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The Equivalence Between Seven Classes of Wavelet Multipliers and Arcwise Connectivity They Induce

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Abstract Let A be a $d \times d$ expansive matrix with $|\det A| = 2$. An A-wavelet is a function $\psi \in L^2(\mathbb{R}^d)$ such that $\{2^{\frac{j}{2}}\psi(A \cdot -k) : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$. A measurable function f is called an A-wavelet multiplier if the inverse Fourier transform of $f\hat{\psi}$ is an A-wavelet whenever ψ is an A-wavelet, where $\hat{\psi}$ denotes the Fourier transform of ψ. A-scaling function multiplier, A-PFW multiplier, semi-orthogonal A-PFW multiplier, **MRA** A-wavelet multiplier, **MRA** A-PFW multiplier and semi-orthogonal **MRA** A-PFW multiplier are defined similarly. In this paper, we prove that the above seven classes of multipliers are equivalent, and obtain a characterization of them. We then prove that if the set of all A-wavelet multipliers acts on some A-scaling function (A-wavelet, A-PFW, semi-orthogonal A-PFW, **MRA** A-wavelet, **MRA** A-PFW, semi-orthogonal **MRA** A-PFW), the orbit is arcwise connected in $L^2(\mathbb{R}^d)$, and that if the generator of an orbit is an **MRA** A-PFW, the orbit is equal to the set of all **MRA** A-PFWs whose Fourier transforms have same module, and is also equal to the set of all **MRA** A-PFWs with corresponding pseudo-scaling functions having the same module of their Fourier transforms.

Keywords Multiplier · Wavelet · Parseval frame wavelet

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1 Introduction

We denote by $\mathbb{Z}$ the set of integers, by $\mathbb{N}$ the set of positive integers, and by $\mathbb{Z}_+$ the set of nonnegative integers. Let A be a $d \times d$ expansive matrix, i.e., an integral matrix with eigenvalues being greater than 1 in module. Define the dilation operator D and translation operator T_k on $L^2(\mathbb{R}^d)$ with $k \in \mathbb{Z}^d$ respectively by

$$Df(\cdot) := |\det A|^{\frac{1}{2}} f(A\cdot) \quad \text{and} \quad T_k f(\cdot) := f(\cdot - k) \quad \text{for } f \in L^2(\mathbb{R}^d),$$

and write $f_{j,k} = D^j T_k f$ for $f \in L^2(\mathbb{R}^d)$, $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$. The Fourier (inverse Fourier) transform is defined by

$$\hat{f}(\cdot) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \cdot \rangle} dx \left(\check{f}(\cdot) = \int_{\mathbb{R}^d} f(x) e^{2\pi i \langle x, \cdot \rangle} dx \right)$$

for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and unitarily extended to $L^2(\mathbb{R}^d)$. Throughout this paper, unless otherwise specified, relations between two measurable sets in $\mathbb{R}^d$, such as equality or inclusion, are always understood up to a set of zero measure. Similarly, equality or inequality between functions is always understood in the "pointwise almost everywhere" sense. We work under the following assumption:

Assumption The matrix A is a $d \times d$ expansive matrix with $|\det A| = 2$.

Proposition 1.1 ([15, Proposition 2])

(i) $f(\cdot + (A^*)^{-1}\varepsilon) = f(\cdot + (A^*)^{-1}\delta)$ *for an arbitrary* $\mathbb{Z}^d$*-periodic function* f, ε *and* δ *with* $\{0, \varepsilon\}$ *and* $\{0, \delta\}$ *being both the sets of representatives of distinct cosets in* $\mathbb{Z}^d / A^*\mathbb{Z}^d$;

(ii) *there exists* $1 \le k_0 \le d$ *such that* $2\langle (A^*)^{-1}\varepsilon, \mathrm{e}_{k_0}\rangle$ *is odd for all* ε *with* $\{0, \varepsilon\}$ *being a set of representatives of distinct cosets in* $\mathbb{Z}^d / A^*\mathbb{Z}^d$, *where* A^* *denotes the transpose of* A, *and* e_{k_0} *denotes the vector in* $\mathbb{R}^d$ *with the* k_0*-th component being* 1 *and the others being zero*.

Definition 1.1 A sequence $\{V_j\}_{j \in \mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^d)$ is called a *multiresolution analysis* (an **MRA**) for $L^2(\mathbb{R}^d)$ associated with A if the following conditions are satisfied:

(i) $V_j \subset V_{j+1}$ for $j \in \mathbb{Z}$;

(ii) $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}^d)$ and $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;

(iii) $f \in V_0$ if and only if $D^j f \in V_j$ for $j \in \mathbb{Z}$;

(iv) there exists $\varphi \in L^2(\mathbb{R}^d)$ such that $\{T_k \varphi : k \in \mathbb{Z}^d\}$ is an orthonormal basis for V_0.

This definition is a natural generalization of one dimensional **MRA** with $A = 2$. Some other "generalized" **MRA**s were introduced in [1, 2, 6, 7, 16, 21] for the construction of wavelet frames in $L^2(\mathbb{R}^d)$. The function φ in (iv) is called a *scaling function* of the **MRA**. By Theorem 1.1 in [4], the condition $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$ in (ii) is trivial, a special case of which can be obtained by Corollary 4.14 in [8]. By the definition, $V_j = \overline{\operatorname{span}}\{ D^j T_k\varphi : k \in \mathbb{Z}^d \}$ (so we say φ generates the **MRA**), and there exists a unique $m \in L^2(\mathbb{T}^d)$ such that $\hat{\varphi}(A^*\cdot) = m(\cdot)\hat{\varphi}(\cdot)$. It is easy to check that

$$\left|m(\cdot)\right|^2 + \left|m\left(\cdot + \left(A^*\right)^{-1}\varepsilon\right)\right|^2 = 1, \tag{1.1}$$

where $\{0, \varepsilon\}$ is a set of representatives of distinct cosets in $\mathbb{Z}^d/A^*\mathbb{Z}^d$. Let k_0 be as in Proposition 1.1. Define $m_1 \in L^2(\mathbb{T}^d)$ by

$$m_1(\cdot) = e^{2\pi i\langle\cdot, \mathrm{e}_{k_0}\rangle}\mu\left(A^*\cdot\right)\overline{m\left(\cdot + \left(A^*\right)^{-1}\varepsilon\right)}, \tag{1.2}$$

and $\psi \in L^2(\mathbb{R}^d)$ via its Fourier transform by

$$\hat{\psi}(\cdot) = m_1\left(\left(A^*\right)^{-1}\cdot\right)\hat{\varphi}\left(\left(A^*\right)^{-1}\cdot\right), \tag{1.3}$$

where μ is an arbitrary $\mathbb{Z}^d$-periodic, unimodular and measurable function. Observe that

$$\begin{pmatrix} m(\cdot) & m(\cdot + (A^*)^{-1}\varepsilon) \\ m_1(\cdot) & m_1(\cdot + (A^*)^{-1}\varepsilon) \end{pmatrix}$$

is unitary. By the same procedure as in [9, Chap. 2, Proposition 2.13], we can prove that $\{T_k\psi : k \in \mathbb{Z}^d\}$ is an orthonormal basis for $W_0 = V_1 \ominus V_0$ (the orthogonal complement of V_0 in V_1), and thus $\{\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$. Such ψ is called an **MRA** A-*wavelet* since it is associated with an **MRA**, which is independent of the choice of ε by Proposition 1.1.

