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The Poisson Operator for Orthogonal Polynomials in the Multidimensional Ball

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Abstract In this paper we define the Poisson operator related to an orthonormal system on the multidimensional ball and we analyze some weighted inequalities for this operator in mixed norm spaces.

Keywords Poisson operator · Mixed norm spaces · Weighted inequalities

Mathematics Subject Classification (2000) Primary 42C10

1 Introduction

Let B^d denote the unit ball in the Euclidean space $\mathbb{R}^d$ for $d \geq 2$. The classical orthogonal polynomials in B^d can be defined with respect to the weight function

$$W_\mu\big(\|x\|\big) = \big(1 - \|x\|^2\big)^{\mu-1/2}, \quad \mu > -1/2,\ x \in B^d,$$

where $\|\cdot\|$ denotes the usual Euclidean norm. For $n \geq 0$, let $\mathcal{V}_n^d$ be the space of d-dimensional polynomials of degree at most n. There are some different orthonormal basis for $\mathcal{V}_n^d$ (see [8, Chap. 2]). One of them is given by

$$P_{n,j,\beta}^{d,\mu}(x) = \big(C_{n,j}^{d,\mu}\big)^{-1} p_j^{(\mu-1/2,n-2j+\frac{d-2}{2})}\big(2\|x\|^2 - 1\big) S_{\beta,n-2j}(x), \quad 0 \leq 2j \leq n,$$

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where $p_j^{(a,b)}$ is the Jacobi polynomial of degree j and order (a,b),

$$(C_{n,j}^{d,\mu})^2 = \frac{\Gamma(j+\mu+\frac{1}{2})\Gamma(n-j+\frac{d}{2})}{2(n+\mu+\frac{d-1}{2})j!\Gamma(n-j+\mu+\frac{d-1}{2})},$$

and $\{S_{\beta,k}\}_{0\le\beta\le d(k)}$ is an orthonormal basis of homogeneous spherical harmonics of degree k (we use $d(k)$ to indicate the dimension of this set). The Jacobi polynomials that we are considering are defined by

$$p_j^{(a,b)}(x) = \frac{(-1)^j}{2^j j!}(1-x)^{-a}(1+x)^{-b}\frac{d^j}{dx^j}\big((1-x)^{a+j}(1+x)^{b+j}\big).$$

They are orthogonal in the space $L^2((-1,1),(1-x)^a(1+x)^b)$ and verify that

$$\int_{-1}^{1} p_j^{(a,b)}(x)p_n^{(a,b)}(x)(1-x)^a(1+x)^b\,dx$$
$$= \delta_{j,n}\frac{2^{a+b+1}\Gamma(j+a+1)\Gamma(j+b+1)}{(2j+a+b+1)j!\Gamma(j+a+b+1)}.$$

The family of polynomials

$$\big\{P_{n,j,\beta}^{d,\mu} : n\ge 0, 0\le 2j\le n, 0\le\beta\le d(n-2j)\big\}$$

is orthonormal and complete in $L^2(B^d, W_\mu(\|x\|))$. Moreover these polynomials satisfy the relation

$$\mathcal{L}^{d,\mu}P_{n,j,\beta}^{d,\mu} = \left(n+\mu+\frac{d-1}{2}\right)^2 P_{n,j,\beta}^{d,\mu}$$

with

$$\mathcal{L}^{d,\mu}f = -\Delta f + \sum_{i=1}^{d}\frac{\partial}{\partial x_i}x_i\left((2\mu-1)f + \sum_{\ell=1}^{d}x_\ell\frac{\partial f}{\partial x_\ell}\right) + \left(\frac{d+1}{2}-\mu\right)^2 f.$$

Fourier expansions for orthogonal polynomials on the ball and on the simplex have been widely studied in the last years, see [6] and the references therein. In other domains (parabolic biangles, hexagonal and triangular domains or cylinders), the analysis of Fourier series is a more recent topic, see [15–17].

