\ne k}\nu_i(x_i - y_i)\right)^2 \\ &= ||\pi_k(x) - \pi_k(y)||^2 + \frac{1}{\nu_k^2}\left(\sum_{i\ne k}\nu_i(x_i - y_i)\right)^2 \\ &\le ||\pi_k(x) - \pi_k(y)||^2 + \frac{1}{\nu_k^2}\sum_{i\ne k}\nu_i^2\sum_{i\ne k}(x_i - y_i)^2 \\ &= ||\pi_k(x) - \pi_k(y)||^2 + ||\pi_k(x) - \pi_k(y)||^2\frac{1}{\nu_k^2}\sum_{i\ne k}\nu_i^2, \end{aligned}$$

and the first inequality in (1.8) follows. Note that the first inequality shows that $\pi_k : \Pi \to \pi_k(\Pi)$ is a bijection.

Now consider the projection $\pi_k(\Pi)$, and let $\mathcal{C} = \{\mathcal{C}_j\}_{j=1}^M$ be a maximal family of lattice cubes of size ρ and vertices from $\pi_k(\rho\mathbb{Z}^d)$, such that $\mathcal{C}_j \subset \pi_k(\Pi)$. Set $\mathcal{S} = \{x \in \pi_k(\Pi) : \mathrm{dist}(x, \partial\pi_k(\Pi)) \le 2\sqrt{d-1}\rho\}$-a strip near the $(d-2)$-dimensional boundary of $\pi_k(\Pi)$. Since the diameter of each $(d-1)$-dimensional cube of size ρ is $\sqrt{d-1}\rho$, it is clear that the set $\pi_k(\Pi)\backslash\mathcal{S}$ is entirely covered by the family of cubes

$\mathcal{C}$. Now set $E_0 = \pi_k(\Pi) \setminus \bigcup_{\mathcal{C}_j \in \mathcal{C}} \mathcal{C}_j$-the part not covered by the cubes, it follows that $E_0 \subset \mathcal{S}$.

We define $E = \pi_k^{-1}(E_0)$, and $\Gamma_j = \pi_k^{-1}(\mathcal{C}_j)$, for $j = 1, 2, ..., M$. Using the fact that π_k is a bijection from Π to $\pi_k(\Pi)$, and the mentioned properties of E_0, and the family of cubes $\mathcal{C}$, the assertions $(i) - (iii)$ follow immediately from inequality (1.8). The proof is now complete. □

1.3 The Poisson Kernel

For $x \in D$ and $y \in \Gamma$ we denote by $P(x, y)$ the Poisson kernel corresponding to operator $\mathcal{L}$ in D. It is proved in Lemma 2 of [5], in a more general setting, that for all $x \in D$,

$$|P(x, y)| \leq C \frac{d(x)}{|x - y|^d}, \ \ y \text{ a.e. in } \Gamma, \tag{1.9}$$

where the y null set is independent of x. We remark here that the estimate (1.9) is proved in the case when the matrix A is periodic. A careful inspection of the proof from [5] shows that the estimate (1.9) continues to hold for non-periodic operators defined on the entire space $\mathbb{R}^d$ with some mild smoothness condition on the coefficients of the operator. We stress that the periodicity condition is used in [5] to get uniform bounds on the Poisson kernel when the coefficients of the operator exhibit oscillations at ε-scale.

Lemma 1.8 *Let $\rho > 0$ be a small number, $x \in D$ be fixed with $d(x) \geq 2\rho$, and let Π be one of the faces of D. Then, there exists a constant C, independent of x and ρ such that*

$$\int_{\Pi_\rho} |P(x, y)| d\sigma(y) \leq C \frac{\rho}{d(x)}.$$

Proof If $d = 2$ then Π is a segment, and Π_ρ is a union of two segments of size ρ. It follows from (1.9) that

$$\int_{\Pi_\rho} |P(x, y)| d\sigma(y) \leq C \frac{1}{d(x)} \int_{\Pi_\rho} d\sigma(y) \leq C \frac{\rho}{d(x)}.$$

We now consider the case $d \geq 3$. After a rotation of the coordinates we may assume that Π is contained in the plane $\{x_d = 0\}$. Note that the boundary of Π is a subset of $(d-2)$-dimensional boundary of D, that is its edges. We let $L_1, ..., L_M$ to be the flat portions of $\partial \Pi$, i.e. the edges of the polygon D that form the $(d-2)$-dimensional boundary of Π. Next, we consider strips of size ρ near each L_i, namely for $i = 1, 2, ..., M$ we set

$$\mathcal{S}_i = \{y = (y_1, ..., y_{d-1}, 0) \in \Pi : \text{dist}(y, L_i) \leq \rho\}.$$

It is clear that $\Pi_\rho = \bigcap_{i=1}^{M} \mathcal{S}_i$, from which we have

$$\int_{\Pi_\rho} |P(x,y)| d\sigma(y) \le \sum_{i=1}^{M} \int_{\mathcal{S}_i} |P(x,y)| d\sigma(y).$$

We now set $\mathcal{S}$ to be one of the $\mathcal{S}_i-$s, and L to be the corresponding edge. It is enough to prove the estimate of the Lemma for $\mathcal{S}$. After a rotation we may assume that L lies in $(d-2)$-dimensional subspace $\{y_{d-1} = y_d = 0\}$, from which we get that

$$\mathcal{S} \subset \{y = (y_1, y_2, ..., y_{d-1}, 0) \in \mathbb{R}^{d-1} : \ |y_{d-1}| \le \rho\}. \tag{1.10}$$