Let $\psi \in L^2(\mathbb{R}^d)$. ψ is called an A-*wavelet* if $\{\psi_{j,k} : j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$; is called an A-*Parseval frame wavelet* (A-PFW) if $\{\psi_{j,k} : j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$ is a Parseval frame for $L^2(\mathbb{R}^d)$, i.e.,

$$\|f\|^2 = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\left|\langle f, \psi_{j,k}\rangle\right|^2 \quad \text{for } f \in L^2\left(\mathbb{R}^d\right).$$

ψ is called a *semi-orthogonal* A-*Parseval frame wavelet* (semi-orthogonal A-PFW) if it is an A-PFW, and $\langle\psi_{j,k}, \psi_{j',k'}\rangle = 0$ for $k, k' \in \mathbb{Z}^d$ and $j, j' \in \mathbb{Z}$ with $j \neq j'$; ψ is called an **MRA** A-*Parseval frame wavelet* (**MRA** A-PFW) if it is an A-PFW, and there exist an A-refinable function φ, a $\mathbb{Z}^d$-periodic measurable function m, a $\mathbb{Z}^d$-periodic, unimodular and measurable function υ such that

$$\hat{\varphi}\left(A^*\cdot\right) = m(\cdot)\hat{\varphi}(\cdot), \qquad \left|m(\cdot)\right|^2 + \left|m\left(\cdot + \left(A^*\right)^{-1}\varepsilon\right)\right|^2 = 1, \tag{1.4}$$

and

$$\hat{\psi}(\cdot) = e^{2\pi i\langle(A^*)^{-1}\cdot, \mathrm{e}_{k_0}\rangle}\upsilon(\cdot)\overline{m\left(\left(A^*\right)^{-1}\cdot + \left(A^*\right)^{-1}\varepsilon\right)}\hat{\varphi}\left(\left(A^*\right)^{-1}\cdot\right), \tag{1.5}$$

where k_0 is as in Proposition 1.1. In this case, m is called a *low pass filter*, and φ is called a *pseudo-scaling function*. ψ is called a *semi-orthogonal* ***MRA*** A-*Parseval frame wavelet* (semi-orthogonal **MRA** A-PFW) if it is an **MRA** A-PFW, and $\langle\psi_{j,k}, \psi_{j',k'}\rangle = 0$ for $k, k' \in \mathbb{Z}^d$ and $j, j' \in \mathbb{Z}$ with $j \neq j'$.

MRA wavelets have many desirable features, but they impose some restrictions. A natural setting for such a theory is provided by frames (see [9, Chap. 8]). In one dimension, this problem was studied in [2, 6, 7]. However, "generalized" **MRA**s therein exclude many useful filters, such as $m(\xi) = \frac{1+e^{-6\pi i\xi}}{2}$, since they involve certain restrictions and technical assumptions such as semi-orthogonality. **MRA** PFWs herein overcome this drawback, and include the filter above. Some results related to **MRA** PFWs can be seen in [15, 19, 20].

The construction of wavelets and wavelet frames is an important issue in wavelet analysis. **MRA**s and generalized **MRA**s in [1, 2, 6, 7, 9, 21] provide us with an approach for the construction of wavelets and wavelet frames. In particular, Bakić, Krishtal and Wilson in [1] first studied a class of **MRA** PFWs associated with a general expansive matrix A with $|\det A| = 2$. Multipliers allow us to obtain new wavelets (frame wavelets) from one wavelet (frame wavelet). A measurable function f defined on $\mathbb{R}^d$ is called an *A-wavelet multiplier* (**MRA** *A-wavelet multiplier*, *A-PFW multiplier*, **MRA** *A-PFW multiplier*, *semi-orthogonal A-PFW multiplier*, *semi-orthogonal* **MRA** *A-PFW multiplier*, *A-scaling function multiplier*) if $(f\hat{\psi})^\vee$ is an A-wavelet (**MRA** A-wavelet, A-PFW, **MRA** A-PFW, semi-orthogonal A-PFW, *semi-orthogonal* **MRA** *A-PFW*, *A-scaling function*) whenever ψ is. The first article on wavelet multipliers can be dated back to [22] in 1998. It is the first of a series of reports describing joint results by two groups consisting of 14 members, one led by Dai and Larson, and the other led by Hernández and Weiss. This article characterized one dimensional 2-wavelet multipliers, as well as the scaling function multipliers and low pass filter multipliers, and proved that the set of **MRA** 2-wavelets is arcwise connected in $L^2(\mathbb{R})$. In 2001, Paluszynski, Šikić, Weiss and Xiao in [19] characterized several classes of 2-PFW multipliers, and proved the arcwise connectivity of several classes of 2-PFW sets in $L^2(\mathbb{R})$. However, these two articles are both of one dimension. In 2002, for $A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, Li (the first author of this paper) in [11] proved the equivalence between A-wavelet multiplier, A-scaling function multiplier and **MRA** A-wavelet multiplier, characterized these three classes of multipliers and low pass A-filter multipliers, and, in terms of multipliers, proved the arcwise connectivity of the set of a class of wavelets. In 2004, D. Li and Cheng in [12] proved that the set of **MRA** A-wavelets is arcwise connected. Using the fact that all 2×2 expansive matrices A with $|\det A| = 2$ can be exactly classified as six integrally similar classes by [10], in 2010, Z. Li, Dai, Diao and Xin in [18] extended the results in [11] to general 2×2 expansive matrices A with $|\det A| = 2$, they also proved the arcwise connectivity of the set of **MRA** A-wavelets. For a general $d \times d$ expansive matrix A with $|\det A| = 2$, in 2010, Z. Li, Dai, Diao and Huang in [17] characterized (**MRA**) A-wavelet multipliers, and proved the arcwise connectivity of the set of **MRA** A-wavelets. Recently, Z. Li and Shi in [14] characterized A-PFW multipliers, and in [13] obtained some conditions for dyadic bivariate wavelet multipliers.

For a general $d \times d$ expansive matrix A with $|\det A| = 2$, in this paper, we prove the equivalence between seven classes of multipliers. The main results of this paper are as follows.

Theorem 1.1 *For a measurable function f defined on $\mathbb{R}^d$, the following are equivalent*:

(1) $|f(\cdot)|=1$ *a.e. on* $\mathbb{R}^d$, *and* $k(\cdot)=\frac{f(A^*\cdot)}{f(\cdot)}$ *is* $\mathbb{Z}^d$*-periodic.*
(2) f *is an* A*-scaling function multiplier.*
(3) f *is an* **MRA** A*-wavelet multiplier.*
(4) f *is an* A*-wavelet multiplier.*
(5) f *is an* A*-PFW multiplier.*
(6) f *is a semi-orthogonal* A*-PFW multiplier.*
(7) f *is an* **MRA** A*-PFW multiplier.*
(8) f *a semi-orthogonal* **MRA** A*-PFW multiplier.*

By Theorem 1.1, a multiplier f always satisfies

$$\begin{aligned} f\big((A^*)^n\cdot\big) &= k\big((A^*)^{n-1}\cdot\big)f\big((A^*)^{n-1}\cdot\big) \\ &= k\big((A^*)^{n-1}\cdot\big)k\big((A^*)^{n-2}\cdot\big)f\big((A^*)^{n-2}\cdot\big) \\ &= \cdots \\ &= k\big((A^*)^{n-1}\cdot\big)k\big((A^*)^{n-2}\cdot\big)\cdots k\big(A^*\cdot\big)k(\cdot)f(\cdot), \end{aligned} \tag{1.6}$$

and

$$f(\cdot)=\overline{k\big((A^*)^{n-1}\cdot\big)k\big((A^*)^{n-2}\cdot\big)\cdots k\big(A^*\cdot\big)k(\cdot)}f\big((A^*)^n\cdot\big) \tag{1.7}$$

for $n\in\mathbb{Z}$ and some $\mathbb{Z}^d$-periodic, unimodular function k. This shows that a multiplier f is determined by its values on a set E with $\{(A^*)^nE: n\in\mathbb{Z}\}$ being a partition of $\mathbb{R}^d$ and a $\mathbb{Z}^d$-periodic, unimodular function k. However, Lemma 2.8 in [1] asserts that an arbitrary $\mathbb{Z}^d$-periodic, unimodular function must satisfy $k(\cdot)=\frac{f(A^*\cdot)}{f(\cdot)}$ for some unimodular function f. This allows us to conjecture that (1.6) and (1.7) determine all multipliers. The following theorem gives a positive answer to this conjecture.

