J Fourier Anal Appl (2014) 20:301–320
DOI 10.1007/s00041-013-9313-2

On Radial Functions and Distributions and Their Fourier Transforms

Ricardo Estrada

Received: 26 July 2013 / Published online: 28 November 2013

Abstract We give formulas relating the Fourier transform of a radial function in $\mathbb{R}^n$ and the Fourier transform of the same function in $\mathbb{R}^{n+1}$, completing the analysis of Grafakos and Teschl (J. Fourier Anal. Appl. 19:167–179, 2013) where the case of $\mathbb{R}^n$ and $\mathbb{R}^{n+2}$ was considered.

Keywords Fourier transforms · Distributions

Mathematics Subject Classification 46F10

1 Introduction

In a recent article, Grafakos and Teschl [9] studied the relationship between the Fourier transforms in different dimensions of the *same* radial function. Indeed, let f be a "nice enough" function of one variable, say, for instance, $f \in \mathcal{S}(\mathbb{R})$ and even; then for each $n \in \mathbb{N}$ the function f yields a radial function $f_n \in \mathcal{S}(\mathbb{R}^n)$ given by $f_n(\mathbf{x}) = f(|\mathbf{x}|)$. If we now apply the Fourier transform[1] $\mathcal{F} = \mathcal{F}_a$, $a \in \mathbb{R}$,

$$\widehat{\phi}(\mathbf{u}) = \mathcal{F}_a\{\phi(\mathbf{x}); \mathbf{u}\} = \int_{\mathbb{R}^n} \phi(\mathbf{x}) e^{ia\mathbf{x}\cdot\mathbf{u}} \mathrm{d}\mathbf{x}, \tag{1.1}$$

[1] In [9] $a = -2\pi$, while in this article we take $a = 1$. The choice $a = -1$ is also popular. Once the formulas are known for a particular value of a the formulas for other choices are easy to derive.

Communicated by Loukas Grafakos.

The author gratefully acknowledges support from NSF, through grant number 0968448.

R. Estrada (✉)
Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA
e-mail: restrada@math.lsu.edu

we obtain *radial* functions $g_n = \widehat{f_n} \in \mathcal{S}(\mathbb{R}^n)$, that is, $g_n(\mathbf{x}) = G_n(|\mathbf{x}|)$. Naturally, there is a relationship between G_n and G_k for any n and k; the formula

$$G_n'(r) = -\frac{a^2 r}{2\pi} G_{n+2}(r), \tag{1.2}$$

was obtained in [9], and this of course gives the formula relating G_n and G_k whenever $n-k$ is *even*. Formula (1.2) holds not only for $f \in \mathcal{S}(\mathbb{R})$ but for more general functions and, actually, for basically *all* relevant tempered distributions.

In this article a formula relating G_n and G_{n+1} is given, namely,

$$G_n(r) = \frac{a}{2\pi} \int_0^\infty G_{n+1}\big(\sqrt{r^2+s^2}\big)\mathrm{d}s, \tag{1.3}$$

which can be iterated to obtain the formula relating G_n and G_k for any n and k. We show that (1.3) holds not only in $\mathcal{S}(\mathbb{R})$ but for many classes of functions and distributions; however, unlike the case of (1.2), it is not possible to extend the integral operator in (1.3) as a continuous operator acting on *all* tempered distributions: the integral must be convergent, in some sense, at infinity!

The plan of the article is as follows. In Sect. 2 we introduce radial test functions and distributions and consider the operator $J_{n,k} : \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^n) \to \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^k)$ that transforms a radial test function in dimension n to the *same* radial function in dimension k. The operator $J_{n,k}$ is quite simple, of course, but its extension to radial distributions is quite interesting, and this we study in Sect. 3, where we show that there are three types of behavior, namely in the case I where $n<k$ and $n-k$ is even a unique distributional extension exists, in case II where $n>k$ and $n-k$ is even there are distributional extensions but they depend on a finite number of parameters, and in the case III where $n-k$ is odd there are no extensions that can be applied to *all* distributions. In the Sect. 3.1 we pay special attention to the case when n or k are 1, employing the spaces $\mathcal{R}_n'$ introduced in [9]. In Sect. 4 we obtain formulas relating two radial test functions in dimensions n and k that are the Fourier transforms of the same radial function, obtaining (1.3), and, more generally,

$$G_n(r) = \left(\frac{a}{2\pi}\right)^q \omega_{q-1} \int_0^\infty G_{n+q}\big(\sqrt{r^2+s^2}\big) s^{q-1}\mathrm{d}s, \tag{1.4}$$

where ω_{q-1} is the surface area of the unit sphere $\mathbb{S}^{q-1}$ of $\mathbb{R}^q$. In Sect. 5 we extend these formulas to distributions. We recover the result [9] that in case I (1.2) and its iterations hold, so that G_{n+2q} is *uniquely* determined by any distribution G_n of the space $\mathcal{R}_n'[0,\infty)$; in case II we show that G_{n-2q} is not uniquely determined by G_n, but that is possible to define continuous operators $\widetilde{Z}_{n,n-2q} : \mathcal{R}_n'[0,\infty) \to \mathcal{R}_{n-2q}'[0,\infty)$ that depend on a finite number of parameters; finally we show that in case III for a given $G_n \in \mathcal{R}_n'[0,\infty)$ there are *so many* corresponding distributions $G_k \in \mathcal{R}_k'[0,\infty)$ that it is impossible to define a continuous association $G_n \mapsto G_k$. In Sect. 5.4 we give several illustrations of our formulas.

It must be pointed out that operators like (1.3) or (1.4), or the one that arises from (1.2) can be written, after a change of variables, as operators of the so-called Erdélyi-Köber type [3], and thus it is possible to employ the distributional theory for such operators [13], [6, Sect. 9.8] in the spaces H'_μ of Zemanian [14, Chap. 5]. In this article we preferred to work directly in the spaces $\mathcal{R}'_n$ of [9] so that we do not use the methods of [6, Chap. 9].

2 Spaces and Transforms

In this section we give the notation for the spaces of functions and distributions employed. We also define several operators needed in our analysis. We have tried to follow the same notation of [9] when possible.

The spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ are well known.[2] The space $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space while we consider the strong dual topology in $\mathcal{S}'(\mathbb{R}^n)$. We usually denote $r = |\mathbf{x}|$ the radial variable in $\mathbb{R}^n$. A test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ is called *radial* if it is a function of r, $\phi(\mathbf{x}) = \varphi(r)$, for some even function $\varphi \in \mathcal{S}(\mathbb{R})$; the space of all radial test functions of $\mathcal{S}(\mathbb{R}^n)$ is denoted as $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^n)$. Similarly, we denote as $\mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n)$ the space of all radial tempered distributions; a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is radial if $f(A\mathbf{x}) = f(\mathbf{x})$ for any orthogonal transformation of $\mathbb{R}^n$, and this actually means, as we shall see, that $f(\mathbf{x}) = f_1(r)$ for some distribution of one variable f_1. Notice, however, that while φ is uniquely determined by ϕ, for a given f there are several possible distributions f_1.

When $n = 1$ then $\mathcal{S}_{\mathrm{rad}}(\mathbb{R})$ and $\mathcal{S}'_{\mathrm{rad}}(\mathbb{R})$ become the spaces of even rapidly decreasing test functions and tempered distributions, respectively, and are also denoted as $\mathcal{S}_{\mathrm{even}}(\mathbb{R})$ and $\mathcal{S}'_{\mathrm{even}}(\mathbb{R})$.

Observe that the space $\mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n)$ is naturally isomorphic to the dual space $(\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^n))'$, that is to say, if the action of a radial distribution is known in all radial test functions, then it can be obtained for arbitrary test functions. Indeed, if $f \in \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$, then

$$\big\langle f(\mathbf{x}), \phi(\mathbf{x})\big\rangle = \big\langle f(\mathbf{x}), \widetilde{\phi}(\mathbf{x})\big\rangle, \tag{2.1}$$

where $\widetilde{\phi} \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R})$ is given as

$$\widetilde{\phi}(\mathbf{x}) = \phi^o\big(|\mathbf{x}|\big), \tag{2.2}$$

$\phi^o \in \mathcal{S}_{\mathrm{even}}(\mathbb{R})$ being defined as

$$\phi^o(r) = \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \phi(r\theta)\mathrm{d}\sigma(\theta). \tag{2.3}$$

Here and in what follows, we denote by $\mathbb{S}^{n-1}$ the unit sphere of $\mathbb{R}^n$, $\mathrm{d}\sigma$ is the Lebesgue measure in $\mathbb{S}^{n-1}$ and $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the sphere. It would be convenient to employ even the value $\omega_0 = 2$.

[2]One can also work in the spaces $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$ without much change, but we chose the framework of $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ because we will study Fourier transforms.

Definition 2.1 Let k, n be strictly positive integers. The operator

$$J_{n,k} : \mathcal{S}_{\text{rad}}\big(\mathbb{R}^n\big) \to \mathcal{S}_{\text{rad}}\big(\mathbb{R}^k\big), \tag{2.4}$$

is defined as follows. If $\phi_n \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n)$ then $\phi_k = J_{n,k}(\phi_n)$ is the radial function given by $\phi_k(\mathbf{x}) = \phi_n(|\mathbf{x}|\mathbf{w})$, $\mathbf{x} \in \mathbb{R}^k$, for any $\mathbf{w} \in \mathbb{S}^{n-1}$.