For $x \in D$ we denote by x_Π the orthogonal projection of x onto the hyperplane containing Π. It is clear that $d(x) \le |x - x_\Pi| = |x_d|$, where x_d is the last coordinate of x. Using this, the estimate (1.9) for the Poisson kernel and (1.10) we get

$$\begin{aligned}
\int_{\mathcal{S}} |P(x,y)| d\sigma(y) &\le C|x_d| \int_{\mathcal{S}} \frac{d\sigma(y)}{[(x_1-y_1)^2 + ... + (x_{d-1}-y_{d-1})^2 + x_d^2]^{d/2}} \\
&\le \frac{C}{|x_d|^{d-1}} \int\limits_{\substack{\overline{y}=(y_1,...,y_{d-1})\in\mathbb{R}^{d-1} \\ |y_{d-1}|\le\rho}} \frac{d\overline{y}}{\left[1 + \left(\frac{x_1-y_1}{x_d}\right)^2 + ... + \left(\frac{x_{d-1}-y_{d-1}}{x_d}\right)^2\right]^{d/2}} \\
&\le (\text{setting } z_i := \frac{x_i - y_i}{x_d}, \ 1 \le i \le d-1) C \\
&\quad \times \int\limits_{\substack{\overline{z}=(z_1,...,z_{d-1})\in\mathbb{R}^{d-1} \\ \left|z_{d-1} - \frac{x_{d-1}}{x_d}\right| \le \frac{\rho}{x_d}}} \frac{d\overline{z}}{(1+|\overline{z}|^2)^{d/2}} \le C\frac{\rho}{|x_d|} \le C\frac{\rho}{d(x)}.
\end{aligned}$$

The proof of the Lemma is complete. □

In the next Lemma we prove certain type of Hölder-smoothness for $P(x,y)$ with respect to its boundary variable y and uniformly in x. We shall also define Γ^* to be the set of singular boundary points (see Appendix).

Lemma 1.9 *Retain the hypothesis of the Standing Assumptions in Section 1.1, and if $\alpha_* > 1$ set $\beta = 1$, otherwise, let $0 < \beta < \alpha_*$ be any number. Fix any $\delta \ge 0$, $x \in D$, and $y_1, y_2 \in \Pi \backslash \Gamma^*$, where Π is a face of D, and $|y_1 - y_2| \le cd(x)$, where c is some universal constant. Then, there exists a constant C depending on β, and δ, and independent of x, y_1, y_2 such that*

$$|P(x,y_1) - P(x,y_2)| \le C\frac{|y_1 - y_2|^\beta}{|x - y_1|^{d-1+\beta+\delta}},$$

where δ can be taken arbitrarily small positive non zero number in dimension two, and zero in dimensions greater than two.

Proof Let $G(x, y)$ be the Green's function corresponding to problem (1.2), then the Poisson kernel has the representation $P(x, y) = n(y)^T A(y)\nabla_y G(x, y)$, where $n(y)$ is the outward unit normal of Γ at y. We will study the regularity properties of the Green's function, which together with smoothness of A will imply the result. We will need the following estimates on the Green's function of $\mathcal{L}$,

$$|G(x, y)| \leq C \begin{cases} \log \frac{C}{|x-y|}, & d = 2, \\ |x - y|^{2-d} & d \geq 3, \end{cases} \tag{1.11}$$

for all $(x, y) \in D \times D$, with $x \neq y$, where for $d = 2$ the estimate is proved in [16], and for $d \geq 3$ in [15]. Now fix any two points $x_0, y_0 \in D$, and set $R = |x_0 - y_0|$, $D_R = \frac{1}{R}(D - x_0)$, and let $G_R(\cdot, \cdot)$ be the Green's function for the scaled domain and the scaled operator. Clearly $G_R(w, z) = R^{d-2}G(Rw + x_0, Rz + x_0)$, where $w, z \in D_R$. Consider $h_R(z) := G_R(0, z)$ in the set $\tilde{D}_R := D_R \cap (B_4(0)\setminus B_{1/4}(0))$. Then h_R is a solution to our PDE in this set and zero on $\partial\tilde{D}_R\setminus(\partial B_4(0) \cup \partial B_{1/4}(0))$. We claim that

$$h_R \in C^{1,\beta}(D_R \cap (B_3(0)\setminus B_{1/2}(0))) \tag{1.12}$$

with uniform norm bounded by constant times the supremum norm of h_R on the set $\tilde{D}_R$. In the sequel, when proving (1.12) we will keep in mind the mentioned relation of constants with the supremum norm of h_R.

We first show that (1.12) with (1.11) would imply the desired estimate. Take any $y_1, y_2 \in \Pi\setminus\Gamma^*$ with $|y_1 - y_2| \leq Cd(x_0)$. Since $n(y_1) = n(y_2)$, from the Poisson representation we have

$$\begin{aligned} |P(x, y_1) - P(x, y_2)| &\leq |n(y_1)^T (A(y_1) - A(y_2))\nabla_y G(x_0, y_1)| \\ &\quad + |n(y_1)^T A(y_2)(\nabla_y G(x_0, y_1) - \nabla_y G(x_0, y_2))|. \end{aligned} \tag{1.13}$$

On the other hand for $R = |x_0 - y_0|$, and $z \in D_R \cap (B_3(0)\setminus B_{1/2}(0))$ we have

$$\nabla h_R(z) = R^{d-1}\nabla_y G(x_0, y), \text{ where } y = Rz + x_0. \tag{1.14}$$

It is then easy to see that (1.13), (1.14), (1.12) and (1.11), together with the smoothness of A would imply the desired estimate. We just remark that in dimension two we may tradeoff the logarithmic singularity in the supremum norm of h_R by slightly increasing the power in the denominator of the estimate in the Lemma by means of the small parameter δ introduced in the formulation, while in dimensions greater than two, the supremum norm of h_R is uniformly bounded away from the origin.

In what follows we prove (1.12). Observe that due to Schauder estimates we locally have

$$h_R \in C^{1,\beta}(D_R \cap (B_3(0)\setminus B_{1/2}(0))). \tag{1.15}$$

It remains to show that when approaching the boundary of $\tilde{D}_R$ the norm does not blow-up.

From boundary regularity for elliptic equations, we also know that solutions are smooth at regular boundaries (see Theorem 6.19 in [13]). In particular in our case we have (at least) C^2 regularity for h_R on the flat boundaries, $\partial D_R \setminus \partial^* D_R$, where $\partial^* D_R$ denotes the set of all points of the boundary of D_R that belong to more than one face of D_R, i.e. the corner points. Again the norm may blow up when approaching the corners $\partial^* D_R$. Since we can approach the corner points both tangentially and non-tangentially, we may consider two cases for $x_j \to \partial^* D_R$:

(i) non-tangential to the boundary, (ii) tangential to the boundary.