Theorem 1.2 *Let* $k(\xi)$ *be a unimodular, measurable and* $\mathbb{Z}^d$*-periodic function defined on* $\mathbb{R}^d$, *let* E *be a measurable set with* $\{(A^*)^nE: n\in\mathbb{Z}\}$ *being a partition of* $\mathbb{R}^d$, *and let* $g(\xi)$ *be a unimodular measurable function defined on* E. *Define*

$$f(\xi)=\begin{cases} g(\xi), & \xi\in E \\ k((A^*)^{-1}\xi)\cdots k((A^*)^{-n}\xi)\cdot g((A^*)^{-n}\xi), & \xi\in(A^*)^nE, n\geq 1 \\ \overline{k(\xi)k(A^*\xi)\cdots k((A^*)^{n-1}\xi)}\cdot g((A^*)^n\xi), & \xi\in(A^*)^{-n}E, n\geq 1 \end{cases}$$

Then f *is any one of seven multipliers in Theorem* 1.1. *Moreover, any one of seven multipliers in Theorem* 1.1 *can be constructed by this way.*

Theorem 1.2 holds for A-wavelet multipliers by Theorem 3.2 in [17] if E is an A-wavelet set. However, by a careful observation to its proof, we find it is enough to require that $\{(A^*)^nE: n\in\mathbb{Z}\}$ forms a partition of $\mathbb{R}^d$. So Theorem 1.2 holds for A-wavelet multipliers. Then we obtain Theorem 1.2 by Theorem 1.1.

Let ψ_0 be an A-wavelet (**MRA** A-wavelet, a semi-orthogonal **MRA** A-PFW, **MRA** A-PFW, a semi-orthogonal A-PFW, A-PFW, A-scaling function). Define

$$M_{\psi_0}=\big\{\psi:\hat{\psi}(\cdot)=f(\cdot)\hat{\psi}_0(\cdot),\ f \text{ is an } A\text{-wavelet multiplier}\big\}. \tag{1.8}$$

Then M_{ψ_0} is a subset of the set of A-wavelets (**MRA** A-wavelets, semi-orthogonal **MRA** A-PFWs, **MRA** A-PFWs, semi-orthogonal A-PFWs, A-PFWs, A-scaling functions) by Theorem 1.1. The following theorems concern the arcwise connectivity of M_{ψ_0} and its characterization.

Theorem 1.3 *Let ψ_0 be an A-wavelet* (**MRA** *A-wavelet, a semi-orthogonal* **MRA** *A-PFW,* **MRA** *A-PFW, a semi-orthogonal A-PFW, A-PFW, A-scaling function*). *Then M_{ψ_0} is arcwise connected, i.e., for an arbitrary $\psi_1 \in M_{\psi_0}$, there exists a continuous mapping $\theta : [0,1] \mapsto L^2(\mathbb{R}^d)$ such that $\theta(0) = \psi_0$, $\theta(1) = \psi_1$ and $\theta(t) \in M_{\psi_0}$ for $t \in [0,1]$.*

Theorem 1.4 *Let ψ_0 be an* **MRA** *A-PFW with φ_0 being a corresponding pseudo-scaling function. Define*

$$W_{\psi_0} = \{\psi : \psi \textit{ is an } \mathbf{MRA}\ A\textit{-PFW such that } |\hat{\psi}(\cdot)| = |\hat{\psi}_0(\cdot)|\},$$

and

$$S_{\psi_0} = \{\psi : \psi \textit{ is an } \mathbf{MRA}\ A\textit{-PFW associated with a pseudo-scaling function } \varphi \textit{ satisfying } |\hat{\varphi}(\cdot)| = |\hat{\varphi}_0(\cdot)|\}.$$

Then $S_{\psi_0} = M_{\psi_0} = W_{\psi_0}$.

Remark 1.1 Let ψ_0 in Theorems 1.3 and 1.4 be an **MRA** A-wavelet with φ_0 being its scaling function. Define

$$\widetilde{W}_{\psi_0} = \{\psi : \psi \text{ is an } A\text{-wavelet such that } |\hat{\psi}(\cdot)| = |\hat{\psi}_0(\cdot)|\},$$

and

$$\widetilde{S}_{\psi_0} = \{\psi : \psi \text{ is an } \mathbf{MRA}\ A\text{-wavelet associated with a scaling function } \varphi \text{ satisfying } |\hat{\varphi}(\cdot)| = |\hat{\varphi}_0(\cdot)|\}.$$

Then $\widetilde{S}_{\psi_0} = M_{\psi_0} = \widetilde{W}_{\psi_0}$, and M_{ψ_0} is arcwise connected. See [22, Theorem 3], [11, Theorem 1.3] and [17, Lemmas 4.1, 4.2] for details. The most interesting thing is the fact that these two results were used effectively for showing that the set of **MRA** A-wavelets is arcwise connected (see [22, Theorem 4] and [17, Theorem 4.1]). But it is unresolved that whether the set of **MRA** A-PFWs is arcwise connected. It is worth expecting that Theorems 1.3 and 1.4 are helpful for solving this problem.

2 Proof of Theorem 1.1

For $\psi \in L^2(\mathbb{R}^d)$, write

$$D_\psi(\cdot) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^d} |\hat{\psi}((A^*)^j(\cdot + k))|^2,$$

$$\Psi_j(\cdot) = \{\hat{\psi}((A^*)^j(\cdot + k)) : k \in \mathbb{Z}^d\} \quad \text{for } j \in \mathbb{Z}_+,$$
$$F_\psi(\cdot) = \overline{\text{span}}\{\Psi_j(\cdot) :\ j \in \mathbb{N}\}.$$

Lemma 2.1 *For* $\psi \in L^2(\mathbb{R}^d)$, $\int_{\mathbb{T}^d} D_\psi(\xi)d\xi = \|\psi\|^2$.

Lemma 2.2 *For an arbitrary* $\varphi \in L^2(\mathbb{R}^d)$, φ *is an* A*-scaling function if and only if*

(1) $\sum_{k\in\mathbb{Z}^d} |\hat{\varphi}(\cdot + k)|^2 = 1$ *a.e. on* $\mathbb{R}^d$;
(2) $\lim_{j\to\infty} |\hat{\varphi}((A^*)^{-j}\cdot)| = 1$ *a.e. on* $\mathbb{R}^d$;
(3) *there exists a* $\mathbb{Z}^d$*-periodic measurable function* m *such that* $\hat{\varphi}(A^*\cdot) = m(\cdot)\hat{\varphi}(\cdot)$ *a.e. on* $\mathbb{R}^d$.

Taking $\Omega = \mathbb{R}^d$ in Theorems 1–3 and Lemmas 5, 6 in [15], we have following five lemmas:

Lemma 2.3 *Let* ψ *be an* A*-PFW. Then* ψ *is a semi-orthogonal* **PFW** *if and only if* $D_\psi(\cdot) \in \mathbb{Z}_+$ *a.e. on* $\mathbb{R}^d$.

Lemma 2.4 *Let* ψ *be an* A*-PFW. Then* ψ *is an* **MAR** A*-PFW if and only if* $\dim F_\psi(\cdot) \in \{0, 1\}$ *a.e. on* $\mathbb{R}^d$.

Lemma 2.5 *Let* ψ *be an* A*-PFW. Then* ψ *is a semi-orthogonal* **MAR** A*-PFW if and only if* $D_\psi(\cdot) \in \{0, 1\}$ *a.e. on* $\mathbb{R}^d$.

Lemma 2.6 *For an* A*-PFW* ψ, *the following are equivalent*:

(i) ψ *is a semi-orthogonal* A*-PFW*;
(ii) $\sum_{k\in\mathbb{Z}^d} |\hat{\psi}(\cdot + k)|^2 = \chi_U(\cdot)$ *a.e. on* $\mathbb{R}^d$, *where* $U = \{\xi \in \mathbb{R}^d : \sum_{k\in\mathbb{Z}^d} |\hat{\psi}(\xi + k)|^2 > 0\}$;
(iii) $\|\psi\|^2 = \sum_{k\in\mathbb{Z}^d} |\langle\psi, T_k\psi\rangle|^2$;
(iv) $\sum_{k\in\mathbb{Z}^d} \hat{\psi}((A^*)^j(\cdot + k))\overline{\hat{\psi}(\cdot + k)} = 0$ *a.e. on* $\mathbb{R}^d$ *for* $j \in \mathbb{N}$.

Lemma 2.7 *Let* ψ *be an* A*-PFW. Define*

$$H_n(\cdot) = \sum_{j=0}^{\infty} \langle\Psi_n(\cdot), \Psi_j(\cdot)\rangle_{\ell^2(\mathbb{Z}^d)} \hat{\psi}((A^*)^j\cdot) \quad \text{a.e. on } \mathbb{R}^d \text{ for } n \in \mathbb{N}.$$

Then

$$H_n(\cdot) = H_{n-1}(A^*\cdot) + \langle\Psi_n(\cdot), \Psi_0(\cdot)\rangle_{\ell^2(\mathbb{Z}^d)} \hat{\psi}(\cdot)$$

a.e. on $\mathbb{R}^d$ *for* $1 < n \in \mathbb{N}$.

