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Mean Value Properties of Harmonic Functions on Sierpinski Gasket Type Fractals

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Abstract In this paper, we establish an analogue of the classical mean value property for both the harmonic functions and some general functions in the domain of the Laplacian on the Sierpinski gasket. Furthermore, we extend the result to some other p.c.f. fractals with Dihedral-3 symmetry.

Keywords Sierpinski gasket · Laplacian · Harmonic function · Mean value property · Analysis on fractals

Mathematics Subject Classification 28A80

1 Introduction

It is well known that harmonic functions (i.e., solutions of the Laplace equation $\Delta u = 0$, where $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$) possess the *mean value property*: Namely, if u is harmonic on a domain $\Omega \subset \mathbb{R}^d$, then for every closed ball $B_r(x) \subset \Omega$ of a center

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$x \in \Omega$ and radius $r > 0$ the average of u over $B_r(x)$ equals to the value of x, i.e.,

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)dy = u(x),$$

where $|B_r(x)|$ is the volume of the ball $B_r(x)$. There is a similar statement for mean values on spheres. More generally, if u is not assumed harmonic but Δu is a continuous function, then

$$\lim_{r\to 0}\frac{1}{r^2}\left(\frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)dy - u(x)\right) = c_n \Delta u(x) \tag{1.1}$$

for the appropriate dimensional constant c_n.

What are the fractal analogs of these results? The analytic theory on p.c.f. fractals was developed by Kigami [3–5] following the work of several probabilists who constructed stochastic processes analogous to Brownian motion, thus obtaining a Laplacian indirectly as the generator of the process. See the book of Barlow [1] for an account of this development. Since analysis on fractals has been made possible by the analytic definition of Laplacian, it is natural to explore the properties of these fractal Laplacians that are natural analogs of results that are known for the usual Laplacian. As for the fractal analog of the mean value property, we won't state the nature of the sets on which we do the averaging here, but will say that if K is a fractal set and $x \in K$, we investigate whether there is a sequence of sets $B_k(x)$ containing x with $\bigcap_k B_k(x) = \{x\}$ such that

$$\frac{1}{\mu(B_k(x))}\int_{B_k(x)} u(y)dy = u(x)$$

for every harmonic function u. Moreover, for general u not assumed harmonic, is there a formula analogous to (1.1)?

In the present paper, we will mainly deal with the Sierpinski gasket $\mathcal{SG}$. This set is a key example of fractals on which a well established theory of Laplacian exists [3–7]. Since the mean value property plays a very important role in the usual theory of harmonic functions, it is of independent interest to understand the similar property of harmonic functions on the Sierpinski gasket. We will prove that for each point $x \in \mathcal{SG} \setminus V_0$, ($V_0$ is the boundary of $\mathcal{SG}$.) there is a sequence of *mean value neighborhoods* $B_k(x)$ depending only on the location of x in $\mathcal{SG}$. $\{B_k(x)\}$ forms a system of neighborhoods of the point x satisfying $\bigcap_k B_k(x) = \{x\}$. On such sequences, we get the fractal analogs of the mean value properties of both the harmonic functions and the general functions which belong to the domain of the fractal Laplacian satisfying some natural continuity assumption. We also investigate the extent to which our method can be applicable to other p.c.f. self-similar sets, but it seems that it strongly depends on the symmetric properties of both the geometric structure and the harmonic structure of the fractals.

The paper is organized as follows: In Sect. 2 we briefly introduce some key notions from analysis on the Sierpinski gasket. In Sects. 3 and 4, we prove the mean value property for harmonic functions and general functions on $\mathcal{SG}$ respectively. Section 5 contains a further extension of the mean value property to p.c.f. self-similar fractals

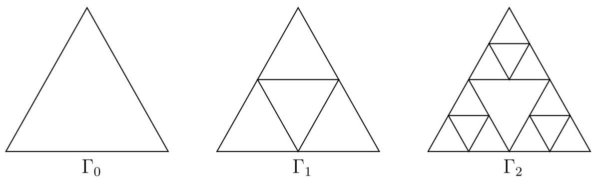

Fig. 1 The first 3 graphs, $\Gamma_0, \Gamma_1, \Gamma_2$ in the approximation to the Sierpinski gasket

with Dihedral-3 symmetry. An interesting open question is to what extent the results of Sect. 4 can be extended to this class of fractals. See [2] for a related result concerning solutions of divergence form elliptic operators.

2 Analysis on the Sierpinski Gasket

For the convenience of the reader, we collect some key facts from analysis on $\mathcal{SG}$ that we need to state and prove our results. These come from Kigami's theory of analysis on fractals, and may be found in [3–5]. An elementary exposition may be found in [6, 7]. Recall that $\mathcal{SG}$ is the attractor of the i.f.s (iterated function system) in the plane consisting of three homotheties $\{F_0, F_1, F_2\}$ with contraction ratio $1/2$ and fixed points equal to the three vertices $\{q_0, q_1, q_2\}$ of an equilateral triangle. Then $\mathcal{SG}$ is the unique nonempty compact set satisfying

$$\mathcal{SG} = \bigcup_{i=0}^{2} F_i(\mathcal{SG}). \tag{2.1}$$

We refer to the sets $F_i(\mathcal{SG})$ as *cells* of level one, and by iterating (2.1) we obtain the splitting of $\mathcal{SG}$ into cells of higher level. For a word $w = (w_1, w_2, \ldots, w_m)$ of length m, the set $F_w(\mathcal{SG}) = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m}(\mathcal{SG})$ with $w_i \in \{0, 1, 2\}$, is called an m-cell. The fractal $\mathcal{SG}$ can be realized as the limit of a sequence of graphs $\Gamma_0, \Gamma_1, \ldots$ with vertices $V_0 \subseteq V_1 \subseteq \cdots$. The initial graph Γ_0 is just the complete graph on $V_0 = \{q_0, q_1, q_2\}$, which is considered the boundary of $\mathcal{SG}$. See Fig. 1. Note that $\mathcal{SG}$ is connected, but just barely: there is a dense set of points $\mathcal{J}$, called *junction points*, defined by the condition that $x \in \mathcal{J}$ if and only if $U \setminus \{x\}$ is disconnected for all sufficiently small neighborhoods U of x. It is easy to see that $\mathcal{J}$ consists of all images of $\{q_0, q_1, q_2\}$ under iterates of the i.f.s. The vertices $\{q_0, q_1, q_2\}$ are not junction points. All other points in $\mathcal{SG}$ will be called *generic points*. In the $\mathcal{SG}$ case, $\mathcal{J} = V_* \setminus V_0$, where $V_* = \bigcup_m V_m$. However, it is not true for general p.c.f. self-similar sets. In all that follows, we assume that $\mathcal{SG}$ is equipped with the self-similar probability measure μ that assigns the measure 3^{-m} to each m-cell.

We define the unrenormalized energy of a function u on Γ_m by

$$E_m(u) = \sum_{x \sim_m y} (u(x) - u(y))^2.$$

The energy renormalization factor is $r = \frac{3}{5}$, so the renormalized graph energy on Γ_m is

$$\mathcal{E}_m(u) = r^{-m} E_m(u),$$

and we can define the *fractal energy* $\mathcal{E}(u) = \lim_{m\to\infty} \mathcal{E}_m(u)$. We define $\operatorname{dom}\mathcal{E}$ as the space of continuous functions with finite energy. Then $\mathcal{E}$ extends by polarization to a bilinear form $\mathcal{E}(u, v)$ which serves as an inner product in this space.