For (i) we consider two points y_i ($i = 1, 2$), with the property that they approach $\partial^* D_R$ non-tangentially, i.e.

$$\operatorname{dist}(y_i, \partial D_R) \geq c_0 \operatorname{dist}(y_i, \partial^* D_R),$$

for some $c_0 > 0$. Then if $|y_1 - y_2| \geq (1/4)\operatorname{dist}(y_1, \partial^* D_R)$ then by Lemma 3.3

$$\begin{aligned} |\nabla h_R(y_1) - \nabla h_R(y_2)| &\leq |\nabla h_R(y_1)| + |\nabla h_R(y_2)| \\ &\leq C \max_{i=1,2} \operatorname{dist}^\beta(y_i, \partial^* D_R) \leq C|y_1 - y_2|^\beta. \end{aligned}$$

If $|y_1 - y_2| \leq (1/3)\operatorname{dist}(y_1, \partial^* D_R)$ then we scale h_R at y_1 with the distance to the corner $\tilde{h}_R(y) = h_R(y_1 + d_1 y)/d_1^{1+\beta}$, where d_1 is the distance from y_1 to $\partial^* D_R$. By Lemma 3.1 we have $\tilde{h}_R$ is uniformly bounded in B_1 and that $\tilde{y}_2 = (y_2 - y_1)/d_1 \in B_{1/3}(0)$. Since in $B_{1/2}(0)$ we have no corner points but only smooth boundary, the elliptic regularity implies that $\tilde{h}_R$ is uniformly C^2, say, (independent of y_1, y_2). But then the $C^{1,\beta}$ norm of $\tilde{h}_R$ is uniformly bounded (independent of y_1, y_2), and we have the same for h_R. In particular

$$|\nabla h_R(y_1) - \nabla h_R(y_2)| = d_1^\beta |\nabla \tilde{h}_R(0) - \nabla \tilde{h}_R(\tilde{y}_2)| \leq C d_1^\beta |\tilde{y}_2|^\beta = C|y_2 - y_1|^\beta.$$

For (ii) we start by taking any point z_0 on the flat boundary and consider the half ball $B_s^+(z_0)$ which is inside the domain $\tilde{D}_R$. For simplicity assume that the flat portion of the boundary, with z_0 on it, is part of the hyperplane $\{x_d = 0\}$, such that $B_s^+ = \{x_d > 0\} \cap B_s(z_0)$. Now we let s denote the largest real number such that $B_{2s}^+(z_0) \subset \tilde{D}_R$. Obviously $\partial^* D_R \cap \overline{B_s^+(z_0)} = \emptyset$, and

$$c_0 s \geq \operatorname{dist}(z_0, \partial^* D_R) \tag{1.16}$$

for some $c_0 > 0$, due to Lipschitz character of the domain. Invoking Lemma 3.1 and using (1.16) we have that for $z \in B_1^+(0)$ the function $v_s(z) := h_R(sz + z_0)/s^{1+\beta}$ satisfies the bound

$$\begin{aligned} 0 \leq v_s(z) &\leq C \frac{(\operatorname{dist}(sz + z_0, \partial^* D_R))^{1+\beta}}{s^{1+\beta}} \\ &\leq C \frac{(\operatorname{dist}(z_0, \partial^* D_R) + s)^{1+\beta}}{s^{1+\beta}} \leq C(c_0 + 1)^{1+\beta} \end{aligned}$$

which is uniformly bounded in $B_1^+(0)$. Hence classical Schauder estimates can be applied to conclude uniform $C^{1,\beta}$-estimates for v_s in $B_{1/2}^+(0)$, i.e.

$$|h_R|_{C^{1,\beta}(B_{s/2}^+(z_0))} = |v_s|_{C^{1,\beta}(B_{1/2}^+(z_0))} \le C_0.$$

This in particular means that the $C^{1,\beta}$ norm is uniformly bounded up to any flat boundary points, which is the desired result. □

2 Proofs of the Theorems

2.1 Proof of Theorem 1.1

By the Poisson representation we have

$$\begin{aligned} u_\varepsilon(x) - u_0(x) &= \int_\Gamma P(x,y)[g_\varepsilon(y) - \overline{g}(y)]d\sigma(y) \\ &= \sum_{j=1}^N \int_{\Pi_j} P(x,y)[g_\varepsilon(y) - \overline{g}(y)]d\sigma(y), \end{aligned}$$

hence it is enough to study the integrals over one particular face. Let Π be one of the faces of Π with Diophantine normal vector $\nu \in \Omega(\tau, c)$. We will assume that the boundary data g is smooth of order greater than $\frac{d-1}{2} + (d-1)\tau$. Since g is smooth and 1-periodic we have

$$g(y) = \sum_{m\in\mathbb{Z}^d} c_m e^{2\pi i m\cdot y},$$

and the order of smoothness of g assures that the series converges absolutely. Define $\mathcal{I}_1 = \{m \in \mathbb{Z}^d : m_1 \neq 0\}$ and for $k = 2, 3, ..., d$ set $\mathcal{I}_k = \{m \in \mathbb{Z}^d : m_k \neq 0\}\backslash(\mathcal{I}_1 \cup ... \cup \mathcal{I}_{k-1})$. We get

$$\int_\Pi P(x,y)[g_\varepsilon(y) - \overline{g}(y)]d\sigma(y) = \sum_{k=1}^d \sum_{m\in\mathcal{I}_k} c_m \int_\Pi P(x,y) e^{\frac{2\pi i}{\varepsilon} m\cdot y} d\sigma(y).$$

We fix $x \in D$, $1 \le k \le d$, and a small parameter $0 < \rho \le cd(x)$, where the constant c will be chosen from (2.1) below. Applying Lemma 1.7 we get a set $E \subset \Pi$, and a family $\{\Gamma_j^\rho\}_{j=1}^M$ with properties $(i) - (iii)$ of the Lemma, and let c_0 be the constant from part (i). Since $E \subset \Pi_{c_0\rho}$ from Lemma 1.8 we get

$$\int_E |P(x,y)|d\sigma(y) \le C\frac{\rho}{d(x)}, \qquad \text{for } x \in D \text{ with } d(x) \ge 2c_0\rho. \tag{2.1}$$