Lemma 2.8 *Let $\{\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ be a Bessel sequence in $L^2(\mathbb{R}^d)$ with Bessel bound B. Then*

$$\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^d}\big|\hat{\psi}\big((A^*)^n(\cdot+k)\big)\big|\big|\hat{\psi}\big((A^*)^j(\cdot+k)\big)\big|\big|\hat{\psi}\big((A^*)^j\cdot\big)\big|$$
$$\le B\sqrt{D_\psi(\cdot)} \quad \text{a.e. on } \mathbb{R}^d \text{ for } n\in\mathbb{Z}_+.$$

Proof Since $\{\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ is a Bessel sequence in $L^2(\mathbb{R}^d)$ with Bessel bound B, we have

$$\sum_{j=1}^{\infty}\big|\hat{\psi}\big((A^*)^j\cdot\big)\big|^2 \le B, \qquad \sum_{k\in\mathbb{Z}^d}\big|\hat{\psi}(\cdot+k)\big|^2 \le B.$$

Then, by Cauchy-Schwartz inequality,

$$\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^d}\big|\hat{\psi}\big((A^*)^n(\cdot+k)\big)\big|\big|\hat{\psi}\big((A^*)^j(\cdot+k)\big)\big|\big|\hat{\psi}\big((A^*)^j\cdot\big)\big|$$
$$\le \left(\sum_{j=1}^{\infty}\big|\hat{\psi}\big((A^*)^j\cdot\big)\big|^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{\infty}\left(\sum_{k\in\mathbb{Z}^d}\big|\hat{\psi}\big((A^*)^n(\cdot+k)\big)\big|\cdot\big|\hat{\psi}\big((A^*)^j(\cdot+k)\big)\big|\right)^2\right)^{\frac{1}{2}}$$
$$\le \sqrt{B}\cdot\left(\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^d}\big|\hat{\psi}\big((A^*)^n(\cdot+k)\big)\big|^2\cdot\sum_{k\in\mathbb{Z}^d}\big|\hat{\psi}\big((A^*)^j(\cdot+k)\big)\big|^2\right)^{\frac{1}{2}}$$
$$= \sqrt{B}\cdot\left(\sum_{k\in\mathbb{Z}^d}\big|\hat{\psi}\big((A^*)^n\cdot+k\big)\big|^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^d}\big|\hat{\psi}\big((A^*)^j(\cdot+k)\big)\big|^2\right)^{\frac{1}{2}}$$
$$\le B\sqrt{D_\psi(\cdot)}.$$

The proof is completed. □

The following two lemmas are borrowed from [3] and [5]:

Lemma 2.9 *For $\psi \in L^2(\mathbb{R}^d)$, ψ is an A-PFW if and only if*

$$\sum_{j\in\mathbb{Z}}\big|\hat{\psi}\big((A^*)^j\cdot\big)\big|^2 = 1,$$
$$\sum_{j=0}^{\infty}\hat{\psi}\big((A^*)^j\cdot\big)\overline{\hat{\psi}\big((A^*)^j(\cdot+k)\big)} = 0 \quad \big(k\in\mathbb{Z}^d\setminus A^*\mathbb{Z}^d\big) \text{ a.e. on } \mathbb{R}^d.$$

Lemma 2.10 *Let ψ be an A-PFW. Then ψ is an A-wavelet if and only if $\|\psi\| = 1$.*

Lemma 2.11 *Let ψ be a semi-orthogonal A-PFW. Then*

$$\chi_U\big((A^*)^n\cdot\big)\hat{\psi}\big((A^*)^n\cdot\big)=\sum_{j=1}^{\infty}\big\langle\Psi_n(\cdot),\Psi_j(\cdot)\big\rangle_{\ell^2(\mathbb{Z}^d)}\hat{\psi}\big((A^*)^j\cdot\big) \tag{2.1}$$

a.e. on $\mathbb{R}^d$ for $n\in\mathbb{N}$, where U is defined as in Lemma 2.6.

Proof By Lemmas 2.6 and 2.7, we have

$$H_n(\cdot)=\sum_{j=1}^{\infty}\big\langle\Psi_n(\cdot),\Psi_j(\cdot)\big\rangle_{\ell^2(\mathbb{Z}^d)}\hat{\psi}\big((A^*)^j\cdot\big)\quad\text{for } n\in\mathbb{N}, \tag{2.2}$$

$H_n(\cdot)=H_{n-1}(A^*\cdot)$ for $1<n\in\mathbb{N}$, and thus $H_n(\cdot)=H_1((A^*)^{n-1}\cdot)$ for $n\in\mathbb{N}$. So, to finish the proof, we only need to prove that $H_1(\cdot)=\chi_U(A^*\cdot)\hat{\psi}(A^*\cdot)$. By Lemmas 2.8 and 2.9,

$$\begin{aligned}
H_1(\cdot)&=\sum_{k\in\mathbb{Z}^d}\hat{\psi}\big(A^*(\cdot+k)\big)\sum_{j=1}^{\infty}\overline{\hat{\psi}\big((A^*)^j(\cdot+k)\big)}\hat{\psi}\big((A^*)^j\cdot\big)\\
&=\sum_{k\in\mathbb{Z}^d}\hat{\psi}\big(A^*(\cdot+k)\big)\sum_{j=0}^{\infty}\overline{\hat{\psi}\big((A^*)^j(A^*\cdot+A^*k)\big)}\hat{\psi}\big((A^*)^{j+1}\cdot\big)\\
&=\sum_{k\in\mathbb{Z}^d}\hat{\psi}\big(A^*\cdot+k\big)\sum_{j=0}^{\infty}\overline{\hat{\psi}\big((A^*)^j(A^*\cdot+k)\big)}\hat{\psi}\big((A^*)^{j+1}\cdot\big).
\end{aligned}$$

Interchanging the order of summation, we obtain $H_1(\cdot)=\chi_U(A^*\cdot)\hat{\psi}(A^*\cdot)$ by Lemma 2.6. The proof is completed. □

When ψ is an A-PFW, and $D_\psi(\cdot)=1$, we have ψ is an A-wavelet and thus

$$\hat{\psi}\big((A^*)^n\cdot\big)=\sum_{j=1}^{\infty}\big\langle\Psi_n(\cdot),\Psi_j(\cdot)\big\rangle_{\ell^2(\mathbb{Z}^d)}\hat{\psi}\big((A^*)^j\cdot\big)\quad\text{for } n\in\mathbb{N}$$

by Lemmas 2.1, 2.10 and 2.11. Then, by standard arguments in [9], we can prove the following lemma:

Lemma 2.12 *For $\psi\in L^2(\mathbb{R}^d)$, ψ is an* **MRA** *A-wavelet if and only if ψ is an A-PFW, and $D_\psi(\cdot)=1$ a.e. on $\mathbb{R}^d$.*

Lemma 2.13 *Given measurable functions f and g defined on $\mathbb{R}^d$, let $g(\cdot)\neq 0$ a.e. on $\mathbb{R}^d$, and let*

$$\begin{aligned} &\sum_{j\in\mathbb{Z}}\bigl|f\bigl((A^*)^j\cdot\bigr)\bigr|^{2n}\bigl|g\bigl((A^*)^j\cdot\bigr)\bigr|^2=1,\\ &\sum_{j\in\mathbb{Z}}\bigl|g\bigl((A^*)^j\cdot\bigr)\bigr|^2=1 \quad \text{a.e. on } \mathbb{R}^d \text{ for } n\in\mathbb{N}. \end{aligned} \tag{2.3}$$

Then $|f(\cdot)|=1$ a.e. on $\mathbb{R}^d$.

