

On Radii of Spheres Determined by Subsets of Euclidean Space

Bochen Liu

Received: 20 April 2013 / Revised: 8 December 2013 / Published online: 18 April 2014
© Springer Science+Business Media New York 2014

Abstract In this paper the author considers the problem of how large the Hausdorff dimension of $E \subset \mathbb{R}^d$ needs to be in order to ensure that the radii set of $(d - 1)$ -dimensional spheres determined by E has positive Lebesgue measure. The author also studies the question of how often can a neighborhood of a given radius repeat. There are two results obtained in this paper. First, by applying a general mechanism developed in Grafakos et al. (2013) for studying Falconer-type problems, the author proves that a neighborhood of a given radius cannot repeat more often than the statistical bound if $\dim_{\mathcal{H}}(E) > d - 1 + \frac{1}{d}$; In \mathbb{R}^2 , the dimensional threshold is sharp. Second, by proving an intersection theorem, the author proves that for a.e $a \in \mathbb{R}^d$, the radii set of $(d - 1)$ -spheres with center a determined by E must have positive Lebesgue measure if $\dim_{\mathcal{H}}(E) > d - 1$, which is a sharp bound for this problem.

Keywords Falconer-type problems · Radii · Intersection

Mathematics Subject Classification 28A75 · 42B20

1 Introduction

The classical Falconer distance conjecture states that if a set $E \subset \mathbb{R}^d$, $d \geq 2$, has Hausdorff dimension greater than $\frac{d}{2}$, then the one-dimensional Lebesgue measure $\mathcal{L}^1(\Delta(E))$ of its distance set,

$$\Delta(E) := \{|x - y| \in \mathbb{R} : x, y \in E\},$$

Communicated by Loukas Grafakos.

B. Liu (✉)
Department of Mathematics, University of Rochester, Rochester, NY 14627, USA
e-mail: bliu19@z.rochester.edu

is positive, where $|\cdot|$ denotes the Euclidean distance. Falconer gave an example based on the integer lattice showing that the exponent $\frac{d}{2}$ is best possible. The best results currently known, culminating almost three decades of efforts by Falconer [3], Mattila [11], Bourgain [1], and others, are due to Wolff [15] for $d = 2$ and Erdoğan [2] for $d \geq 3$. They prove that $\mathcal{L}^1(\Delta(E)) > 0$ if

$$\dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{3}.$$

It is natural to consider related problems, where distances determined by E may be replaced by other geometric objects, such as triangles [5], volumes [6], angles [7], [8], and others. Generally, if

$$\Phi : (\mathbb{R}^d)^{k+1} \rightarrow \mathbb{R}^m$$

for some $1 \leq m \leq \binom{d+1}{2}$, one can define a configuration set

$$\Delta_{\Phi}(E) = \{\Phi(x^1, \dots, x^{k+1}) : x^j \in E\}$$

and ask how large $\dim_{\mathcal{H}}(E)$ needs to be to ensure $\mathcal{L}^m(\Delta_{\Phi}(E)) > 0$. For example, in the distance problem, $k = 1$, $\Phi(x^1, x^2) = |x^1 - x^2|$; in the volume problem, $k = d$, $\Phi(x^1, \dots, x^{d+1}) = |\det(x^{d+1} - x^1, \dots, x^{d+1} - x^d)|$; in the angle problem, $k = 1$, $\Phi(x^1, x^2) = \frac{x^1 \cdot x^2}{|x^1||x^2|}$.

In this paper, the author considers the problem of the distribution of radii of spheres determined by $(d + 1)$ -tuples of points from E .

Definition 1.1 A sphere is said to be determined by a $(d + 1)$ -tuple (x^1, \dots, x^{d+1}) if it is the unique sphere passing through all the points x^1, \dots, x^{d+1} . A sphere is said to be determined by a set E if it is determined by a $(d + 1)$ -tuple $(x^1, \dots, x^{d+1}) \in E \times \dots \times E$.

Let $R(x^1, \dots, x^{d+1})$ be the radius of the unique $(d - 1)$ -dimensional sphere determined by (x^1, \dots, x^{d+1}) and it equals 0 if such a sphere does not exist or it is not unique. The author obtains two types of results. First, the author estimates how often a neighborhood of a given radius occurs, in the sense defined below. Second, the author finds the optimal dimension such that the Lebesgue measure of the set of radii is positive. The main geometric results are the following.