Notice that (2.3) yields $J_{n,1}(\phi_n) = \phi_n^o$. Also, $J_{k,q} J_{n,k} = J_{n,q}$, and thus $\phi_k(\mathbf{x}) = \phi_n^o(|\mathbf{x}|)$. Furthermore, $\phi_k^o = \phi_n^o$ for any n and k. It is convenient to have a formula for the adjoint operator $(J_{n,k})^T : \mathcal{S}'_{\text{rad}}(\mathbb{R}^k) \to \mathcal{S}'_{\text{rad}}(\mathbb{R}^n)$ when applied to "nice" functions.

Proposition 2.2 *Let* $\phi_n \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n)$ *and* $\psi_k \in \mathcal{S}_{\text{rad}}(\mathbb{R}^k)$. *Then*

$$\omega_{n-1}\big\langle \psi_k, J_{n,k}(\phi_n)\big\rangle_{\mathcal{S}'(\mathbb{R}^k)\times\mathcal{S}(\mathbb{R}^k)} = \omega_{k-1}\big\langle r^{k-n} J_{k,n}(\psi_k), \phi_n\big\rangle_{\mathcal{S}'(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)}. \tag{2.5}$$

Therefore

$$(J_{n,k})^T(\psi_k) = \left(\frac{\omega_{k-1}}{\omega_{n-1}}\right) r^{k-n} J_{k,n}(\psi_k). \tag{2.6}$$

Proof Let $\phi_k = J_{n,k}(\phi_n)$ and $\psi_n = J_{k,n}(\psi_k)$. Then

$$\begin{aligned}
\int_{\mathbb{R}^k} \psi_k(\mathbf{x})\phi_k(\mathbf{x})\mathrm{d}\mathbf{x} &= \omega_{k-1}\int_0^\infty \psi_k^o(r)\phi_k^o(r) r^{k-1}\mathrm{d}r \\
&= \omega_{k-1}\int_0^\infty r^{k-n}\psi_n^o(r)\phi_n^o(r) r^{n-1}\mathrm{d}r \\
&= \frac{\omega_{k-1}}{\omega_{n-1}}\int_{\mathbb{R}^n} |\mathbf{x}|^{k-n}\psi_n(\mathbf{x})\phi_n(\mathbf{x})\mathrm{d}\mathbf{x},
\end{aligned}$$

and (2.5) follows. Since $\langle (J_{n,k})^T(\psi_k), \phi_n\rangle = \langle \psi_k, J_{n,k}(\phi_n)\rangle$, we also obtain (2.6). □

Observe that the function $|\mathbf{x}|^{k-n}$ is locally integrable in $\mathbb{R}^n$ since $k \geq 1$, and thus it defines a regular distribution of $\mathcal{S}'_{\text{rad}}(\mathbb{R}^n)$.

It is not always possible to extend the operator $J_{n,k}$ to *all* tempered distributions, that is, as an operator from $\mathcal{S}'_{\text{rad}}(\mathbb{R}^n)$ to $\mathcal{S}'_{\text{rad}}(\mathbb{R}^k)$, but there is a natural extension to several classes of radial functions, for instance to the radial continuous functions, or to the radial distributions whose support does not contain the origin. A related operator is defined in the ensuing definition.

Definition 2.3 Let $\mathcal{S}_{\text{rad}}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ be the closed subspace of $\mathcal{S}_{\text{rad}}(\mathbb{R}^n)$ formed by those radial test functions all of whose derivatives vanish at $\mathbf{0}$. Denote by $\mathcal{S}'_{\text{rad}}(\mathbb{R}^n \setminus \{\mathbf{0}\}) \cong (\mathcal{S}_{\text{rad}}(\mathbb{R}^n \setminus \{\mathbf{0}\}))'$ the corresponding dual space. The restriction of the operator $J_{n,k}$ to

$\mathcal{S}_{\text{rad}}(\mathbb{R}^n) \cap \mathcal{S}'_{\text{rad}}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ admits a continuous[3] extension

$$J^0_{n,k} : \mathcal{S}'_{\text{rad}}\big(\mathbb{R}^n \setminus \{\mathbf{0}\}\big) \to \mathcal{S}'_{\text{rad}}\big(\mathbb{R}^k \setminus \{\mathbf{0}\}\big). \tag{2.7}$$

Actually Proposition 2.2 yields

$$J^0_{n,k}(f_n) = \left(\frac{\omega_{k-1}}{\omega_{n-1}}\right) r^{k-n} (J_{k,n})^T (f_n). \tag{2.8}$$

Notice that multiplication by r^λ is a well defined operation of the space $\mathcal{S}_{\text{rad}}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ to itself and from $\mathcal{S}'_{\text{rad}}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ to itself for any $\lambda \in \mathbb{C}$, but it is an operator from $\mathcal{S}_{\text{rad}}(\mathbb{R}^n)$ to itself or from $\mathcal{S}'_{\text{rad}}(\mathbb{R}^n)$ to itself only when $\lambda = 0, 2, 4, \ldots$.

Observe also that the distributions of $\mathcal{S}'_{\text{rad}}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ are naturally defined in $\mathbb{R}^n \setminus \{\mathbf{0}\}$, not in $\mathbb{R}^n$. In fact the restriction gives a projection $\pi : \mathcal{S}'_{\text{rad}}(\mathbb{R}^n) \to \mathcal{S}'_{\text{rad}}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ whose kernel is, exactly, the radial distributions with support $\{\mathbf{0}\}$. We may observe at this point that a radial distribution has support at the origin if and only if it is of the form $\sum_{j=0}^{N} a_j \nabla^{2j} \delta(\mathbf{x})$ for some N and some constants a_j, $0 \le j \le N$, where ∇^2 is the Laplacian.

3 Extension of $J_{n,k}$ to Distributions

When $k > n$ and $k - n$ is even then $|\mathbf{x}|^{k-n}$ is smooth at the origin, and thus we may define an extension of $J_{n,k}$ from $\mathcal{S}'_{\text{rad}}(\mathbb{R}^n)$ to $\mathcal{S}'_{\text{rad}}(\mathbb{R}^k)$.

Definition 3.1 If $k > n$ and $k - n$ is even, the operator

$$\overline{J}_{n,k} : \mathcal{S}'_{\text{rad}}\big(\mathbb{R}^n\big) \to \mathcal{S}'_{\text{rad}}\big(\mathbb{R}^k\big), \tag{3.1}$$

the unique continuous extension of $J_{n,k}$, is defined as

$$\overline{J}_{n,k}(f_n) = \left(\frac{\omega_{k-1}}{\omega_{n-1}}\right) r^{k-n} (J_{k,n})^T (f_n). \tag{3.2}$$

We follow the convention introduced by the late Professor Farassat [8] of denoting distributional operators with an overbar.

We shall show that the operator $\overline{J}_{n,k}$ can be defined only when $k > n$ and $k - n$ is even. In order to do so we need the following easy result.

Proposition 3.2 *If $k > n$ and $k - n$ is even, the operator $\overline{J}_{n,k}$ is onto and has a kernel of dimension $(k-n)/2$, consisting of the distributions of the form*

$$\sum_{j=0}^{(k-n-2)/2} a_j \nabla^{2j} \delta(\mathbf{x}), \tag{3.3}$$

where a_j, $0 \le j \le (k-n-2)/2$, are arbitrary constants.

[3] We shall usually consider the strong topology on dual spaces, but most results also hold for the weak topology as well since in spaces of distributions weak and strong convergence of sequences are equivalent [10, 12].

Observe that for any k and n we have $(J_{n,k})^{-1} = J_{k,n}$. Hence the last proposition yields that if $k < n$ and $k - n$ is even then no continuous distributional extension of $J_{n,k}$ can exist since if it did it would be *the* inverse of $\overline{J}_{k,n}$, but such an inverse does not exist. However, continuous distributional operators that are what can be called generalized extensions of $J_{n,k}$ do exist.

