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# Sharp Spectral Multipliers for a New Class of Grushin Type Operators

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**Abstract** We describe weighted restriction (Plancherel) type estimates and sharp Hebisch-Müller-Stein type spectral multiplier result for a new class of Grushin type operators. We also discuss the optimal exponent for Bochner-Riesz summability in this setting.

**Keywords** Grushin operators · Spectral multipliers · Bochner-Riesz analysis

**Mathematics Subject Classification** 35J70 · 35H20 · 35L05 · 58J35

## 1 Introduction

On the space $L^2(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ with the standard Lebesgue measure consider a class of Grushin type operators defined by the formula

$$L_\sigma = -\sum_{j=1}^{d_1} \partial_{x'_j}^2 - \left(\sum_{j=1}^{d_1} |x'_j|^\sigma\right) \sum_{k=1}^{d_2} \partial_{x''_k}^2 \tag{1}$$

where exponent $\sigma > 0$. In the case $\sigma = 2$, the spectral properties of these operators were studied by Martini and the second author in [29] where sharp spectral multiplier and optimal Bochner-Riesz summability results were obtained. The aim of this paper is to obtain analogous results for the class of Grushin operators corresponding to the

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exponent $\sigma = 1$. The general strategy of the proof of the sharp spectral multiplier result for $\sigma = 1$ is the same as one described in [29] for $\sigma = 2$. However, the proofs of the two most crucial estimates (Proposition 2.2 and Lemma 3.4 below) are new and significantly more difficult. The spectral decompositions of operators $L_1$ and $L_2$ have similar structure, which can be described in substantially explicit way for both operators. Nevertheless the asymptotic behaviour of "eigenvalues and eigenfunctions" is quite different and essentially new techniques are required to obtain crucial weighted estimates for spectral multipliers of operator $L_1$. We use results derived in [16] to obtain a description of the spectral decomposition of the operator $L_1$ necessary for the proof of Proposition 2.2 and Lemma 3.4.

The closure of operator $L_\sigma$, $\sigma > 0$ initially defined on $C_c^\infty(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ is a non-negative self-adjoint operator and it admits a spectral resolution $E_{L_\sigma}(\lambda)$ for all $\lambda \geq 0$, see e.g. [33]. By spectral theorem for every bounded Borel function $F : \mathbb{R} \to \mathbb{C}$, one can define the operator

$$F(L_\sigma) = \int_{\mathbb{R}} F(\lambda)\, dE_{L_\sigma}(\lambda) \tag{2}$$

which is bounded on $L^2(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$. This paper is devoted to spectral multipliers that is we investigate sufficient conditions on function $F$ under which the operator $F(L_1)$ extends to bounded operator acting on spaces $L^p(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ for some range of $p$. We also study closely related question of critical exponent $\kappa$ for which the Bochner-Riesz means $(1 - tL_1)_+^\kappa$ are bounded on $L^p(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ uniformly in $t \in [0, \infty)$. In the sequel we shall only discuss the Grushin operator $L_1$ which for simplicity we denote just by $L$.

The essential motivation and rationale for spectral multiplier results of the type, which we consider here come from the problem of convergence of eigenfunction expansion of differential operators. Arguably convergence of eigenfunction expansion is the most significant issue in harmonic analysis. Initially convergence of the Fourier series was the question, which defined the research area of Fourier and harmonic analysis two centuries ago. The problem can be easily formulated and generalised to the language of spectral theory and spectral resolution of self-adjoint operators. Verifying some form of convergence of spectral resolutions is a necessary step to validated use of Fourier series and more general eigenfunction expansions in the theory of Partial Differential Equations (PDE). Both pointwise and $L^p$ norm convergence is considered in this context, see for example [3, 14, 15] and [38, pp. 386–395]. The celebrated result of Fefferman, see [13], shows that, except of straightforward case $p = 2$, in most of the situations one cannot expect direct convergence of eigenfunction expansion in $L^p$ spaces and to obtain positive results one has to consider Bochner Riesz means or other spectral multipliers. The full description of convergence of Bochner-Riesz means in classical Fourier analysis became one of most intriguing open problem in mathematics, see [38, pp. 386–395]. An example of recent progress can be found in [3].

Convergence of Bochner-Riesz means in $L^p$ space is essentially equivalent with boundedness of these operators on the same space and we formulate our result this way, see Theorem 1.2 below. The critical order of Bochner-Riesz summability is usually expressed in terms of homogeneous (doubling) dimension of the considered ambient space which for the standard Laplace operator coincides with the Euclidean dimension of $\mathbb{R}^n$. In Fourier analysis this problem remains open only for $n > 2$ and for $p$ close to $2n/(n+1)$, see [3]. However, for more general differential operators the problem could be nontrivial on whole range of $1 \le p \le \infty$. In fact the results obtained by Thangavelu, Askey and Wainger shows that the Bochner-Riesz profile of the harmonic oscillator $-d_x^2 + x^2$ essentially differs from the standard Laplace operator and classical Fourier series or transform, see [2, 41]. In most of the cases full description of Bochner-Riesz profile of general differential operators is an open problem. Some results of that type were obtained by Christ, Seeger and Sogge see [7, 34, 36, 37].

Our study focuses on one especially intriguing and surprising direction in the theory of spectral multipliers devoted to investigation of sub-elliptic or degenerate operators. The main idea in this area is that the sharp results are expected to be determined by the topological dimension of underling ambient space rather than the homogeneous dimension of the space. This part of spectral multipliers theory was initiated by results obtained by Hebisch [21], Müller and Stein [32]. Other examples of papers devoted to sharp spectral multipliers for sub-elliptic or degenerate operators include [4, 9, 10, 22, 24, 29].

Almost all sharp results in the theory of Bochner-Riesz summability and spectral multipliers are based on relevant Stein-Tomas restriction (or discrete restriction) type estimates see for example [3, 5, 12, 18, 39, 40] or [38, pp. 386–395]. For the standard Laplace operator $(1,2)$ restriction estimates can be reformulated in terms of $L^2$ norm of the convolution kernel and follows from the Plancherel equality. Again for more general differential operators obtaining $(1,2)$ restriction estimates (Plancherel estimates in terminology of [12]) is far from being trivial, see [12]. It turns out that in the case of sub-elliptic and degenerate operators situation is more complex and the standard $(1,2)$ restriction estimates, like the ones considered in [27], do not lead to sharp Bochner-Riesz summability results. Instead one has to consider weighted version of such estimates. In our case the required version of $(1,2)$ weighted restriction estimates which is necessary to obtain sharp spectral multiplier result is described in Proposition 3.5 below.

The close relation between $L^p$ convergence of Bochner-Riesz means and restriction estimates provides another motivation for this research area. It is so because the restriction type estimates are more directly relevant to the theory of PDE via their relation to the resolvent of differential operator, Helmholtz equation and Strichartz estimates, see for example [5, 17, 18, 25].

A more general and possibly more natural problem than the Bochner-Riesz summability is to consider all possible functions of self-adjoint differential operators. This idea leads naturally to the subject of spectral multipliers. As we explain it above spectral multipliers theory investigate sufficient (differentiability) conditions on function $F$ under which the spectral multiplier $F(L)$, defined initially on $L^2$ extends to bounded operator acting on $L^p$ spaces for some range of $p$. Modelled on celebrated results of Hörmander and Mikhlin [19, 31] conditions on $F$ are most often expressed in the following form

$$\sup_{t>0} \|\eta \delta_t F\|_{W_p^s} < \infty. \tag{3}$$

Here $\eta \in C_c^\infty(0,\infty)$ is a non-trivial auxiliary function, $W_p^s$ is $L^p$ Sobolev space of order $s$ and for a function $F:\mathbb{R}\to\mathbb{C}$ we define $\delta_t F(x)=F(tx)$. For last forty or so years the theory of spectral multipliers has attracted a lot of attention. A huge amount of literature is devoted to the problem and we refer readers to [5, 10, 12, 32, 34] for detailed discussion and further relevant references. The optimal Bochner-Riesz summability problem which we describe above translates to the language of condition (3) in the following way. In the standard multiplier results the order of differentiability is required to be larger than half of the homogeneous dimension, see [35]. In the result, which we discuss here, this required order is reduced to half of the Euclidean dimension of the ambient space. In addition if one considers $p=\infty$ in (3) this corresponds to universal Bochner-Riesz summability and the result is not sharp. To obtain sharp results one has to show that it suffices to impose condition (3) with order $s$ larger than half of the Euclidean dimension and $p=2$. Such result is at least formally stronger than Bochner-Riesz result so we state it in such terms, see Theorem 1.1 below.