Theorem 1.2 Let $E \subset \mathbb{R}^d$ be compact and ν be a Frostman measure on E . Then $\dim_{\mathcal{H}}(E) > d - 1 + \frac{1}{d}$ implies

$$(\nu \times \dots \times \nu) \left(\{(x^1, \dots, x^{d+1}) : |R(x^1, \dots, x^{d+1}) - t| < \epsilon\} \right) \lesssim \epsilon \tag{1.1}$$

where the implicit constant is uniform in t on any bounded set.

Moreover, $\dim_{\mathcal{H}}(E) > d - 1 + \frac{1}{d}$ implies

$$\mathcal{L}(R(E \times \cdots \times E)) > 0.$$

When $d = 2$, this result is sharp in the sense that (1.1) does not generally hold when $\dim_{\mathcal{H}}(E) < \frac{3}{2}$.

In contrast to the incidence result, using Mattila’s classical estimate on the dimension of intersections (see Theorem 5.1), one can see that

$$\dim_{\mathcal{H}}(E) > d - 1 \implies R(E \times \cdots \times E) = \mathbb{R}^+. \tag{1.2}$$

This is the optimal result because there is no unique $(d - 1)$ -dimensional sphere passing through $d + 1$ points in a hyperplane. However, one can never get this bound by improving the incidence result above due to its sharpness.

The author also proves an intersection result where rotations and translations in Theorem 5.1 are replaced by dilations and translations (see Theorem 5.4). From this intersection theorem, one can conclude the following.

Theorem 1.3 *Given $E \subset \mathbb{R}^d$ with $\dim_{\mathcal{H}}(E) > d - 1$, then for a.e. $a \in \mathbb{R}^d$,*

$$\mathcal{L}(\{r > 0 : E \text{ determines } S_{a,r}^{d-1}\}) > 0,$$

where $S_{a,r}^{d-1}$ is the $(d - 1)$ -dimensional sphere with center a and radius r .

The bound $d - 1$ is sharp because a hyperplane can never determine a $(d - 1)$ -dimensional sphere of finite radius.

Notation. Throughout the paper,

$X \lesssim Y$ means that there exists $C > 0$ such that $X \leq CY$.

$S^{d-1} \approx \{x \in \mathbb{R}^d : |x| = 1\}$.

For $A \subset \mathbb{R}^d$, $A_{a,r} = \{rx + a : x \in A\}$.

For a measure μ on \mathbb{R}^d and a function $q(x)$ on \mathbb{R}^d , $q_*\mu$ is the measure on \mathbb{R} induced by $q(x)$, i.e. $\int_{\mathbb{R}} f d(q_*\mu) = \int_{\mathbb{R}^d} f \circ q d\mu$.

$\hat{\mu}(\xi) = \int e^{-2\pi i x \cdot \xi} d\mu(x)$ is the Fourier transform of measure μ .

2 The Incidence Result

In [4], Grafakos et al. develop the following general mechanism to solve Falconer-type problems.

Φ is said to be translation invariant if it can be written as

$$\Phi_0(x^{k+1} - x^1, \dots, x^{k+1} - x^k).$$

Theorem 2.1 (Grafakos et al. [4]) *Suppose Φ is translation invariant and for some $\gamma > 0$,*

$$\begin{aligned}
 |\widehat{\mu}_t(-\xi, \xi, 0, \dots, 0)| &\lesssim (1 + |\xi|)^{-\gamma}, \\
 |\widehat{\mu}_t(0, \xi, 0, \dots, 0)| &\lesssim (1 + |\xi|)^{-\gamma}, \\
 |\widehat{\mu}_t(\xi, 0, 0, \dots, 0)| &\lesssim (1 + |\xi|)^{-\gamma},
 \end{aligned}
 \tag{2.1}$$

where μ_t is the natural measure on $\{u : \Phi_0(u) = t\}$ and the implicit constant is uniform in t on any bounded set. Then $\dim_{\mathcal{H}}(E) > d - \frac{\gamma}{k}$ implies

$$(\nu \times \dots \times \nu) \left(\{(x^1, \dots, x^{k+1}) : |\Phi(x^1, \dots, x^{k+1}) - t| < \epsilon\} \right) \lesssim \epsilon^m \tag{2.2}$$

where ν is a Frostman measure on E and the implicit constant is uniform in t in any bounded set. It follows that the m -dimensional Lebesgue measure $\mathcal{L}^m(\Delta_\Phi(E)) > 0$.