Now for $j = 1, 2, ..., M$ fix some $y_j \in \Gamma_j^\rho$, and outside E we have

$$\int\limits_{\Pi\setminus E} P(x,y)e^{\frac{2\pi i}{\varepsilon}m\cdot y}d\sigma(y) = \sum_{j=1}^{M}\int\limits_{\Gamma_j^\rho}[P(x,y)-P(x,y_j)]e^{\frac{2\pi i}{\varepsilon}m\cdot y}d\sigma(y)$$
$$+\sum_{j=1}^{M}P(x,y_j)\int\limits_{\Gamma_j^\rho}e^{\frac{2\pi i}{\varepsilon}m\cdot y}d\sigma(y)$$
$$:= A_1(x)+A_2(x).$$

Estimate of A_1. Since $\operatorname{diam}(\Gamma_j^\rho) \leq Cd(x)$, for any $y \in \Gamma_j^\rho$ from Lemma 1.9 we obtain

$$|P(x,y)-P(x,y_j)| \leq C\frac{|y-y_j|^\beta}{|x-y_j|^{d-1+\beta+\delta/2}}.$$

In view of $|y-y_j| \leq \operatorname{diam}(\Gamma_j^\rho) \leq C\rho$, the last estimate implies

$$|A_1(x)| \leq C\sum_j\int\limits_{\Gamma_j^\rho}\frac{|y-y_j|^\beta}{|x-y_j|^{d-1+\beta+\delta/2}}d\sigma(y) \leq C\frac{\rho^\beta}{d(x)^{\beta+\delta}}\sum_j\frac{|\Gamma_j^\rho|}{|x-y_j|^{d-1-\delta/2}}, \tag{2.2}$$

where $\delta > 0$ is any small number. The sum in (2.2) is bounded up to multiplication by some constant depending on $\delta > 0$ by the integral $\int_\Gamma \frac{d\sigma(y)}{|x-y|^{d-1-\delta/2}}$, and hence is uniformly bounded with respect to x. We conclude that

$$|A_1(x)| \leq C_\delta\frac{\rho^\beta}{d(x)^{\beta+\delta}}. \tag{2.3}$$

Estimate of A_2. Observe that $m_k \neq 0$, and $\pi_k(\Gamma_j^\rho)$ is a $(d-1)$-dimensional rectangle with sides parallel to the coordinate axes, hence we may apply Lemma 1.6, and using the fact that $\mathcal{H}^{d-1}(\Gamma_j^\rho) \approx \rho^{d-1}$ we get

$$\left|\int\limits_{\Gamma_j^\rho}e^{\frac{2\pi i}{\varepsilon}m\cdot y}d\sigma(y)\right| \leq C\varepsilon^{d-1}||m||^{(d-1)\tau(\nu)} \leq C\left(\frac{\varepsilon}{\rho}\right)^{d-1}\mathcal{H}^{d-1}(\Gamma_j^\rho)||m||^{(d-1)\tau(\nu)}.$$

Using this for A_2 we have

$$|A_2(x)| \leq C\left(\frac{\varepsilon}{\rho}\right)^{d-1}||m||^{(d-1)\tau}\sum_j|P(x,y_j)|\mathcal{H}^{d-1}(\Gamma_j^\rho) \leq C\left(\frac{\varepsilon}{\rho}\right)^{d-1}||m||^{(d-1)\tau}.$$

Combining the estimates for A_1 and A_2, for the integral on $\Pi\backslash E$ we get

$$\left|\int_{\Pi\backslash E} P(x,y)[g_\varepsilon(y)-\overline{g}]d\sigma(y)\right| \leq C\sum_{k=1}^{d}\sum_{m\in\mathcal{I}_k}|c_m|\left(\frac{\rho^\beta}{d(x)^{\beta+\delta}}+\left(\frac{\varepsilon}{\rho}\right)^{d-1}||m||^{(d-1)\tau}\right)$$
$$\leq C\left(\frac{\rho^\beta}{d(x)^{\beta+\delta}}+\left(\frac{\varepsilon}{\rho}\right)^{d-1}\right), \tag{2.4}$$

where the convergence of series with Fourier coefficients is due to the smoothness of g of order greater than $\frac{d}{2}+(d-1)\tau$ (see Lemma 2.3, [1]). Since $\beta\leq 1$ clearly the estimate (2.1) is better than (2.4), thus we have

$$|u_\varepsilon(x)-u_0(x)|\leq C_\delta\left(\frac{\rho^\beta}{d(x)^{\beta+\delta}}+\left(\frac{\varepsilon}{\rho}\right)^{d-1}\right), \tag{2.5}$$

for all $x\in D$ satisfying $d(x)\geq 2c_0\rho$. Equalizing the estimates we obtain

$$\frac{\rho^\beta}{d(x)^{\beta+\delta}}=\left(\frac{\varepsilon}{\rho}\right)^{d-1} \Longleftrightarrow \rho=\varepsilon^{\frac{d-1}{d-1+\beta}}d(x)^{\frac{\beta+\delta}{d-1+\beta}}.$$

Comparing this with $d(x)\geq 2c_0\rho$, we get that (2.5) holds true if $d(x)\geq C\varepsilon^{\frac{d-1}{d-1-\delta}}$, where C is some absolute constant, thus we conclude that

$$|u_\varepsilon(x)-u_0(x)|\leq C_\delta\left(\frac{\varepsilon^\beta}{d(x)^{\beta+\delta}}\right)^{\frac{d-1}{d-1+\beta}}.$$

When $d(x)<C\varepsilon^{\frac{d-1}{d-1-\delta}}$ the estimate of the theorem follows by the uniform bound $|u_\varepsilon(x)-u_0(x)|\leq C||g||_{L^\infty(\mathbb{T}^d)}$ which is due to the maximum principle. Theorem 1.1 is proved.