Example of a spectral multiplier results with $L^\infty$ type condition, which is relevant to our study, can be found in [1]. Then an example of $L^2$ type condition is described in [6] and [30]. All papers [1, 6, 30] are devoted to invariant operators acting on Lie groups. In the case of Heisenberg type groups the sharp version of these results were described in the mentioned above papers [21, 32], which in turn provide model and rationale for our study.

Our two main results, the sharp spectral multiplier and the corresponding optimal results for convergence of Bochner-Riesz means, are stated in Theorems 1.1 and 1.2 below. We set $D=\max\{d_1+d_2, 3d_2/2\}$ and as above by $W_2^s$ we denote $L^2$ Sobolev space that is $\|F\|_{W_2^s}=\|(I-d_x^2)^{s/2}F\|_2$.

**Theorem 1.1** *Suppose that function $F:\mathbb{R}\to\mathbb{C}$ satisfies*

$$\sup_{t>0}\|\eta\delta_t F\|_{W_2^s}<\infty$$

*for some $s>D/2$. Then the spectral multiplier operator $F(L)$ is of weak type $(1,1)$ and bounded on $L^p(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$ for all $p\in(1,\infty)$. In addition*

$$\big\|F(L)\big\|_{L^1\to L^{1,w}}\le C\sup_{t>0}\|\eta\delta_t F\|_{W_2^s}\quad \text{and}\quad \big\|F(L)\big\|_{L^p\to L^p}\le C_p\sup_{t>0}\|\eta\delta_t F\|_{W_2^s}.$$

The above result is sharp if $d_1\ge d_2/2$, see discussion in Sect. 5 below. A version of result essentially equivalent to Theorem 1.1 can be expressed in terms of Bochner-Riesz summability of the operator $L$. Our approach allows us to obtain the following result which is again optimal if $d_1\ge d_2/2$.

**Theorem 1.2** *Suppose that $\kappa>(D-1)/2$ and $p\in[1,\infty]$. Then the Bochner-Riesz means $(1-tL)_+^\kappa$ are bounded on $L^p(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$ uniformly in $t\in[0,\infty)$.*

Proofs of Theorems 1.1 and 1.2 are concluded in Sect. 4. Similarly as in [29] the key point of proving Theorems 1.1 and 1.2 is to obtain "weighted Plancherel estimate" for spectral multipliers of the considered Grushin type operators. A proof of such estimates is described in Sect. 3 and constitutes a main original contribution of this paper to the discussed research area. A part of the proofs of Theorems 1.1 and 1.2 described in Sect. 4 below is essentially the same as in [29]. We repeat the short argument here for the sake of completeness. To make it easier to compare the results obtained in [29] and in this paper we try to use the same notation as in [29] whenever it is possible.

## 2 Notation and Preliminaries

A more general class of Grushin type operators which includes operators $L_\sigma$ for $\sigma > 0$ defined above was studied in [33]. In what follows we will need the basic results concerning the Riemannian distance corresponding to Grushin type operators and the standard Gaussian bounds for the corresponding heat kernels, which were obtained in [33]. Recall that the Riemannian distance corresponding to the operator $L_\sigma$ (which is often also called the control distance) can be defined by

$$\rho_\sigma(x, y) = \sup_{\psi \in D} \big(\psi(x) - \psi(y)\big)$$

for all $x, y \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ where

$$D = \left\{ \psi \in W^{1,\infty}\big(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}\big) : \left( \sum_{j=1}^{d_1} |\partial_{x'_j} \psi|^2 + \left( \sum_{j=1}^{d_1} |x'_j|^\sigma \right) \sum_{k=1}^{d_2} |\partial_{x''_k} \psi|^2 \right) \le 1 \right\}.$$

There is a range of the equivalent definitions of the Riemannian distances associated to the operators $L_\sigma$. An important example is a "shortest path" definition originated in Riemannian geometry. We refer the reader to [23] for a survey and comparison of various possibilities of the distances corresponding to subelliptic operators. The relevance of the distance $\rho_\sigma$ for the operator $L_\sigma$ is explained by the fact that this is *optimal* distance for which the finite speed of propagation of the corresponding wave equation or Davies-Gaffney and Gaussian estimates for the heat equation hold. See [33, Proposition 4.1] for more detailed discussion and further references.

In the sequel we shall need the following estimates for the distance $\rho = \rho_1$ established also in [33].

**Proposition 2.1** *Let $\rho$ be the Riemannian distance corresponding to the Grushin operator $L$ and let $B(x, r)$ be the ball with centre at $x$ and radius $r$. Then*

$$\rho(x, y) \sim |x' - y'| + \begin{cases} \frac{|x''-y''|}{(|x'|+|y'|)^{1/2}} & \text{if } |x'' - y''| \le (|x'| + |y'|)^{3/2}, \\ |x'' - y''|^{2/3} & \text{if } |x'' - y''| \ge (|x'| + |y'|)^{3/2}. \end{cases} \tag{4}$$

*Moreover the volume of $B(x, r)$ satisfies following estimates*

$$|B(x, r)| \sim r^{d_1+d_2} \max\{r, |x'|\}^{d_2/2}, \tag{5}$$

*and in particular, for all $\lambda \ge 0$,*

$$|B(x, \lambda r)| \le C(1 + \lambda)^Q |B(x, r)| \tag{6}$$

*where* $Q = d_1 + \frac{3d_2}{2}$ *is the homogeneous dimension of the considered metric space. Next, there exist constants* $b, C > 0$ *such that, for all* $t > 0$, *the integral kernel* $p_t$ *of the operator* $\exp(-tL)$ *satisfies the following Gaussian bounds*

$$\left|p_t(x,y)\right| \le C\left|B\left(y, t^{1/2}\right)\right|^{-1} e^{-b\rho(x,y)^2/t} \tag{7}$$

*for all* $x, y \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$.

*Proof* For the proof, we refer readers to [33, Proposition 5.1 and Corollary 6.6]. □

Next, let $\mathcal{F} : L^2(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) \to L^2(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ be the partial Fourier transform in variables $x''$ defined by

$$\mathcal{F}\phi\left(x', \xi\right) = (2\pi)^{-d_2/2} \int_{\mathbb{R}^{d_2}} \phi\left(x', x''\right) e^{-i\xi\cdot x''}\, dx''.$$

Then

$$\mathcal{F}L\phi\left(x', \xi\right) = \widetilde{L}_\xi \mathcal{F}\phi\left(x', \xi\right)$$

where $\widetilde{L}_\xi$ are Schrödinger type operators defined by

$$\widetilde{L}_\xi = -\Delta_{d_1} + \left(\sum_{j=1}^{d_1} \left|x'_j\right|\right)|\xi|^2$$

acting on $L^2(\mathbb{R}^{d_1})$ where $\xi \in \mathbb{R}^{d_2}$. In what fallows we will need the following estimates for the operator $\widetilde{L}_\xi$, compare [9, 10, 21] and [29].