Our target in this paper is the analysis of the Poisson operator related to the orthonormal system $P_{n,j,\beta}^{d,\mu}$. This operator is defined in a spectral way by the identity

$$P_t^{d,\mu}f = e^{-t\sqrt{\mathcal{L}^{d,\mu}}}f.$$

Then for an appropriate function f,

$$P_t^{d,\mu}f(x) = \sum_{n=0}^{\infty}\sum_{j=0}^{[\frac{n}{2}]}\sum_{\beta=0}^{d(n-2j)} e^{-(n+\mu+\frac{d-1}{2})t}a_{n,j,\beta}(f)P_{n,j,\beta}^{d,\mu}(x),$$

where

$$a_{n,j,\beta}(f)=\int_{B^d} f(t)\overline{P_{n,j,\beta}^{d,\mu}}(t)W_\mu(t)\,dt.$$

Several authors have studied the Poisson operator for different orthonormal systems. In the case of the simplex (where a product of Jacobi polynomials of different orders appears) a first approach to the problem was done in [3]. In this setting, the problem has been analyzed more recently in [11, 12]. In some way, we will follow some of the ideas used in these papers but properly adapted to our context.

The most adequate spaces to analyze the convergence of the Poisson operator $P_t^{d,\mu}$ are the mixed norm spaces. Given a non-negative function w, we will consider the weighted mixed norm spaces

$$L^p_{\mathrm{rad}}L^2_{\mathrm{ang}}\big(B^d, w(\|x\|)\big)=\big\{f:\|f\|_{L^p_{\mathrm{rad}}L^2_{\mathrm{ang}}(B^d,w(\|x\|))}<\infty\big\},$$

with

$$\|f\|_{L^p_{\mathrm{rad}}L^2_{\mathrm{ang}}(B^d,w(\|x\|))}:=\left(\int_0^1\left(\int_{\mathbb{S}^{d-1}}\big|f(rx')\big|^2dx'\right)^{p/2}r^{d-1}w(r)\,dr\right)^{1/p}.$$

These spaces are used in harmonic analysis when spherical harmonics are involved. For example, they appear in the analysis of the disc multiplier and the Bochner-Riesz means for the Fourier transform [4, 5, 9, 10, 14]. Mixed norm spaces have been also used to treat the mean convergence of the Fourier-Bessel expansions in the multidimensional ball [2].

In these spaces, the boundedness of the Poisson operator will be reduced to a vector-valued inequality for the Poisson operator related to the Jacobi polynomials. In order to obtain this inequality we will need a very precise control of this operator in terms of all the parameters involved. The main estimate will be done by means of an improved version of a result in [13, Lemma 5.8], also used in [12].

In our first result we show a uniform weighted inequality for the Poisson operator. To this end we consider weights in the Muckenhoupt A_p class. Taking into account that we will reduce the inequality to an estimate for the Hardy-Littlewood maximal function on $(0,\pi)$ we give some definition properly adapted to our setting. A non-negative locally integrable function w on $(0,\pi)$ belongs to $A_p(0,\pi)$, $1<p<\infty$, when

$$\left(\frac{1}{|I|}\int_I w\right)\left(\frac{1}{|I|}\int_I w^{-\frac{1}{p-1}}\right)^{p-1}\le C,\qquad I\subseteq(0,\pi).$$

Being M the usual Hardy-Littlewood maximal function on $(0,\pi)$, i.e.,

$$Mf(x)=\sup_{r>0}\frac{1}{|\mathbf{B}_r(x)|}\int_{\mathbf{B}_r(x)}\big|f(y)\big|\,dy,$$

with $\mathbf{B}_r(x)=\{y\in(0,\pi):|x-y|<r\}$, it is well known that

$$\left\|\left(\sum_{k=0}^\infty Mf_k^2\right)^{1/2}\right\|_{L^p((0,\pi),w)}\le C_{p,w}\left\|\left(\sum_{k=0}^\infty f_k^2\right)^{1/2}\right\|_{L^p((0,\pi),w)},\tag{1}$$

for each weight $w \in A_p(0,\pi)$.