3 Proof of Theorem 1.2

Since $R(x^1, \dots, x^{d+1})$ is translation invariant and it can be written as $R_0(x^{d+1} - x^1, \dots, x^{d+1} - x^d)$, by Theorem 2.1 it suffices to show that the three inequalities in (2.1) hold for the natural measure μ_1 on $\{(u_0, \dots, u_d) \in \mathbb{R}^d : R_0(u^1, \dots, u^d) = 1\}$. Observe that

$$\begin{aligned}
 &\text{a } (d + 1)\text{-tuple } (x^1, \dots, x^{d+1}) \text{ determines a sphere of radius } 1 \\
 &\Leftrightarrow (0, x^{d+1} - x^1, \dots, x^{d+1} - x^d) \text{ determines a sphere of radius } \\
 &\quad 1 \text{ passing through the origin} \\
 &\Leftrightarrow x^{d+1} - x^i \in a + S^{d-1} \text{ for some } a \in S^{d-1} \\
 &\quad \text{and } 0 \neq x^{d+1} - x^i \neq x^{d+1} - x^j, \forall i \neq j.
 \end{aligned}$$

Hence

$$\begin{aligned}
 &\{(u^1, \dots, u^d) : R_0(u^1, \dots, u^d) = 1\} \\
 &= \{(\sigma_0 + \sigma_1, \sigma_0 + \sigma_2, \dots, \sigma_0 + \sigma_d) : \sigma_i \in S^{d-1}\} - N,
 \end{aligned}
 \tag{3.1}$$

where N is a set of measure 0.

Let ψ be a smooth cut-off function which may vary from line to line. By changing variables as (3.1),

$$\begin{aligned}
 |\widehat{\mu}_1(\xi, -\xi, 0, \dots, 0)| &= \left| \int_{\{u:R_0(u)=1\}} e^{-2\pi i u \cdot (\xi, -\xi, 0, \dots, 0)} du \right| \\
 &= \left| \int_{S^{d-1}} \dots \int_{S^{d-1}} e^{-2\pi i \xi \cdot (\sigma_1 - \sigma_2)} \psi d\sigma_0 \dots d\sigma_d \right|
 \end{aligned}$$

$$\begin{aligned}
 &\lesssim \left| \int_{S^{d-1}} \int_{S^{d-1}} e^{-2\pi i \xi \cdot (\sigma_1 - \sigma_2)} \psi \, d\sigma_1 \, d\sigma_2 \right| \\
 &= \left| \int_{S^{d-1}} e^{-2\pi i \xi \cdot \sigma_1} \left(\int_{S^{d-1}} e^{2\pi i \xi \cdot \sigma_2} \psi \, d\sigma_2 \right) d\sigma_1 \right| \\
 &= \left| \int_{S^{d-1}} e^{-2\pi i \xi \cdot \sigma_1} a(\xi, \sigma_1) \, d\sigma_1 \right|. \tag{3.2}
 \end{aligned}$$

By stationary phase (see, e.g. [13]), $|a(\xi, \sigma_1)| \lesssim (1 + |\xi|)^{-\frac{d-1}{2}}$ and

$$\left| \int_{S^{d-1}} e^{-2\pi i \xi \cdot \sigma_1} \left((1 + |\xi|)^{\frac{d-1}{2}} a(\xi, \sigma_1) \right) d\sigma_1 \right| \lesssim (1 + |\xi|)^{-\frac{d-1}{2}}. \tag{3.3}$$

Hence

$$|\widehat{\mu}_1(\xi, -\xi, 0, \dots, 0)| \lesssim (1 + |\xi|)^{-(d-1)}.$$

For $\widehat{\mu}_1(\xi, 0, \dots, 0)$ and $\widehat{\mu}_1(0, \xi, 0, \dots, 0)$, after changing variables as above,

$$\begin{aligned}
 \widehat{\mu}_1(\xi, 0, \dots, 0) &= \int_{S^{d-1}} \dots \int_{S^{d-1}} e^{-2\pi i \xi \cdot (\sigma_0 + \sigma_1)} \psi \, d\sigma_0 \dots d\sigma_d, \\
 \widehat{\mu}_1(0, \xi, 0, \dots, 0) &= \int_{S^{d-1}} \dots \int_{S^{d-1}} e^{-2\pi i \xi \cdot (\sigma_0 + \sigma_2)} \psi \, d\sigma_0 \dots d\sigma_d.
 \end{aligned}$$

By a similar argument,

$$|\widehat{\mu}_1(\xi, 0, \dots, 0)|, |\widehat{\mu}_1(0, \xi, 0, \dots, 0)| \lesssim (1 + |\xi|)^{-(d-1)},$$

which completes the proof of Theorem 1.2.