**Proposition 2.2** *For all* $\gamma \in [0, \infty)$ *and* $f \in L^2(\mathbb{R}^{d_1})$,

$$\left\| \left(\sum_{j=1}^{d_1} \left|x'_j\right|\right)^\gamma |\xi|^{2\gamma} f \right\|_2 \le C_\gamma \left\|\widetilde{L}_\xi^\gamma f\right\|_2. \tag{8}$$

*Proof* Set $\widetilde{L} = -\Delta_{d_1} + \sum_{j=1}^{d_1} |x'_j|$ and next define operator $L_{x'_i}$ by the following formula

$$L_{x'_i} = -\partial_{x'_i}^2 + \left|x'_i\right|.$$

By Proposition 3.4 of [16]

$$\left\|\left|x'_i\right|^k f\right\|_2 \le C'_k \left\|L_{x'_i}^k f\right\|_2$$

for all positive natural numbers $k \in \mathbb{N}$. Hence

$$\left\| \left(\sum_{j=1}^{d_1} \left|x'_j\right|\right)^k f \right\|_2^2 \le C \sum_{j=1}^{d_1} \left\|\left|x'_j\right|^k f\right\|_2^2 \le C_k \sum_{j=1}^{d_1} \left\|L_{x'_j}^k f\right\|_2^2.$$

Note that all $L_{x'_i}$ are non-negative self-adjoint operators and commute strongly, that is, their resolvents commute. Therefore for all $\ell_i \in \mathbb{Z}_+$, operators $\prod_{i=1}^n L_{x'_i}^{\ell_i}$ are self-adjoint and non-negative. Hence

$$\sum_{j=1}^{d_1} L_{x'_j}^{2k} \le \left(\sum_{j=1}^{d_1} L_{x'_j}\right)^{2k}$$

for all $k \in \mathbb{N}$ and

$$\begin{aligned}\left\| \left(\sum_{j=1}^{d_1} |x'_j|\right)^k f \right\|_2^2 &\le C_k \sum_{j=1}^{d_1} \| L_{x'_j}^k f \|_2^2 = C_k \left\langle \sum_{j=1}^{d_1} L_{x'_j}^{2k} f, f \right\rangle \\ &\le C_k \left\langle \left(\sum_{j=1}^{d_1} L_{x'_j}\right)^{2k} f, f \right\rangle = C_k \| \widetilde{L}^k f \|_2^2.\end{aligned}$$

Next, for a function $f \in C_c^\infty(\mathbb{R}^{d_1})$ we define function $\delta_t f$ by the formula $\delta_t f(x) = f(tx)$. Note that if $t = |\xi|^{-2/3}$ then

$$\widetilde{L}_\xi^k = \left(-\Delta_{d_1} + \left(\sum_{j=1}^{d_1} |x'_j|\right) t^{-3}\right)^k = t^{-2k} \delta_{t^{-1}} \widetilde{L}^k \delta_t.$$

Hence

$$\begin{aligned}\| \widetilde{L}_\xi^k f \|_2 &= \| t^{-2k} \delta_{t^{-1}} \widetilde{L}^k \delta_t f \|_2 \\ &= t^{-2k} t^{d_1/2} \| \widetilde{L}^k \delta_t f \|_2 \\ &\ge C''_k t^{-2k} t^{d_1/2} \left\| \left(\sum_{j=1}^{d_1} |x'_j|\right)^k \delta_t f \right\|_2 \\ &= C''_k |\xi|^{2k} \left\| \left(\sum_{j=1}^{d_1} |x'_j|\right)^k f \right\|_2.\end{aligned}$$

This proves Proposition 2.2 for all $\gamma = k \in \mathbb{N}$. Now in virtue of Löwner-Heinz inequality (see, e.g., [8, Sect. I.5]) we can extend these estimates to all $\gamma \in [0, \infty)$. □

## 3 Crucial Estimates

To be able to obtain a required description of spectral decomposition of the operators $\widetilde{L}_\xi$ we need the following properties of spectral decomposition of operator $A = -\frac{d^2}{dx^2} + |x|$ acting on $L^2(\mathbb{R})$ which are essentially based on results from [16].

**Proposition 3.1** *Let $\lambda_n$ and $h_n$ be the $n$-th eigenvalue and normalised eigenfunction of the operator $A = -\frac{d^2}{dx^2} + |x|$. Then its spectral decomposition satisfies following properties*:

(i) *The operator $A$ has only a pointwise spectrum and its eigenvalues belong to $(1, \infty)$. In particular the first eigenvalue is larger than 1.*
(ii) *Every eigenvalue of $A$ is simple and the only point of accumulation of the eigenvalue sequence is $\infty$. Thus $\{h_n\}_{n\in\mathbb{N}}$ is a complete orthonormal system of $L^2(\mathbb{R})$.*
(iii) *The eigenvalues $\lambda_n$ satisfy the following estimates*:

$$C_1\left(\frac{3\pi}{4}n\right)^{2/3} \le \lambda_n \le C_2\left(\frac{3\pi}{4}n\right)^{2/3}, \tag{9}$$

$$\frac{\pi}{2}\lambda_{n+1}^{-1/2} \le \lambda_{n+1} - \lambda_n \le \frac{\pi}{2}\lambda_n^{-1/2}, \tag{10}$$

*where $C_2 \ge C_1 > 0$ are constants.*
(iv) *For the eigenfunction $h_n$ corresponding to the eigenvalue $\lambda_n$,*

$$h_n(u) \le \begin{cases} C\lambda_n^{-\frac{1}{4}}(1 + ||u| - \lambda_n|)^{-\frac{1}{4}}, & u \in \mathbb{R}, \\ C\exp(-c|u|^{\frac{3}{2}}), & u \ge 2\lambda_n. \end{cases} \tag{11}$$

*Proof* (i), (ii) and (iii) are just reformulation of Proposition 2.1, Corollary 2.2, Facts 2.3, 2.7 and 2.8 of [16]. (iv) is an easy consequence of Theorem 2.6 of [16] and estimates for Airy function (see for example [20], pp. 213–215). □

Now we are able to describe spectral resolutions of Grushin operator $L = L_1$ and operators $\widetilde{L}_\xi$ defined in Sect. 2. It is interesting to compare it with spectral decomposition of the operator $L_2$ obtained in [29]. From a point of view of obtaining weighted Plancherel estimates required for the proof of our multiplier results the spectral decompositions of $L_1$ and $L_2$ are significantly different even though they share many common features. We also have to investigate integral kernels of spectral multipliers of $L$ and $\widetilde{L}_\xi$. For $T = F(L)$ or $T = F(\widetilde{L}_\xi)$, by $K_T$ we denote the integral kernel of the operator $T$, defined by the identity

$$Tf(x) = \int_X K_T(x, y) f(y)\, dy$$

where $X = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ for $L$ and $X = \mathbb{R}^{d_1}$ for $\widetilde{L}_\xi$.

In terms of the eigenvalues and eigenfunctions of the operator $A = -\frac{d^2}{dx^2} + |x|$, one can obtain explicit formula for the integral kernel of the operator $F(L)$, compare also [29, Proposition 5]. Let $\lambda_n$ and $h_n$ be the $n$-th eigenvalue and eigenfunction of the operator $-\frac{d^2}{dx^2} + |x|$ on $L^2(\mathbb{R})$. We know that $\{h_n\}_{n\in\mathbb{N}}$ is a complete orthonormal system of $L^2(\mathbb{R})$. For all positive integers $d_1$, all $\mathbf{n} \in \mathbb{N}^{d_1}$ and all $\xi \in \mathbb{R}^{d_2}$, we define function $\tilde{h}_{d_1,\mathbf{n}} \colon \mathbb{R}^{d_1} \to \mathbb{R}$ by the formula

$$\tilde{h}_{d_1,\mathbf{n}}(x', \xi) = |\xi|^{d_1/3} h_{n_1}\left(|\xi|^{2/3} x_1'\right) \cdots h_{n_{d_1}}\left(|\xi|^{2/3} x_{d_1}'\right).$$

We are now able to describe the kernel $K_{F(L)}$.

**Proposition 3.2** *For all bounded compactly supported Borel functions* $F:\mathbb{R}\to\mathbb{C}$

$$K_{F(L)}(x,y)=(2\pi)^{-d_2}\int_{\mathbb{R}^{d_2}}K_{F(\widetilde{L}_\xi)}\big(x',y'\big)e^{i\xi(x''-y'')}\,d\xi$$

$$=(2\pi)^{-d_2}\int_{\mathbb{R}^{d_2}}\sum_{\mathbf{n}\in\mathbb{N}^{d_1}}F\Bigg(\sum_{i=1}^{d_1}|\xi|^{\frac{4}{3}}\lambda_{n_i}\Bigg)\tilde{h}_{d_1,\mathbf{n}}\big(y',\xi\big)$$

$$\times\,\tilde{h}_{d_1,\mathbf{n}}\big(x',\xi\big)e^{i\xi\cdot(x''-y'')}\,d\xi$$

*for almost all* $x=(x',x'')$, $y=(y',y'')\in\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$.