Proposition 3.3 *If $k < n$ and $k - n$ is even, there are continuous operators*

$$\widetilde{J}_{n,k} : \mathcal{S}'_{\mathrm{rad}}\bigl(\mathbb{R}^n\bigr) \to \mathcal{S}'_{\mathrm{rad}}\bigl(\mathbb{R}^k\bigr), \tag{3.4}$$

such that if $f_n \in \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n)$ then

$$\overline{J}_{k,n}\widetilde{J}_{n,k}(f_n) = f_n, \tag{3.5}$$

while if $f_k \in \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^k)$ then there exists $g \in \operatorname{Ker} \overline{J}_{k,n}$ such that

$$\widetilde{J}_{n,k}\overline{J}_{k,n}(f_k) = f_k + g. \tag{3.6}$$

Proof Indeed, since $\operatorname{Ker} \overline{J}_{k,n}$ is finite dimensional, there exist closed subspaces E of $\mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^k)$ such that $\mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^k) = E \oplus \operatorname{Ker} \overline{J}_{k,n}$. Then the restriction of $\overline{J}_{k,n}$ to E is a bi-continuous isomorphism of E to $\mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n)$; if we define $\widetilde{J}_{n,k}$ as $i \circ (\overline{J}_{k,n}|_E)^{-1}$, where $i : E \to \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^k)$ is the natural injection, then $\widetilde{J}_{n,k}$ satisfies the required properties, and, clearly, any operator that satisfies (3.5) and (3.6) should be of this form. □

Notice that if $k < n$ and $k - n$ is even then the operators that satisfy the conditions of Proposition 3.3 are not unique, since if $\widetilde{J}_{n,k}$ is one such operator, then so is $\widetilde{J}_{n,k} + T$ for any continuous operator T from $\mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n)$ to the finite dimensional space $\operatorname{Ker} \overline{J}_{k,n}$. Observe also that if $\pi : \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n) \to \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ is the projection, then

$$\pi \widetilde{J}_{n,k} = J^0_{n,k}\pi. \tag{3.7}$$

If $f \in \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n)$ has support $\operatorname{supp} f \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$ then one can naturally identify f and $\pi(f)$, however, in general, $\mathbf{0}$ belongs to the support of $\widetilde{J}_{n,k}(f)$, and thus $\widetilde{J}_{n,k}(f)$ cannot be identified with $J^0_{n,k}(f)$:

$$\widetilde{J}_{n,k}(f) = J^0_{n,k}(f) + h_f, \tag{3.8}$$

for some $h_f \in \operatorname{Ker} \overline{J}_{k,n}$ if $\operatorname{supp} f \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$; it is *not* possible to construct $\widetilde{J}_{n,k}$ in such a way that $h_f = 0$ for all $f \in \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n)$ with $\operatorname{supp} f \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$.[4]

We now show that the operator $J_{n,k}$ does not have any continuous distributional generalized extensions that satisfy (3.7) if $k - n$ is odd.

Proposition 3.4 *If $k - n$ is odd then there are no continuous operators*

$$J : \mathcal{S}'_{\mathrm{rad}}\bigl(\mathbb{R}^n\bigr) \to \mathcal{S}'_{\mathrm{rad}}\bigl(\mathbb{R}^k\bigr), \tag{3.9}$$

[4] A proof can be given following the ideas of the proof of Proposition 3.4.

such that

$$\pi J = J^0_{n,k}\pi, \tag{3.10}$$

where π is the restriction to $\mathbb{R}^n \setminus \{\mathbf{0}\}$.

Proof We shall suppose that such an operator exists and find a contradiction. It is enough to do it when $k = n - 1$, since if J satisfies (3.10) for a general pair n, k with $k - n$ odd then $\widetilde{J}_{k,n-1}J$ or $\overline{J}_{k,n-1}J$ would be an operator that satisfies (3.10) from $\mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n)$ to $\mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^{n-1})$.

Now, condition (3.10) yields that if $\operatorname{supp} f \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$ then $J(f)$ is a radial distribution in $\mathbb{R}^k$ equal to the sum of two parts, namely, $J^0_{n,k}(f)$, whose support is contained in $\mathbb{R}^k \setminus \{\mathbf{0}\}$, plus a distribution of the form $\sum_{j=0}^N a_j(f)\nabla^{2j}\delta(\mathbf{x})$ for some N and some continuous functionals a_j, $0 \le j < N$. Since the space of radial distributions with support $\{\mathbf{0}\}$ is the *LF* limit of finite dimensional spaces,[5] it follows [4] that one can choose an N that works for *all* such distributions f. Hence

$$r^{2N+2}J(f) = J^0_{n,k}\big(r^{2N+2}f\big), \quad \text{if } \operatorname{supp} f \subset \mathbb{R}^n \setminus \{\mathbf{0}\}. \tag{3.11}$$

Next we apply this to $f_\varepsilon = \varepsilon^{1-n}\nabla^{2M}\delta(r - \varepsilon)$, where $M > N + 1$. Observe that in the space $\mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n)$ we have that $f_0(\mathbf{x}) = \lim_{\varepsilon\to 0^+} f_\varepsilon(\mathbf{x}) = \omega_{n-1}\nabla^{2M}\delta(\mathbf{x})$, and, by continuity,

$$\begin{aligned} r^{2N+2}J(f_0) &= r^{2N+2}\lim_{\varepsilon\to 0^+} J(f_\varepsilon) \\ &= \lim_{\varepsilon\to 0^+} J^0_{n,n-1}\big(r^{2N+2}f_\varepsilon\big) \\ &= \lim_{\varepsilon\to 0^+} \varepsilon^{1-n}r^{2N+2}\nabla^{2M}\delta(r - \varepsilon), \end{aligned} \tag{3.12}$$

but in $\mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^{n-1})$,

$$\begin{aligned} \varepsilon^{1-n}r^{2N+2}\nabla^{2M}\delta(r - \varepsilon) &\sim \varepsilon^{-1}\varepsilon^{1-(n-1)}r^{2N+2}\nabla^{2M}\delta(r - \varepsilon) \\ &\sim \varepsilon^{-1}\omega_{n-2}|\mathbf{x}|^{2N+2}\nabla^{2M}\delta(\mathbf{x}), \end{aligned} \tag{3.13}$$

and it follows that the limit $\lim_{\varepsilon\to 0^+} \varepsilon^{1-n}r^{2N+2}\nabla^{2M}\delta(r - \varepsilon)$ in (3.12) does not exist since $|\mathbf{x}|^{2N+2}\nabla^{2M}\delta(\mathbf{x}) \neq 0$. □

It is important to understand what Proposition 3.4 says and what it does not say. Suppose that $k - n$ is odd. Let $f_n \in \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n)$. Then we can consider the restriction of f to $\mathbb{R}^n \setminus \{\mathbf{0}\}$, $g_n = \pi(f_n)$, and compute $g_k = J^0_{n,k}(g_n)$. Are there distributions $F_k \in \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^k)$ whose restriction to $\mathbb{R}^k \setminus \{\mathbf{0}\}$ is g_k? The answer is yes, but the problem is that there are infinitely many. If we were able to choose one such F_k in a continuous way—as is the case when $k - n$ is even—then an operator that satisfies (3.9) would

[5] For properties of *LF* spaces see [10] or [12].

be constructed, but we cannot do this because there are just too many choices and no way to choose the "correct" one. This does not mean that for a *particular* distribution f_n one cannot choose a particular distribution f_k, one of the many F_k, in a *natural* way. Indeed, if for instance f_n is continuous at the origin, then there is a unique F_k that is continuous at the origin, and that would be our f_k. Or, for instance, one can use analytic continuation techniques to choose the extension F_k that one would call f_k. The example given by $f_{n,\lambda}(\mathbf{x}) = |\mathbf{x}|^\lambda$ illustrates these ideas: if $\Re e\lambda > 0$ then $f_{n,\lambda}$ is continuous at the origin and thus one can define in a natural way $f_{k,\lambda}(\mathbf{x}) = |\mathbf{x}|^\lambda$ in $\mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^k)$; then we can use analytic continuation, since $f_{n,\lambda}$ is analytic for $\lambda \neq -n, -n-2, -n-4, \dots$ while $f_{k,\lambda}$ admits an analytic continuation for $\lambda \neq -k, -k-2, -k-4, \dots$, so that one has the natural association $f_{n,\lambda} \to f_{k,\lambda}$ for any λ that satisfies both restrictions; interestingly, if $k-n$ is odd, then $f_{n,-k}(\mathbf{x}) = |\mathbf{x}|^{-k}$ is a well defined element of $\mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n)$ but we cannot associate to it a unique *natural* radial distribution of $\mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^k)$.

3.1 The Cases when $n = 1$ or $k = 1$

Our results on the distributional extension of $J_{n,k}$ hold even when $n = 1$ or $k = 1$, but there is a better approach in such cases. Following [9], we shall denote by $\mathcal{R}_n = r^{n-1}\mathcal{S}_{\mathrm{even}}(\mathbb{R})$. If $\mathcal{A}$ is a subspace of $\mathcal{S}(\mathbb{R})$ we shall denote by $\mathcal{A}[0,\infty)$ the space of restrictions of elements of $\mathcal{A}$ to $[0,\infty)$, and by p the restriction map, $p(\Phi) = \Phi|_{[0,\infty)}$. In general p is not an isomorphism of $\mathcal{A}$ onto $\mathcal{A}[0,\infty)$, for instance when $\mathcal{A} = \mathcal{S}(\mathbb{R})$, but sometimes p is an isomorphism, as when $\mathcal{A} = \mathcal{R}_n$, in particular when $\mathcal{A} = \mathcal{R}_0 = \mathcal{S}_{\mathrm{even}}(\mathbb{R})$; in those cases the transpose $p^T : \mathcal{A}'[0,\infty) \to \mathcal{A}'$ is also an isomorphism.

Any radial distribution $f \in \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n)$ gives an element $f_\diamondsuit \in \mathcal{R}'_n$ by the formula

$$\big\langle f(\mathbf{x}), \phi(\mathbf{x})\big\rangle_{\mathcal{S}'(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)} = \frac{\omega_{n-1}}{2}\big\langle f_\diamondsuit(r), \phi^o(r) r^{n-1}\big\rangle_{\mathcal{R}'_n\times\mathcal{R}_n}, \tag{3.14}$$

and, conversely, any element of $\mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n)$ is of this form [9].