The standard Laplacian may then be defined using the weak formulation: $u \in \operatorname{dom}\Delta$ with $\Delta u = f$ if f is continuous, $u \in \operatorname{dom}\mathcal{E}$, and

$$\mathcal{E}(u, v) = -\int f v \, d\mu$$

for all $v \in \operatorname{dom}_0\mathcal{E}$, where $\operatorname{dom}_0\mathcal{E} = \{v \in \mathcal{E} : v|_{V_0} = 0\}$. There is also a pointwise formula (which is proven to be equivalent in [7]) which, for points in $V_* \setminus V_0$ computes

$$\Delta u(x) = \frac{3}{2} \lim_{m\to\infty} 5^m \Delta_m u(x),$$

where Δ_m is a discrete Laplacian associated to the graph Γ_m, defined by

$$\Delta_m u(x) = \sum_{y \sim_m x} \big(u(y) - u(x)\big)$$

for x not on the boundary.

It is not necessary to invoke the measure to define *harmonic functions*, although it is true that these are just the solutions of $\Delta h = 0$. The more direct definition is that

$$h(x) = \frac{1}{4} \sum_{y \sim_m x} h(y)$$

for every nonboundary point and every m. This can be viewed as a mean value property of h at the junction points. The space of harmonic functions is 3-dimensional and the values at the 3 boundary points may be freely assigned. Moreover, there is a simple efficient algorithm, the "$\frac{1}{5} - \frac{2}{5}$ rule", for computing the values of a harmonic function exactly at all vertex points in terms of the boundary values. The harmonic functions satisfy the *maximum principle*, i.e., the maximum and minimum are attained on the boundary and only on the boundary if the function is not constant. We call a continuous function h a *piecewise harmonic spline* of level m if $h \circ F_w$ is harmonic for all $|w| = m$.

The Laplacian satisfies the scaling property

$$\Delta(u \circ F_i) = \frac{1}{5}(\Delta u) \circ F_i$$

and by iteration

$$\Delta(u \circ F_w) = \frac{1}{5^m}(\Delta u) \circ F_w$$

for $F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m}$.

Although there is no satisfactory analogue of gradient, there is a *normal derivative* $\partial_n u(q_i)$ defined at boundary points by

$$\partial_n u(q_i) = \lim_{m\to\infty} \sum_{y \sim_m q_i} r^{-m} \big(u(q_i) - u(y)\big),$$

the limit existing for all $u \in \operatorname{dom}\Delta$. The definition may be localized to boundary points of cells: for each point $x \in V_m \setminus V_0$, there are two cells containing x as a boundary point, hence two normal derivatives at x. For $u \in \operatorname{dom}\Delta$, the normal derivatives at x satisfy the *matching condition* that their sum is zero. The matching conditions allow us to glue together local solutions to $\Delta u = f$.

As is shown in [3, 4, 7], the Dirichlet problem for the Laplacian can be solved by integrating against an explicitly given *Green's function*. Recall that the Green's function $G(x, y)$ is a uniform limit of $G_M(x, y)$ as M goes to the infinity, with G_M defined by

$$G_M(x,y) = \sum_{m=0}^{M} \sum_{z,z' \in V_{m+1}\setminus V_m} g(z,z')\psi_z^{(m+1)}(x)\psi_{z'}^{(m+1)}(y)$$

and

$$\begin{cases} g(z,z) = \frac{9}{50} r^m & \text{for } z \in V_{m+1} \setminus V_m, \\ g(z,z') = \frac{3}{50} r^m & \text{for } z, z' \in V_{m+1} \setminus V_m \\ & \quad \text{with } z, z' \in F_w(\mathcal{SG}) \text{ for } |w| = m, \text{ and } z \neq z', \end{cases}$$

where $\psi_z^m(x)$ denotes a piecewise harmonic spline of level m satisfying $\psi_z^{(m)}(x) = \delta_z(x)$ for $x \in V_m$.

3 Mean Value Property of Harmonic Functions on $\mathcal{SG}$

Lemma 3.1 (a) *Let C be any cell with boundary points p_0, p_1, p_2, and h any harmonic function. Then*

$$\frac{1}{\mu(C)} \int_C h d\mu = \frac{1}{3}(h(p_0) + h(p_1) + h(p_2)).$$

(b) *Let p be any junction point, and C_1, C_2 the two m-cells containing p. Then*

$$\frac{1}{\mu(C_1 \cup C_2)} \int_{C_1 \cup C_2} h d\mu = h(p).$$

Proof The space of harmonic functions on C is three-dimensional. A simple basis $\{h_0, h_1, h_2\}$ is obtained by taking $h_j(p_j) = 1$ and $h_j(p_k) = 0$ for $k \neq j$. Noticing that $h_0 + h_1 + h_2$ is identically 1 on C, by symmetry, $\int_C h_i d\mu = \frac{1}{3}\mu(C)$ for each i. Hence (a) follows. (b) follows by combining (a) for $C = C_1$ and $C = C_2$ with the mean value property of h at p. □

Note that (b) gives a trivial solution to the problem of finding mean value neighborhoods for junction points.

Given a point x in $\mathcal{SG} \setminus V_0$, consider any cell $F_w(\mathcal{SG})$ (denote it by C_w) containing the point x, with boundary points $F_w q_i = p_i$. Choose the cell C_w small enough, such

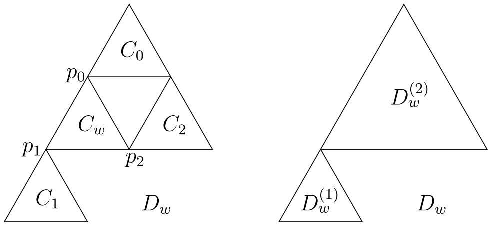

Fig. 2 C_w and its three neighboring cells. The *right* part of the figure refers to the proof of Lemma 4.1

that it does not intersect V_0. Then it must have three neighboring cells C_0, C_1 and C_2 of the same level with C_i intersecting C_w at p_i. Denote by D_w the union of C_w and its three neighbors. See Fig. 2. In this section, we will describe a method to find a subset B of D_w, containing C_w, such that for any harmonic function h, the mean value of h over B is equal to its value at x, i.e., $M_B(h) = h(x)$ where $M_B(h)$ is defined by

$$M_B(h) = \frac{1}{\mu(B)} \int_B h d\mu.$$

Then we will call the set B a *k level mean value neighborhood* of x associated to C_w where k is the length of w.

Let h be a harmonic function on $\mathcal{SG}$. The harmonic extension algorithm implies that there exist coefficients $\{a_i(x)\}$ depending only on the relative position of x and C_w such that

$$h(x) = \sum_i a_i(x) h(p_i).$$

Moreover, since constants are harmonic we must have

$$\sum_i a_i(x) = 1$$

and by the maximum principle all $a_i(x) \geq 0$. Let W denote the triangle in $\mathbb{R}^3$ with boundary points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ and π_W the plane in $\mathbb{R}^3$ containing W. So $\{(a_0(x), a_1(x), a_2(x))\} \in W$ for any $x \in C_w$. However, not every point in W occurs in this way.

On the other hand, given a set B such that $C_w \subset B \subset D_w$, by linearity we have

$$M_B(h) = \sum_i a_i h(p_i) \tag{3.1}$$

for some coefficients (a_0, a_1, a_2) depending only on the relative geometry of B and C_w. Again we must have $\sum a_i = 1$ by considering $h \equiv 1$. So $(a_0, a_1, a_2) \in \pi_W$. (Later we will show that (a_0, a_1, a_2) does not have to belong to W for some sets B.) Thus we have a map, denoted by $\mathcal{T}$ from the collection of B's to π_W. If we can show that the image of the map $\mathcal{T}$ covers the triangle W for some reasonable class of sets B, then we can get a set B over which the mean value property holds for all harmonic functions. Moreover, if we can prove $\mathcal{T}$ is one-to-one, then we get a mean value neighborhood B of x associated to C_w, that is unique within the collection of sets we are considering.