4 Sharpness of Theorem 1.2 in \mathbb{R}^2

When $d = 2$, Theorem 1.2 says $\dim_{\mathcal{H}}(E) > \frac{3}{2}$ implies (2.2), more precisely

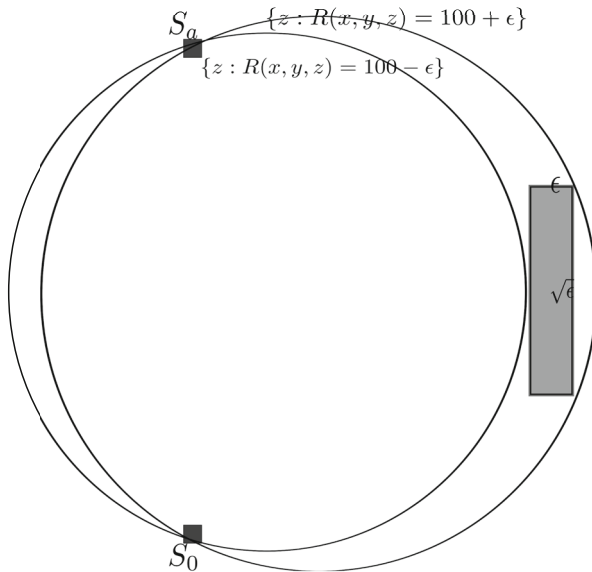
$$(\nu \times \nu \times \nu) \left(\{(x^1, x^2, x^3) : |R(x^1, x^2, x^3) - t| < \epsilon\} \right) \lesssim \epsilon \tag{4.1}$$

where ν is a Frostman measure on the compact set $E \subset \mathbb{R}^2$ and the implicit constant is uniform in t on any bounded set. The following example motivated by Mattila [11] shows that $\frac{3}{2}$ is sharp for (4.1).

Without loss of generality, fix $t = 100$. Let $C_\alpha \subset [0, 1]$ denote the Cantor set of dimension α and ν denote the natural probability measure on C_α . Let

$$E = \left(\bigcup_{k=-200}^{200} (C_\alpha + k) \right) \times [-200, 200] \tag{4.2}$$

and extend ν to E in the natural way.



Let

$$S_a = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, a \leq y \leq a + 1\}.$$

When $a \gg 1$, for all $x \in S_0, y \in S_a$, the vector $y - x$ is almost perpendicular to the x -axis. Therefore one can pick a $\sqrt{\epsilon} \times \epsilon$ rectangle in

$$\{z \in \mathbb{R}^2 : |R(x, y, z) - 100| < \epsilon\}$$

such that its $\sqrt{\epsilon}$ -edges are perpendicular to the x -axis. (see the figure above).

If a varies continuously, the $\sqrt{\epsilon} \times \epsilon$ rectangle translates continuously. Then there exists an $a_0 \in (a - 2, a)$ such that one of the $\sqrt{\epsilon}$ -edges lies on the line $x = n$ for some $n \in \mathbb{Z}$. By the construction of E , for all $x \in S_0, y \in S_{a_0}$,

$$\nu \left(\{z \in \mathbb{R}^2 : |R(x, y, z) - 100| < \epsilon\} \right) \gtrsim \epsilon^{\frac{1}{2}} \cdot \epsilon^\alpha = \epsilon^{\frac{1}{2} + \alpha}. \tag{4.3}$$

Hence

$$\begin{aligned}
 & (\nu \times \nu \times \nu) (\{(x, y, z) : |R(x, y, z) - 100| < \epsilon\}) \\
 & \gtrsim (\nu \times \nu \times \nu) (\{(x, y, z) : x \in S_0, y \in S_a, |R(x, y, z) - 100| < \epsilon\}) \quad (4.4) \\
 & \gtrsim \epsilon^{\frac{1}{2} + \alpha},
 \end{aligned}$$

which implies that when $\alpha < \frac{1}{2}$, (4.1) fails.