2.2 Proof of Theorem 1.2

For $\beta>0$ we set $\kappa=\frac{d-1}{d-1+\beta}$. By Theorem 1.1 we have

$$|u_\varepsilon(x)-u_0(x)|\leq C\frac{\varepsilon^{\beta\kappa}}{d(x)^{(\beta+\delta)\kappa}},\ x\in D. \tag{2.6}$$

Set $p_0=\frac{1}{\beta\kappa}$, and fix $1\leq p<p_0$. Then for $\delta>0$ small enough we have $p(\beta+\delta)\kappa=p\beta\kappa+\delta p\kappa<1$. This, together with (2.6) implies that

$$||u_\varepsilon-u_0||_{L^p(D)}\leq C\varepsilon^{\beta\kappa},\ 1\leq p<p_0. \tag{2.7}$$

Now fix $p_0\leq r<\infty$, and let $1\leq p<p_0$. Using the uniform boundedness of $|u_\varepsilon-u_0|$, and estimate (2.7) we obtain

$$||u_\varepsilon - u_0||_{L^r(D)} = \left(\int\limits_D |u_\varepsilon - u_0|^{r-p} |u_\varepsilon - u_0|^p \right)^{\frac{1}{r}}$$
$$\leq C||u_\varepsilon - u_0||_{L^p(D)}^{\frac{p}{r}} \leq C\varepsilon^{\frac{\beta\kappa p}{r}}.$$

Now take $p = p_0 - \delta$, where $\delta > 0$ is small enough. Since $p_0\beta\kappa = 1$, from the last estimate we get

$$||u_\varepsilon - u_0||_{L^r(D)} \leq C\varepsilon^{\beta\kappa \frac{p_0-\delta}{r}} = C\varepsilon^{\frac{1-\beta\kappa\delta}{r}} = C\varepsilon^{\frac{1}{r}-\delta_1},$$

where $\delta_1 = \frac{\beta\kappa\delta}{r}$. Combining this with (2.7), for $1 \leq p < \infty$ we get

$$||u_\varepsilon - u_0||_{L^p(D)} \leq C\varepsilon^{\min\{\beta\kappa, \frac{1}{p}\}-\delta}. \tag{2.8}$$

Now if $\beta = 1$, then we are done, otherwise we have $\alpha_* \leq 1$, and (2.8) holds true for each $0 < \beta < \alpha_*$, and $\delta > 0$. Observe that for all $d \geq 2$ we have

$$0 < \alpha_*\kappa - \beta\kappa < \alpha_* - \beta, \text{ where } 0 < \beta < \alpha_* \leq 1.$$

Using this, for each $\delta > 0$ we choose $0 < \beta < \alpha_*$ such that $\alpha_* - \beta < \delta/2$, and from (2.8) we get

$$||u_\varepsilon - u_0||_{L^p(D)} \leq C\varepsilon^{\min\{\gamma, \frac{1}{p}\}-\frac{3}{2}\delta},$$

completing the proof.

2.3 Proof of Theorem 1.3

For the proof we will follow the same strategy as in Sect. 3 of [2]. The only part that needs to be modified in this setting is Lemma 3.2 of [2], which proves certain type of equidistribution result for the family $\lambda\Gamma \mod 1$, as $\lambda \to \infty$, where for $x \in \mathbb{R}^d$, and $x \mod 1$ denotes the unique point $y \in \mathbb{T}^d$, with $x - y \in \mathbb{Z}^d$. On the other hand, the proof of Lemma 3.2 of [2] is based on the following fact: for any smooth function $g : \mathbb{T}^d \to \mathbb{C}$ one has

$$\int\limits_{\mathbb{T}^d} g(x)dx = \lim_{\lambda\to\infty} \frac{1}{\mathcal{H}^{d-1}(\Gamma)} \int\limits_\Gamma g(\lambda y)d\sigma(y). \tag{2.9}$$

So, to complete the proof of the Theorem we need to prove (2.9), which is now due to the Diophantine property of the faces of D. Observe that since the linear combinations of exponentials $e_m(y) := e^{2\pi i m\cdot y}$, $m \in \mathbb{Z}^d$, $y \in \mathbb{T}^d$ are dense in the uniform metric in the space of smooth functions on $\mathbb{T}^d$, it is enough to prove (2.9) for each e_m, $m \in \mathbb{Z}^d$. When $m = 0$ then (2.9) is trivial, now fix some non zero $m \in \mathbb{Z}^d$. We need to show that the limit in (2.9) is 0, which is enough to establish on each face of D. Let Π be a face of D with a normal vector $\nu \in \Omega(\tau, c)$. The proof will be complete once we show that

$$\mathcal{J}_\lambda := \int_\Pi e_m(\lambda y) d\sigma(y) \to 0, \text{ as } \lambda \to \infty. \tag{2.10}$$

Since $m \neq 0$, then $m_k \neq 0$ for some $1 \leq k \leq d$. Take $\varepsilon > 0$ small and apply Lemma 1.7 for k and ε. We will get a partition of Π into a set E, and a finite family of sets $\{\Gamma_j\}_{j=1}^M$ with properties (i)–(iii) of Lemma 1.7. It is easy to see from the definition of sets Π_ε that $\mathcal{H}^{d-1}(\Pi_\varepsilon) \leq C\varepsilon$, and since $E \subset \Pi_{c_0\varepsilon}$, for some absolute constant c_0, we have $\mathcal{H}^{d-1}(E) \leq C\varepsilon$. We then use the properties of the partition and applying Lemma 1.6 on each of the Γ_j's we get

$$|\mathcal{J}_\lambda| = \left| \int_E + \sum_{j=1}^M \int_{\Gamma_j} \right| \leq C\varepsilon + C\lambda^{-(d-1)} ||m||^{(d-1)\tau} M \leq C\varepsilon,$$

if λ is large enough, which proves (2.10), completing the proof of the Theorem.

3 Appendix: PDE tools

In this appendix we shall prove some basic estimates for Green's function for a given second order elliptic linear operator $\mathcal{L}$, in polygonal domains. The estimates are standard but hard to find in literatures, therefore for the readers' convenience we have chosen to give proofs of these estimates.

Our starting point will be to fix the domain D and the operator $\mathcal{L}$, as defined in Section 1, along with the corresponding Green's function $G(x, y)$ defined on $D \times D \setminus \{(x, x) : x \in D\}$.