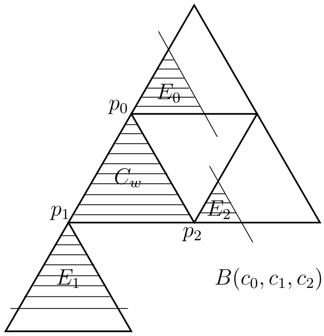

Fig. 3 The relative geometry of $B(c_0, c_1, c_2)$ and C_w

The above is the basic idea of our method. Hence, the remaining task in this section is to find a suitable class $\mathcal{B}$ of sets B such that there is a map $\mathcal{T}$ from $\mathcal{B}$ to π_W, such that $\mathcal{T}(\mathcal{B})$ covers the triangle W. Comparing with the usual mean value neighborhoods (they are just balls in the Euclidean case), it is reasonable to require B to be as simple as possible. They should be connected, possess some symmetry properties, depend only on the relative geometry of x and C_w, and be independent of the level of C_w and the location of C_w.

In the following, we use ρ to denote the distance from p_0 to the line containing p_1 and p_2, namely, ρ is the length of the height of the minimal equilateral triangle containing C_w. Call ρ the *size* of C_w.

Definition 3.1 Let c_0, c_1, c_2 be three real numbers satisfying $0 \le c_i \le 1$, denote by $B(c_0, c_1, c_2)$ the set

$$B(c_0, c_1, c_2) = C_w \cup E_0 \cup E_1 \cup E_2,$$

where each E_i is a sub-triangle domain in C_i obtained by cutting C_i symmetrically with a line at the distance $c_i\rho$ away from the vertex p_i.

Remark See Fig. 3 for a sketch of $B(c_0, c_1, c_2)$. For example, $B(0,0,0) = C_w$ and $B(1,1,1) = D_w$. Denote by

$$\mathcal{B} = \{B(c_0, c_1, c_2) : 0 \le c_i \le 1\}$$

the natural 3-parameter family of all such sets. Each member of $\mathcal{B}$ contains C_w and is contained in D_w. Denote by

$$\sigma : \mathcal{B} \mapsto \Lambda$$

the natural one-to-one projection with $\sigma(B(c_0, c_1, c_2)) = (c_0, c_1, c_2)$, where $\Lambda = \{(c_0, c_1, c_2) : 0 \le c_i \le 1\}$.

For each vector $(c_0, c_1, c_2) \in \Lambda$, there is a unique vector $(a_0, a_1, a_2) \in \pi_W$ corresponding to the set $B(c_0, c_1, c_2)$, satisfying (3.1) where B is replaced by $B(c_0, c_1, c_2)$. This defines a map T from Λ to π_W. Then $\mathcal{T}$ described above from $\mathcal{B}$ to π_W is exactly $\mathcal{T} = T \circ \sigma$.

The following lemma shows that the value $T(c_0, c_1, c_2)$ is independent of the particular choice of C_w, which benefits from the symmetric properties of the set $B(c_0, c_1, c_2)$.

Lemma 3.2 *$T(c_0, c_1, c_2)$ is independent of the particular choice of C_w.*

Proof Let h be a harmonic function. First we consider the integral $\int_{E_i} h d\mu$. Denote by $\{s_i, t_i, p_i\}$ the boundary points of C_i. By linearity, $\frac{1}{\mu(C_w)} \int_{E_i} h d\mu$ can be expressed as a non-negative linear combination of $\{h(s_i), h(t_i), h(p_i)\}$, which by symmetry must have the form

$$\int_{E_i} h d\mu = \big(m_i h(p_i) + n_i\big(h(s_i) + h(t_i)\big)\big)\mu(C_w), \tag{3.2}$$

for some appropriate non-negative coefficients m_i, n_i. Notice that in (3.2), the coefficients m_i, n_i are independent of the location of C_i in $\mathcal{SG}$. Actually, they only depend on the relative position of E_i in C_i, i.e., m_i, n_i depend only on c_i. Using the mean value property at p_i, namely

$$4h(p_i) = h(p_{i-1}) + h(p_{i+1}) + h(s_i) + h(t_i),$$

we obtain

$$\begin{aligned}\int_{E_i} h d\mu &= \big(m_i h(p_i) + n_i\big(4h(p_i) - h(p_{i-1}) - h(p_{i+1})\big)\big)\mu(C_w)\\ &= \big((4n_i + m_i)h(p_i) - n_i\big(h(p_{i-1}) + h(p_{i+1})\big)\big)\mu(C_w).\end{aligned}$$

Notice that the ratio of $\mu(E_i)$ to $\mu(C_i)$ also depends only on c_i. Combined with Lemma 3.1(a), we see that $(a_0, a_1, a_2) = T(c_0, c_1, c_2)$ is independent of the particular choice of C_w, depending only on (c_0, c_1, c_2). □

We will show the image of the map $\mathcal{T}$ covers the triangle W. More precisely, $T(c_0, c_1, c_2)$ will fill out a set $\widetilde{W}$ which is a bit larger than W. Denote by $P_0 = (1, 0, 0)$, $P_1 = (0, 1, 0)$ and $P_2 = (0, 0, 1)$ the three boundary points of the triangle W in $\mathbb{R}^3$ and by O the center point of W.

Lemma 3.3 *$T(0, 0, 1) = P_2$ and $T(0, 1, 1) = Q_0$ where $Q_0 = \{-\frac{1}{9}, \frac{5}{9}, \frac{5}{9}\}$ is a point in π_W located outside of W.*

Proof From Definition 3.1, $B(0, 0, 1) = C_w \cup C_2$. Hence by Lemma 3.1(b), for any harmonic function h, we have $M_{B(0,0,1)}(h) = h(p_2)$. This implies $T(0, 0, 1) = P_2$. Similarly, $B(0, 1, 1) = C_w \cup C_1 \cup C_2$, then for any harmonic function h, still using Lemma 3.1, we get

$$\begin{aligned}M_{B(0,1,1)}(h) &= \frac{1}{3\mu(C)}\left(\int_{C_w \cup C_1} h d\mu + \int_{C_w \cup C_2} h d\mu - \int_{C_w} h d\mu\right)\\ &= -\frac{1}{9}h(p_0) + \frac{5}{9}h(p_1) + \frac{5}{9}h(p_2),\end{aligned}$$

which gives $T(0, 1, 1) = Q_0$. □

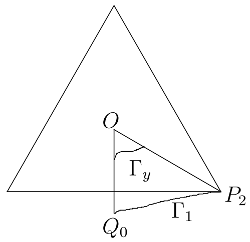

Fig. 4 A 1/6 region of $\widetilde{W}$ surrounded by $\overline{OQ_0}$, $\overline{OP_2}$ and $\widehat{P_2Q_0}$

Lemma 3.4 *$T(\{(0,c,1): 0 \le c \le 1\})$ is a continuous curve lying outside of W, joining P_2 and Q_0. (See Fig. 4.)*

Proof From Lemma 3.3, by varying c continuously between 0 and 1 we trace a continuous curve $\widehat{P_2Q_0}$ joining P_2 and Q_0. So we only need to prove the curve $\widehat{P_2Q_0}$ lies outside of W. To prove this, we consider the set $B = B(0,c,1)$ for $0 \le c \le 1$. In this case

$$B = C_w \cup E_1 \cup C_2.$$

Given a harmonic function h, by the proof of Lemma 3.2, we have

$$\int_{E_1} h d\mu = \big((4n_1 + m_1)h(p_1) - n_1\big(h(p_0) + h(p_2)\big)\big)\mu(C_w),$$

for some appropriate non-negative coefficients m_1, n_1 depending only on c.

On the other hand, we have

$$\int_{C_w \cup C_2} h d\mu = 2h(p_2)\mu(C_w),$$

by Lemma 3.1(b).