It follows that for $\dim_{\mathcal{H}}(E) < \frac{3}{2}$, (4.1) is not generally true, which proves the sharpness argument of Theorem 1.2.

Remark 4.1 The sharpness of (4.1) doesn't mean $\frac{3}{2}$ is the best possible for the radii problem, but for the method used above, it cannot be, generally, improved.

Remark 4.2 The example above cannot show the sharpness in higher dimensions. For example, in \mathbb{R}^3 , the rectangle one can pick will become a $\sqrt{\epsilon} \times \sqrt{\epsilon} \times \epsilon$ rectangle. For any set like $C_\alpha \times C_\beta \times C_\gamma$, the lower bound obtained for (2.2) is $\epsilon^{\frac{\alpha}{2} + \frac{\beta}{2} + \gamma}$. In nontrivial cases (i.e. $\dim_{\mathcal{H}}(E) > 2$) it is always true that $\frac{\alpha}{2} + \frac{\beta}{2} + \gamma > \frac{1}{2}(\alpha + \beta + \gamma) > 1$, which does not contradict the upper bound. In fact, in many relevant problems, a similar example can show the sharpness in $d = 2$ but fails in $d \geq 3$ (see, e.g. [5], [11]).

5 An Intersection Theorem

Given $A, B \subset \mathbb{R}^d$, the author considers the behavior of the intersection $A \cap T_\alpha(B)$, where $\{T_\alpha\}_\alpha$ is some family of transformations. Mattila ([11], [10]) proves a general intersection theorem for orthogonal transformations.

Theorem 5.1 (Mattila [10]) *In \mathbb{R}^d , let $s, t > 0, s + t > d$ and $t > \frac{d+1}{2}$. If $A, B \subset \mathbb{R}^d$ are Borel sets with $\mathcal{H}^s(A) > 0, \mathcal{H}^t(B) > 0$, then for θ_d almost all $g \in O(d)$,*

$$\mathcal{L}(\{a \in \mathbb{R}^d : \dim_{\mathcal{H}}(A \cap (gB + a)) \geq s + t - d\}) > 0,$$

where θ_d is the Haar measure on $O(d)$.

There are also intersection theorems on larger transformation groups, e.g. similarities ([9], [10]).

Theorem 5.2 (Kahane [9]) *Let G be a closed subgroup of $GL(n, \mathbb{R})$ and let τ be a Haar measure on G . Let E and F be two σ -compact subsets of \mathbb{R}^d . Then for τ -almost all $g \in G$ and $\epsilon > 0$,*

$$\mathcal{L}(\{a \in \mathbb{R}^d : \dim_{\mathcal{H}}(E \cap (gF + a)) \geq \dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F) - d - \epsilon\}) > 0.$$

From Theorem 5.1, it follows that when $d = 2$, for every $r > 0$, there exists some $a \in \mathbb{R}^2$ such that $\dim_{\mathcal{H}}(E \cap S_{a,r}^{d-1}) \geq \dim_{\mathcal{H}}(E) - 1 > 0$. Hence (1.2) holds for $d = 2$. In higher dimensions it follows by the following lemma.

Lemma 5.3 *Suppose $F \subset S^{d-1} \subset \mathbb{R}^d$ and $\dim_{\mathcal{H}}(F) > d - 2$, then F uniquely determines S^{d-1} , i.e. S^{d-1} is the only sphere which can be determined by F .*

Proof It follows by induction. For $d = 2$, it’s trivial because F has at least three points.

Suppose it holds in \mathbb{R}^{k-1} . For $F \subset S^{k-1} \subset \mathbb{R}^k$ with $\dim_{\mathcal{H}}(F) > k - 2$, there exists a hyperplane P such that $\dim_{\mathcal{H}}(P \cap F) > k - 3$ (see, e.g. [12]). Then $P \cap F$ determines the lower dimensional sphere $P \cap S^{k-1}$. Hence F determines S^{k-1} because there is at least one point in $F \setminus (P \cap F)$.

Note that all the intersection results above try to determine when the translation set has positive Lebesgue measure. The following theorem considers when the dilation set has positive Lebesgue measure.