*Proof* We noticed in Sect. 2 that $\mathcal{F}L\phi(x',\xi)=\widetilde{L}_\xi\mathcal{F}\phi(x',\xi)$ where $\mathcal{F}$ is the partial Fourier transform in variables $x''$. Next note that for all $\xi\neq 0$

$$\widetilde{L}_\xi\tilde{h}_{d_1,\mathbf{n}}\big(x',\xi\big)=\Bigg(\sum_{j=1}^{d_1}|\xi|^{\frac{4}{3}}\lambda_{n_j}\Bigg)\tilde{h}_{d_1,\mathbf{n}}\big(x',\xi\big).$$

Moreover by Proposition 3.1(ii), the set $\{\tilde{h}_{d_1,\mathbf{n}}(x',\xi)\}_{\mathbf{n}\in\mathbb{N}^{d_1}}$ is a complete orthonormal system of $L^2(\mathbb{R}^{d_1})$. Hence if $\mathcal{G}:L^2(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})\to L^2(\mathbb{N}^{d_1}\times\mathbb{R}^{d_2})$ is the isometry defined by

$$\mathcal{G}\psi(\mathbf{n},\xi)=\int_{\mathbb{R}^{d_1}}\psi\big(x',\xi\big)\tilde{h}_{d_1,\mathbf{n}}\big(x',\xi\big)\,dx',$$

then

$$\mathcal{G}\mathcal{F}L\phi(\mathbf{n},\xi)=\sum_{j=1}^{d_1}|\xi|^{\frac{4}{3}}\lambda_{n_j}\mathcal{G}\mathcal{F}\phi(\mathbf{n},\xi)$$

and

$$\mathcal{G}\mathcal{F}F(L)\phi(\mathbf{n},\xi)=F\Bigg(\sum_{j=1}^{d_1}|\xi|^{\frac{4}{3}}\lambda_{n_j}\Bigg)\mathcal{G}\mathcal{F}\phi(\mathbf{n},\xi).\tag{12}$$

However, the inverse of $\mathcal{G}$ is given by

$$\mathcal{G}^{-1}\varphi\big(x',\xi\big)=\sum_{\mathbf{n}\in\mathbb{N}^{d_1}}\varphi(\mathbf{n},\xi)\tilde{h}_{d_1,\mathbf{n}}\big(x',\xi\big)$$

and inverse of $\mathcal{F}$ can be expressed in terms of partial inverse Fourier transform in $x''$. Applying $\mathcal{G}^{-1}$ and $\mathcal{F}^{-1}$ to both sides of equality (12) shows Proposition 3.2. □

Next, for all positive integers $d_1$ and all $\mathbf{n}\in\mathbb{N}^{d_1}$ we define function $H_{d_1,\mathbf{n}}:\mathbb{R}^{d_1}\to\mathbb{R}$ by the formula

$$H_{d_1,\mathbf{n}}\big(x'\big)=h^2_{n_1}\big(x'_1\big)\cdots h^2_{n_{d_1}}\big(x'_{d_1}\big).$$

As a simple consequence of Proposition 3.2 we obtain following estimates.

**Proposition 3.3** *For all $\gamma \geq 0$ and for every compactly supported bounded Borel function $F : \mathbb{R} \to \mathbb{C}$,*

$$\left\| \left( \sum_{i=1}^{d_1} |x_i'| \right)^{\gamma} K_{F(L)}(\cdot, y) \right\|_2^2 \leq C_\gamma \int_0^\infty |F(\theta)|^2 \sum_{\mathbf{n} \in \mathbb{N}^{d_1}} \frac{\theta^{Q/2-\gamma}}{N_{\mathbf{n}}^{Q/2-3\gamma}} H_{d_1,\mathbf{n}} \left( \frac{\theta^{1/2} y'}{N_{\mathbf{n}}^{1/2}} \right) \frac{d\theta}{\theta}$$

*for almost all $y = (y', y'') \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ where $N_{\mathbf{n}} = \sum_{i=1}^{d_1} \lambda_{n_i}$ and $\lambda_{n_i}$ is the eigenvalue corresponding to eigenfunction $h_{n_i}$.*

*Proof* By Propositions 2.2 and 3.2

$$\left\| \left( \sum_{i=1}^{d_1} |x_i'| \right)^{\gamma} K_{F(L)}(\cdot, y) \right\|_2^2 = \int_{\mathbb{R}^{d_2}} \left\| \left( \sum_{i=1}^{d_1} |x_i'| \right)^{\gamma} K_{F(\widetilde{L}_\xi)}(x', y') \right\|_{L^2(\mathbb{R}^{d_1})}^2 d\xi$$
$$\leq \int_{\mathbb{R}^{d_2}} |\xi|^{-4\gamma} \left\| \widetilde{L}_\xi^\gamma K_{F(\widetilde{L}_\xi)}(x', y') \right\|_{L^2(\mathbb{R}^{d_1})}^2 d\xi. \quad (13)$$

Next note that for all $\gamma \geq 0$ and $y' \in \mathbb{R}^{d_1}$

$$\widetilde{L}_\xi^\gamma \left( K_{F(\widetilde{L}_\xi)}(\cdot, y') \right) = K_{\widetilde{L}_\xi^\gamma F(\widetilde{L}_\xi)}(\cdot, y').$$

Hence

$$\left\| \widetilde{L}_\xi^\gamma K_{F(\widetilde{L}_\xi)}(x', y') \right\|_{L^2(\mathbb{R}^{d_1})}^2 \leq \left\| K_{\widetilde{L}_\xi^\gamma F(\widetilde{L}_\xi)}(x', y') \right\|_{L^2(\mathbb{R}^{d_1})}^2$$
$$\leq \sum_{\mathbf{n} \in \mathbb{N}^{d_1}} \left| \left( \sum_{i=1}^{d_1} |\xi|^{\frac{4}{3}} \lambda_{n_i} \right)^{\gamma} F \left( \sum_{i=1}^{d_1} |\xi|^{\frac{4}{3}} \lambda_{n_i} \right) \right|^2$$
$$\times \left| \tilde{h}_{d_1,\mathbf{n}}(y', \xi) \right|^2$$
$$\leq C |\xi|^{\frac{2d_1}{3} + \frac{8\gamma}{3}} \sum_{\mathbf{n} \in \mathbb{N}^{d_1}} N_{\mathbf{n}}^{2\gamma} \left| F\left( |\xi|^{\frac{4}{3}} N_{\mathbf{n}} \right) \right|^2 H_{d_1,\mathbf{n}}\left( |\xi|^{\frac{2}{3}} y' \right).$$
$$(14)$$

Now substituting (14) to (13) and simple change of variables proves Proposition 3.3 □

The following lemma is a version of Lemma 9 of [29]. However the proof is more complex and requires a new approach especially when $d_1 \geq 2$. It is the most essential part of the proof of our main spectral multiplier results.

**Lemma 3.4** *For all* $\varepsilon > 0$ *there exists a constant* $C > 0$ *which does not depend on* $x' \in \mathbb{R}^{d_1}$ *such that*

$$\sum_{\mathbf{n}\in\mathbb{N}^{d_1}} \frac{\max\{1,|x'|\}^{2\varepsilon}}{N_{\mathbf{n}}^{d_1/2+3\varepsilon}} H_{d_1,\mathbf{n}}\left(\frac{x'}{N_{\mathbf{n}}^{1/2}}\right) < C < \infty \tag{15}$$

*where* $N_{\mathbf{n}} = \sum_{i=1}^{d_1} \lambda_{n_i}$ *and* $\lambda_{n_i}$ *is the eigenvalue corresponding to eigenfunction* $h_{n_i}$.