Hence

$$\int_B h d\mu = \int_{E_1} h d\mu + \int_{C_w \cup C_2} h d\mu$$
$$= \big(-n_1 h(p_0) + (4n_1 + m_1)h(p_1) + (2 - n_1)h(p_2)\big)\mu(C_w).$$

The coefficient of $h(p_0)$ is always less than 0. Moreover, it equals to 0 if and only if $E_1 = \emptyset$ ($c = 0$). Hence $T(0,c,1)$ will always lie on the outside of the triangle W as c varies between 0 and 1. □

Now we come to the main result of this section.

Theorem 3.1 *The map $\mathcal{T}$ from $\mathcal{B}$ to π_W fills out a region $\widetilde{W}$ which contains the triangle W.*

Proof We only need to prove the map $\mathcal{T}$ from $\mathcal{B}$ to π_W fills out a 1/6 region surrounded by the line segments $\overline{OQ_0}$, $\overline{OP_2}$ and the curve $\widehat{P_2Q_0}$ as shown in Fig. 4. Then we will get the desired result by exploiting the symmetry.

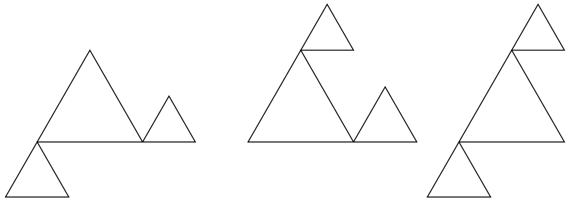

Fig. 5 The 3 shapes of $B \in \mathcal{B}^*$ associated to C_w shown in Fig. 2

Consider a subfamily $\mathcal{B}_1 = \{B(0,0,c) : 0 \le c \le 1\}$ of $\mathcal{B}$. If we restrict the map $\mathcal{T}$ to $\mathcal{B}_1$, by varying c continuously between 0 and 1 we trace a curve (it is a line segment, which follows from the symmetry of E_2) in W joining the center O and the vertex point P_2.

Consider another subfamily $\mathcal{B}_2 = \{B(0,c,c) : 0 \le c \le 1\}$ of $\mathcal{B}$. If we restrict the map $\mathcal{T}$ to $\mathcal{B}_2$, by varying c continuously between 0 and 1 we trace a curve (it is also a line segment, which follows from the symmetric effect of E_1 and E_2) in W joining the center O and the point Q_0 across the boundary line $\overline{P_1P_2}$ with Q_0 located outside of W, where Q_0 is the point defined in Lemma 3.3.

Fix a number $0 \le y \le 1$. Consider a subfamily $\mathcal{C}_y = \{B(0,c,y) : 0 \le c \le y\}$ of $\mathcal{B}$. If we restrict the map $\mathcal{T}$ to $\mathcal{C}_y$, by varying c continuously between 0 and y we trace a curve Γ_y joining the two points $T(0,0,y)$ and $T(0,y,y)$. The first endpoint $T(0,0,y)$ lies on the line segment $\overline{OP_2}$ and the second endpoint $T(0,y,y)$ lies on the line segment $\overline{OQ_0}$. (See Fig. 4 for Γ_y.) When $y=0$, the curve Γ_0 draws back to the single center point O. When $y=1$, by Lemma 3.4, the curve Γ_1 is a continuous curve located outside of the triangle W. Moreover, P_2 is the only common points of Γ_1 and W. Hence if we vary y continuously between 0 and 1, we can fill out the 1/6 region surrounded by the line segments $\overline{OQ_0}$, $\overline{OP_2}$ and the curve $\widehat{P_2Q_0}$. □

Remark In the proof of the above theorem, we actually only consider those sets B in $\mathcal{B}$ which are contained in the union of C_w and subsets of only two neighbors. See Fig. 5. Of course, the map $\mathcal{T}$ restricted to this subfamily is one-to-one, which can be easily seen from the proof. Hence instead of $\mathcal{B}$, the map $\mathcal{T}$ is one-to-one from $\mathcal{B}^*$ onto $\widetilde{W}$, where

$$\mathcal{B}^* = \{B(0,c_1,c_2) : 0 \le c_i \le 1\} \cup \{B(c_0,0,c_2) : 0 \le c_i \le 1\} \\ \cup \{B(c_0,c_1,0) : 0 \le c_i \le 1\}.$$

Based on the discussion in the beginning of this section, we then have

Theorem 3.2 *For each point $x \in \mathcal{SG} \setminus V_0$, there exists a system of mean value neighborhoods $B_k(x)$ with $\bigcap_k B_k(x) = \{x\}$.*

Proof Let k_0 be the smallest value of k such that there exists a k level cell C_w containing x but not intersecting V_0. (k_0 depends on the location of x in $\mathcal{SG}$.) Then by using Theorem 3.1 we can find a sequence of words $w^{(k)}$ of length k ($k \ge k_0$) and a sequence of mean value neighborhoods $B_k(x)$ associated to $C_{w^{(k)}}$. Obviously, $\{B_k(x)\}_{k \ge k_0}$ will form a system of neighborhoods of the point x satisfying $\bigcap_{k \ge k_0} B_k(x) = \{x\}$. □

4 Mean Value Property of General Functions on $\mathcal{SG}$

In this section, we extend the mean value property to more general functions on $\mathcal{SG}$. Given a point x in $\mathcal{SG} \setminus V_0$ and a cell $C_w = F_w(\mathcal{SG})$ containing x, for each mean value neighborhood B of x associated to C_w, we assign a constant c_B to B. We want

$$M_B(u) - u(x) \approx c_B \Delta u(x)$$

for u in dom Δ. More precisely, let $\{B_k(x)\}_{k \geq k_0}$ be the system of mean value neighborhoods of the point x; we want

$$\lim_{k \to \infty} \frac{1}{c_{B_k(x)}} \big(M_{B_k(x)} - u(x) \big) = \Delta u(x) \tag{4.1}$$

for appropriate functions in the domain of Δ, which is the desired fractal analog of (1.1).

For this purpose, let v be a function on $\mathcal{SG}$ satisfying $\Delta v \equiv 1$. For each point x in $\mathcal{SG} \setminus V_0$, and each mean value neighborhood B of x, define c_B by

$$c_B = M_B(v) - v(x).$$

Note that the result is independent of which v, because any two such functions differ by a harmonic function and the equality $M_B(h) - h(x) = 0$ always holds for any harmonic function h. So we can choose

$$v(x) = -\int G(x, y) d\mu(y),$$

which vanishes on the boundary of $\mathcal{SG}$. Here G is Green's function.

We will prove that c_B is controlled by the size of C_w. More precisely, we will prove:

Theorem 4.1 *Let $x \in \mathcal{SG} \setminus V_0$ and B be a k level mean value neighborhood of x. Then*

$$c_0 \frac{1}{5^k} \leq c_B \leq c_1 \frac{1}{5^k}$$

for some constant c_0, c_1 which are independent of x.