Theorem 5.4 *Suppose that $E \subset \mathbb{R}^d$ is compact with $\dim_{\mathcal{H}}(E) = s > 1$ and $\Gamma \subset \mathbb{R}^d$ is a smooth hypersurface with nonzero Gaussian curvature. Then for a.e. $a \in \{z \in \mathbb{R}^d : E \subset \bigcup_{r \in \mathbb{R}} \Gamma_{z,r}\}$,*

$$\mathcal{L}(\{r \in \mathbb{R} : \dim_{\mathcal{H}}(E \cap \Gamma_{a,r}) \geq s - 1\}) > 0 \tag{5.1}$$

In particular, letting $\Gamma = S^{d-1}$, Theorem 1.3 follows from Theorem 5.4 and Lemma 5.3.

6 Proof of Theorem 5.4

Without loss of generality, one can assume that Γ is bounded and on any line passing through the origin there is at most one point of Γ .

Let σ denote the surface measure on Γ . Let $\{p_i\}$ be a partition of unity on Γ such that in the support of each p_i , Γ has a local coordinate system $u^i = (u^i_1, \dots, u^i_{d-1})$. Thus there is a well-defined coordinate system for the cone $C_i = \{rx : x \in \text{supp } p_i, r \in \mathbb{R} - \{0\}\}$. One can also extend p_i to C_i by setting $p_i(rx) = p_i(x), x \in \Gamma$. By changing variables $x = rx(u^i) + a$ on each $C_i + a$, it follows that

$$\begin{aligned} \int_{\bigcup_r \Gamma_{a,r}} f(x) dx &= \sum_i \int_{C_i+a} p_i(x-a) f(x) dx \\ &= \sum_i \iint_{\mathbb{R}} p_i(rx(u^i)) f(rx(u^i) + a) |r|^{d-1} \phi_i(u^i) du^i dr \\ &= \sum_i \int_{\mathbb{R}} \int_{\text{supp } p_i} p_i(x) f(rx+a) |r|^{d-1} \tilde{\phi}_i(x) d\sigma(x) dr \\ &= \int_{\mathbb{R}} \int_{\Gamma} f(rx+a) |r|^{d-1} \psi(x) d\sigma(x) dr, \end{aligned} \tag{6.1}$$

where $\phi_i, \tilde{\phi}_i, \psi$ are smooth cut-off functions.

Since $\dim_{\mathcal{H}}(E) = s > 1$, for every $\epsilon > 0$, there exists a measure μ on E such that the $(s - \epsilon)$ -energy $I_{s-\epsilon}(\mu) < \infty$ (see, e.g. [12]). Let $g(x) = rx + a$. Define measures $\sigma_{a,r}$ on $\Gamma_{a,r}$ and $\mu_{a,r}$ on $E \cap \Gamma_{a,r}$ by

$$\begin{aligned} \sigma_{a,r} &= q_*(\psi\sigma), \\ \mu_{a,r} &= \lim_{\delta \rightarrow 0} \mu * \rho_\delta d\sigma_{a,r}, \end{aligned} \tag{6.2}$$

where $\rho_\delta(x) = \delta^{-d} \rho(\frac{x}{\delta})$, $\rho \in C_0^\infty$ and $\int \rho = 1$.

Let

$$g(r) = \begin{cases} |r|^{d-1} \Phi(r) & \text{if } -1 < r < 1 \\ \Phi(r) & \text{otherwise} \end{cases} \tag{6.3}$$

where $\Phi \in L^1$ and $\Phi > 0$ everywhere.

Lemma 6.1 *Under notations above,*

$$\iint g(r) I_{s-1-\epsilon}(\mu_{a,r}) dr da < \infty. \tag{6.4}$$

Denote $\nu_{a,r} = \frac{g(r)^{\frac{1}{2}} \mu_{a,r}}{|r|^{d-1} \mu_{a,r}(\mathbb{R}^d)}$, $G_a = \{r \in \mathbb{R} : \mu_{a,r}(\mathbb{R}^d) > 0\}$ and $H = \{z \in \mathbb{R}^d : E \subset \bigcup_{r \in \mathbb{R}} \Gamma_{z,r}\}$. From Lemma 6.1, $I_{s-1-\epsilon}(\mu_{a,r}) < \infty$ for a.e. $(a, r) \in \mathbb{R}^d \times \mathbb{R}$. Then by (6.1), (6.2), for $a \in H$,