With the previous notation, the first result that we prove is the following

Theorem 1 *Assume that* $d \geq 2$, $\mu > 0$, $1 < p < \infty$, *and let* u *be a non-negative function on* $(0,1)$, *and*

$$U(\phi) = (\cos\phi/2)^{2\mu-\mu p}(\sin\phi/2)^{d-1-p(d/2-1/2)}u(\sin\phi/2).$$

If $U(\phi) \in A_p(0,\pi)$, *then*

$$\sup_{t>0}\left\| P_t^{d,\mu} f\right\|_{L^p_{\mathrm{rad}}L^2_{\mathrm{ang}}(B^d,u(\|x\|)W_\mu(\|x\|))} \leq C_{d,p,\mu,u}\|f\|_{L^p_{\mathrm{rad}}L^2_{\mathrm{ang}}(B^d,u(\|x\|)W_\mu(\|x\|))}.$$

As a standard consequence of this, we have

Corollary 2 *Assume that* $d \geq 2$, $\mu > 0$, $1 < p < \infty$, *and*

$$\max\left\{\frac{2d}{d+1}, \frac{2\mu+1}{\mu+1}\right\} < p < \min\left\{\frac{2d}{d-1}, \frac{2\mu+1}{\mu}\right\}. \tag{2}$$

Then

$$\sup_{t>0}\left\| P_t^{d,\mu} f\right\|_{L^p_{\mathrm{rad}}L^2_{\mathrm{ang}}(B^d,W_\mu(\|x\|))} \leq C_{d,p,\mu}\|f\|_{L^p_{\mathrm{rad}}L^2_{\mathrm{ang}}(B^d,W_\mu(\|x\|))}.$$

Of course, we can also deduce the convergence $P_t^{d,\mu} f \to f$, $t \to 0$, in the $L^p_{\mathrm{rad}}L^2_{\mathrm{ang}}(B^d, W_\mu(\|x\|))$-norm in the range of p given in (2).

Unfortunately, we cannot study the operator $\sup_{t>0}|P_t^{d,\mu} f|$. The reason is the following. If we first take the supremum, then we are missing the orthogonality of the spherical harmonics. Instead of this we prove

Theorem 3 *Assume that* $d \geq 2$, $\mu > 0$, $1 < p < \infty$, *and let* u *be a non-negative function on* $(0,1)$, *and*

$$U(\phi) = u(\sin\phi/2)(\cos\phi/2)^{2\mu-\mu p}(\sin\phi/2)^{d-1-p(d/2-1/2)}.$$

If $U(\phi) \in A_p(0,\pi)$, *then*

$$\begin{aligned}\left(\int_0^1\left(\sup_{t>0}\int_{\mathbb{S}^{d-1}}\left|P_t^{d,\mu} f\left(rx'\right)\right|^2 dx'\right)^{p/2} u(r)W_\mu(r)r^{d-1}\,dr\right)^{1/p} \\ \leq C_{d,p,\mu,u}\|f\|_{L^p_{rad}L^2_{\mathrm{ang}}(B^d,u(\|x\|)W_\mu(\|x\|))}.\end{aligned} \tag{3}$$

Note that the left hand side of (3) is bounded below by

$$\sup_{t>0}\left\| P_t^{d,\mu} f\right\|_{L^p_{\mathrm{rad}}L^2_{\mathrm{ang}}(B^d,u(\|x\|)W_\mu(\|x\|))}$$

and above by

$$\left\| \sup_{t>0} \left| P_t^{d,\mu} f \right| \right\|_{L^p_{\mathrm{rad}} L^2_{\mathrm{ang}}(B^d, W_\mu(\|x\|))},$$

therefore Theorem 3 implies Theorem 1.

The proof of Theorem 3 is contained in the next section. We will use two lemmas which will be proved in the last section.

2 Proof of Theorem 3

Replacing the index n throughout by $i = n - 2j$, we have

$$P_t^{d,\mu} f(x) = \sum_{i=0}^{\infty} \sum_{\beta=0}^{d(i)} \sum_{j=0}^{\infty} e^{-t(i+2j+\mu+\frac{d-1}{2})} a_{i+2j,j,\beta}(f) P_{i+2j,j,\beta}^{d,\mu}(x).$$

Each appropriate function defined on B^d can be written as

$$f(x) = \sum_{m=0}^{\infty} \sum_{\beta=0}^{d(m)} f_{\beta,m}(r) S_{\beta,m}\left(x'\right),$$

where $x = rx'$, with $x' \in \mathbb{S}^{d-1}$, $r \in (0, 1)$, and

$$f_{\beta,m}(r) = \int_{\mathbb{S}^{d-1}} f\left(rx'\right) \overline{S_{\beta,m}}\left(x'\right) d\sigma\left(x'\right). \tag{4}$$