Proposition 3.5 *The operator $pJ_{n,1} : \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^n) \to \mathcal{R}_n[0,\infty)$ has a unique continuous extension*

$$\mathfrak{J}_{n,1} : \mathcal{S}'_{\mathrm{rad}}\big(\mathbb{R}^n\big) \to \mathcal{R}'_n[0,\infty), \tag{3.15}$$

given by $F = \mathfrak{J}_{n,1}(f) = (p^T)^{-1} f_\diamondsuit$, that is

$$\big\langle F(r), \phi^o(r) r^{n-1}\big\rangle_{\mathcal{R}'_n[0,\infty)\times\mathcal{R}_n[0,\infty)} = \frac{1}{\omega_{n-1}}\big\langle f(\mathbf{x}), \phi(\mathbf{x})\big\rangle_{\mathcal{S}'(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)}. \tag{3.16}$$

The operator $\mathfrak{J}_{n,1}$ is an isomorphism and its inverse $\mathfrak{J}_{1,n} = \mathfrak{J}_{n,1}^{-1}$ is the unique continuous extension of $J_{1,n}p^{-1} : \mathcal{S}_{\mathrm{even}}[0,\infty) \to \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^n)$ as an operator from $\mathcal{R}'_n[0,\infty)$ to $\mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n)$.

It is interesting that one has a natural injection $\mathcal{R}_k[0,\infty) \to \mathcal{R}_n[0,\infty)$ when $k-n$ is even and $k > n$, and this gives a projection $\mathcal{R}'_n[0,\infty) \to \mathcal{R}'_k[0,\infty)$; this projection is exactly $\mathfrak{J}_{k,1}\overline{J}_{n,k}\mathfrak{J}_{1,n}$; it has a kernel of finite dimension and has many generalized inverses $\mathcal{R}'_k[0,\infty) \to \mathcal{R}'_n[0,\infty)$, which are precisely of the form $\mathfrak{J}_{n,1}\widetilde{J}_{k,n}\mathfrak{J}_{1,k}$. On the other hand, Proposition 3.4 immediately yields that if $k-n$ is odd then there are no continuous operators H from $\mathcal{R}'_k[0,\infty)$ to $\mathcal{R}'_n[0,\infty)$ such that $H(f)|_{(0,\infty)} = f|_{(0,\infty)}$ for all $f \in \mathcal{R}'_k[0,\infty)$.

Notice that while $\mathcal{S}_{\text{even}}[0,\infty)$ is naturally imbedded in $\mathcal{R}'_n[0,\infty)$ for any n, the space $\mathcal{S}_{\text{even}}(\mathbb{R})$ can be imbedded in $\mathcal{R}'_n$ only when n is odd, since the elements of $\mathcal{R}'_n$ are even distributions when n is odd, but they are odd distributions when n is even. Therefore it is easier to work in the spaces $\mathcal{R}'_n[0,\infty)$ instead of in the spaces $\mathcal{R}'_n$.

On the other hand, we can also define an operator $J_n : \mathcal{S}'[0,\infty) \to \mathcal{S}'_{\text{rad}}(\mathbb{R}^n)$ as $J_n(f) = \mathfrak{J}_{1,n}(g)$, where $g = f|_{\mathcal{R}_n[0,\infty)} \in \mathcal{R}'_n[0,\infty)$. The operator J_n is onto, but has a non trivial kernel, since $J_n(\delta^{(q)}(r)) = 0$ when $0 \le q < n-1$, or when $n-q$ is even.

4 Fourier Transforms

Let $\phi_n \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n)$ and $\phi_k = J_{n,k}(\phi_n) \in \mathcal{S}_{\text{rad}}(\mathbb{R}^k)$. Let $\psi_n = \widehat{\phi_n}$ and $\psi_k = \widehat{\phi_k}$ be their corresponding Fourier transforms and let $\Psi_n = pJ_{n,1}(\psi_n)$ and $\Psi_k = pJ_{k,1}(\psi_k)$. Our aim is to find a formula for the operator $Z_{n,k} : \mathcal{S}_{\text{even}}[0,\infty) \to \mathcal{S}_{\text{even}}[0,\infty)$ that sends Ψ_n to Ψ_k, that is,

$$Z_{n,k} = pJ_{k,1}\mathcal{F}J_{n,k}\mathcal{F}^{-1}J_{1,n}p^{-1}. \tag{4.1}$$

We will also consider the related operator $\mathfrak{Z}_{n,k} : \mathcal{S}_{\text{rad}}(\mathbb{R}^n) \to \mathcal{S}_{\text{rad}}(\mathbb{R}^k)$ given by $\mathfrak{Z}_{n,k}(\psi_n) = \psi_k$, that is,

$$\mathfrak{Z}_{n,k} = \mathcal{F}J_{n,k}\mathcal{F}^{-1}. \tag{4.2}$$

Let us start with the operator $Z_{n+1,n}$.

Proposition 4.1 *If $\Psi_n, \Psi_{n+1} = Z_{n,n+1}(\Psi_n) \in \mathcal{S}_{\text{even}}[0,\infty)$ then*

$$\Psi_n(r) = \frac{a}{\pi}\int_0^\infty \Psi_{n+1}\big(\sqrt{r^2+s^2}\big)\mathrm{d}s. \tag{4.3}$$

Proof It is enough to do the proof when $a=1$. Let $\phi_n \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n)$. Let us define $h \in \mathcal{S}'(\mathbb{R}^{n+1})$ as

$$h(\mathbf{x}, x_{n+1}) = \phi_n(\mathbf{x})\delta(x_{n+1}), \qquad (\mathbf{x}, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}. \tag{4.4}$$

Then

$$\begin{aligned}\widehat{h}(\mathbf{u}, u_{n+1}) &= \int_{\mathbb{R}^n}\int_{-\infty}^{\infty} \phi_n(\mathbf{x})\delta(x_{n+1})e^{i(\mathbf{x}\cdot\mathbf{u}+x_{n+1}u_{n+1})}\mathrm{d}x_{n+1}\mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^n}\phi_n(\mathbf{x})e^{i\mathbf{x}\cdot\mathbf{u}}\mathrm{d}\mathbf{x} \\ &= \widehat{\phi_n}(\mathbf{u}) = \psi_n(\mathbf{u}) = \Psi_n\big(|\mathbf{u}|\big).\end{aligned} \tag{4.5}$$

On the other hand, we also have that

$$h(\mathbf{x}, x_{n+1}) = \phi_{n+1}(\mathbf{x}, x_{n+1})\delta(x_{n+1}), \tag{4.6}$$

and thus

$$\begin{aligned}\widehat{h}(\mathbf{u}, u_{n+1}) &= \frac{1}{(2\pi)^{n+1}}\widehat{\phi}_{n+1} * \widehat{\delta(x_{n+1})} \\ &= \frac{1}{(2\pi)^{n+1}}\psi_{n+1}(\mathbf{u}, u_{n+1}) * (2\pi)^n \delta(\mathbf{u}) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_{n+1}(\mathbf{u}, u_{n+1} - s)\mathrm{d}s. \end{aligned} \tag{4.7}$$

Hence if $r = |\mathbf{u}|$,

$$\begin{aligned}\Psi_n(r) = \Psi_n\big(|\mathbf{u}|\big) &= \widehat{h}(\mathbf{u}, 0) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_{n+1}(\mathbf{u}, s)\mathrm{d}s \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi_{n+1}\big(\sqrt{|\mathbf{u}|^2 + s^2}\big)\mathrm{d}s \\ &= \frac{1}{\pi}\int_{0}^{\infty} \Psi_{n+1}\big(\sqrt{r^2 + s^2}\big)\mathrm{d}s, \end{aligned} \tag{4.8}$$

as required. □

Let us consider an illustration.

Example 4.2 Let us take $\phi_n(\mathbf{x}) = e^{-t|\mathbf{x}|^2}$ where $t > 0$. Then $\psi_n(\mathbf{u}) = (\pi/t)^{n/2}e^{-|\mathbf{u}|^2/4t}$ and $\Psi_n(r) = (\pi/t)^{n/2}e^{-r^2/4t}$, an expression that equals

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi_{n+1}\big(\sqrt{r^2 + s^2}\big)\mathrm{d}s &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (\pi/t)^{(n+1)/2}e^{-(r^2+s^2)/4t}\mathrm{d}s \\ &= \frac{\pi^{(n-1)/2}}{2t^{(n+1)/2}}e^{-r^2/4t}\int_{-\infty}^{\infty} e^{-s^2/4t}\mathrm{d}s,\end{aligned}$$

and this yields the well known formula $\int_{-\infty}^{\infty} e^{-s^2/4t}\mathrm{d}s = 2\sqrt{\pi t}$.

If we iterate the (4.3) we can express Ψ_n in terms of Ψ_{n+q}, for $q > 0$.