*Proof* We split the sum into two parts,

$$\begin{aligned}\sum_{\mathbf{n}\in\mathbb{N}^{d_1}} &\frac{\max\{1,|x'|\}^{2\varepsilon}}{N_{\mathbf{n}}^{d_1/2+3\varepsilon}} H_{d_1,\mathbf{n}}\left(\frac{x'}{N_{\mathbf{n}}^{1/2}}\right)\\ &\le \left(\sum_{N_{\mathbf{n}}^{3/2}\le|x'|/(2d_1)} + \sum_{N_{\mathbf{n}}^{3/2}>|x'|/(2d_1)}\right)\frac{\max\{1,|x'|\}^{2\varepsilon}}{N_{\mathbf{n}}^{d_1/2+3\varepsilon}} H_{d_1,\mathbf{n}}\left(\frac{x'}{N_{\mathbf{n}}^{1/2}}\right).\end{aligned} \tag{16}$$

*Part 1:* $N_{\mathbf{n}}^{3/2} \le |x'|/(2d_1)$. By Proposition 3.1 $\lambda_{n_i} \ge 1$ so $N_{\mathbf{n}} > 1$. Hence this part is empty unless $|x'| > 1$. Note that

$$\frac{|x'|_\infty}{N_{\mathbf{n}}^{1/2}} \ge \frac{|x'|}{d_1 N_{\mathbf{n}}^{1/2}} \ge 2N_{\mathbf{n}}$$

where $|x'|_\infty = \max\{x'_1,\dots,x'_{d_1}\}$. By (11) for every natural number $N \le |x'|/(2d_1)$

$$\sum_{(N-1)^{2/3}<N_{\mathbf{n}}\le N^{2/3}} H_{d_1,\mathbf{n}}\left(\frac{x'}{N_{\mathbf{n}}^{1/2}}\right) \le C\exp\left(-c|x'|_\infty^{\frac{3}{2}}/N^{\frac{1}{2}}\right) \le C\exp\left(-c|x'|^{\frac{3}{2}}/N^{\frac{1}{2}}\right).$$

Thus

$$\begin{aligned}\sum_{N_{\mathbf{n}}^{3/2}\le|x'|/(2d_1)} &\frac{\max\{1,|x'|\}^{2\varepsilon}}{N_{\mathbf{n}}^{d_1/2+3\varepsilon}} H_{d_1,\mathbf{n}}\left(\frac{x'}{N_{\mathbf{n}}^{1/2}}\right)\\ &\le \sum_{N\le|x'|/(2d_1)} \sum_{(N-1)^{2/3}<N_{\mathbf{n}}\le N^{2/3}} \frac{\max\{1,|x'|\}^{2\varepsilon}}{N_{\mathbf{n}}^{d_1/2+3\varepsilon}} H_{d_1,\mathbf{n}}\left(\frac{x'}{N_{\mathbf{n}}^{1/2}}\right)\\ &\le C\sum_{N\le|x'|/2} |x'|^{2\varepsilon} N^{-d_1/3-2\varepsilon}\exp\left(-c|x'|^{\frac{3}{2}}/N^{\frac{1}{2}}\right)\\ &\le C\sum_{N\in\mathbb{N}} \sup_{t\ge 2N} t^{4\varepsilon/3}\exp(-ct) \le C.\end{aligned} \tag{17}$$

*Part 2:* $N_{\mathbf{n}}^{3/2} > |x'|/(2d_1)$. Again by (11)

$$H_{d_1,\mathbf{n}}\left(\frac{x'}{N_{\mathbf{n}}^{1/2}}\right)=\prod_{i=1}^{d_1}h_{n_i}^2\left(\frac{x_i'}{N_{\mathbf{n}}^{1/2}}\right)$$
$$\le C\prod_{i=1}^{d_1}\lambda_{n_i}^{-\frac{1}{2}}\left(1+\left|\frac{|x_i'|}{N_{\mathbf{n}}^{1/2}}-\lambda_{n_i}\right|\right)^{-\frac{1}{2}}.$$

Hence

$$\sum_{N_{\mathbf{n}}^{3/2}>|x'|/(2d_1)}\frac{\max\{1,|x'|\}^{2\varepsilon}}{N_{\mathbf{n}}^{d_1/2+3\varepsilon}}H_{d_1,\mathbf{n}}\left(\frac{x'}{N_{\mathbf{n}}^{1/2}}\right)$$
$$\le C\sum_{N_{\mathbf{n}}^{3/2}>|x'|/(2d_1)}\frac{\max\{1,|x'|\}^{2\varepsilon}}{N_{\mathbf{n}}^{d_1/2+3\varepsilon}}\prod_{i=1}^{d_1}\lambda_{n_i}^{-\frac{1}{2}}\left(1+\left|\frac{|x_i'|}{N_{\mathbf{n}}^{1/2}}-\lambda_{n_i}\right|\right)^{-\frac{1}{2}}. \tag{18}$$

Next, define function $g\colon [1,\infty)^{d_1}\to\mathbb{R}_+$ by the formula

$$g(\mu)=g(\mu_1,\dots,\mu_{d_1})=\frac{\max\{1,|x'|\}^{2\varepsilon}}{N_\mu^{d_1/2+3\varepsilon}}\prod_{i=1}^{d_1}\mu_i^{-\frac{1}{3}}\left(1+\left|\frac{|x_i'|}{N_\mu^{1/2}}-\mu_i^{2/3}\right|\right)^{-\frac{1}{2}}$$

where $N_\mu=\sum_{i=1}^{d_1}\mu_i^{2/3}$. Note that $g(\mu_1,\dots,\mu_{d_1})>0$ and there exists a constant $C>0$ such that

$$\left|\nabla g(\mu_1,\dots,\mu_{d_1})\right|\le Cg(\mu_1,\dots,\mu_{d_1})$$

when $\mu=(\mu_1,\dots\mu_{d_1})\in[1,\infty)^{d_1}$ and $N_\mu=\sum_{i=1}^{d_1}\mu_i^{2/3}\ge(|x'|/(2d_1))^{2/3}$. By the above estimate for the gradient of $g$

$$e^{-C|\mu-\bar{\mu}|}\le\left|\frac{g(\mu)}{g(\bar{\mu})}\right|\le e^{C|\mu-\bar{\mu}|}$$

for all $\mu,\bar{\mu}$ in the region described above. Hence

$$g(\mu_1,\dots,\mu_{d_1})\le C\int_{\prod_{i=1}^{d_1}[\mu_i,\mu_i+1]}g(\xi_1,\dots,\xi_{d_1})d\xi_1\dots d\xi_{d_1}. \tag{19}$$

Set $\mu_{n_i}=\lambda_{n_i}^{3/2}$. By (18),

$$\sum_{N_{\mathbf{n}}^{3/2}>|x'|/(2d_1)}\frac{\max\{1,|x'|\}^{2\varepsilon}}{N_{\mathbf{n}}^{d_1/2+3\varepsilon}}H_{d_1,\mathbf{n}}\left(\frac{x'}{N_{\mathbf{n}}^{1/2}}\right)\le\sum_{N_{\mathbf{n}}^{3/2}>|x'|/(2d_1)}g(\mu_{n_1},\dots,\mu_{n_{d_1}}). \tag{20}$$

However , by (10) and mean value theorem for each $1\le i\le d_1$,

$$\mu_{n_i}-\mu_{n_i-1}=\lambda_{n_i}^{3/2}-\lambda_{n_i-1}^{3/2}\ge\frac{3\pi}{4}\lambda_{n_i}^{-1/2}\lambda_{n_i-1}^{1/2}$$

$$\geq \frac{3\pi}{4}\left(\frac{\lambda_{n_i-1}^{3/2}}{\frac{\pi}{2}+\lambda_{n_i-1}^{3/2}}\right)^{1/2}$$
$$\geq \frac{3\pi}{8} > 1 \tag{21}$$

which means that for all $\mathbf{n} \in \mathbb{N}^{d_1}$, cubes $\prod_{i=1}^{d_1}[\mu_{n_i}, \mu_{n_i}+1]$ are mutually disjoint. Note again that by Proposition 3.1 $\lambda_{n_i} \geq 1$ so $N_\mathbf{n} > 1$. Hence by (19), (20) and (21)

$$\sum_{N_\mathbf{n}^{3/2} > |x'|/(2d_1)} \frac{\max\{1,|x'|\}^{2\varepsilon}}{N_\mathbf{n}^{d_1/2+3\varepsilon}} H_{d_1,\mathbf{n}}\left(\frac{x'}{N_\mathbf{n}^{1/2}}\right)$$
$$\leq C \int_{N_\mu > \max\{(|x'|/(2d_1))^{2/3},1\}} g(\mu_1,\ldots,\mu_{d_1})\, d\mu_1\ldots d\mu_{d_1}.$$