Moreover

$$\|f\|^p_{L^p_{\mathrm{rad}} L^2_{\mathrm{ang}}(B^d, w(\|x\|))} = \int_0^1 \left(\sum_{m=0}^{\infty} \sum_{\beta=0}^{d(m)} \left| f_{\beta,m}(r) \right|^2 \right)^{p/2} w(r) r^{d-1}\, dr.$$

So, from (4), it is clear that

$$P_t^{d,\mu} f\left(rx'\right) = \sum_{i=0}^{\infty} \sum_{\beta=0}^{d(i)} \sum_{j=0}^{\infty} e^{-t(i+2j+\mu+\frac{d-1}{2})} b_{i,j}(f_{\beta,j}) r^j \mathcal{P}_j^{(\mu-1/2, i+\frac{d-2}{2})}(r) S_{\beta,j}\left(x'\right)$$

where $\mathcal{P}_j^{(\mu-1/2, i+\frac{d-2}{2})}$ are the normalized polynomials

$$\mathcal{P}_j^{(\mu-1/2, i+\frac{d-2}{2})}(r) = \left(C_{i+2j,j}^{d,\mu}\right)^{-1} p_j^{(\mu-1/2, i+\frac{d-2}{2})}\left(2r^2 - 1\right)$$

and

$$b_{i,j}(f) = \int_0^1 f(v) v^i \mathcal{P}_j^{(\mu-1/2, i+\frac{d-2}{2})}(v) W_\mu(v) v^{d-1}\, dv.$$

Now, for a function g defined on $(0,1)$, we take the operator

$$P_t^{d,\mu,i}g(r) := \sum_{j=0}^{\infty} e^{-t(i+2j+\mu+\frac{d-1}{2})} b_{i,j}(g) r^i \mathcal{P}_j^{(\mu-1/2,i+\frac{d-2}{2})}(r).$$

Thus the inequality in Theorem 3 will follow from the estimate

$$\int_0^1 \left(\sum_{i=0}^{\infty}\sum_{\beta=0}^{d(i)} \sup_{t>0} \left|P_t^{d,\mu,i} f_{\beta,i}(r)\right|^2\right)^{p/2} u(r) W_\mu(r) r^{d-1}\,dr$$
$$\leq C_{d,p,\mu,u} \int_0^1 \left(\sum_{i=0}^{\infty}\sum_{\beta=0}^{d(i)} \left|f_{\beta,i}(r)\right|^2\right)^{p/2} u(r) W_\mu(r) r^{d-1}\,dr. \tag{5}$$

The operator $P_t^{d,\mu,i}$ can be written as

$$P_t^{d,\mu,i} f(r) = \int_0^1 f(v) K_i^{\mu-1/2,\frac{d-2}{2}}(v,r) W_\mu(v) v^{d-1}\,dv$$

where

$$K_i^{a,b}(v,r) = (vr)^i \sum_{j=0}^{\infty} e^{-t(i+2j+a+b+1)} \mathcal{P}_j^{(a,i+b)}(v) \mathcal{P}_j^{(a,i+b)}(r).$$

In this manner, the proof of the inequality (5) will be a consequence of Lemmas 4 and 5. The first one shows an integral expression for the kernel $K_i^{a,b}$ and the second one provides an estimate for the integrals appearing in the first lemma.

Lemma 4 *For $a,b > -1/2$ and $i \geq 0$, we have the identity*

$$K_i^{a,b}(v,r) = (vr)^i \frac{\Gamma(i+a+b+2)\sinh t}{2^{i+a+b+2}\pi\,\Gamma(a+1/2)\Gamma(b+i+1/2)}$$
$$\times \int_{-1}^1\int_{-1}^1 f_{i+a+b+1}(t,v,r,x,y)\left(1-x^2\right)^{a-1/2}\left(1-y^2\right)^{i+b-1/2}\,dx\,dy,$$

where

$$f_\lambda(t,v,r,x,y) = \frac{1}{(\cosh t - x\sqrt{1-v^2}\sqrt{1-r^2} - yvr)^{\lambda+1}}.$$

Lemma 5 *For $\gamma,\lambda > -1/2$ and $A > B > 0$ it is verified that*

$$\int_{-1}^1 \frac{(1-x^2)^{\gamma-1/2}}{(A-Bx)^{\gamma+\lambda+1}}\,dx \leq \frac{\Gamma(\gamma+1/2)\Gamma(\lambda+1/2)}{\Gamma(\gamma+\lambda+1)} \frac{2^{\gamma+1/2}}{B^{\gamma+1/2}(A-B)^{\lambda+1/2}}.$$

Lemma 5 is an improved version of Lemma 5.8 in [13]. The improvement of our version lies on the control over *all* the constants involved in the estimate.