To prove Theorem 4.1, we need the explicit expression for the function v. Recall from Sect. 2 that $v(x)$ is the uniform limit of $v_M(x)$ for

$$v_M(x) = -\int G_M(x, y) d\mu(y).$$

Interchanging the integral and summation,

$$v_M(x) = -\sum_{m=0}^{M} \sum_{z, z' \in V_{m+1} \setminus V_m} g(z, z') \int \psi_{z'}^{(m+1)}(y) d\mu(y) \psi_z^{(m+1)}(x).$$

Notice that for each $z \in V_{m+1} \setminus V_m$, $\psi_z^{(m+1)}$ is a piecewise harmonic spline of level $(m+1)$ satisfying $\psi_z^{(m+1)}(y) = \delta_z(y)$ for $y \in V_{m+1}$. More precisely, $\psi_z^{(m+1)}$ is supported in the two $(m+1)$-cells meeting at z. If $F_\tau(\mathcal{SG})$ is one of these cells with vertices z, z_1 and z_2, then $\psi_z^{(m+1)} + \psi_{z_1}^{(m+1)} + \psi_{z_2}^{(m+1)}$ restricted to $F_\tau(\mathcal{SG})$ is identically 1. Thus

$$\int_{F_\tau(\mathcal{SG})} (\psi_z^{(m+1)} + \psi_{z_1}^{(m+1)} + \psi_{z_2}^{(m+1)}) d\mu = \mu(F_\tau(\mathcal{SG})) = \frac{1}{3^{m+1}}.$$

By symmetry all three summands have the same integral, so $\int_{F_\tau(\mathcal{SG})} \psi_z^{(m+1)} d\mu = \frac{1}{3^{m+2}}$. Together with the contribution from the other $(m+1)$-cell we find for each $z \in V_{m+1} \setminus V_m$,

$$\int \psi_z^{(m+1)}(y) d\mu(y) = \frac{2}{3^{m+2}}. \tag{4.2}$$

Hence

$$v_M(x) = -\frac{2}{9} \sum_{m=0}^{M} \frac{1}{3^m} \sum_{z,z' \in V_{m+1} \setminus V_m} g(z,z') \psi_z^{(m+1)}(x).$$

Substituting the exact value of $g(z,z')$ (see Sect. 2 and details in [7] p. 50) into it, we get

$$\begin{aligned} v_M(x) &= -\frac{2}{9} \sum_{m=0}^{M} \frac{1}{3^m} \Bigg(\sum_{|w|=m} \sum_{z,z' \in F_w(V_0) \setminus F_w(V_1)} g\big(z,z'\big) \psi_z^{(m+1)}(x) \Bigg) \\ &= -\frac{2}{9} \sum_{m=0}^{M} \frac{1}{3^m} \Bigg(\sum_{|w|=m} \sum_{z \in F_w(V_0) \setminus F_w(V_1)} \left(\frac{9}{50} r^m + 2\frac{3}{50} r^m \right) \psi_z^{(m+1)}(x) \Bigg) \\ &= -\frac{1}{15} \sum_{m=0}^{M} \frac{1}{5^m} \phi_m(x) \end{aligned}$$

for

$$\phi_m(x) = \sum_{z \in V_{m+1} \setminus V_m} \psi_z^{(m+1)}(x).$$

Thus

$$v(x) = -\frac{1}{15} \sum_{m=0}^{\infty} \frac{1}{5^m} \phi_m(x).$$

Remark The function v is invariant under Dihedral-3 symmetry.

This is a direct corollary of the fact that each $\phi_m(x)$ is invariant under D_3 symmetry.

Due to the above remark, we may assume that D_w associated to C_w has a fixed shape as shown in Fig. 2 without loss of generality. We now show that although c_B depends on the relative position of x in C_w, it does not depend on the location of x or C_w in $\mathcal{SG}$.

Lemma 4.1 *Let x, x' be two distinct points in $\mathcal{SG} \setminus V_0$. Let C_w and $C_{w'}$ be two k and k' level neighboring cells of x and x' respectively. Denote by B and B' two mean value neighborhoods of x and x' respectively. If B and B' have the same shapes (the same relative locations associated to C_w and $C_{w'}$ respectively), then*

$$c_B = 5^{k'-k} c_{B'}.$$

In particular, if B and B' have the same levels and same shapes, then $c_B = c_{B'}$.

Proof D_w can be decomposed into a union of a k level cell $D_w^{(1)}$ and a $(k-1)$ level cell $D_w^{(2)}$ as shown in Fig. 2. Denote by q the junction point connecting $D_w^{(1)}$ and $D_w^{(2)}$. Similarly, $D_{w'}$ can also be written as a union of a k' cell $D_{w'}^{(1)}$ and a $(k'-1)$ cell $D_{w'}^{(2)}$ with a junction point q' connecting them.

Let τ be the linear function mapping D_w onto $D_{w'}$. Suppose $D_w^{(1)} = F_\alpha(\mathcal{SG})$ and $D_w^{(2)} = F_\beta(\mathcal{SG})$ where α and β are the corresponding words of $D_w^{(1)}$ and $D_w^{(2)}$ respectively. Similarly, denote by α' and β' the corresponding words of $D_{w'}^{(1)}$ and $D_{w'}^{(2)}$. Hence we can write τ as $\tau(z) = F_{\alpha'} \circ F_\alpha^{-1}(z)$ if $z \in D_w^{(1)}$, and $\tau(z) = F_{\beta'} \circ F_\beta^{-1}(z)$ if $z \in D_w^{(2)}$. In particular, $\tau(q) = q'$ and $\tau(x) = x'$.

Consider the function $(v \circ F_\alpha - 5^{k'-k} v \circ F_{\alpha'})$ defined on $\mathcal{SG}$. Noting that $|\alpha| = k$ and $|\alpha'| = k'$, using the scaling property of Δ(see details in [7], p. 33), we have

$$\Delta(v \circ F_\alpha - 5^{k'-k} v \circ F_{\alpha'}) = r^{|\alpha|} \frac{1}{3^{|\alpha|}} \Delta v \circ F_\alpha - 5^{k'-k} r^{|\alpha'|} \frac{1}{3^{|\alpha'|}} \Delta v \circ F_{\alpha'} = 0,$$

which shows that the difference between $v \circ F_\alpha$ and $5^{k'-k} v \circ F_{\alpha'}$ is a harmonic function. Hence the difference between v and $5^{k'-k} v \circ \tau$ on $D_w^{(1)}$ is harmonic. A similar discussion will show that the difference between v and $5^{k'-k} v \circ \tau$ on $D_w^{(2)}$ is also harmonic. Since the matching condition on normal derivatives of $(v - 5^{k'-k} v \circ \tau)$ at q holds obviously, we have proved that $\Delta(v - 5^{k'-k} v \circ \tau) = 0$ on D_w, i.e., the function $(v - 5^{k'-k} v \circ \tau)$ is harmonic on D_w.

By the definition $c_B = M_B(v) - v(x)$ and $c_{B'} = M_{B'}(v) - v(x')$. Notice that for the second equality, by changing variables we can write $c_{B'} = M_B(v \circ \tau) - v \circ \tau(x)$. Hence

$$c_B - 5^{k'-k} c_{B'} = M_B(v - 5^{k'-k} v \circ \tau) - (v - 5^{k'-k} v \circ \tau)(x) = 0,$$

since $(v - 5^{k'-k} v \circ \tau)$ is a harmonic function on D_w. □

Proof of Theorem 4.1 (Estimate of c_B from above.) From Lemma 4.1, since c_B depends only on the relative geometry of B and C_w and the size of C_w, but not on the location of C_w, we may assume that D_w is contained in a $(k-2)$ level cell C in $\mathcal{SG}$ without loss of generality.

By the definition of c_B, we may write

$$c_B = M_B(v) - v(x) = \lim_{M\to\infty}\left(\frac{1}{\mu(B)}\int_B v_M d\mu - v_M(x)\right).$$

Substituting the exact formula of v_M into it, we get

$$c_B = -\frac{1}{15}\sum_{m=0}^{\infty}\frac{1}{5^m}\left(M_B(\phi_m) - \phi_m(x)\right),$$

for

$$\phi_m = \sum_{z\in V_{m+1}\setminus V_m}\psi_z^{(m+1)}.$$

Notice that each ϕ_m is a piecewise harmonic spline of level $m+1$. So when $m+1\le k-2$, ϕ_m is harmonic in the cell C, which yields that $M_B(\phi_m)-\phi_m(x)=0$. So the first $k-2$ terms in the infinite series of v will contribute 0 to c_B. Hence

$$c_B = -\frac{1}{15}\sum_{m=k-2}^{\infty}\frac{1}{5^m}\left(M_B(\phi_m) - \phi_m(x)\right).$$

It is easy to see that this implies

$$|c_B| \le \frac{1}{15}\sum_{m=k-2}^{\infty}\frac{1}{5^m}\frac{1}{\mu(B)}\int_B|\phi_m(y)-\phi_m(x)|d\mu(y).$$

Then by the maximum principle, we finally get

$$|c_B| \le \frac{1}{15}\sum_{m=k-2}^{\infty}\frac{1}{5^m} = \frac{25}{12}\cdot\frac{1}{5^k}.$$

(Estimate of c_B from below.) Without loss of generality, we assume that x is located in the $1/3$ region of C_w as shown in Fig. 6, i.e., x is contained in the triangle $T_{p_1,p_2,o}$, where o is the geometric center of C_w. Then by the proof of Theorem 3.1, B is a subset of the union of C_w and two of its neighbors C_1 and C_2. Hence we can write $B = C_w\cup E_1\cup E_2$, where $E_i = B\cap C_i$.