Proof Suppose $|f(\cdot)|>1$ on some set E with positive measure. Then $|f(\cdot)|^{2n}|g(\cdot)|^2 \le 1$ a.e. on E by (2.3), which implies that $|g(\cdot)|^2\le |f(\cdot)|^{-2n}\to 0$ as $n\to\infty$ a.e. on E. This is a contradiction. So

$$\bigl|f(\cdot)\bigr|\le 1 \quad \text{a.e. on } \mathbb{R}^d. \tag{2.4}$$

By (2.3), we also have

$$\sum_{j\in\mathbb{Z}}\bigl|g\bigl((A^*)^j\cdot\bigr)\bigr|^2\bigl(1-\bigl|f\bigl((A^*)^j\cdot\bigr)\bigr|^{2n}\bigr)=0.$$

It follows that $|f(\cdot)|=1$ a.e. on $\mathbb{R}^d$ by (2.4) and the fact that $g\neq 0$ a.e. on $\mathbb{R}^d$. The proof is completed. □

Lemma 2.14 *For an arbitrary multiplier f of* (3)–(8) *in Theorem* 1.1, *$|f(\cdot)|=1$ a.e. on $\mathbb{R}^d$.*

Proof Choose ψ as one **MRA** A-wavelet satisfying $\hat\psi(\cdot)\neq 0$ a.e. on $\mathbb{R}^d$ ([1, Example 5.14]). Then $(f^n\hat\psi)^\vee$ is an A-PFW for every $n\in\mathbb{N}$ by Theorem 1.1. So

$$\sum_{j\in\mathbb{Z}}\bigl|f\bigl((A^*)^j\cdot\bigr)\bigr|^{2n}\bigl|\hat\psi\bigl((A^*)^j\cdot\bigr)\bigr|^2=1, \qquad \sum_{j\in\mathbb{Z}}\bigl|\hat\psi\bigl((A^*)^j\cdot\bigr)\bigr|^2=1 \quad \text{a.e. on } \mathbb{R}^d$$

by Lemma 2.9. This implies that $|f(\cdot)|=1$ a.e. on $\mathbb{R}^d$ by Lemma 2.13. The proof is completed. □

Lemma 2.15 *For an arbitrary A-scaling function multiplier f, $|f(\cdot)|=1$ a.e. on $\mathbb{R}^d$.*

Proof Choose φ as one A-scaling function satisfying $\hat\varphi(\cdot)\neq 0$ a.e. on $\mathbb{R}^d$ ([1, Example 5.14]). Then $(f^n\hat\varphi)^\vee$ is an A-scaling function for every $n\in\mathbb{N}$. So

$$\sum_{k\in\mathbb{Z}^d}\bigl|f(\cdot+k)\bigr|^{2n}\bigl|\hat\varphi(\cdot+k)\bigr|^2=1, \qquad \sum_{k\in\mathbb{Z}^d}\bigl|\hat\varphi(\cdot+k)\bigr|^2=1 \quad \text{a.e. on } \mathbb{R}^d \tag{2.5}$$

by Lemma 2.2. It follows that

$$\sum_{k\in\mathbb{Z}^d}\bigl|\hat\varphi(\cdot+k)\bigr|^2\bigl(1-\bigl|f(\cdot+k)\bigr|^{2n}\bigr)=0, \tag{2.6}$$

$$\left|\hat{\varphi}(\cdot)\right|^2 \le \left|f(\cdot)\right|^{-2n} \tag{2.7}$$

a.e. on $\mathbb{R}^d$. Suppose $|f(\cdot)| > 1$ on some set E with positive measure. Then $|\hat{\varphi}(\cdot)|^2 \le |f(\cdot)|^{-2n} \to 0$ as $n \to \infty$ a.e. on E, which is a contradiction. So $|f(\cdot)| \le 1$ a.e. on $\mathbb{R}^d$. This leads to $|f(\cdot)| = 1$ a.e. on $\mathbb{R}^d$ by (2.6) and the fact that $\hat{\varphi} \ne 0$ a.e. on $\mathbb{R}^d$. The proof is completed. □

Proof of Theorem 1.1 By Lemmas 2.14 and 2.15, we may as well assume that $|f(\cdot)| = 1$ a.e. on $\mathbb{R}^d$. Then (1), (3), (4) and (5) are equivalent by [17, Theorem 3.1, Corollary 3.1] and [14, Theorem 3.2], and

$$D_{(f\hat{\psi})^\vee}(\cdot) = D_\psi(\cdot) \quad \text{a.e. on } \mathbb{R}^d \text{ for } \psi \in L^2(\mathbb{R}^d). \tag{2.8}$$

Suppose f is an arbitrary one of (6)–(8), and ψ is an **MRA** A-wavelet. Then $(f\hat{\psi})^\vee$ is an A-PFW, and $D_\psi(\cdot) = 1$ by Lemma 2.12. So $D_{(f\hat{\psi})^\vee}(\cdot) = 1$ a.e. on $\mathbb{R}^d$ by (2.8). This implies that $(f\hat{\psi})^\vee$ is an **MRA** A-wavelet by Lemma 2.12, and thus (3) holds. To finish the proof, next we prove that (1) and (2) are equivalent, and that (1) implies every one of (6)–(8).

(1)⇒(2): Suppose (1) holds, and φ is an A-scaling function satisfying $\hat{\varphi}(A^*\cdot) = m(\cdot)\hat{\varphi}(\cdot)$ for some $\mathbb{Z}^d$-periodic function m. Then we have

$$f(A^*\cdot)\hat{\varphi}(A^*\cdot) = k(\cdot)m(\cdot)f(\cdot)\hat{\varphi}(\cdot), \tag{2.9}$$

and $k(\cdot)m(\cdot)$ is $\mathbb{Z}^d$-periodic by (1), and

$$\sum_{k\in\mathbb{Z}^d} \left|\hat{f}(\cdot + k)\right|^2 \left|\hat{\varphi}(\cdot + k)\right|^2 = 1, \qquad \lim_{j\to\infty} \left|f\left((A^*)^{-j}\cdot\right)\right|\left|\hat{\varphi}\left((A^*)^{-j}\cdot\right)\right| = 1 \tag{2.10}$$

by (1) and Lemma 2.2. So $(f\hat{\varphi})^\vee$ is an A-scaling function by (2.9), (2.10) and Lemma 2.2, and thus f is an A-scaling function multiplier.

(2)⇒(1): Suppose f is an A-scaling function multiplier, and φ is an A-scaling function satisfying $\hat{\varphi}(\cdot) \ne 0$ a.e. on $\mathbb{R}^d$ ([1, Example 5.14]). Then $\varphi_1 = (f\hat{\varphi})^\vee$ is an A-scaling function satisfying $\hat{\varphi_1}(\cdot) \ne 0$ a.e. on $\mathbb{R}^d$. So there exists $\mathbb{Z}^d$-periodic functions m and m_1 satisfying

$$\hat{\varphi}(A^*\cdot) = m(\cdot)\hat{\varphi}(\cdot), \qquad \hat{\varphi_1}(A^*\cdot) = m_1(\cdot)\hat{\varphi_1}(\cdot),$$

which implies that

$$\hat{\varphi_1}(A^*\cdot) = f(A^*\cdot)m(\cdot)\hat{\varphi}(\cdot) = f(A^*\cdot)m(\cdot)\overline{f(\cdot)}\hat{\varphi_1}(\cdot).$$

Therefore,

$$k(\cdot) = \frac{f(A^*\cdot)}{f(\cdot)} = \frac{m_1(\cdot)}{m(\cdot)},$$

which is $\mathbb{Z}^d$-periodic by periodicity of m and m_1.

(1)⇒(6): Suppose (1) holds, and ψ is a semi-orthogonal A-PFW. Then $D_\psi(\cdot) \in \mathbb{Z}_+$ by Lemma 2.3, and $(f\hat{\psi})^\vee$ is an A-PFW by the equivalence between (1) and (5).

Since $|f(\cdot)|=1$, we have $D_\psi(\cdot)=D_{(f\hat{\psi})^\vee}(\cdot)$, and thus $D_{(f\hat{\psi})^\vee}(\cdot)\in\mathbb{Z}_+$. So $(f\hat{\psi})^\vee$ is a semi-orthogonal A-PFW by Lemma 2.3, and thus (6) holds.