$$\begin{aligned} 1 &= (\mu(E))^2 \\ &= \left(\lim_{\delta \rightarrow 0} \int_{\bigcup_r \Gamma_{a,r}} \mu * \rho_\delta(x) dx \right)^2 \\ &= \left(\lim_{\delta \rightarrow 0} \iint \mu * \rho_\delta(rx + a) |r|^{d-1} \psi(x) d\sigma(x) dr \right)^2 \\ &= \left(\int |r|^{d-1} \mu_{a,r}(\mathbb{R}^d) dr \right)^2 \\ &= \left(\int_{G_a} |r|^{d-1} \mu_{a,r}(\mathbb{R}^d) I_{s-1-\epsilon}^{-\frac{1}{2}}(\mu_{a,r}) I_{s-1-\epsilon}^{\frac{1}{2}}(\mu_{a,r}) dr \right)^2 \\ &= \left(\int_{G_a} I_{s-1-\epsilon}^{-\frac{1}{2}}(\nu_{a,r}) g(r)^{\frac{1}{2}} I_{s-1-\epsilon}^{-\frac{1}{2}}(\mu_{a,r}) dr \right)^2 \\ &\leq \int_{G_a} I_{s-1-\epsilon}^{-1}(\nu_{a,r}) dr \int_{G_a} g(r) I_{s-1-\epsilon}(\mu_{a,r}) dr. \end{aligned} \tag{6.5}$$

Therefore,

$$\begin{aligned} \int_H \left(\int_{G_a} I_{s-1-\epsilon}^{-1}(v_{a,r}) dr \right)^{-1} da &\leq \int_H \int_{G_a} g(r) I_{s-1-\epsilon}(\mu_{a,r}) dr da \\ &\leq \iint g(r) I_{s-1-\epsilon}(\mu_{a,r}) dr da \\ &< \infty. \end{aligned} \tag{6.6}$$

It follows that for a.e. $a \in H$, $\int_{G_a} I_{s-1-\epsilon}^{-1}(v_{a,r}) dr > 0$. Hence for a.e. $a \in H$, $\mathcal{L}^1(G_a) > 0$ and for a.e. $r \in G_a$, $I_{s-1-\epsilon}(v_{a,r}) < \infty$. As a consequence, $\dim_{\mathcal{H}}(E \cap \Gamma_{a,r}) \geq s - 1 - \epsilon$ for a.e. $r \in G_a$ (see, e.g. [12]). Since G_a is independent of ϵ , by choosing a sequence $\epsilon_j \rightarrow 0$, Theorem 5.4 follows.

7 Proof of Lemma 6.1

From the the well-known equality $I_\alpha(\mu) = c_{\alpha,d} \int |\hat{\mu}(\xi)| |\xi|^{-d+\alpha} d\xi$ (see, e.g. [12]) and Plancherel,

$$\begin{aligned} &\iint g(r) I_{s-1-\epsilon}(\mu_{a,r}) dr da \\ &= c_{\alpha,d} \iiint g(r) |\hat{\mu} * \widehat{\sigma_{a,r}}(\xi)|^2 |\xi|^{-d-1+s-\epsilon} d\xi dr da \\ &= c_{\alpha,d} \iiint g(r) \left| \int \hat{\mu}(\eta) \widehat{\psi\sigma}(r(\xi - \eta)) e^{-2\pi i a \cdot (\xi - \eta)} d\eta \right|^2 |\xi|^{-d-1+s-\epsilon} d\xi dr da \\ &= c_{\alpha,d} \iint \left(\int \left| \int \hat{\mu}(\eta) \widehat{\psi\sigma}(r(\xi - \eta)) e^{2\pi i a \cdot \eta} d\eta \right|^2 da \right) g(r) |\xi|^{-d-1+s-\epsilon} d\xi dr \\ &= c_{\alpha,d} \iint \left(\int |\hat{\mu}(a) \widehat{\psi\sigma}(r(\xi - a))|^2 da \right) g(r) |\xi|^{-d-1+s-\epsilon} d\xi dr \\ &= c_{\alpha,d} \iint \left(\int |\widehat{\psi\sigma}(r(\xi - a))|^2 g(r) dr \right) |\hat{\mu}(a)|^2 |\xi|^{-d-1+s-\epsilon} da d\xi. \end{aligned} \tag{7.1}$$