By Lemma 4 and applying Lemma 5 twice (firstly with $\gamma = i + b$, $\lambda = a + 1$, $A = \cosh t - x\sqrt{1-v^2}\sqrt{1-r^2}$, and $B = vr$, and secondly with $\gamma = a$, $\lambda = 1/2$, $A = \cosh t - vr$, and $B = \sqrt{1-v^2}\sqrt{1-r^2}$), we deduce the following estimate

$$K_i^{a,b}(v,r) \le \frac{1}{2\pi} \frac{1}{(vr)^{b+1/2}(\sqrt{1-v^2}\sqrt{1-r^2})^{a+1/2}} \frac{\sinh t}{\cosh t - vr - \sqrt{1-v^2}\sqrt{1-r^2}},$$

for $a, b > -1/2$ and $i \ge 0$. Now, with the changes of variable $v = \sin\phi/2$ and $r = \sin\theta/2$ and taking the function $g_{\beta,i}(\phi) = (\sin\phi/2)^{d/2-1/2}(\cos\phi/2)^{\mu} f_{\beta,i}(\sin\phi/2)$, the vector-valued inequality (5) follows from

$$\int_0^{\pi}\left(\sum_{i=0}^{\infty}\sum_{\beta=0}^{d(i)}\sup_{t>0}\left|\int_0^{\pi}\frac{g_{\beta,i}(\theta)\sinh t}{\cosh t - 1 + 2(\sin\frac{\phi-\theta}{4})^2}\,d\theta\right|^2\right)^{p/2} U(\phi)\,d\phi$$
$$\le C_{d,p,\mu,u}\int_0^{\pi}\left(\sum_{i=0}^{\infty}\sum_{\beta=0}^{d(i)}\left|g_{\beta,i}(\phi)\right|^2\right)^{p/2} U(\phi)\,d\phi, \tag{6}$$

with $U(\phi) = (\cos\phi/2)^{2\mu-\mu p}(\sin\phi/2)^{d-1-p(d/2-1/2)}u(\sin\phi/2)$.

Finally, (6) is a consequence of the pointwise inequality

$$\sup_{t>0}\left|\int_0^{\pi}\frac{g(\theta)\sinh t}{\cosh t - 1 + 2(\sin\frac{\phi-\theta}{4})^2}\,d\theta\right| \le CMg(\theta) \tag{7}$$

and the vector-valued inequality (1). The estimate in (7) follows by using the bounds

$$\frac{\sinh t}{\cosh t - 1 + 2(\sin\frac{\phi-\theta}{4})^2} \le C\frac{t}{t^2 + |\phi-\theta|^2},$$

for $0 < t < T$, with $T > 1$, and

$$\frac{\sinh t}{\cosh t - 1 + 2(\sin\frac{\phi-\theta}{4})^2} \le C,$$

for $t \ge T$.

3 Proof of Lemmas 4 and 5

Proof of Lemma 4 In order to obtain the expression for the kernel, we will apply the product formula due to Dijksma and Koornwinder [7]

$$p_k^{(a,b)}(2v^2-1)p_k^{(a,b)}(2r^2-1)$$
$$= \frac{\Gamma(a+b+1)\Gamma(k+a+1)\Gamma(k+b+1)}{\pi k!\Gamma(k+a+b+1)\Gamma(a+1/2)\Gamma(b+1/2)}$$

$$\times \int_{-1}^{1}\int_{-1}^{1} C_{2k}^{a+b+1}\big(x\sqrt{1-v^2}\sqrt{1-r^2}+yvr\big)$$
$$\times \big(1-x^2\big)^{a-1/2}\big(1-y^2\big)^{b-1/2}\,dx\,dy$$