Claim 1 *Let $\widetilde{B} = F_0(\mathcal{SG})\cup\widetilde{E}_1\cup\widetilde{E}_2$, where $\widetilde{E}_i$ is a triangle obtained by cutting $F_i(\mathcal{SG})$ symmetrically with a line below the top vertex F_iq_0. (See Fig. 7.) If $\widetilde{B}$ and B have the same shapes, then*

$$c_B = 5^{1-k}c_{\widetilde{B}}.$$

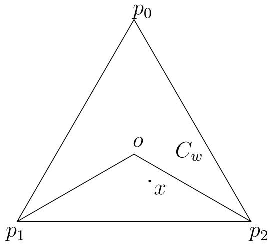

Fig. 6 A 1/3 region of C_w

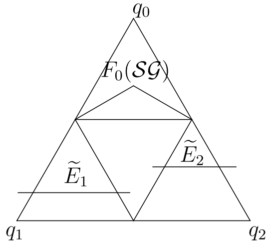

Fig. 7 A sketch of $\widetilde{B}$

This is a direct corollary of Lemma 4.1.

We only need to prove that $c_{\widetilde{B}}$ for $\widetilde{B}$ defined in Claim 1 has a positive lower bound. For simplicity of notation, in all that follows, we write B instead of $\widetilde{B}$. In other words, we only need to consider B whose associate cell C_w is $F_0(\mathcal{SG})$. In this setting, $p_i = F_0 q_i$, $C_1 = F_1(\mathcal{SG})$ and $C_2 = F_2(\mathcal{SG})$.

We write $v = -\frac{1}{15}\widetilde{v}$ where $\widetilde{v}$ is the non-negative function defined by

$$\widetilde{v} = \sum_{m=0}^{\infty} \frac{1}{5^m}\phi_m.$$

For each $M \geq 0$, denote by

$$\widetilde{v}_M = \sum_{m=0}^{M} \frac{1}{5^m}\phi_m$$

the partial sum of the first $M+1$ terms of $\widetilde{v}$. Then $\widetilde{v}_M$ converges to $\widetilde{v}$ uniformly as $M \to \infty$.

We have the following three claims on $\widetilde{v}$.

Claim 2 $0 \leq \widetilde{v} \leq 1$ *on* $\mathcal{SG}$ *and* $\widetilde{v}$ *takes constant* 1 *along the maximal inner upside-down triangle contained in* $\mathcal{SG}$.

Proof Consider the partial sum function $\widetilde{v}_M$. Obviously, $\widetilde{v}_M$ is a $(M+1)$-level piecewise harmonic function on $\mathcal{SG}$. For convenience, denote by ∇ the maximal inner upside-down triangle contained in $\mathcal{SG}$. We divide the vertices V_{M+1} into three parts, V'_{M+1}, V''_{M+1} and V'''_{M+1}, where V'_{M+1} consists of those vertices lying along ∇, V''_{M+1} consists of those vertices at distance $2^{-(M+1)}$ from ∇, and V'''_{M+1} consists of the remain vertices. Then by using the "$\frac{1}{5} - \frac{2}{5}$" rule, an inductive argument shows that

$\widetilde{v}_M \equiv 1$ on V'_{M+1}, $\widetilde{v}_M \equiv 1 - \frac{1}{5^M}$ on V''_{M+1}, and $\widetilde{v}_M \le 1 - \frac{1}{5^M}$ on V'''_{M+1}. Since $\widetilde{v}$ is the uniform limit of $\widetilde{v}_M$ and V'_{M+1} goes to ∇ as M goes to the infinity, we then have $0 \le \widetilde{v} \le 1$ on $\mathcal{SG}$ and $\widetilde{v} \equiv 1$ on ∇. □

Claim 3 *For each x contained in the triangle $T_{p_1,p_2,o}$, $\widetilde{v}(x) \ge \frac{24}{25}$.*

Proof For $\tau = (0,1,1), (0,1,2), (0,2,1)$ and $(0,2,2)$, by using the harmonic extension algorithm, namely, the "$\frac{1}{5} - \frac{2}{5}$" rule, we get that

$$\widetilde{v}(F_\tau q_0) = \widetilde{v}_2(F_\tau q_0) = \sum_{m=0}^{2} \frac{1}{5^m}\phi_m(F_\tau q_0) = 1 \cdot \frac{4}{5} + \frac{1}{5} \cdot \frac{3}{5} + \frac{1}{25} \cdot 1 = \frac{24}{25},$$

where $\frac{4}{5}$, $\frac{3}{5}$ and 1 are the values of ϕ_0, ϕ_1 and ϕ_2 at $F_\tau q_0$ respectively. Also, for those τ, by Claim 2, we have

$$\widetilde{v}(F_\tau q_1) = \widetilde{v}_2(F_\tau q_1) = \widetilde{v}(F_\tau q_2) = \widetilde{v}_2(F_\tau q_2) = 1.$$

Notice that for each point x in the triangle $T_{p_1,p_2,o}$, x is contained in one of the four 3-level cells $F_{011}(\mathcal{SG})$, $F_{012}(\mathcal{SG})$, $F_{021}(\mathcal{SG})$ and $F_{022}(\mathcal{SG})$. Since $\widetilde{v}_2$ is harmonic in each such cell, by using the maximal principle, we get that

$$\widetilde{v}_2(x) \ge \frac{24}{25}.$$

Hence $\widetilde{v}(x) \ge \frac{24}{25}$ since each term in the infinite series of $\widetilde{v}$ is non-negative. □

Claim 4 $M_B(\widetilde{v}) \le \frac{17}{18}$.

Proof First of all we prove that

$$\int_{F_0(\mathcal{SG})} \widetilde{v}(y)d\mu(y) = \frac{5}{18}.$$

We need to compute $\int_{F_0(\mathcal{SG})} \phi_m(y)d\mu(y)$ for each non-negative integer m. For each $m \ge 0$,

$$\int_{F_0(\mathcal{SG})} \phi_m(y)d\mu(y) = \frac{1}{3} \cdot 3^{m+1} \cdot \frac{2}{3^{m+2}} = \frac{2}{9},$$

by using (4.2) and the fact that $\phi_m = \sum_{z \in V_{m+1} \setminus V_m} \psi_z^{(m+1)}$. Hence

$$\int_{F_0(\mathcal{SG})} \widetilde{v}(y)d\mu(y) = \frac{2}{9} \sum_{m=0}^{\infty} \frac{1}{5^m} = \frac{5}{18}.$$