By Γ^* we denote the "singular" boundary of D, i.e. the set of all points of Γ that belong to more than one face of D. Let us fix a boundary point $z \in \Gamma^*$, and let Π_1 and Π_2 be any two supporting hyperplanes of D at z. Choose $\alpha > 0$ so that the angle between these two planes, i.e. $\arccos(\nu_1 \cdot \nu_2)$ equals $\pi/(1 + \alpha)$, where ν_i denotes the outward unit normal to Π_i. Then obviously a rotated and translated version of the function $\mathbf{Im}(x_1 + \mathbf{i}x_2)^{1+\alpha}$ will be harmonic in the convex cylindrical cone generated by the two bounding halfspaces of D whose boundaries are Π_1 and Π_2 correspondingly. It is well known that positive harmonic functions in cone like domains (with zero boundary values) behave as r^λ where λ is the first eigenvalue to the Laplace-Beltrami operator of surface which is the intersection of the cone with the unit sphere (see e.g. [4]). This fact can be used along with freezing coefficient techniques to show similar behavior for the solutions to variable coefficients elliptic equations.

We formalize the discussion above in the next lemma. Let D be a given convex polygonal domain, and fix $x_0 \in \Gamma^*$. Choose $\alpha > 0$ so that $\pi/(1 + \alpha)$ be the maximal angle between any two supporting planes of D at the point x_0.

Lemma 3.1 *With the above notation, consider any (nonnegative) solution h to $\mathcal{L}h = 0$ in $D \cap B_1(x_0)$ with zero boundary data on $B_1(x_0) \cap \partial D$, and non-negative on $D \cap \partial B_1(x_0)$. Then for any $\beta < \alpha$ there exists a constant C depending on β such that*

$$0 \le h(x) \le CM|x - x_0|^{1+\beta}, \qquad \forall\, x \in D \cap B_1,$$

where $M = \sup_{B_1(x_0)\cap D} h$, and $x_0 \in \Gamma^*$.

Remark 3.2 This estimate is well-known, but not easy to find a reference to (at least we could not!). Indeed, the estimate should be sharper than what we present here, but that will not affect our results, as the estimate deteriorates at faces of $(d-2)$-dimension (facets), and the only optimality we loose (by our statement) is that we do not allow $\beta = \alpha$. The latter is due to our proof. Variations of this lemma can be found in [4,20].

Proof The proof is based on scaling and Phragmén-Lindelöf type argument. After a translation we may assume $x_0 = 0$. Next, if A is the matrix of the operator $\mathcal{L}$, then after a change of variables by $x = By$, where B is an invertible matrix of size d, the matrix, corresponding to the new operator will be $|\det B|^{-1} B^T AB$. Also, note that the matrix $\frac{1}{2}(A(0) + A(0)^T)$ is positive definite and symmetric, hence by a composition of orthogonal transformation and scaling we may bring it to a scalar multiple of an identity matrix, i.e. the symmetric component of the new operator will be a scalar multiple of Laplacian at the origin. Since the orthogonal transformation and scaling will transform D to a new polygonal domain, with the same angles between its faces, as the original one, without loss of generality, we will assume that $\frac{1}{2}(A(0) + A(0)^T)$ is the identity matrix.

Let $\Pi_i = \{x \in \mathbb{R}^d : x \cdot \nu_i = 0\}$, $i = 1, 2$ be two supporting planes to D at the origin, so that the angle between Π_1 and Π_2 is $\pi/(1 + \alpha)$. Set $D_\alpha = \{x \in \mathbb{R}^d : x \cdot \nu_i > 0,\ i = 1, 2\}$, then clearly $D \subset D_\alpha$. Now, for any $\gamma \in (\beta, \alpha)$ we denote by D_γ a convex region containing D_α, bounded by two hyperplanes passing through the origin and forming an angle equal to $\pi/(1+\gamma)$. Let us finally set H_γ to be the positive barrier function supported in D_γ, which is a rotation of $\mathbf{Im}(x_1 + \mathbf{i}x_2)^{1+\gamma}$. Clearly for some constant C we have

$$\sup_{B_R \cap D_\gamma} H_\gamma(x) = CR^{1+\gamma}. \tag{3.1}$$

Also, to simplify notation we define the solutions h to be zero outside D. After this preliminary set up, we claim now that there exists a constant $C_0 > 0$ such that

$$\sup_{B_r} h(x) \le C_0 M r^{1+\beta}, \quad \forall r \in (0, 1], \text{ where } M = \sup_{B_1} h. \tag{3.2}$$

If this fails, then there exists a sequence of points $r_j \searrow 0$, positive numbers $c_j \to \infty$, and solutions h_j to our equation such that

$$\sup_{B_{r_j}} h_j = c_j M_j r_j^{1+\beta}, \tag{3.3}$$

and

$$\sup_{B_r} h_j < c_j M_j r^{1+\beta}, \quad \forall r \in (r_j, 1], \tag{3.4}$$

where $M_j = \sup_{B_1} h_j$. To show this, we proceed by induction. Indeed, if (3.2) is false, then for $c_1 = 2$ there exists a solution h_1 with $\sup_{B_r} h_1 \geq c_1 M_1 r^{1+\beta}$, for some $0 < r < 1$. We now take r_1 to be the largest of these r, hence we get

$$\sup_{B_r} h_1 \leq c_1 M_1 r^{1+\beta}, \ \forall r \in (r_1, 1],$$

and

$$\sup_{B_{r_1}} h_1 = c_1 M_1 r_1^{1+\beta}.$$

Now if we have chosen r_j, c_j, and h_j satisfying (3.3) and (3.4), for $j = 1, 2, ..., n$, we take $c_{n+1} > c_n + 1$ so that $c_{n+1}\left(\frac{1}{2}r_n\right)^{1+\beta} > 1$. Then we proceed as in the case $n = 1$. Clearly we will get r_j decreasing to 0.

Scaling h_j by r_j through $\tilde{h}_j(x) = h_j(r_j x)/c_j M_j r_j^{1+\beta}$, we see from (3.3) and (3.4) that

$$1 \leq \sup_{B_R} \tilde{h}_j \leq R^{1+\beta} \qquad \forall 1 \leq R \leq \frac{1}{r_j}. \tag{3.5}$$

Furthermore, $\tilde{h}_j$ satisfies the scaled equation $\mathcal{L}_j \tilde{h}_j = 0$ in the scaled domain $\frac{1}{r_j}(B_1 \cap D)$, and with zero boundary data on $\frac{1}{r_j}(\partial D \cap B_1)$.