Proposition 4.3 *If* $\Psi_n, \Psi_{n+q} = Z_{n,n+q}(\Psi_n) \in \mathcal{S}_{\mathrm{even}}[0, \infty)$ *then*

$$\Psi_n(r) = \left(\frac{a}{2\pi}\right)^q \omega_{q-1}\int_0^{\infty} \Psi_{n+q}\big(\sqrt{r^2 + s^2}\big)s^{q-1}\mathrm{d}s, \tag{4.9}$$

Proof Indeed, (4.3) yields

$$\Psi_n(r) = \left(\frac{a}{2\pi}\right)^q \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Psi_{n+q}\left(\sqrt{r^2 + s_1^2 + \cdots s_q^2}\right) \mathrm{d}s_q \cdots \mathrm{d}s_1, \tag{4.10}$$

and using polar coordinates $(s_1, \ldots, s_q) = \mathbf{s} = s\mathbf{v}$, $|\mathbf{v}| = 1$, we obtain

$$\begin{aligned} \Psi_n(r) &= \left(\frac{a}{2\pi}\right)^q \int_0^{\infty} \int_{\mathbb{S}^{q-1}} \Psi_{n+q}\left(\sqrt{r^2 + s^2}\right) s^{q-1} \mathrm{d}\sigma(\mathbf{v}) \mathrm{d}s \\ &= \left(\frac{a}{2\pi}\right)^q \omega_{q-1} \int_0^{\infty} \Psi_{n+q}\left(\sqrt{r^2 + s^2}\right) s^{q-1} \mathrm{d}s, \end{aligned}$$

that is, (4.9). □

Formula (4.9) allows us to find formulas for certain integrals over $\mathbb{S}^{k-1}$.

Example 4.4 The integral

$$\int_{\mathbb{S}^{k-1}} v_1^{2p_1} \cdots v_k^{2p_k} \mathrm{d}\sigma(\mathbf{v}) = \frac{2\Gamma(p_1 + 1/2) \cdots \Gamma(p_k + 1/2)}{\Gamma(p_1 + \cdots + p_k + k/2)}, \tag{4.11}$$

where $(p_1, \ldots, p_k) \in \mathbb{N}^k$, appears in several computations involving the potential of an inverse-square field [1, Appendix A], [5, (3.13)]. We can obtain this and similar integrals in the following way. Since

$$Z_{n+q_1+\cdots+q_k,n} = Z_{n+q_1,n} \cdots Z_{n+q_1+\cdots+q_k,n+q_1+\cdots+q_{k-1}},$$

we obtain

$$\begin{aligned} &\omega_{q_1+\cdots+q_k-1} \int_0^{\infty} \Psi_{n+q_1+\cdots+q_k}\left(\sqrt{r^2 + s^2}\right) s^{q_1+\cdots+q_k-1} \mathrm{d}s \\ &\quad = \omega_{q_1-1} \cdots \omega_{q_k-1} \\ &\qquad \times \int_0^{\infty} \cdots \int_0^{\infty} \Psi_{n+q_1+\cdots+q_k}\left(\sqrt{r^2 + s_1^2 \cdots + s_k^2}\right) s_1^{q_1-1} \cdots s_k^{q_k-1} \mathrm{d}s_1 \cdots \mathrm{d}s_k, \end{aligned}$$

so that

$$\int_{\mathbb{S}_+^{k-1}} v_1^{q_1-1} \cdots v_k^{q_k-1} \mathrm{d}\sigma(\mathbf{v}) = \frac{\omega_{q_1+\cdots+q_k-1}}{\omega_{q_1-1} \cdots \omega_{q_k-1}}, \tag{4.12}$$

if $q_j \geq 1$, where $\mathbb{S}_+^{k-1} = \{\mathbf{v} \in \mathbb{S}^{k-1} : v_j \geq 0, 1 \leq j \leq k\}$. This yields, of course, the integral $\int_{\mathbb{S}^{k-1}} |v_1|^{q_1-1} \cdots |v_k|^{q_k-1} \mathrm{d}\sigma(\mathbf{v})$, equal to 2^{-k} the integral in (4.12), and thus (4.11). The particular case

$$\int_0^{\pi/2} \cos^{q_1-1}\theta \sin^{q_2-1}\theta \mathrm{d}\theta = \frac{\omega_{q_1+q_2-1}}{\omega_{q_1-1}\omega_{q_2-1}} = \frac{\Gamma(q_1/2)\Gamma(q_2/2)}{2\Gamma((q_1+q_2)/2)}, \tag{4.13}$$

follows from (4.12) when $k = 2$.

Next we give formulas to obtain Ψ_n from Ψ_{n-q}, for $q > 0$. We start with the case when q is even and recover the formula for $Z_{n,n-2}$ from [9].

Proposition 4.5 *If* $\Psi_n, \Psi_{n-2} = Z_{n,n-2}(\Psi_n) \in \mathcal{S}_{\mathrm{even}}[0,\infty)$ *then*

$$\Psi_n(r) = -\frac{2\pi}{a^2 r}\Psi'_{n-2}(r). \tag{4.14}$$

Proof Let us employ (4.9) with $q = 2$ to obtain

$$\Psi_{n-2}(r) = \frac{a^2}{2\pi}\int_0^\infty \Psi_n\big(\sqrt{r^2+s^2}\big)s\,\mathrm{d}s. \tag{4.15}$$

Differentiation of this relation gives (4.14):

$$\begin{aligned}\Psi'_{n-2}(r) &= \frac{a^2}{2\pi}\int_0^\infty \frac{\Psi'_n(\sqrt{r^2+s^2})rs}{\sqrt{r^2+s^2}}\mathrm{d}s \\ &= \frac{a^2 r}{2\pi}\Psi_n\big(\sqrt{r^2+s^2}\big)\big|_{s=0}^{\infty} \\ &= -\frac{a^2 r}{2\pi}\Psi_n(r),\end{aligned}$$

since $\Psi_n \in \mathcal{S}(\mathbb{R})$. □

Let us use the notation

$$L = -\frac{2\pi}{a^2 r}\frac{\mathrm{d}}{\mathrm{d}r}. \tag{4.16}$$

Notice that L sends $\mathcal{S}_{\mathrm{even}}[0,\infty)$ to itself.

Proposition 4.6 *If* $\Psi_n, \Psi_{n-2p} = Z_{n,n-2p}(\Psi_n) \in \mathcal{S}_{\mathrm{even}}[0,\infty)$, $p > 0$, *then*

$$\Psi_n(r) = L^p\big(\Psi_{n-2p}(r)\big). \tag{4.17}$$

We can also express Ψ_n in terms of Ψ_{n-q} when $q > 0$ is odd.

Proposition 4.7 *If* $\Psi_n, \Psi_{n-2p+1} = Z_{n,n-2p+1}(\Psi_n) \in \mathcal{S}_{\mathrm{even}}[0,\infty)$, $p > 0$, *then*

$$\Psi_n(r) = \frac{a}{\pi}L^p\int_0^\infty \Psi_{n-2p+1}\big(\sqrt{r^2+s^2}\big)\mathrm{d}s. \tag{4.18}$$

In particular

$$\Psi_n(r) = -\frac{2}{ar}\frac{\mathrm{d}}{\mathrm{d}r}\int_0^\infty \Psi_{n-1}\big(\sqrt{r^2+s^2}\big)\mathrm{d}s. \tag{4.19}$$

Proof We just need to combine (4.17) and (4.3). □

5 Distributional Formulas

We shall now consider the extension of the operators $Z_{n,k}$ and the operators $\mathfrak{Z}_{n,k}$ to distributions. As it was the case with the operators $J_{n,k}$, there are three cases, according to whether $n-k$ is even and positive, even and negative, or odd.

5.1 Case I: $k-n$ Even, $k>n$

Since the operator $J_{n,k}$ has a unique distributional extension $\overline{J}_{n,k}$ we immediately obtain from (4.2) the following results.

Proposition 5.1 *If $k-n$ is even, $k>n$ then the operator $\mathfrak{Z}_{n,k}:\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^n)\to\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^k)$ has a unique continuous extension $\overline{\mathfrak{Z}}_{n,k}:\mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n)\to\mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^k)$. The operator $\overline{\mathfrak{Z}}_{n,k}$ is onto and has a kernel of dimension $q=(k-n)/2$, formed by the distributions of the form $\sum_{j=0}^{q-1}a_j|\mathbf{x}|^{2j}$.*

Observe that the operator L defined in (4.16) can be considered as an operator from $\mathcal{R}'_n[0,\infty)$ to $\mathcal{R}'_{n+2}[0,\infty)$. Then we have the ensuing result [9].

Proposition 5.2 *If $k-n$ is even, $k>n$ then the operator $Z_{n,k}:\mathcal{S}_{\mathrm{even}}[0,\infty)\to\mathcal{S}_{\mathrm{even}}[0,\infty)$, has a unique continuous extension $\overline{Z}_{n,k}:\mathcal{R}'_n[0,\infty)\to\mathcal{R}'_k[0,\infty)$. Actually,*

$$\overline{Z}_{n,k}=L^{(k-n)/2}. \tag{5.1}$$

The operator $\overline{Z}_{n,k}$ is onto and has a kernel of dimension $q=(k-n)/2$, formed by the distributions of the form $\sum_{j=0}^{q-1}a_jr^{2j}$.