valid for $a, b > -1/2$, here C_n^λ is the Gegenbauer polynomial of degree n and order λ. Thus, we have

$$K_i^{a,b}(v,r) = \frac{\Gamma(i+a+b+1)}{2\pi\Gamma(a+1/2)\Gamma(b+i+1/2)} \times \int_{-1}^{1}\int_{-1}^{1} S_{i+a+b+1}(t,v,r,x,y)\big(1-x^2\big)^{a-1/2}\big(1-y^2\big)^{i+b-1/2}\,dx\,dy,$$

with

$$S_\lambda(t,v,r,x,y) = \sum_{k=0}^{\infty} e^{-t(2k+\lambda)}(2k+\lambda)C_{2k}^{\lambda}\big(x\sqrt{1-v^2}\sqrt{1-r^2}+yvr\big).$$

Now recall the identity [1, 1.27]

$$\sum_{n=0}^{\infty}(n+\lambda)C_n^\lambda(x)z^n = \frac{\lambda(1-z^2)}{(1-2xz+z^2)^{\lambda+1}}, \quad |z|<1,\ \lambda>0.$$

On the other hand, note that Gegenbauer polynomials of even (respectively, odd) orders are even (respectively, odd) functions. Hence, we get

$$S_\lambda(t,v,r,x,y) = \frac{\lambda\sinh t}{2^{\lambda+1}}\big(f_\lambda(t,v,r,x,y)+f_\lambda(t,v,r,-x,-y)\big).$$

By the symmetry of the integrals defining $K_j^{a,b}$, we conclude that

$$K_i^{a,b}(v,r) = \frac{\Gamma(i+a+b+2)\sinh t}{2^{i+a+b+2}\pi\Gamma(a+1/2)\Gamma(b+i+1/2)} \times \int_{-1}^{1}\int_{-1}^{1} f_{i+a+b+1}(t,v,r,x,y)\big(1-x^2\big)^{a-1/2}\big(1-y^2\big)^{i+b-1/2}\,dx\,dy.$$

□

Proof of Lemma 5 Denote

$$I_{\gamma,\lambda} = \int_{-1}^{1}\frac{(1-x^2)^{\gamma-1/2}}{(A-Bx)^{\gamma+\lambda+1}}\,dx.$$

We have

$$I_{\gamma,\lambda} \le 2\int_0^1 \frac{(1-x^2)^{\gamma-1/2}}{(A-Bx)^{\gamma+\lambda+1}}\,dx \le 2^{\gamma+\frac{1}{2}}\int_0^1\frac{(1-x)^{\gamma-1/2}}{(A-Bx)^{\gamma+\lambda+1}}\,dx.$$

With the change of variable $1 - x = t$, we obtain

$$I_{\gamma,\lambda} \le 2^{\gamma+1/2} \int_0^1 \frac{t^{\gamma-1/2}}{(A - B + Bt)^{\gamma+\lambda+1}}\, dt$$
$$= \frac{2^{\gamma+1/2}}{(A-B)^{\gamma+\lambda+1}} \int_0^1 \frac{t^{\gamma-1/2}}{(1 + \frac{B}{A-B}t)^{\gamma+\lambda+1}}\, dt.$$

Finally, we conclude with the change of variable $\frac{B}{A-B}t = z$. Indeed,

$$I_{\gamma,\lambda} \le \frac{2^{\gamma+1/2}}{B^{\gamma+1/2}(A-B)^{\lambda+1/2}} \int_0^{\frac{A-B}{A}} \frac{z^{\gamma-1/2}}{(1+z)^{\gamma+\lambda+1}}\, dz$$
$$\le \frac{2^{\gamma+1/2}}{B^{\gamma+1/2}(A-B)^{\lambda+1/2}} \int_0^{\infty} \frac{z^{\gamma-1/2}}{(1+z)^{\gamma+\lambda+1}}\, dz$$
$$= \frac{2^{\gamma+1/2}}{B^{\gamma+1/2}(A-B)^{\lambda+1/2}} \frac{\Gamma(\gamma+1/2)\Gamma(\lambda+1/2)}{\Gamma(\gamma+\lambda+1)}.$$ □

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