Using the changes of variables $\mu_i = \xi_i^{3/2}$ we get

$$\sum_{N_\mathbf{n}^{3/2} > |x'|/(2d_1)} \frac{\max\{1,|x'|\}^{2\varepsilon}}{N_\mathbf{n}^{d_1/2+3\varepsilon}} H_{d_1,\mathbf{n}}\left(\frac{x'}{N_\mathbf{n}^{1/2}}\right)$$
$$\leq C \int_{N_\xi > \max\{(|x'|/(2d_1))^{2/3},1\}} \left[\frac{\max\{1,|x'|\}^{2\varepsilon}}{N_\xi^{d_1/2+3\varepsilon}} N_\xi^{\frac{d_1}{4}} \prod_{i=1}^{d_1} \xi_i^{-\frac{1}{2}} \left||x_i'| - \xi_i N_\xi^{\frac{1}{2}}\right|^{-\frac{1}{2}}\right]$$
$$\times\, d\xi_1^{\frac{3}{2}}\ldots d\xi_{d_1}^{\frac{3}{2}}$$
$$\leq C \int_{N_\xi > \max\{(|x'|/(2d_1))^{2/3},1\}} \left[\frac{\max\{1,|x'|\}^{2\varepsilon}}{N_\xi^{d_1/4+3\varepsilon}} \prod_{i=1}^{d_1} \left||x_i'| - \xi_i N_\xi^{\frac{1}{2}}\right|^{-\frac{1}{2}}\right]$$
$$\times\, d\xi_1\ldots d\xi_{d_1} = I \tag{22}$$

where $N_\xi = \sum_{i=1}^{d_1} \xi_i$. To estimate this integral we use the following decomposition

$$\left\{\xi : N_\xi \geq \max\left\{\left(\left|x'\right|/(2d_1)\right)^{2/3}, 1\right\}\right\}$$
$$= \bigcup_{j=1}^{d_1} E_j = \bigcup_{j=1}^{d_1} \left\{\xi : N_\xi \geq \max\left\{\left(\left|x'\right|/(2d_1)\right)^{2/3}, 1\right\}, N_\xi/d_1 \leq \xi_j \leq N_\xi\right\}.$$

Now on each of set $E_j$ we introduce new coordinates

$$\nu_1 = \xi_1, \ldots, \nu_{j-1} = \xi_{j-1}, \quad \nu_j = N_\xi, \quad \nu_{j+1} = \xi_{j+1}, \ldots, \nu_{d_1} = \xi_{d_1}.$$

Then

$$I \leq C \sum_{j=1}^{d_1} \int_{\max\{(|x'|/(2d_1))^{2/3},1\}}^{\infty} \frac{\max\{1,|x'|\}^{2\varepsilon}}{\nu_j^{d_1/4+3\varepsilon}}$$

$$\times \int_{S_j} \prod_{i \neq j} \big||x_i'| - \nu_i \nu_j^{\frac{1}{2}}\big|^{-\frac{1}{2}} \big||x_j'| - \bar{\nu}_j \nu_j^{\frac{1}{2}}\big|^{-\frac{1}{2}} d\nu_1 \dots d\nu_{d_1} \tag{23}$$

where $\bar{\nu}_j = \nu_j - \sum_{i \neq j} \nu_i$ and $S_j = \{\nu\colon \nu_j/d_1 \leq \bar{\nu}_j \leq \nu_j, 0 \leq \nu_i \leq \nu_j, \forall i \neq j\}$.

Next we split the integral into two parts: $\nu_j > \max\{(2d_1|x'|)^{2/3}, 1\}$ and $(|x'|/(2d_1))^{2/3} \leq \nu_j \leq (2d_1|x'|)^{2/3}$. Note that if $\nu_j \geq (2d_1|x'|)^{2/3}$ and $\nu_j/d_1 \leq \bar{\nu}_j \leq \nu_j$ then

$$\big||x_j'| - \bar{\nu}_j \nu_j^{\frac{1}{2}}\big|^{-\frac{1}{2}} \leq C \nu_j^{-3/4}.$$

Note also that there exists a constant $C$ such that for all $A, N > 0$

$$\int_0^N |A - x|^{-1/2} dx \leq C N^{1/2}.$$

Hence for $\nu_j > \max\{(2d_1|x'|)^{2/3}, 1\}$,

$$\begin{aligned}
&\int_{S_j} \prod_{i \neq j} \big||x_i'| - \nu_i \nu_j^{\frac{1}{2}}\big|^{-\frac{1}{2}} \big||x_j'| - \bar{\nu}_j \nu_j^{\frac{1}{2}}\big|^{-\frac{1}{2}} d\nu_1 \dots d\nu_{j-1} d\nu_{j+1} \dots d\nu_{d_1} \\
&\quad \leq C \nu_j^{-3/4} \prod_{i \neq j} \int_0^{\nu_j} \big||x_i'| - \nu_i \nu_j^{\frac{1}{2}}\big|^{-\frac{1}{2}} d\nu_i \\
&\quad \leq C \nu_j^{-3/4} \nu_j^{\frac{d_1 - 1}{4}} \leq C \nu_j^{d_1/4 - 1}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\max\{(2d_1|x'|)^{2/3}, 1\}}^{\infty} \frac{\max\{1, |x'|\}^{2\varepsilon}}{\nu_j^{d_1/4 + 3\varepsilon}} \int_{S_j} \prod_{i \neq j} \big||x_i'| - \nu_i \nu_j^{\frac{1}{2}}\big|^{-\frac{1}{2}} \big||x_j'| - \bar{\nu}_j \nu_j^{\frac{1}{2}}\big|^{-\frac{1}{2}} d\nu_1 \dots d\nu_{d_1} \\
&\quad \leq C \int_{\max\{(2d_1|x'|)^{2/3}, 1\}}^{\infty} \frac{\max\{1, |x'|\}^{2\varepsilon}}{\nu_j^{d_1/4 + 3\varepsilon}} \nu_j^{d_1/4 - 1} d\nu_j \\
&\quad \leq C \int_{\max\{(2d_1|x'|)^{2/3}, 1\}}^{\infty} \frac{\max\{1, |x'|\}^{2\varepsilon}}{\nu_j^{1 + 3\varepsilon}} d\nu_j \leq C.
\end{aligned} \tag{24}$$