In particular,

$$\overline{Z}_{n,n+2}=L, \tag{5.2}$$

which means that if $G\in\mathcal{R}'_n[0,\infty)$, $H\in\mathcal{R}'_{n+2}[0,\infty)$ and $H=L(G)$, then $g=\mathfrak{J}_{1,n}(G)$ and $h=\mathfrak{J}_{1,n+2}(H)$ are the Fourier transforms of the *same* radial function, F, one in dimension n and the other in dimension $n+2$:

$$g=\mathcal{F}(f_n),\qquad h=\mathcal{F}(f_{n+2}),\qquad f_n=J_n(F),\qquad f_{n+2}=J_{n+2}(F). \tag{5.3}$$

5.2 Case II: $k-n$ Even, $k<n$

If $q=(n-k)/2>0$, then the operator $Z_{n,k}$ given by $Z_{n,n-2q}(\Psi_n)=\Psi_{n-2q}$,

$$\Psi_{n-2q}(r)=\left(\frac{a}{2\pi}\right)^{2q}\omega_{2q-1}\int_0^\infty\Psi_n\bigl(\sqrt{r^2+s^2}\bigr)s^{2q-1}\mathrm{d}s, \tag{5.4}$$

sends $\mathcal{S}_{\mathrm{even}}[0,\infty)$ to itself but it does not have a continuous extension from $\mathcal{R}'_n[0,\infty)$ to $\mathcal{R}'_k[0,\infty)$. However, a continuous extension into a related space can be constructed.

Definition 5.3 The space $\mathcal{R}_{k,q}[0,\infty)$ consists of those elements $\Phi \in \mathcal{R}_k[0,\infty)$ that satisfy

$$\mu_{2j}(\Phi) = \int_0^\infty r^{2j}\Phi(r)\mathrm{d}r = 0, \quad 0 \le j \le q-1. \tag{5.5}$$

The space $\mathcal{R}_{k,q}[0,\infty)$ is a subspace of codimension q of $\mathcal{R}_k[0,\infty)$.

Observe that since $k < n$ the operator $\overline{Z}_{k,n} : \mathcal{R}'_k[0,\infty) \to \mathcal{R}'_n[0,\infty)$ is onto, and has a kernel of dimension q, generated by the distributions r^{2j}, $0 \le j \le q-1$. The adjoint operator $\overline{Z}^T_{k,n} : \mathcal{R}_n[0,\infty) \to \mathcal{R}_k[0,\infty)$ is injective and has an image of codimension q, namely $\mathcal{R}_{k,q}[0,\infty)$; therefore we can define an inverse operator $W : \mathcal{R}_{k,q}[0,\infty) \to \mathcal{R}_n[0,\infty)$, $W\overline{Z}^T_{k,n} = I$. Hence the adjoint $W^T : \mathcal{R}'_n[0,\infty) \to \mathcal{R}'_{k,q}[0,\infty)$ is well defined, continuous, and coincides with $Z_{n,n-2q}$ in $\mathcal{S}_{\text{even}}[0,\infty)$.

Definition 5.4 The operator $\widetilde{Z}^0_{n,n-2q} : \mathcal{R}'_n[0,\infty) \to \mathcal{R}'_{k,q}[0,\infty)$, is defined as

$$\langle \widetilde{Z}^0_{n,n-2q}(F), \Phi \rangle = \langle F, W(\Phi) \rangle, \tag{5.6}$$

for $F \in \mathcal{R}'_n[0,\infty)$ and $\Phi \in \mathcal{R}_{k,q}[0,\infty)$, where

$$W\{\Phi(r); s\} = \left(\frac{a}{2\pi}\right)^{2q} \omega_{2q-1} s \int_0^s (s^2 - r^2)^{q-1}\Phi(r)\mathrm{d}r. \tag{5.7}$$

It is interesting to observe that if $\Phi \in \mathcal{R}_k[0,\infty)$ then $W\{\Phi(r); s\}$ is a well defined smooth function in $[0,\infty)$, which has the behaviour of the elements of $\mathcal{R}_n[0,\infty)$ as $s \to 0^+$, but which at infinity has a development of the form $\chi + \Psi$, where Ψ is of rapid decay while χ is a polynomial that depends linearly on μ_{2j}, $0 \le j \le q-1$. Hence $W\{\Phi(r); s\}$ belongs to $\mathcal{R}_n[0,\infty)$ exactly when $\Phi \in \mathcal{R}_{k,q}[0,\infty)$.

Proposition 5.5 *The operator* $Z_{n,n-2q} : \mathcal{S}_{\text{even}}[0,\infty) \to \mathcal{S}_{\text{even}}[0,\infty)$, *given by the integral* (5.4) *has a unique continuous extension* $\widetilde{Z}^0_{n,n-2q} : \mathcal{R}'_n[0,\infty) \to \mathcal{R}'_{k,q}[0,\infty)$, *that is, if* $\Phi \in \mathcal{R}_{k,q}[0,\infty)$, $F \in \mathcal{R}'_n[0,\infty)$, *and* $\{\Psi_n^{\{j\}}\}_{j=1}^\infty$ *is a sequence of* $\mathcal{S}_{\text{even}}[0,\infty)$ *that converges to* F *in* $\mathcal{R}'_n[0,\infty)$ *then the limit*

$$\lim_{j\to\infty} \int_0^\infty \Psi^{\{j\}}_{n-2q}(r)\Phi(r)\mathrm{d}r, \tag{5.8}$$

exists, independently of the sequence, and equals $\langle \widetilde{Z}^0_{n,n-2q}(F), \Phi \rangle$.

Since the space $\mathcal{R}_{k,q}[0,\infty)$ has finite codimension, there will exist operators $\widetilde{Z}_{n,n-2q} : \mathcal{R}'_n[0,\infty) \to \mathcal{R}'_k[0,\infty)$ such that $\widetilde{Z}_{n,n-2q}(F)|_{\mathcal{R}_{k,q}[0,\infty)} = \widetilde{Z}^0_{n,n-2q}(F)$ for all $F \in \mathcal{R}'_n[0,\infty)$. These operators are all of the following form,

$$\langle \widetilde{Z}_{n,n-2q}(F), \Phi \rangle = \left\langle \widetilde{Z}^{0}_{n,n-2q}(F), \Phi - \sum_{j=0}^{q-1} \mu_{2j}(\Phi)\vartheta_j \right\rangle$$

$$+ \sum_{j=0}^{q-1} \mu_{2j}(\Phi)\langle F, \Omega_j \rangle, \tag{5.9}$$

for $\Phi \in \mathcal{R}_k[0,\infty)$. Here $\vartheta_1, \dots, \vartheta_{q-1} \in \mathcal{R}_k[0,\infty)$ satisfy $\mu_{2j}(\vartheta_l) = \delta_{jl}$, while $\Omega_1, \dots, \Omega_{q-1} \in \mathcal{R}_n[0,\infty)$ are arbitrary. This form of the operators $\widetilde{Z}_{n,n-2q}$ can be obtained from the following general result from linear algebra.

Lemma 5.6 *Let E, F, and G be locally convex topological vector spaces. Let $E_0 = \{x \in E : \langle x_j^*, x\rangle = 0, 1 \le j \le q\}$ be a subspace of finite codimension of E, where $x_1^*, \dots, x_q^* \in E'$ are linearly independent.*

(1) *There exist $w_1, \dots, w_q \in E$ such that $\langle x_j^*, w_l \rangle = \delta_{jl}$, for $1 \le j, l \le q$.*
(2) *If $Z_0 : E_0 \to F$ is a continuous linear map, then it has continuous linear extensions $Z : E \to F$, all of which are of the form*

$$Z(x) = Z_0\left(x - \sum_{j=1}^{q} \langle x_j^*, x\rangle w_j\right) + \sum_{j=1}^{q} \langle x_j^*, x\rangle a_j, \tag{5.10}$$

where $a_1, \dots, a_q \in F$ are arbitrary.
(3) *If $T : G \to E_0'$ is a continuous linear operator then there exist continuous operators $\widetilde{T} : G \to E'$ such that $\widetilde{T}(z)|_{E_0} = T(z)$ for all $z \in G$, where the dual spaces have both the strong topology (or both the weak topology), and all such operators are of the form*

$$\langle \widetilde{T}(z), x \rangle = \left\langle T(z), x - \sum_{j=1}^{q} \langle x_j^*, x\rangle w_j \right\rangle + \sum_{j=1}^{q} \langle x_j^*, x\rangle \langle b_j^*, z\rangle, \tag{5.11}$$

where $b_1^*, \dots, b_q^* \in G'$ are arbitrary.

Naturally, one can also construct the operators $\widetilde{Z}_{n,n-2q}$ by employing the operators $\widetilde{J}_{n,n-2q}$ from the Proposition 3.3.

Proposition 5.7 *Any operator $\widetilde{Z}_{n,n-2q}$ constructed as in (5.9) is of the form*

$$\widetilde{Z}_{n,n-2q} = p\mathfrak{J}_{n-2q,1}\mathcal{F}\widetilde{J}_{n,n-2q}\mathcal{F}^{-1}\mathfrak{J}_{1,n}p^{-1}, \tag{5.12}$$

where $\widetilde{J}_{n,n-2q}$ satisfies the conditions of Proposition 3.3.