Since Γ is smooth with nonzero Gaussian curvature everywhere, by stationary phase (see, e.g. [14]) and the construction of g (see (6.3)),

$$\int |\widehat{\psi\sigma}(r(\xi - a))|^2 g(r) dr \lesssim |\xi - a|^{-(d-1)}.$$

Thus, to prove Lemma 6.1, it suffices to show

$$\iint |\hat{\mu}(a)|^2 |\xi - a|^{-(d-1)} |\xi|^{-d-1+s-\epsilon} da d\xi < \infty. \tag{7.2}$$

For each $a \neq 0$, let $\xi = |a|\zeta$, then

$$\int |\xi - a|^{-(d-1)} |\xi|^{-d-1+s-\epsilon} d\xi = |a|^{-d+s-\epsilon} \int \left| \zeta - \frac{a}{|a|} \right|^{-(d-1)} |\zeta|^{-d-1+s-\epsilon} d\zeta.$$

When ϵ is small, $1 < s-\epsilon < d$, then $\int \left| \zeta - \frac{a}{|a|} \right|^{-(d-1)} |\zeta|^{-d-1+s-\epsilon} d\zeta < \infty$ uniformly because $\frac{a}{|a|} \in S^{d-1}$ which is compact. Hence

$$\begin{aligned} \iint |\hat{\mu}(a)|^2 |\xi - a|^{-(d-1)} |\xi|^{-d-1+s-\epsilon} da d\xi &\lesssim \int |\hat{\mu}(a)|^2 |a|^{-d+s-\epsilon} da \\ &< \infty, \end{aligned} \quad (7.3)$$

which proves (7.2) and completes the proof of Lemma 6.1.

Acknowledgments The author wishes to thank Professor Alex Iosevich for suggesting this problem and comments that helped improve the manuscript.

References

1. Bourgain, J.: Hausdorff dimension and distance sets. *Israel J. Math.* **87**(1–3), 193–201 (1994)
2. Erdoğan, B.: A bilinear Fourier extension theorem and applications to the distance set problem. *Int. Math. Res. Not.* **23**, 1411–1425 (2005) NULL.
3. Falconer, K.J.: On the Hausdorff dimensions of distance sets. *Mathematika* **32**, 206–212 (1986)
4. Grafakos, L., Greenleaf, A., Iosevich A., Palsson, E.: Multilinear generalized Radon transforms and point configurations. *Forum Math.* (2013). doi:[10.1515/forum-2013-0128](https://doi.org/10.1515/forum-2013-0128)
5. Greenleaf, A., Iosevich, A.: On triangles determined by subsets of the Euclidean plane, the associated bilinear operators and applications to discrete geometry. *Anal. PDE* **5**(2), 397–409 (2012)
6. Greenleaf, A., Iosevich, A., Mouroglou, M.: On volumes determined by subsets of Euclidean space. *Forum Math.* (2013). doi:[10.1515/forum-2011-0126](https://doi.org/10.1515/forum-2011-0126)
7. Harangi, V., Keleti, T., Kiss, G., Maga, P., Mathe, A., Mattila, P., Strenner, B.: How large dimension guarantees a given angle? *Monatsh. Math.* **171**(2), 169–187 (2013)
8. Iosevich, A., Mouroglou, M., Palsson, E.: On angles determined by fractal subsets of the Euclidean space via Sobolev bounds for bilinear operators. [arXiv:1110.6792](https://arxiv.org/abs/1110.6792), submitted (2011).
9. Kahane, J.: Sur la dimension des intersections. *Aspects Math. Appl.* **34**, 419–430 (1986)
10. Mattila, P.: Hausdorff dimension and capacities of intersections of sets in n-space. *Acta Math.* **152**, 77–105 (1984)
11. Mattila, P.: Spherical averages of Fourier transforms of measures with finite energy: dimensions of intersections and distance sets. *Mathematika* **34**, 207–228 (1987)
12. Mattila, P.: *Geometry of Sets and Measures in Euclidean Spaces*. Cambridge University Press, Cambridge (1995)
13. Sogge, C.: *Fourier Integrals in Classical Analysis*. Cambridge University Press, Cambridge (1993)
14. Stein, E.M.: *Harmonic Analysis*. Princeton University Press, New Jersey (1993)
15. Wolff, T.: Decay of circular means of Fourier transforms of measures. *Int. Math. Res. Not.* **10**, 547–567 (1999)