(1)⇒(7): Suppose (1) holds, and ψ is an **MRA** A-PFW. Then $(f\hat{\psi})^\vee$ is an A-PFW by the equivalence between (1) and (5). Write

$$F_{(f\hat{\psi})^\vee}(\cdot)=\overline{\operatorname{span}}\{\widetilde{\Psi}_j(\cdot),\, j\in\mathbb{N}\},$$

where

$$\widetilde{\Psi}_j(\cdot)=\{f\hat{\psi}((A^*)^j(\cdot+k)):k\in\mathbb{Z}^d\}\quad\text{for } j\in\mathbb{N}.$$

By Lemma 2.4, to prove (7) we only need to prove that $\dim F_{(f\hat{\psi})^\vee}(\cdot)\in\{0,1\}$. By Lemma 2.4, we have $\dim F_\psi(\cdot)\in\{0,1\}$. So there exist functions $j_0:\mathbb{R}^d\to\mathbb{N}$ and $c_j:\mathbb{R}^d\to\mathbb{C}$ with $j\in\mathbb{N}$ such that $\Psi_j(\cdot)=c_j(\cdot)\Psi_{j_0(\cdot)}(\cdot)$, i.e.,

$$\hat{\psi}((A^*)^j(\cdot+k))=c_j(\cdot)\hat{\psi}((A^*)^{j_0(\cdot)}(\cdot+k))\quad\text{for } j\in\mathbb{N}\text{ and } k\in\mathbb{Z}^d.$$

By (1), we also have

$$\begin{aligned}f((A^*)^j(\cdot+l))&=k((A^*)^{j-1}(\cdot+l))k((A^*)^{j-2}(\cdot+l))\cdots k(\cdot+l)f(\cdot+l)\\&=k((A^*)^{j-1}\cdot)k((A^*)^{j-2}\cdot)\cdots k(\cdot)f(\cdot+l)\\&=f((A^*)^j\cdot)\overline{f(\cdot)}f(\cdot+l)\end{aligned}\tag{2.11}$$

for $j\in\mathbb{N}$ and $l\in\mathbb{Z}^d$, which implies that

$$f((A^*)^j(\cdot+k))\overline{f((A^*)^{j_0(\cdot)}(\cdot+k))}=f((A^*)^j\cdot)\overline{f((A^*)^{j_0(\cdot)}\cdot)}$$

for $j\in\mathbb{N}$ and $k\in\mathbb{Z}^d$. It follows that

$$\begin{aligned}\widetilde{\Psi}_j(\cdot)&=\{f((A^*)^j(\cdot+k))c_j(\cdot)\hat{\psi}((A^*)^{j_0(\cdot)}(\cdot+k)):k\in\mathbb{Z}^d\}\\&=\{c_j(\cdot)f((A^*)^j(\cdot+k))\overline{f((A^*)^{j_0(\cdot)}(\cdot+k))}(f\hat{\psi})((A^*)^{j_0(\cdot)}(\cdot+k)):k\in\mathbb{Z}^d\}\\&=c_j(\cdot)f((A^*)^j\cdot)\overline{f((A^*)^{j_0(\cdot)}\cdot)}\widetilde{\Psi}_{j_0(\cdot)}\end{aligned}$$

for $j\in\mathbb{N}$, and thus $\dim F_{(f\hat{\psi})^\vee}\in\{0,1\}$.

(1)⇒(8): Suppose (1) holds, and ψ is a semi-orthogonal **MRA** A-PFW. Then $D_\psi(\cdot)\in\{0,1\}$ by Lemma 2.5, $(f\hat{\psi})^\vee$ is an A-PFW by the equivalence between (1) and (5). Since $|f(\cdot)|=1$, we have $D_{(f\hat{\psi})^\vee}=D_\psi$, and thus $D_{(f\hat{\psi})^\vee}(\cdot)\in\{0,1\}$. So $(f\hat{\psi})^\vee$ is a semi-orthogonal **MRA** A-PFW by Lemma 2.5. Therefore (8) holds. The proof is completed. □

3 Proof of Theorems 1.3 and 1.4

Lemma 3.1 *Let ψ be an* **MRA** *A-PFW, and let φ be a corresponding pseudo-scaling function. Then*

$$\left|\hat{\varphi}(\cdot)\right|^2 = \sum_{j=1}^{\infty}\left|\hat{\psi}\left(\left(A^*\right)^j\cdot\right)\right| \quad a.e.\ on\ \mathbb{R}^d. \tag{3.1}$$

Proof By the definition of **MRA** A-PFW, $|\hat{\varphi}(A^*\cdot)|^2+|\hat{\psi}(A^*\cdot)|^2=|\hat{\varphi}(\cdot)|^2$. It follows that

$$\left|\hat{\varphi}(\cdot)\right|^2 = \left|\hat{\varphi}\left(\left(A^*\right)^n\cdot\right)\right|^2+\sum_{j=1}^{n}\left|\hat{\psi}\left(\left(A^*\right)^j\cdot\right)\right|^2,$$

and thus

$$\left|\hat{\varphi}\left(\left(A^*\right)^n\cdot\right)\right|^2 = \left|\hat{\varphi}(\cdot)\right|^2-\sum_{j=1}^{n}\left|\hat{\psi}\left(\left(A^*\right)^j\cdot\right)\right|^2 \tag{3.2}$$

for $n\in\mathbb{N}$. Observe that $\{\sum_{j=1}^{n}|\hat{\psi}((A^*)^j\cdot)|^2\}$ is an increasing sequence. It follows that $\lim_{n\to\infty}|\hat{\varphi}((A^*)^n\cdot)|^2$ exists, and thus

$$\begin{aligned}\int_{\mathbb{R}^d}\lim_{n\to\infty}\left|\hat{\varphi}\left(\left(A^*\right)^n\xi\right)\right|^2 d\xi &= \lim_{n\to\infty}\int_{\mathbb{R}^d}\left|\hat{\varphi}\left(\left(A^*\right)^n\xi\right)\right|^2 d\xi\\ &= \lim_{n\to\infty} 2^{-n}\int_{\mathbb{R}^d}\left|\hat{\varphi}(\xi)\right|^2 d\xi\\ &= 0.\end{aligned}$$

So $\lim_{n\to\infty}|\hat{\varphi}((A^*)^n\cdot)|^2=0$, which implies (3.1) by (3.2). The proof is completed. □

Lemma 3.2 *Let φ be a pseudo-scaling function. Define*

$$E=\left\{\xi\in\mathbb{R}^d:\hat{\varphi}(\xi)\neq 0\right\},\qquad \Delta_0=E,\qquad \Delta_n=\left(A^*\right)^n E\setminus\left(A^*\right)^{n-1}E \quad for\ n\geq 1.$$

Then $\{\Delta_n:\ n\geq 0\}$ is a partition of $\mathbb{R}^d$.

Proof Suppose ψ and m are respectively an **MRA** A-PFW and a low pass filter corresponding to φ, and they are related as in (1.4) and (1.5). By (1.5), we have $\mathrm{supp}(\hat{\psi})\subset A^*E$, where $\mathrm{supp}(f)=\{\xi\in\mathbb{R}^d:\ \hat{f}(\xi)\neq 0\}$ for a measurable function f. It follows that

$$\mathrm{supp}(\widehat{\psi_{j,k}})\subset\left(A^*\right)^{j+1}E \quad \text{for } j\in\mathbb{Z} \text{ and } k\in\mathbb{Z}^d. \tag{3.3}$$

Since ψ is an A-PFW, $\{\widehat{\psi_{j,k}}:\ j\in\mathbb{Z},\ k\in\mathbb{Z}^d\}$ is a Parseval frame for $L^2(\mathbb{R}^d)$. If $\bigcup_{j\in\mathbb{Z}}(A^*)^jE\neq\mathbb{R}^d$, then, by (3.3), there exists a set S with positive and finite measure such that

$$S \cap \left(\bigcup_{j \in \mathbb{Z}} \bigcup_{k \in \mathbb{Z}^d} \operatorname{supp}(\widehat{\psi_{j,k}}) \right) = \emptyset.$$

It is obvious that χ_S does not belong to the closed linear span of $\{\widehat{\psi_{j,k}} : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$, which contradicts the fact that $\{\widehat{\psi_{j,k}} : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ is a Parseval frame for $L^2(\mathbb{R}^d)$. So $\bigcup_{j \in \mathbb{Z}} (A^*)^j E = \mathbb{R}^d$. Also by the refinable property of φ, we have $E \subset A^* E$. Thus

$$(A^*)^j E \subset (A^*)^{j+1} E \quad \text{for } j \in \mathbb{Z} \quad \text{and} \quad \bigcup_{j=0}^{\infty} (A^*)^j E = \mathbb{R}^d.$$