We can also define operators $\widetilde{\mathfrak{Z}}_{n,k} : \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n) \to \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^k)$ in case II,

$$\widetilde{\mathfrak{Z}}_{n,k} = \mathcal{F}\widetilde{J}_{n,k}\mathcal{F}^{-1}, \tag{5.13}$$

which are generalized inverses of the distributional operator $\overline{\mathfrak{Z}}_{k,n}$, namely, $\overline{\mathfrak{Z}}_{k,n}\widetilde{\mathfrak{Z}}_{n,k}(f_n) = f_n$, while $\widetilde{\mathfrak{Z}}_{n,k}\overline{\mathfrak{Z}}_{k,n}(f_k) = f_k + g$, where $g \in \operatorname{Ker}\overline{\mathfrak{Z}}_{k,n}$. Of course, any operator constructed as in (5.9) is of the form $\widetilde{Z}_{n,k} = p\mathfrak{I}_{k,1}\widetilde{\mathfrak{Z}}_{n,k}\mathfrak{I}_{1,n}p^{-1}$.

5.3 Case III: $n - k$ Odd

Let us start with some definitions.

Definition 5.8 Let $\mathcal{S}_{\mathrm{rad},0}(\mathbb{R}^n)$ be the subset of radial test functions whose moments of all orders vanish, $\phi \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^n)$ belongs to $\mathcal{S}_{\mathrm{rad},0}(\mathbb{R}^n)$ if and only

$$\int_{\mathbb{R}^n} \mathbf{x}^\alpha \phi(\mathbf{x})\mathrm{d}\mathbf{x} = 0, \quad \forall \alpha \in \mathbb{N}^n, \tag{5.14}$$

and let $\mathcal{S}'_{\mathrm{rad},0}(\mathbb{R}^n) = (\mathcal{S}_{\mathrm{rad},0}(\mathbb{R}^n))'$ be the corresponding dual space.

Notice that $\phi \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^n)$ belongs to $\mathcal{S}_{\mathrm{rad},0}(\mathbb{R}^n)$ if and only

$$\int_{\mathbb{R}^n} |\mathbf{x}|^{2j} \phi(\mathbf{x})\mathrm{d}\mathbf{x} = 0, \quad \forall j \in \mathbb{N}. \tag{5.15}$$

Definition 5.9 The space $\mathcal{R}_{n,\infty}[0,\infty)$ consists of those elements $\Phi \in \mathcal{R}_n[0,\infty)$ that satisfy

$$\mu_{2j}(\Phi) = \int_0^\infty r^{2j}\Phi(r)\mathrm{d}r = 0, \quad \forall j \in \mathbb{N}. \tag{5.16}$$

The spaces $\mathcal{R}_{n,\infty}[0,\infty)$ and $\mathcal{S}_{\mathrm{rad},0}(\mathbb{R}^n)$ as well as the dual spaces $\mathcal{R}'_{n,\infty}[0,\infty)$ and $\mathcal{S}'_{\mathrm{rad},0}(\mathbb{R}^n)$ are naturally isomorphic; we shall denote by $\mathfrak{I}^0_{n,1} : \mathcal{S}'_{\mathrm{rad},0}(\mathbb{R}^n) \to \mathcal{R}'_{n,\infty}[0,\infty)$ this isomorphism and its inverse by $\mathfrak{I}^0_{1,n} = (\mathfrak{I}^0_{n,1})^{-1}$. Furthermore, both the Fourier transform and its inverse are isomorphisms of $\mathcal{S}_{\mathrm{rad},0}(\mathbb{R}^n)$ to $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ and isomorphisms of the corresponding dual spaces.

Employing the operator $J^0_{n,k} : \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n \setminus \{\mathbf{0}\}) \to \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^k \setminus \{\mathbf{0}\})$ given in Definition 2.3 we can define the following operators.

Definition 5.10 The operator $\mathfrak{Z}^0_{n,k} : \mathcal{S}'_{\mathrm{rad},0}(\mathbb{R}^n) \to \mathcal{S}'_{\mathrm{rad},0}(\mathbb{R}^k)$ is defined as

$$\mathfrak{Z}^0_{n,k} = \mathcal{F} J^0_{n,k} \mathcal{F}^{-1}. \tag{5.17}$$

The operator $Z^0_{n,k} : \mathcal{R}'_{n,\infty}[0,\infty) \to \mathcal{R}'_{k,\infty}[0,\infty)$ is defined as

$$Z^0_{n,k} = \mathfrak{I}^0_{k,1}\mathcal{F} J^0_{n,k}\mathcal{F}^{-1}\mathfrak{I}^0_{1,n} = \mathfrak{I}^0_{k,1}\mathfrak{Z}^0_{n,k}\mathfrak{I}^0_{1,n}. \tag{5.18}$$

The operators $\mathfrak{Z}^0_{n,k}$ and $Z^0_{n,k}$ are distributional versions of $\mathfrak{Z}_{n,k}$ and $Z_{n,k}$ for *any* integers n and k.

Proposition 5.11 *The operator $\mathfrak{Z}_{n,k} : \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^n) \to \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^k)$ has a unique continuous extension as an operator from $\mathcal{S}'_{\mathrm{rad},0}(\mathbb{R}^n)$ to $\mathcal{S}'_{\mathrm{rad},0}(\mathbb{R}^k)$, namely $\mathfrak{Z}^0_{n,k}$. The operator $Z_{n,k} : \mathcal{S}_{\mathrm{even}}[0,\infty) \to \mathcal{S}_{\mathrm{even}}[0,\infty)$ has a unique extension as an operator from $\mathcal{R}'_{n,\infty}[0,\infty)$ to $\mathcal{R}'_{k,\infty}[0,\infty)$, namely $Z^0_{n,k}$.*

Let us denote by π the canonical projection from $\mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n)$ to $\mathcal{S}'_{\mathrm{rad},0}(\mathbb{R}^n)$. Then in case I, we have

$$\pi\overline{\mathfrak{Z}}_{n,k} = \mathfrak{Z}^0_{n,k}\pi, \tag{5.19}$$

while in case II,

$$\pi\widetilde{\mathfrak{Z}}_{n,k} = \mathfrak{Z}^0_{n,k}\pi, \tag{5.20}$$

for any generalized inverse operator $\widetilde{\mathfrak{Z}}_{n,k}$. Similarly, if we now denote by π the projection of $\mathcal{R}'_n[0,\infty)$ onto $\mathcal{R}'_{n,\infty}[0,\infty)$, then

$$\pi\overline{Z}_{n,k} = Z^0_{n,k}\pi, \tag{5.21}$$

in case I, and

$$\pi\widetilde{Z}_{n,k} = Z^0_{n,k}\pi, \tag{5.22}$$

for any of the operators $\widetilde{Z}_{n,k}$ in case II. On the other hand, Proposition 3.4 immediately yields the following result.

Proposition 5.12 *If $k-n$ is odd then there are no continuous operators $\mathfrak{Z} : \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n) \to \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^k)$ such that*

$$\pi\mathfrak{Z} = \mathfrak{Z}^0_{n,k}\pi, \tag{5.23}$$

where π is the canonical projection from $\mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n)$ to $\mathcal{S}'_{\mathrm{rad},0}(\mathbb{R}^n)$. There are no continuous operators $Z : \mathcal{R}'_n[0,\infty) \to \mathcal{R}'_k[0,\infty)$ such that

$$\pi Z = Z^0_{n,k}\pi, \tag{5.24}$$

where π is the projection of $\mathcal{R}'_n[0,\infty)$ onto $\mathcal{R}'_{n,\infty}[0,\infty)$.

Let us consider the case of $Z_{n+1,n}$, that is, $\Psi_n = Z_{n+1,n}(\Psi_{n+1})$ where

$$\Psi_n(r) = \frac{a}{\pi}\int_0^\infty \Psi_{n+1}\left(\sqrt{r^2+s^2}\right)\mathrm{d}s. \tag{5.25}$$

What Proposition 5.12 tell us is that we cannot extend in a *canonical, continuous way* the formula (5.25) if we want to replace Ψ_{n+1} by $G_{n+1} \in \mathcal{R}'_{n+1}[0,\infty)$ and Ψ_n by $G_n \in \mathcal{R}'_n[0,\infty)$. There are some subclasses of $\mathcal{R}'_{n+1}[0,\infty)$ for which the formula can be extended, yielding an element of $\mathcal{R}'_n[0,\infty)$: we need to require G_{n+1} to be not too big at infinity so that the integral makes sense, but we cannot do it for *all* distributions $G_{n+1} \in \mathcal{R}'_{n+1}[0,\infty)$. In particular, if G_{n+1} has compact support, then $Z_{n+1,n}(G_{n+1})$ can be obtained from the integral formula; more generally, if G_{n+1} is an ordinary function of r for $r > A$ for some A and $G_{n+1}(r) = O(r^{\alpha})$ for some $\alpha < -1$, then the integral would be convergent, yielding a canonical $Z_{n+1,n}(G_{n+1})$. Actually it is enough to ask that the order relation $G_{n+1}(r) = O(r^{\alpha})$, for some $\alpha < -1$, holds in the Cesàro sense.[6]

5.4 Illustrations

We shall now give several examples of the formulas. We shall take $a = 1$.