By our assumption, the mean value neighborhood B can be written as

$$B = F_0(\mathcal{SG}) \cup E_1 \cup E_2,$$

where $E_i = B \cap C_i$. Hence we have

$$\begin{aligned} M_B(\widetilde{v}) &= \frac{1}{\mu(B)}\left(\int_{F_0(\mathcal{SG})} \widetilde{v}(y)d\mu(y) + \int_{E_1} \widetilde{v}(y)d\mu(y) + \int_{E_2} \widetilde{v}(y)d\mu(y)\right) \\ &\le \frac{1}{\mu(B)}\left(\int_{F_0(\mathcal{SG})} \widetilde{v}(y)d\mu(y) + \int_{E_1} 1 \cdot d\mu(y) + \int_{E_2} 1 \cdot d\mu(y)\right) \\ &= \frac{5/18 + \mu(E_1) + \mu(E_2)}{1/3 + \mu(E_1) + \mu(E_2)}, \end{aligned}$$

where the inequality follows from Claim 2. Since $0 \le \mu(E_1) + \mu(E_2) \le \frac{2}{3}$, $\frac{\frac{5}{18}+x}{\frac{1}{3}+x}$ is increasing in $x \ge 0$,

$$\frac{5/18 + \mu(E_1) + \mu(E_2)}{1/3 + \mu(E_1) + \mu(E_2)} \le \frac{5/18 + 2/3}{1/3 + 2/3} = \frac{17}{18}.$$

Hence we always have

$$M_B(\widetilde{v}) \le \frac{17}{18}. \qquad \square$$

Now we turn to estimate c_B. Obviously,

$$c_B = M_B(v) - v(x) = -\frac{1}{15}\big(M_B(\widetilde{v}) - \widetilde{v}(x)\big).$$

By Claims 3 and 4, we notice that $M_B(\widetilde{v}) - \widetilde{v}(x) \le \frac{17}{18} - \frac{24}{25} = -\frac{7}{450}$. Hence

$$c_B \ge \frac{1}{15} \cdot \frac{7}{450} > 0. \qquad \square$$

On the other hand, given a point x and $C_w = F_w(\mathcal{SG})$ a k level neighborhood of x, for any $u \in \operatorname{dom} \Delta$, we write

$$u = h^{(k)} + (\Delta u(x))v + R^{(k)}$$

on C_w, where $h^{(k)}$ is a harmonic function defined by

$$h^{(k)} + (\Delta u(x))v|_{\partial C_w} = u|_{\partial C_w}.$$

It is not hard to prove the following estimate:

Lemma 4.2 *Let $u \in \operatorname{dom} \Delta$ with $g = \Delta u$ satisfying the following Hölder condition*

$$|g(y) - g(x)| \le c\gamma^k \quad (0 < \gamma < 1)$$

for all $y \in C_w$. Then the remainder satisfies

$$R^{(k)} = O\left(\left(\frac{\gamma}{5}\right)^k\right)$$

on C_w (hence also on $B_k(x)$).

Proof It is easy to check that $\Delta R^{(k)}(y) = \Delta u(y) - \Delta u(x)$ and $R^{(k)}(y)$ vanishes on the boundary of C_w. Hence $R^{(k)}$ is given by the integral of $\Delta u(y) - \Delta u(x)$ on C_w against a scaled Green's function. Noticing that the scaling factor is $(\frac{1}{5})^k$ and

$$|\Delta u(y) - \Delta u(x)| \le c\gamma^k,$$

we then get the desired result. □

This looks like a Taylor expansion remainder estimate of u at x. See more details on this topic in [8].

Remark If we require $u \in \operatorname{dom} \Delta^2$, then the remainder $R^{(k)}$ satisfies

$$R^{(k)} = O\left(\left(\frac{3}{5} \cdot \frac{1}{5}\right)^k\right)$$

on C_w(hence also on $B_k(x)$). The reason is that in this case Δu satisfies the Hölder condition that $|\Delta u(y) - \Delta u(x)| \le c(\frac{3}{5})^k$ for all $y \in C_w$, because $\Delta^2 u$ is assumed continuous, see [8], Theorem 8.4.

Using the above lemma and Theorem 4.1, we then have the following main result of this section.

Theorem 4.2 *Let $u \in \operatorname{dom} \Delta$ with $g = \Delta u$ satisfying the Hölder condition $|g(y) - g(x)| \le c\gamma^k$ for some γ with $0 < \gamma < 1$, for all x, y belonging to the same k level cell. Then*

$$\lim_{k\to\infty} \frac{1}{c_{B_k(x)}} (M_{B_k(x)}(u) - u(x)) = \Delta u(x).$$

Proof Using Taylor expansion of u and noticing that $M_{B_k(x)}(h^{(k)}) - h^{(k)}(x) = 0$, $M_{B_k(x)}(v) - v(x) = c_{B_k(x)}$, we have

$$\begin{aligned}\frac{1}{c_{B_k(x)}}\big(M_{B_k(x)}(u) - u(x)\big) - \Delta u(x) &= \frac{1}{c_{B_k(x)}}\big(M_{B_k(x)}\big(R^{(k)}\big) - R^{(k)}(x)\big) \\ &= \frac{1}{c_{B_k(x)}} O\left(\left(\frac{\gamma}{5}\right)^k\right) = O\big(\gamma^k\big).\end{aligned}$$

Hence letting $k \to \infty$, we get the desired result. □

5 p.c.f. Fractals with Dihedral-3 Symmetry

The results for $\mathcal{SG}$ should extend to other p.c.f. fractals which possess symmetric properties of both the geometric structure and the harmonic structure. We assume that a regular harmonic structure is given on a p.c.f. self-similar fractal K. The reader is referred to [4, 7] for exact definitions and any unexplained notations. We assume

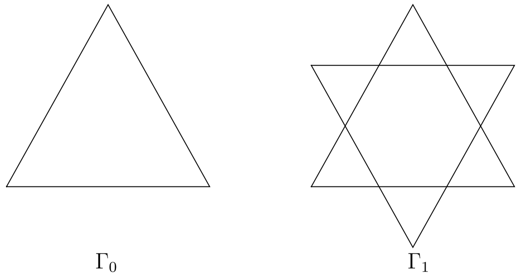

Fig. 8 The first 2 graphs, Γ_0, Γ_1 in the approximation to the hexagasket

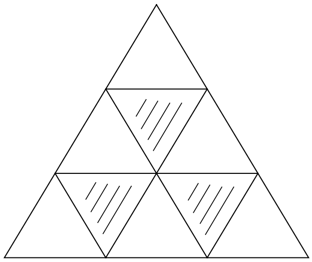

Fig. 9 The graph of the V_1 vertices of the level 3 Sierpinski gasket

now that $\sharp V_0 = 3$ and all structures possess full D_3 symmetry. This means there exists a group $\mathcal{G}$ of homeomorphisms of K isomorphic to D_3 that acts as permutations on V_0, and $\mathcal{G}$ preserves the harmonic structures and the self-similar measure.

Assume that the fractal K is the invariant set of a finite iterated function system of contractive similarities. We denote these maps $\{F_i\}_{i=1,\dots,N}$ with $N \geq 3$. Let r_i denote the i-th resistance renormalization factor and μ_i denote the i-th weight of the self-similar measure μ on K. In general, it is not necessary that all r_i's and all μ_i's be the same, but here we must have $r_0 = r_1 = \cdots = r_N$ and $\mu_0 = \mu_1 = \cdots = \mu_N$ from the above Dihedral-3 symmetry assumption. We denote $V_0 = \{q_0, q_1, q_2\}$ the set of boundary points.

Examples (i) The Sierpinski gasket $\mathcal{SG}$. In this case all $r_i = 3/5$ and all $\mu_i = 1/3$.

(ii) The hexagasket, or fractal Star of David, can be generated by 6 maps with simultaneously rotate and contract by a factor of $1/3$ in the plane. Thus V_0 consists of 3 points of an equilateral triangle, and V_1 consists of the vertices of the Star of David, as shown in Fig. 8. Although the same geometric fractal can be constructed by using contractions which do not rotate, this gives rise to a different self-similar structure (in particular with $\sharp V_0 = 6$). Our choice of self-similar structure destroys the D_6 symmetry of the geometric fractal, but it has the advantage of easier computation. In this case, all $r_i = 3/7$ and all $\mu_i = 1/6$. Note that in this example there exist points in V_1 that are not junction points.