This easily leads to the lemma. The proof is completed. □

Proof of Theorem 1.3 Choose E such that $\{(A^*)^j E : j \in \mathbb{Z}\}$ is a partition of $\mathbb{R}^d$. Suppose $\psi_1 \in M_{\psi_0}$. Then there exists an A-wavelet multiplier f such that $\hat{\psi}_1 = f \hat{\psi}_0$. Define a function λ on E such that $f(\xi) = e^{2\pi i \lambda(\xi)}$ and $0 \le \lambda(\xi) < 1$ for $\xi \in E$. Since f is an A-wavelet multiplier, there exists a $\mathbb{Z}^d$-periodic real function β such that $\frac{f(A^*\xi)}{f(\xi)} = e^{2\pi i \beta(\xi)}$. Extend λ to $\mathbb{R}^d$ in the following way:

$$\lambda(\xi) = \begin{cases} \lambda((A^*)^{-1}\xi) + \beta((A^*)^{-1}\xi), & \xi \in (A^*)^{j+1} E, \ j \ge 0; \\ \lambda(A^*\xi) - \beta(\xi), & \xi \in (A^*)^j E, \ j < 0. \end{cases}$$

Then $f(\xi) = e^{2\pi i \lambda(\xi)}$ for a.e. $\xi \in \mathbb{R}^d$. Define $\theta : [0, 1] \to L^2(\mathbb{R}^d)$ by

$$\theta(t) = (f_t \hat{\psi}_0)^{\vee} \quad \text{for } t \in [0, 1]$$

where $f_t(\xi) = e^{2\pi i t \lambda(\xi)}$. Then

$$\theta(0) = \psi_0, \qquad \theta(1) = \psi_1, \tag{3.4}$$

and f_t is an A-wavelet multiplier by Theorem 1.1. It follows that

$$\theta(t) \in M_{\psi_0} \quad \text{for } 0 \le t \le 1. \tag{3.5}$$

Observe that $|\widehat{\theta(t)}(\xi) - \widehat{\theta(s)}(\xi)|^2 \le 4|\hat{\psi}_0(\xi)|^2$ for $0 \le t, s \le 1$. By Lebesgue dominated theorem and Plancheral theorem, we have $\lim_{t \to s} \|\theta(t) - \theta(s)\|_2 = 0$, and thus θ is continuous. This implies that M_{ψ_0} is arcwise connected by (3.4) and (3.5). The proof is completed. □

Proof of Theorem 1.4 By Theorem 1.1 and Lemma 3.1, we have $M_{\psi_0} \subset W_{\psi_0} = S_{\psi_0}$. Now we prove that $S_{\psi_0} \subset M_{\psi_0}$. Suppose $\psi \in S_{\psi_0}$ with φ being a corresponding pseudo-scaling function, m and m_0 are respectively low pass filters corresponding to φ and φ_0, and υ and υ_0 are unimodular $\mathbb{Z}^d$-periodic functions such that

$$\hat{\psi}(\cdot) = e^{2\pi i \langle (A^*)^{-1}\cdot, e_{k_0} \rangle} \upsilon(\cdot) \overline{m\big((A^*)^{-1} \cdot + (A^*)^{-1}\varepsilon\big)} \hat{\varphi}\big((A^*)^{-1}\cdot\big), \tag{3.6}$$

$$\hat{\psi}_0(\cdot) = e^{2\pi i\langle (A^*)^{-1}\cdot, e_{k_0}\rangle} v_0(\cdot)\overline{m_0\big((A^*)^{-1}\cdot + (A^*)^{-1}\varepsilon\big)}\hat{\varphi}_0\big((A^*)^{-1}\cdot\big). \tag{3.7}$$

To prove that $\psi \in M_{\psi_0}$, we only need to prove that there exists an A-wavelet multiplier f such that

$$\hat{\psi}(\cdot) = f(\cdot)\hat{\psi}_0(\cdot). \tag{3.8}$$

Since $\hat{\varphi}(A^*\cdot) = m(\cdot)\hat{\varphi}(\cdot)$, $\hat{\varphi}_0(A^*\cdot) = m_0(\cdot)\hat{\varphi}_0(\cdot)$, and $|\hat{\varphi}(\cdot)| = |\hat{\varphi}_0(\cdot)|$, we have

$$\begin{aligned}
\big|m_0(\cdot)\big|^2 \sum_{k\in\mathbb{Z}^d} \big|\hat{\varphi}_0(\cdot+k)\big|^2 &= \sum_{k\in\mathbb{Z}^d} \big|m_0(\cdot+k)\big|^2\big|\hat{\varphi}_0(\cdot+k)\big|^2 \\
&= \sum_{k\in\mathbb{Z}^d} \big|\hat{\varphi}_0\big(A^*(\cdot+k)\big)\big|^2 \\
&= \sum_{k\in\mathbb{Z}^d} \big|\hat{\varphi}\big(A^*(\cdot+k)\big)\big|^2 \\
&= \sum_{k\in\mathbb{Z}^d} \big|m(\cdot+k)\big|^2\big|\hat{\varphi}(\cdot+k)\big|^2 \\
&= \big|m(\cdot)\big|^2 \sum_{k\in\mathbb{Z}^d} \big|\hat{\varphi}(\cdot+k)\big|^2.
\end{aligned}$$

This implies that

$$\big|m_0(\cdot)\big| = \big|m(\cdot)\big| \quad \text{on } \Big\{\xi \in \mathbb{R}^d : \sum_{k\in\mathbb{Z}^d} \big|\hat{\varphi}_0(\xi+k)\big|^2 > 0\Big\}. \tag{3.9}$$

Next we divide two cases to construct an A-wavelet multiplier f satisfying (3.8).

Case 1. $\{\xi \in \mathbb{R}^d : \hat{\varphi}_0(\xi) \neq 0\} = \mathbb{R}^d$.

In this case, $\frac{\hat{\varphi}_1(\cdot)}{\hat{\varphi}_0(\cdot)}$ is an unimodular function, and $|m_0(\cdot)| = |m(\cdot)| \neq 0$ by (3.9) and refinable property of φ and φ_0. Put

$$f(\xi) := \frac{\overline{m((A^*)^{-1}\xi + (A^*)^{-1}\varepsilon)}\hat{\varphi}((A^*)^{-1}\xi)}{\overline{m_0((A^*)^{-1}\xi + (A^*)^{-1}\varepsilon)}\hat{\varphi}_0((A^*)^{-1}\xi)} \cdot \frac{v(\xi)}{v_0(\xi)}.$$

Then $|f(\cdot)| = 1$, and $\hat{\psi}(\cdot) = f(\cdot)\hat{\psi}_0(\cdot)$. By refinable property of φ_0, φ_1 and the fact that $|m_0(\cdot)| = |m(\cdot)| \neq 0$, we have

$$\begin{aligned}
\frac{f(A^*\cdot)}{f(\cdot)} &= \frac{\overline{m(\cdot + (A^*)^{-1}\epsilon) m_0((A^*)^{-1}\cdot + (A^*)^{-1}\epsilon)} m((A^*)^{-1}\cdot)}{\overline{m_0(\cdot + (A^*)^{-1}\epsilon) m((A^*)^{-1}\cdot + (A^*)^{-1}\epsilon)} m_0((A^*)^{-1}\cdot)} \cdot \frac{v(A^*\cdot)v_0(\cdot)}{v_0(A^*\cdot)v_0(\cdot)} \\
&= \frac{\overline{m(\cdot + (A^*)^{-1}\epsilon)}}{\overline{m_0(\cdot + (A^*)^{-1}\epsilon)}} \cdot \frac{m((A^*)^{-1}\cdot) m((A^*)^{-1}\cdot + (A^*)^{-1}\epsilon)}{m_0((A^*)^{-1}\cdot) m_0((A^*)^{-1}\cdot + (A^*)^{-1}\epsilon)} \cdot \frac{v(A^*\cdot)v_0(\cdot)}{v_0(A^*\cdot)v(\xi)}.
\end{aligned}$$

This implies that $\frac{f(A^*\cdot)}{f(\cdot)}$ is $\mathbb{Z}^d$-periodic by $\mathbb{Z}^d$-periodicity of m, m_0, v and v_0, and the fact that $\mathbb{Z}^d = A^*\mathbb{Z}^d + \{0, \epsilon\}$. Therefore, f is a multiplier satisfying (3.8) by Theorem 1.1.