By compactness (or Arzelà-Ascoli type theorem) we can take a locally converging subsequence (again labeled r_j) such that

$$\tilde{h}_j \to \tilde{h}_0, \text{ and } \mathcal{L}_j \to \mathcal{L}_0,$$

where $\mathcal{L}_0$ is the operator with the constant matrix $A(0)$, and $\mathcal{L}_0 \tilde{h}_0 = 0$, in the cone $D_0 := \bigcup_{j=1}^{\infty} \frac{1}{r_j}(D \cap B_1)$. Since $\frac{1}{2}(A(0) + A(0)^T)$ is the identity matrix, we get that $\tilde{h}_0$ is harmonic in D_0. Moreover by (3.5) we also have

$$1 \leq \sup_{B_R \cap D_0} \tilde{h}_0 \leq R^{1+\beta}, \quad \forall R \geq 1. \tag{3.6}$$

Now the blow-up cone D_0 (with vertex at the origin) whose boundary consists of k-hyperplanes, (for some positive integer k) may be cylindrical (i.e. translation invariant) in some directions. In this case we want to reduce the dimension by showing that the function $\tilde{h}_0$ is independent of the cylindrical directions. It should be remarked that such a reduction is needed only because of our barrier argument to follow; the argument does not work with cylindrical domains, and needs the cone to have only one vertex.

One may see this as asking for the the intersection of the cone and the unit sphere to be a proper subset of the upper hemisphere (after rotation).

To this end we claim that positive harmonic functions in cones (with vertex at the origin) with zero Dirichlet data on the boundary of the cone must be homogeneous of some fixed positive degree if the cone is NTA-domain (non-tangentially accessible). This is proved in Theorem 1 of [20] and since Lipschitz domains are NTA, we get the claim for D_0 (for NTA-domains see [20], and the references therein). Next, we show that the solution $\tilde{h}_0$ is independent of the cylindrical directions. For simplicity, assume that D_0 is cylindrical with respect to the last coordinate. Set $e_d = (0, ..., 0, 1) \in \mathbb{R}^d$, then for any $a > 0$ we have that $\tilde{h}_1(x) := \tilde{h}_0(x + ae_d)$ is also a positive harmonic function in D_0 with zero Dirichlet data on the boundary, and hence is homogeneous of the same degree as $\tilde{h}_0$, say $p > 0$. Now for any $\lambda > 0$ we get

$$\lambda^p \tilde{h}_1(x) = \tilde{h}_1(\lambda x) = \tilde{h}_0(\lambda x + ae_d) = \lambda^p \tilde{h}_0\left(x + \frac{a}{\lambda}e_d\right),$$

hence $\tilde{h}_0(x + ae_d) = \tilde{h}_0(x + \frac{a}{\lambda}e_d) \to \tilde{h}_0(x)$, as $\lambda \to \infty$. Thus $\tilde{h}_0$ is independent of the cylindrical directions. In particular, and without loss of generality, we may assume that our cone D_0 has the origin as the only vertex. This means that

$$\partial B_1(0) \cap \overline{D}_0 \subset \partial B_1(0) \cap \overline{D}_\alpha \subset \partial B_1(0) \cap D_\gamma. \tag{3.7}$$

Let us now take the two-dimensional barrier H_γ in the convex (cylindrical) cone D_γ introduced in (3.1). Now choose $\varepsilon > 0$ such that $\beta + \varepsilon < \gamma$. Define a new function $H_\gamma^\varepsilon := R^{-\varepsilon} H_\gamma$, and observe that by (3.7) there is a $c_0 > 0$ such that $H_\gamma(x) \geq c_0$ over the set $\partial B_1(0) \cap \overline{D}_0$ (by Harnack's inequality). From this we infer that for R sufficiently large

$$\inf_{D_0 \cap \partial B_R} H_\gamma^\varepsilon(x) = R^{1+\gamma-\varepsilon} \inf_{D_0 \cap \partial B_1} H_\gamma^\varepsilon(x) \geq c_0 R^{1+\gamma-\varepsilon} > R^{1+\beta} \geq \sup_{D_0 \cap \partial B_R} \tilde{h}_0.$$

Hence by the maximum principle (both functions are harmonic) we conclude that $H_\gamma^\varepsilon \geq \tilde{h}_0$ in the truncated cone $D_0 \cap B_R$. In particular as R becomes large we arrive at $1 = \sup_{B_1} \tilde{h}_0 \leq \sup_{B_1} H_\gamma^\varepsilon \leq R^{-\varepsilon} \sup_{B_1} H_\gamma < 1/2$ (say). This is a contradiction and we conclude that our claim (3.2) must be true. The proof of the lemma is complete. □

Using this lemma we can now estimate the gradient of the Green's function.

Lemma 3.3 *Let D, and h be as in Lemma 3.1. Then, for any $\beta < \alpha$ there exists a constant C depending on β, so that*

$$|\nabla h(x)| \leq C_0 M d_*(x)^\beta, \ \forall x \in D \cap B_{1/2},$$

where $M = \sup_{D \cap B_1(x_0)} h(x)$, and $d_*(x)$ is the distance from x to Γ^*-the singular boundary of D.

Proof By dividing the function h by its supremum norm, we may assume that h is bounded by 1. We shall prove the lemma by contradiction. Suppose the claim fails. Then there exists a sequence of solutions h_j to our problem and $x_j \in D \cap B_{1/2}$ with $d_*(x_j) \to 0$, such that

$$|\nabla h_j(x_j)| \geq j d_*(x^j)^\beta. \tag{3.8}$$

Now defining $d_j = d_*(x_j)$ and

$$v_j(x) = \frac{h_j(d_j x + x_j)}{d_j|\nabla h_j(x_j)|}, \text{ in } D_j := \frac{1}{d_j}(D - x_j),$$

we see that v_j solves the scaled version of our problem in the scaled domain:

$$\mathcal{L}_j v_j = 0 \text{ in } D_j, \text{ and } |\nabla v_j(0)| = 1,$$

and moreover v_j has the following properties:

$$0 \leq v_j(x) = \frac{h_j(d_j x + x_j)}{d_j|\nabla h_j(x_j)|} \leq \frac{C_0|d_j x + x_j - y_j|^{1+\beta}}{j d_j^{1+\beta}},$$

where $y_j \in \Gamma^*$ is the closest singular point to x_j, and in the second inequality above we have used the estimate in Lemma 3.1, and estimate (3.8). Now, for $j = 1, 2, \ldots$ we arrive at

$$0 \leq v_j(x) \leq \frac{C d_j^{1+\beta}}{j d_j^{1+\beta}} \leq \frac{C}{j}, \text{ for all } x \in D_j \cap B_2. \tag{3.9}$$