If we assume now that $(|x'|/(2d_1))^{2/3} \leq \nu_j \leq (2d_1|x'|)^{2/3}$ then by the change of variables $\nu_i \nu_j^{\frac{1}{2}} = u_i$ one gets

$$\begin{aligned}
&\int_{S_j} \prod_{i \neq j} \big||x_i'| - \nu_i \nu_j^{\frac{1}{2}}\big|^{-\frac{1}{2}} \big||x_j'| - \bar{\nu}_j \nu_j^{\frac{1}{2}}\big|^{-\frac{1}{2}} d\nu_1 \dots d\nu_{j-1} d\nu_{j+1} \dots d\nu_{d_1} \\
&\quad \leq C \nu_j^{\frac{1 - d_1}{2}} \int_{[0, \nu_j^{3/2}]^{d_1 - 1}} \big||x_i'| - u_i\big|^{-\frac{1}{2}} \left||x_j'| + \sum_{i \neq j} u_i - \nu_j^{\frac{3}{2}}\right|^{-\frac{1}{2}} d\mathbf{u}
\end{aligned}$$

where $d\mathbf{u} = du_1 \cdots du_{j-1} du_{j+1} \cdots du_{d_1}$. Hence,

$$\int_{(|x'|/(2d_1))^{2/3}}^{(2d_1|x'|)^{2/3}} \frac{\max\{1,|x'|\}^{2\varepsilon}}{\nu_j^{d_1/4+3\varepsilon}} \int_{S_j} \prod_{i\neq j} \big||x_i'| - \nu_i \nu_j^{\frac{1}{2}}\big|^{-\frac{1}{2}} \big||x_j'| - \bar{\nu}_j \nu_j^{\frac{1}{2}}\big|^{-\frac{1}{2}} d\nu_1 \ldots d\nu_{d_1}$$

$$\le C \int_{(|x'|/(2d_1))^{2/3}}^{(2d_1|x'|)^{2/3}} \nu_j^{\frac{2-3d_1}{4}} \int_{[0,\nu_j^{3/2}]^{d_1-1}} \big||x_i'| - u_i\big|^{-\frac{1}{2}} \Big||x_j'| + \sum_{i\neq j} u_i - \nu_j^{\frac{3}{2}}\Big|^{-\frac{1}{2}} d\mathbf{u}\, d\nu_j$$

$$\le C|x'|^{\frac{2-3d_1}{6}} \int_{[0,2d_1|x'|]^{d_1-1}} \prod_{i\neq j} \big||x_i'| - u_i\big|^{-\frac{1}{2}}$$

$$\times \int_{(|x'|/(2d_1))^{2/3}}^{(2d_1|x'|)^{2/3}} \Big||x_j'| + \sum_{i\neq j} u_i - \nu_j^{\frac{3}{2}}\Big|^{-\frac{1}{2}} d\nu_j\, d\mathbf{u}$$

$$\le C|x'|^{\frac{2-3d_1}{6}} |x'|^{1/6} \prod_{i\neq j} \int_0^{2d_1|x'|} \big||x_i'| - u_i\big|^{-\frac{1}{2}} du_i$$

$$\le C|x'|^{\frac{2-3d_1}{6}} |x'|^{1/6} |x'|^{(d_1-1)/2}$$

$$\le C.$$

Now (22), (23), (24) and the above estimates yield

$$\sum_{N_{\mathbf{n}}^{3/2} > |x'|/(2d_1)} \frac{\max\{1,|x'|\}^{2\varepsilon}}{N_{\mathbf{n}}^{d_1/2+3\varepsilon}} H_{d_1,\mathbf{n}}\Big(\frac{x'}{N_{\mathbf{n}}^{1/2}}\Big) \le C.$$

□

Next, for all $R > 0$ we define the weight function $w_R : (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})^2 \to \mathbb{R}_+$ by the formula

$$w_R(x,y) = \min\{R, |y'|^{-1}\}|x'|.$$

The estimates obtained in this section can be summarised in the following proposition.

**Proposition 3.5** *For all $\gamma \in [0, d_2/4)$ and all bounded compactly supported Borel functions $F : \mathbb{R} \to \mathbb{C}$,*

$$\bigg\| \Big|\sum_{i=1}^{d_1} |x_i'|\Big|^{\gamma} K_{F(L)}(\cdot, y)\bigg\|_2^2 \le C_\gamma \int_0^\infty |F(\lambda)|^2 \lambda^{(d_1+d_2)/2} \min\{\lambda^{d_2/4-\gamma}, |y'|^{2\gamma-d_2/2}\} \frac{d\lambda}{\lambda}$$

*for almost all $y = (y', y'') \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. In particular, for all $R > 0$, if $\operatorname{supp} F \subseteq [R^2, 4R^2]$, then*

$$\operatorname*{ess\,sup}_{y\in\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} |B(y, R^{-1})|^{1/2} \|w_R(\cdot, y)^{\gamma} K_{F(L)}(\cdot, y)\|_2 \le C_\gamma \|\delta_{R^2} F\|_{L^2},$$

*where the constant $C_\gamma$ does not depend on $R$.*

*Proof* We obtain the first inequality by Proposition 3.3 and Lemma 3.4 with $\varepsilon = d_2/4 - \gamma$. Next if we assume that $\operatorname{supp} F \subseteq [R^2, 4R^2]$, then in virtue of the definition of the weight $w_R$ and estimate (5), the first inequality implies the second one. □

## 4 The Multiplier Theorems

In the following section we show that Theorems 1.1 and 1.2 are straightforward consequence of Proposition 3.5. The argument is essential the same as in Sect. 5 of [29] with an obvious adjustment of exponents in some calculations and we quote it here for sake of completeness. An alternative proof based on the wave equation technique can be obtain by a simple modification of the proof of [10, Lemma 3.4].

**Proposition 4.1** *For all $R > 0$, $\alpha \ge 0$, $\beta > \alpha$, and for all functions $F : \mathbb{R} \to \mathbb{C}$ such that* $\operatorname{supp} F \subseteq [-4R^2, 4R^2]$,

$$\operatorname*{ess\,sup}_{y\in\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} \left|B\left(y, R^{-1}\right)\right|^{1/2} \left\|\left(1 + R\rho(\cdot, y)\right)^{\alpha} K_F(\cdot, y)\right\|_2 \le C_{\alpha,\beta} \|\delta_{R^2} F\|_{W^{\beta}_{\infty}}, \tag{25}$$

*where the constant $C_{\alpha,\beta}$ does not depend on $R$. If in addition $\beta > \alpha + Q/2$, then*

$$\operatorname*{ess\,sup}_{y\in\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} \left\|\left(1 + R\rho(\cdot, y)\right)^{\alpha} K_F(\cdot, y)\right\|_1 \le C_{\alpha,\beta} \|\delta_{R^2} F\|_{W^{\beta}_{\infty}}, \tag{26}$$

*where again $C_{\alpha,\beta}$ does not depend on $R$.*

*Proof* Note that the heat kernel of the operator $L$ satisfies Gaussian bounds (7) so Proposition 4.1 is a straightforward consequence of [12, Lemmas 4.3 and 4.4]. □

Recall that the homogeneous dimension of the ambient space is given by $Q = d_1 + 3d_2/2$.

**Lemma 4.2** *Suppose that $0 \le \gamma < \min\{d_1/2, d_2/4\}$ and $\beta > Q/2 - \gamma$. For all $y \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and $R > 0$,*

$$\int_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} \left(1 + w_R(x, y)\right)^{-2\gamma} \left(1 + R\rho(x, y)\right)^{-2\beta} dx \le C_{\gamma,\beta} \left|B\left(y, R^{-1}\right)\right|. \tag{27}$$

*Moreover, for all $x, y \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and $R > 0$,*

$$w_R(x, y) \le C\left(1 + R\rho(x, y)\right). \tag{28}$$

*Proof* By the homogeneity properties of the distance $\rho$ and the weights $w_R$, we only prove the case $R = 1$. For other case, one just dilate them by $\delta_t(x', x'') = (tx', t^{3/2}x'')$. By (4),

$$\min\left\{1, \left|y'\right|^{-1}\right\}\left|x'\right| \le 1 + \left|x' - y'\right| \le C\left(1 + \rho(x, y)\right),$$

which proves (28).