Example 5.13 Let us consider the distribution $f_n \in \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n)$ given by $f_n(\mathbf{x}) = e^{-t|\mathbf{x}|}$, and let $g_n = \widehat{f_n}$, $g_n(\mathbf{x}) = G_n(|\mathbf{x}|)$. We have that

$$G_1(r) = \int_{-\infty}^{\infty} e^{-t|x|+irx}\mathrm{d}r = \frac{2t}{t^2 + r^2}. \tag{5.26}$$

Thus

$$\begin{aligned} G_2(r) &= -\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\int_0^{\infty} G_1\left(\sqrt{r^2+s^2}\right)\mathrm{d}s \\ &= -\frac{4t}{r}\frac{\mathrm{d}}{\mathrm{d}r}\int_0^{\infty}\frac{\mathrm{d}s}{t^2+r^2+s^2} \\ &= 2t\pi\left(t^2+r^2\right)^{-3/2}. \end{aligned} \tag{5.27}$$

Applying (5.1) to G_1 we obtain G_n for n odd, and applying (5.1) to G_2 we obtain G_n for n even; the result is

$$G_n(r) = 2^n \pi^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right) t\left(t^2 + r^2\right)^{-(n+1)/2}, \tag{5.28}$$

as obtained by other methods [11, Chap. 6, Example 14].

Example 5.14 Take $f_n(\mathbf{x}) = |\mathbf{x}|^{\lambda}$, $g_n = \widehat{f_n}$ where λ is not an integer; since g_n is radial and homogeneous of degree $\lambda - n$, we obtain $g_n(\mathbf{x}) = G_n(|\mathbf{x}|) = C_\lambda^n |\mathbf{x}|^{-\lambda-n}$, for a certain constant C_λ^n. Employing the integral formula (5.25) for $\Re e\lambda > 2n$ we obtain

$$C_\lambda^n r^{-\lambda-n} = \frac{1}{\pi}\int_0^{\infty} C_\lambda^{n+1}\left(r^2+s^2\right)^{-(\lambda-n-1)/2}\mathrm{d}s, \tag{5.29}$$

or

$$\frac{C_\lambda^n}{C_\lambda^{n+1}} = \frac{1}{\pi}\int_0^{\infty}\left(1+v^2\right)^{-(\lambda-n-1)/2}\mathrm{d}v = \frac{\Gamma(\frac{\lambda+n}{2})}{2\sqrt{\pi}\,\Gamma(\frac{\lambda+n+1}{2})}, \tag{5.30}$$

[6] See [2, 7] for the theory of order relation in the Cesàro sense for distributions.

and since $C_\lambda^1 = 2^{\lambda+1}\sqrt{\pi}\Gamma(\frac{\lambda+1}{2})/\Gamma(\frac{-\lambda}{2})$, we obtain the well known result [11, Chap. 6, Eq. (64)]

$$C_\lambda^n = \frac{2^{\lambda+n}\pi^{n/2}\Gamma(\frac{\lambda+n}{2})}{\Gamma(\frac{-\lambda}{2})}, \tag{5.31}$$

initially for $\Re e\lambda > 2n$ and for all non-integral λ by analytic continuation.

Example 5.15 Take $f_n(\mathbf{x}) = \delta(|\mathbf{x}| - 1)$, $g_n = \widehat{f_n}$ so that [11] $g_n(\mathbf{x}) = G_n(|\mathbf{x}|) = (2\pi)^{n/2}J_{(n-2)/2}(|\mathbf{x}|)|\mathbf{x}|^{-(n-2)/2}$, where J_ν is the Bessel function of the first kind. Then replacing $n-2$ by m, our formulas yield the integral representation

$$J_{m/2}(r) = \frac{r^{m/2}}{(2\pi)^{q/2}}\omega_{q-1}\int_0^\infty \frac{J_{(m+q)/2}(\sqrt{r^2+s^2})s^{q-1}}{(r^2+s^2)^{(m+q)/4}}\mathrm{d}s. \tag{5.32}$$

Example 5.16 This example illustrates the problems with the distributional extension of $\mathfrak{Z}_{n,k}$ and $Z_{n,k}$ when $n-k$ is odd. Consider the distribution $G_1(r) = \alpha$, a constant; clearly one cannot compute $Z_{1,2}(G_1)$ because it is given in terms of a divergent integral. However one may proceed as follows. Let $F(r) = \alpha\delta(r) + \beta\delta'(r)$, a distribution of $\mathcal{S}'[0,\infty)$. Then we obtain corresponding radial distributions $f_n \in \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n)$ given by $f_1(x) = \alpha\delta(x)$, $f_2(\mathbf{x}) = -2\pi\beta\delta(\mathbf{x})$, $f_n(\mathbf{x}) = 0$ if $n \geq 3$. Thus if $g_n = \widehat{f_n}$, $g_n(\mathbf{x}) = G_n(|\mathbf{x}|)$, we obtain $g_1 = \alpha$, $g_2 = -2\pi\beta$, $g_n = 0$ for $n \geq 3$. Thus, since they are the Fourier transforms of the *same* radial function, in dimensions 1 and 2, one could think that $\mathfrak{Z}_{1,2}(\alpha) = -2\pi\beta$, and, similarly, that $Z_{1,2}(\alpha) = -2\pi\beta$; however, of course, this is absurd, since α and β are *arbitrary* constants.

A summary is in order. In case I, if $g_n \in \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n)$, corresponding to $G_n \in \mathcal{R}'_n[0,\infty)$, $g_n(\mathbf{x}) = G_n(|\mathbf{x}|)$, then there is a unique radial distribution $g_k \in \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^k)$, corresponding to $G_k \in \mathcal{R}'_k[0,\infty)$, such that g_n and g_k are the Fourier transform of the same radial distribution, in their corresponding dimensions; the associations $g_n \longmapsto g_k$ and $G_n \longmapsto G_k$ are given by the operators $\overline{\mathfrak{Z}}_{n,k} : \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^n) \to \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^k)$ and $\overline{Z}_{n,k} : \mathcal{R}'_n[0,\infty) \to \mathcal{R}'_k[0,\infty)$, respectively. In case II, for a given g_n, or G_n, there are many possible g_k and G_k, but they depend on a finite number of parameters, and for each set of such parameters we obtain a continuous operator, $\widetilde{\mathfrak{Z}}_{n,k}$ or $\widetilde{Z}_{n,k}$. In case III, for each g_n, or G_n, there are *so many* possible g_k and G_k, that it is impossible to have a continuous association $g_n \longmapsto g_k$ or $G_n \longmapsto G_k$. A useful analogy is the following: case I is like multiplying elements of $\mathcal{S}'[0,\infty)$ by x^q for some $q \in \mathbb{N}$, case II is like multiplying by x^{-q} for some $q \in \mathbb{N}$, $q \geq 1$, while case III is like multiplication with x^α for some $\alpha \in \mathbb{R} \setminus \mathbb{Z}$.

References

1. Bowen, J.M.: Delta function terms arising from classical point-source fields. Am. J. Phys. **62**, 511–515 (1994)
2. Campos Ferreira, J.: Introduction to the Theory of Distributions. Longman, London (1997)
3. Erdélyi, A., Köber, H.: Some remarks on Hankel transforms. Q. J. Math. Oxford Ser. **11**, 212–221 (1940)

4. Estrada, R.: The non-existence of regularization operators. J. Math. Anal. Appl. **286**, 1–10 (2003)
5. Estrada, R., Kanwal, R.P.: Regularization and distributional derivatives of $(x_1^2 + \cdots + x_p^2)^{n/2}$ in $\mathcal{D}'(\mathbb{R}^p)$. Proc. R. Soc. Lond. A **401**, 281–297 (1985)
6. Estrada, R., Kanwal, R.P.: Singular Integral Equations. Birkhäuser, Boston (2000)
7. Estrada, R., Kanwal, R.P.: A Distributional Approach to Asymptotics. Theory and Applications, 2nd edn. Birkhäuser, Boston (2002)
8. Farassat, F.: Introduction to generalized functions with applications in aerodynamics and aeroacoustics. NASA Technical Paper 3248, Hampton, VA: NASA Langley Research Center (1996). http://ntrs.nasa.gov
9. Grafakos, L., Teschl, G.: On Fourier transforms of radial functions and distributions. J. Fourier Anal. Appl. **19**, 167–179 (2013)
10. Horváth, J.: Topological Vector Spaces and Distributions, vol. I. Addison-Wesley, Reading (1966)
11. Kanwal, R.P.: Generalized Functions: Theory and Technique, 3rd edn. Birkhäuser, Boston (2004)
12. Trèves, F.: Topological Vector Spaces, Distributions, and Kernels. Academic Press, New York (1967)
13. Walton, J.R.: A distributional approach to dual integral equations of Titchmarsh type. SIAM J. Math. Anal. **6**, 628–643 (1975)
14. Zemanian, A.H.: Generalized Integral Transforms. Wiley, New York (1965)