Next, and on the other hand, we have by the definition of v_j that $|\nabla v_j(0)| = 1$. Also $\partial D_j \cap B_{1/2}$ consists of separated hyperplane or is empty, and therefore v_j will be uniformly C^{1,α_0}, for some $\alpha_0 > 0$ up to the boundary $\partial D_j \cap B_{1/2}$. This would then imply (by elliptic estimates)

$$1 = |\nabla v_j(0)| \leq C \sup_{D_j \cap B_2} v_j \leq \frac{C}{j},$$

where the last inequality is due to (3.9). This gives a contradiction as $j \to \infty$, completing the proof of the Lemma. □

4 Further Horizon

In this section we shall discuss some further aspects of the homogenization problem as well as the Fourier approach chosen here, and in previous papers of the authors [1,2]. Our approach actually works in very general setting, and can be adapted to a regular domains, which are not necessarily convex, but with some control on the vanishing order of the curvature. It should also be noted here that one can not analyze the speed of convergence relying merely on the smoothness of domain without any restriction on the geometry of the boundary. Indeed, as some simple examples show even without singular kernels the integrals of the form $\int_\Gamma g(x/\varepsilon)d\sigma(x)$, where Γ is a smooth curve, and g is a smooth and 1-periodic function, may converge to its limit with a speed slower than any given rate. This kind of examples are not difficult to construct if one allows the curvature of a surface to be vanishing of infinite order at some point.

Below we shall discuss a few cases that our technique from [1,2], and the current paper can be used to derive speed of convergence for the homogenization problem. It should be remarked that the speed deteriorates when the boundary looses convexity or regularity. The departing point for our arguments below will be the setting of this paper, with a second degree divergence type operator, of scalar type. It seems plausible that the ideas can be worked out (with some efforts) for systems, but that would require a better understanding of the behavior of solutions to systems.

In the next few subsections we line up several possible directions, towards which our results can be generalized. We also suggest some more specific possible approach. Nevertheless, we stress that the reader may see these suggestions as conjectures and not statements or claims of proofs of the ideas.

4.1 Intersection of Finite Number of Smooth Convex Domains

Here we no longer have the smoothness of the domain, and hence the regularity for the Poisson kernel required in [2] does not hold, but one will still have the estimate (1.9) according to [5]. In this regard one may try to do a fine covering of the surface to be able to combine our approach from Lemma 1.8 to treat the singular parts of the boundary, with the approach from [1], and [2] for smooth boundaries. We believe that this should give some speed of convergence, though worse than the smooth case.

4.2 General Polygonal Domains

The astute reader may have already noticed that the stationary phase analysis part of the paper works out for any polygonal domains, and convexity is not necessary; the Diophantine condition, nevertheless, is indispensable for our analysis. The convexity was used to hold a good grip on the behavior of the Poisson kernel. For non-convex (or generally Lipschitz) domains we still have (a deteriorated) control of the Green's function and thus of the Poisson kernel. Indeed, in the estimate (3.1) we lose one degree, and the Green's function in non-convex case becomes C^β close to Lipschitz points, with $\beta < 1$.

It is thus unclear what happens at edges where the Green's function is not as regular as in the convex case. There is a possibility that the pointwise convergence breaks down at such corners (we could not verify this). Since the pointwise convergence takes place at other points (as before) one may then conclude L^p-convergence locally (away from such points).

4.3 Not Strictly Convex Domains

For smooth domains, one may replace the strict convexity requirement of [1,2], by the condition that the principal curvatures do not vanish at certain directions. This should still give some speed of convergence, depending on the number of non-vanishing curvatures, though the speed will be lower than in strict convex case.

4.4 Local Behavior

A careful inspection of the proof of Theorem 1.1 shows that the pointwise convergence of solutions to (1.2) exhibit local behavior. This is due to the fact, that the Poisson kernel has a better regularity at flat boundaries, or near the corners with smaller angle than the worst case, and consequently we will have a better rate for pointwise convergence if we consider $u_\varepsilon(x)$ when x is close to these well-behaved boundary points. This in particular indicates that it should be plausible to combine the methods for smooth and strictly convex domains from [1], and those of the current paper, to treat the case of convex C^2-regular domains whose boundaries do not have flat portions of positive surface measure with normal vector from $\mathbb{R}\mathbb{Q}^d$ (i.e. rational directions). We remark that for such domains it follows from [18] that u_ε has a pointwise limit, although the methods of [18] do not imply effective statements about the convergence rate.

4.5 Domains with Inner Boundaries

A further generalization, worthy of mention, are related to domains with disconnected boundary. One may consider the case of a ring between two convex domain, or even schlicht-type domains; e.g. the unit disc minus a line with Diophantine normal direction. Other cases can be a half-plane with normal of the boundary being Diophantine. All these cases will work perfectly well but will need some more care and work than a few words we use to explain.

4.6 Other Type of Oscillation

There are further form of oscillations that can be treated with our method above, and in our earlier works. One such problem is the oscillation of the source/sink given by

$$\Delta u_\varepsilon(x) = -\mu_\varepsilon \qquad \text{in } D$$

with some given boundary data (either oscillating or fixed). Here $\mu_\varepsilon := f(x/\varepsilon)d\sigma_T$ with $d\sigma_T$ being surfaces measure over a surface $T \subset D$, and f a 1-periodic function. One may still put some restrictions on the geometry of T to allow the Fourier analysis technique above to work. The representation of such a solution through Green's and Poisson kernel can be used along with the arguments in our papers. Observe that the speed for this problem (when the boundary data is fixed) should be faster, due to the fact that Green's function has a less singular behavior than the Poisson kernel. It would also be interesting to investigate the problem for surfaces of codimension greater than one.

4.7 Other Type of Operators

Generally, any type of problems where an integral representation is available can be treated by this method. This includes for example higher order operators, and equations of divergence type. The above boundary homogenization for parabolic operators is also one further possible direction to be developed.

One may naturally try to generalize the results here to systems, but that would require good sources of references (or a carrying-out analysis) for the PDE part of the current paper.

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