Because of the translation invariance, to prove (27), it is enough to consider the case $y''=0$. By (5) it suffices to show that

$$\int_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} \left(1+\frac{|x'-y'|}{1+|y'|}\right)^{-2\gamma} \big(1+\rho(x,y)\big)^{-2\beta}\, dx \le C_{\gamma,\beta}\big(1+|y'|\big)^{d_2/2}.$$

Again we split the integral into two parts, according to the asymptotic behaviour (4). In the region $X_1=\{x\in\mathbb{R}^{d_1}\times\mathbb{R}^{d_2} : |x''|\ge(|x'|+|y'|)^{3/2}\}$, we choose $\beta_1$ and $\beta_2$ in such a way that $\beta=\beta_1+\beta_2$, $\beta_1>d_1/2-\gamma$ and $\beta_2>3d_2/4$. Then

$$\begin{aligned}
&\int_{X_1} \left(1+\frac{|x'-y'|}{1+|y'|}\right)^{-2\gamma} \big(1+\rho(x,y)\big)^{-2\beta}\, dx \\
&\quad\le C\big(1+|y'|\big)^{2\gamma}\int_{\mathbb{R}^{d_1}}\big(1+|x'-y'|\big)^{-2(\gamma+\beta_1)}\,dx'\int_{\mathbb{R}^{d_2}}\big(1+|x''|^{2/3}\big)^{-2\beta_2}\,dx'' \\
&\quad\le C_{\gamma,\beta}\big(1+|y'|\big)^{d_2/2}.
\end{aligned}$$

In the region $X_2=\{x\in\mathbb{R}^{d_1}\times\mathbb{R}^{d_2} : |x''|<(|x'|+|y'|)^{3/2}\}$, instead, we choose $\beta_1$ and $\beta_2$ in such a way that $\beta=\tilde\beta_1+\tilde\beta_2$, $\tilde\beta_1>d_1/2+d_2/4-\gamma$ and $\tilde\beta_2>d_2/2$. Then the integral over $X_2$ is estimated by

$$\begin{aligned}
&\int_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} \left(1+\frac{|x'-y'|}{1+|y'|}\right)^{-2\gamma}\big(1+|x'-y'|\big)^{-2\tilde\beta_1}\left(1+\frac{|x''|}{(|x'|+|y'|)^{1/2}}\right)^{-2\tilde\beta_2} dx \\
&\quad\le C_{\gamma,\beta}\int_{\mathbb{R}^{d_1}}\left(1+\frac{|u|}{1+|y'|}\right)^{-2\gamma}\big(1+|u|\big)^{-2\tilde\beta_1}\big(|u+y'|+|y'|\big)^{d_2/2}\,du \\
&\quad\le C_{\gamma,\beta}\left(\big(1+|y'|\big)^{2\gamma}\int_{\mathbb{R}^{d_1}}\big(1+|u|\big)^{-2\nu}\,du+|y'|^{d_2/2}\int_{\mathbb{R}^{d_1}}\big(1+|u|\big)^{-2\tilde\beta_1}\,du\right),
\end{aligned}$$

where $\nu=\tilde\beta_1+\gamma-d_2/4>d_1/2$. The conclusion follows. □

**Proposition 4.3** *For all* $R>0$, $\alpha\ge 0$, $\beta>\alpha$, $\gamma\in[0,d_2/4)$, *and for all functions* $F:\mathbb{R}\to\mathbb{C}$ *such that* $\operatorname{supp}F\subseteq[R^2,4R^2]$,

$$\begin{aligned}
&\operatorname*{ess\,sup}_{y\in\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} \big|B\big(y,R^{-1}\big)\big|^{1/2}\big\|\big(1+R\rho(\cdot,y)\big)^{\alpha}\big(1+w_R(\cdot,y)\big)^{\gamma}K_{F(L)}(\cdot,y)\big\|_2 \\
&\quad\le C_{\alpha,\beta,\gamma}\|\delta_{R^2}F\|_{W_2^\beta},
\end{aligned}$$

*where the constant* $C_{\alpha,\beta,\gamma}$ *does not depend on* $R$.

*Proof* The estimate (25), together with (28) and a Sobolev embedding, immediately implies Proposition 4.3 in the case $\beta>\alpha+d_2/2+1/2$. On the other hand, in the case $\alpha=0$, Proposition 4.3 follows from Proposition 3.5 for all $\beta>0$. We obtain now Proposition 4.3 for the whole range of exponents by interpolation (see [11] and also [12, Lemma 4.3] for similar methods). □

For the purpose of the next statement we set $D = Q - \min\{d_1, d_2/2\} = \max\{d_1 + d_2, 3d_2/2\}$.

**Corollary 4.4** *For all $R > 0$, $\alpha \geq 0$, $\beta > \alpha + D/2$, and for all functions $F : \mathbb{R} \to \mathbb{C}$ such that* $\operatorname{supp} F \subseteq [R^2, 4R^2]$,

$$\operatorname*{ess\,sup}_{y \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \left\| \big(1 + R\rho(\cdot, y)\big)^{\alpha} K_{F(L)}(\cdot, y) \right\|_1 \leq C_{\alpha,\beta} \|\delta_{R^2} F\|_{W_2^{\beta}}, \tag{29}$$

*where the constant $C_{\alpha,\beta}$ does not depend on $R$. In particular, under the same hypotheses,*

$$\operatorname*{ess\,sup}_{y \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \setminus B(y,r)} \left| K_{F(L)}(x, y) \right| dx \leq C_{\alpha,\beta} (1 + rR)^{-\alpha} \|\delta_{R^2} F\|_{W_2^{\beta}}. \tag{30}$$

*Proof* Corollary 4.4 follows from Proposition 4.3, together with (27) and Hölder's inequality. □

We are finally able to prove our main results.

*Proofs of Theorems 1.1 and 1.2* To prove Theorem 1.1 we can follow the lines of the proof of [12, Theorem 3.1], where the inequality (4.18) there is replaced by our (30). Next we can use that same argument as in [12, Sect. 6] to conclude the proof of Theorem 1.2, see also [29]. □

## 5 Final Remarks

Now we shall show that, if $d_1 \geq d_2/2$, then the result in Theorem 1.1 is sharp. More precisely, if $d_1 \geq d_2/2$ and $s < D/2 = (d_1 + d_2)/2$, then the weak type (1, 1) estimates in Theorem 1.1 cannot hold. Indeed, if we consider the functions $H_t(\lambda) = \lambda^{it}$, then, for $t > 1$, and any $\eta \in C_c^{\infty}(\mathbb{R}_+)$

$$\|\eta H_t\|_{W_2^s} \sim t^s.$$

On the other hand, we make the following observation.

**Proposition 5.1** *Suppose that $L$ is the Grushin operator acting on $X = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then the following lower bounds holds*:

$$\left\| H_t(L) \right\|_{L^1 \to L^{1,w}} = \left\| L^{it} \right\|_{L^1 \to L^{1,w}} \geq C\big(1 + |t|\big)^{(d_1+d_2)/2}$$

*for all $t > 0$.*

*Proof* Because the Grushin operator is elliptic on $X_0 = \{x \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} : x' \neq 0\}$, one can use the same argument as in [35] to prove that, for all $y \in X_0$,

$$\left| p_t(x, y) - \left| y' \right|^{-d_2} (4\pi t)^{-(d_1+d_2)/2} \mathrm{e}^{-\rho(x,y)^2/4t} \right| \leq C t^{1/2} t^{-(d_1+d_2)/2}$$

for all $x$ in a small neighbourhood of $y$ and all $t \in (0, 1)$. Here $p_t = K_{\exp(-tL)}$ is the heat kernel corresponding to the Grushin operator. The rest of the argument is the same as in [35], so we skip it here. □

To show that Theorems 1.1 and 1.2 are sharp one can also use the results described in [26].

Finally we would like to mention a few natural open problems related to the sharp spectral multiplier results, which we prove in this paper. First such problem is to extend Theorem 1.1 to the class of all operators $L_\sigma$ defined by (1) for all $\sigma > 0$. Especially intuitions coming from interpolation techniques suggest the results described in Theorem 1.1 above and in [29, Theorem 1] can be extended at least for all $1 < \sigma < 2$. However we want to point out that it seems that interpolation approach including Stein's complex interpolation cannot be used here. Another interesting question, which arises is to obtain possible precise description of the spectral decompositions of operators $L_\sigma$. Third interesting problem is to remove the condition $d_1 \geq d_2/2$ from the Theorem 1.1. That is to show that half of the Euclidean dimension $(d_1 + d_2)/2$ is the critical exponent for $L^1$ spectral multiplier for all values of $d_1$ and $d_2$. At this point it worth to mention that in the resent paper [28] similar restriction is removed from [29, Theorem 1] and it is shown that for $\sigma = 2$ half of the Euclidean dimension of the considered ambient space is the critical exponent for all values of $d_1$ and $d_2$. In this situation, it is a natural question whether the analogue of Theorem 1.1 can be obtained for all values of $\sigma > 0$ and dimensions $d_1, d_2$.

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