Case 2. $\{\xi \in \mathbb{R}^d : \hat{\varphi}_0(\xi) \neq 0\} \neq \mathbb{R}^d$.

Define

$$\mu(\xi) = \begin{cases} \frac{m(\xi)}{m_0(\xi)} & \text{if } m_0(\xi)\sum_{k\in\mathbb{Z}^d}|\hat{\varphi}_0(\xi-(A^*)^{-1}\epsilon+k)|^2 \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

For ξ satisfying $m_0(\xi)\sum_{k\in\mathbb{Z}^d}|\hat{\varphi}_0(\xi-(A^*)^{-1}\epsilon+k)|^2 \neq 0$, there exists $k_\xi \in \mathbb{Z}^d$ such that

$$m_0(\xi)\hat{\varphi}_0\big(\xi - \big(A^*\big)^{-1}\epsilon + k_\xi\big) \neq 0.$$

Then

$$\hat{\psi}\big(A^*\xi - \epsilon + A^*k_\xi\big) = e^{2\pi i\langle \xi-(A^*)^{-1}\epsilon,\, e_{k_0}\rangle} v\big(A^*\xi\big)\overline{m(\xi)}\hat{\varphi}\big(\xi - \big(A^*\big)^{-1}\epsilon + k_\xi\big),$$
$$\hat{\psi}_0\big(A^*\xi - \epsilon + A^*k_\xi\big) = e^{2\pi i\langle \xi-(A^*)^{-1}\epsilon,\, e_{k_0}\rangle} v_0\big(A^*\xi\big)\overline{m_0(\xi)}\hat{\varphi}_0\big(\xi - \big(A^*\big)^{-1}\epsilon + k_\xi\big)$$

by (3.6) and (3.7). Observe that $|\hat{\psi}(A^*\xi - \epsilon + A^*k_\xi)| = |\hat{\psi}_0(A^*\xi - \epsilon + A^*k_\xi)|$ due to the fact that $\psi \in S_{\psi_0} = W_{\psi_0}$. It follows that $|\overline{m(\xi)}\hat{\varphi}(\xi - (A^*)^{-1}\epsilon + k_\xi)| = |\overline{m_0(\xi)}\hat{\varphi}_0(\xi - (A^*)^{-1}\epsilon + k_\xi)|$, which implies that $|m(\xi)| = |m_0(\xi)| \neq 0$ since $|\hat{\varphi}| = |\hat{\varphi}_0|$. So $\frac{m(\xi)}{m_0(\xi)}$ is unimodular, and thus μ is. It is obvious that μ is $\mathbb{Z}^d$-periodic. So μ is unimodular and $\mathbb{Z}^d$-periodic.

To obtain an A-wavelet multiplier f satisfying (3.8), we only need to construct a measurable function t such that

$$\big|t(\xi)\big| = 1, \tag{3.10}$$
$$\hat{\varphi}_0(\xi) = t(\xi)\hat{\varphi}(\xi), \tag{3.11}$$
$$\overline{\mu(\xi)} = t\big(A^*\xi\big)\overline{t(\xi)}. \tag{3.12}$$

Indeed, if (3.10)–(3.12) hold, define $f(\xi) = \overline{\mu((A^*)^{-1}\xi + (A^*)^{-1}\epsilon)t((A^*)^{-1}\xi)} \cdot \frac{v(\xi)}{v_0(\xi)}$. Then f is unimodular, and

$$\frac{f(A^*\xi)}{f(\xi)} = \overline{\mu\big(\xi + \big(A^*\big)^{-1}\epsilon\big)}\mu\big(\big(A^*\big)^{-1}\xi + \big(A^*\big)^{-1}\epsilon\big)\mu\big(\big(A^*\big)^{-1}\xi\big) \cdot \frac{v(A^*\xi)v_0(\xi)}{v_0(A^*\xi)v(\xi)},$$

which implies that $\frac{f(A^*\xi)}{f(\xi)}$ is $\mathbb{Z}^d$-periodic. So f is a wavelet multiplier by Theorem 1.1. It is easy to check that

$$f(\xi)\hat{\psi}_0(\xi) = e^{2\pi i\langle (A^*)^{-1}\xi,\, e_{k_0}\rangle} v(\xi)\overline{\mu\big(\big(A^*\big)^{-1}\xi + \big(A^*\big)^{-1}\varepsilon\big)m_0\big(\big(A^*\big)^{-1}\xi + \big(A^*\big)^{-1}\varepsilon\big)}$$
$$\times\, \hat{\varphi}\big(\big(A^*\big)^{-1}\xi\big). \tag{3.13}$$

When $m_0((A^*)^{-1}\xi + (A^*)^{-1}\varepsilon)\hat{\varphi}((A^*)^{-1}\xi) = 0$, we have $f(\xi)\hat{\psi}_0(\xi) = 0$ by (3.13). We also have $\hat{\psi}_0(\xi) = 0$ by (3.7), which implies that $\hat{\psi}(\xi) = 0$ due to the fact that $\psi \in S_{\psi_0} = W_{\psi_0}$. So $\hat{\psi}(\xi) = f(\xi)\hat{\psi}_0(\xi)$. When $m_0((A^*)^{-1}\xi + (A^*)^{-1}\varepsilon)\hat{\varphi}((A^*)^{-1}\xi) \neq 0$, we have

$$\overline{\mu\big((A^*)^{-1}\xi+(A^*)^{-1}\varepsilon\big)m_0\big((A^*)^{-1}\xi+(A^*)^{-1}\varepsilon\big)}=\overline{m\big((A^*)^{-1}\xi+(A^*)^{-1}\varepsilon\big)}$$

by the definition of μ, which implies that $\hat{\psi}(\xi)=f(\xi)\hat{\psi}_0(\xi)$ by (3.13). Therefore (3.8) holds.

Next we construct t satisfying (3.10)–(3.12) to finish the proof. Replacing φ in Lemma 3.2 by φ_0, we get a partition $\{\Delta_n:\ n\geq 0\}$ of $\mathbb{R}^d$. Define t by

$$t(\xi)=\frac{\hat{\varphi}_0(\xi)}{\hat{\varphi}(\xi)}\quad \text{if } \xi\in\Delta_0,\qquad t(\xi)=\overline{\mu\big((A^*)^{-1}\xi\big)}t\big((A^*)^{-1}\xi\big)\quad \text{if } \xi\in\Delta_n,\ n\geq 1.$$

It is obvious that t satisfies (3.10) and (3.12). Now we prove that t satisfies (3.11) by induction. It is obvious that (3.11) holds when $\xi\in\Delta_0$. Suppose (3.11) holds for $\xi\in\Delta_n$. Let $\xi\in\Delta_{n+1}$. When $\hat{\varphi}_0(\xi)=0$, we have $\hat{\varphi}(\xi)=0$ since $|\hat{\varphi}|=|\hat{\varphi}_0|$. When $\hat{\varphi}_0(\xi)\neq 0$, we have

$$\hat{\varphi}_0(\xi)=m_0\big((A^*)^{-1}\xi\big)t\big((A^*)^{-1}\xi\big)\hat{\varphi}\big((A^*)^{-1}\xi\big),$$

and $|m_0((A^*)^{-1}\xi)|=|m((A^*)^{-1}\xi)|\neq 0$ by (3.9). So $m_0((A^*)^{-1}\xi)=m((A^*)^{-1}\xi)\times\overline{\mu((A^*)^{-1}\xi)}$ by the definition of μ and its unimodular property. It follows that

$$\begin{aligned}\hat{\varphi}_0(\xi)&=m\big((A^*)^{-1}\xi\big)\overline{\mu\big((A^*)^{-1}\xi\big)}t\big((A^*)^{-1}\xi\big)\hat{\varphi}\big((A^*)^{-1}\xi\big)\\&=m\big((A^*)^{-1}\xi\big)t(\xi)\hat{\varphi}\big((A^*)^{-1}\xi\big)\\&=t(\xi)\hat{\varphi}(\xi).\end{aligned}$$

Therefore, (3.11) holds. The proof is completed. □

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