(iii) The level 3 Sierpinski gasket $\mathcal{SG}_3$, obtained by taking 6 contractions of ratio $1/3$ as shown in Fig. 9. Here we have all $r_i = 7/15$ and $\mu_i = 1/6$. Note that all seven vertices in $V_1 \setminus V_0$ are junction points, but the one in the middle intersects three 1-cells. In a similar manner we could define $\mathcal{SG}_n$ for any value of $n \geq 2$.

We prove that there are results analogous to Theorem 3.1, which yield the existence of mean value neighborhoods associated to K.

Given a point x in $K \setminus V_0$, consider any cell $F_w K = C_w$ with boundary points p_0, p_1, p_2 containing the point x. Without losing of generality, we may require that the cell C_w does not intersect V_0. For each i, denote by $C_{i,1}, \dots, C_{i,l_i}$ the neighboring cells of C_w of the same size, intersecting C_w at p_i, where l_i is the number of such cells. It is possible that $l_i = 0$ for some i since p_i may be a non-junction point. If this is true, the matching condition says that the normal derivative of any harmonic function h must be zero at this point, which yields that the value of h at this point is the mean value of the values of h at the other two boundary points of C_w. In other words, the restriction of all global harmonic functions in C_w is two dimensional. Denote by D_w the union of C_w and all its neighboring cells, i.e.,

$$D_w = C_w \cup \bigcup_{i,j} C_{i,j}.$$

Two cells C_w and $C_{w'}$ are said to have the same *neighborhood type* if they have the same relative geometry with respect to D_w and $D_{w'}$ respectively. It is obvious that there only exist finitely many distinct types. For example, for $\mathcal{SG}$, all cells have exactly only one neighborhood type. For $\mathcal{SG}_3$, the number of the finite types is 3. For $\mathcal{SG}_n$ $(n \geq 4)$, the number of the finite types becomes 4. For the hexagasket gasket, the number of the finite types is 2.

Let h be a harmonic function on K. Given a set B containing C_w, define

$$M_B(h) = \frac{1}{\mu(B)} \int_B h d\mu$$

the *mean value* of h over B. We are interested in an identity

$$M_B(h) = \sum_i a_i h(p_i) \tag{5.1}$$

for some coefficients (a_0, a_1, a_2) satisfying $\sum a_i = 1$. Notice that this is true for $\mathcal{SG}$. In that setting, a harmonic function is uniquely determined by its values on the boundary of any given cell C_w because the harmonic extension matrix associated with C_w is invertible. However, in the general case, the harmonic extension matrices may not be invertible. So we can not prove (5.1) for every set B simply by linearity. However, it will suffice to show that the equality (5.1) holds for certain specified sets B.

Consider a set B which is a subset of D_w, containing C_w. Then B must be made up of four parts, i.e.,

$$B = C_w \cup E_0 \cup E_1 \cup E_2$$

where $E_i = B \cap C_i$ with $C_i = \bigcup_{j=1}^{l_i} C_{i,j}$. It is possible that C_i may be empty since p_i may be a nonjunction point. We can also subdivide each E_i into l_i small pieces, i.e., $E_i = \bigcup_j E_{i,j}$ for $E_{i,j} = E_i \cap C_{i,j}$. For each i, we require that $E_{i,1}, \dots, E_{i,l_i}$ be of the same size and shape. Moreover, in analogy with the $\mathcal{SG}$ case, we require that each $E_{i,j}$ to be a symmetric (under the reflection symmetry that fixes p_i) cutoff sub-triangle of $C_{i,j}$, containing p_i as one of its vertex points. This means that there is a straight line $L_{i,j}$, symmetric under the reflection symmetry fixing p_i, cutting $C_{i,j}$ into two parts, and $E_{i,j}$ is the one containing p_i. For each $E_{i,j}$, define the distance between p_i and the line $L_{i,j}$ the size of $E_{i,j}$. Of course, for each fixed i, $E_{i,1}, \dots, E_{i,l_i}$ have the same sizes. We call the common value the size of E_i. Suppose the size of every $C_{i,j}$ is ρ. (Of course, they are all equal.) Then for each i, the size of E_i is $c_i \rho$ where the coefficient $0 \le c_i \le 1$. Hence we can write the set $B = B(c_0, c_1, c_2)$. (If p_i is a nonjunction point, then c_i should always be 0.) For example, suppose that the boundary points of C_w consist of junction points, then $B(0,0,0) = C_w$ and $B(1,1,1) = D_w$. Denote by

$$\mathcal{B} = \{B(c_0, c_1, c_2) : 0 \le c_i \le 1\}$$

the family of all such sets. Then we can show that the formula (5.1) holds for each $B \in \mathcal{B}$.

Proposition 5.1 *Let $B \in \mathcal{B}$, then for any harmonic function h, we have (5.1) for some coefficients (a_0, a_1, a_2) independent of h. Moreover, $\sum_i a_i = 1$.*

Proof Each $B \in \mathcal{B}$ can be written as $B = C_w \cup E_0 \cup E_1 \cup E_2$. Given a harmonic function h on K, for fixed i, we first consider the integral $\int_{E_i} h d\mu$. Obviously,

$$\int_{E_i} h d\mu = \sum_j \int_{E_{i,j}} h d\mu.$$

For each $1 \le j \le l_i$, denote by $\{z_{i,j}, w_{i,j}, p_i\}$ the boundary points of $C_{i,j}$. Since each $E_{i,j}$ is contained in $C_{i,j}$, $\frac{1}{\mu(C_w)} \int_{E_{i,j}} h d\mu$ can be expressed as a linear combination of $h(p_i), h(z_{i,j})$ and $h(w_{i,j})$ with non-negative coefficients independent of the harmonic function h. Since the set $E_{i,j}$ is symmetric under the reflection symmetry fixing p_i, the two coefficients with respect to $h(z_{i,j})$ and $h(w_{i,j})$ must be equal. In other words, we can write

$$\int_{E_{i,j}} h d\mu = (m_{i,j} h(p_i) + n_{i,j} h(z_{i,j}) + n_{i,j} h(w_{i,j}))\mu(C_w)$$

for $m_{i,j}, n_{i,j} \ge 0$. Moreover, since for each fixed i, $E_{i,j}$ are in the same relative position associated to $C_{i,j}$ for different j's, $\int_{E_{i,j}} h d\mu$ can be expressed as a linear combination of $h(p_i), h(z_{i,j}), h(w_{i,j})$ with the same coefficients for different j's. Hence we can write

$$\int_{E_i} h d\mu = (m_i h(p_i) + n_i \sum_j (h(z_{i,j}) + h(w_{i,j})))\mu(C_w),$$

for suitable coefficients $m_i, n_i \ge 0$. The mean value property at the point p_i says that

$$\sum_j (h(z_{i,j}) + h(w_{i,j})) = (2l_i + 2)h(p_i) - (h(p_{i-1}) + h(p_{i+1})).$$

Combining the above two equalities, we get

$$\int_{E_i} h d\mu = ((m_i + 2l_i n_i + 2n_i)h(p_i) - n_i h(p_{i-1}) - n_i h(p_{i+1}))\mu(C_w).$$

On the other hand, by the linearities and symmetries of both the harmonic structure and the self-similar measure,

$$\int_{C_w} h d\mu = \frac{\mu(C_w)}{3}(h(p_0) + h(p_1) + h(p_2)).$$

Since the ratio of $\mu(E_{i,j})$ to $\mu(C_w)$ depends only on c_i, we have proved that $M_B(h)$ can be viewed as a linear combination of the values of h on the boundary points of C_w, i.e.,

$$M_B(h) = \sum_i a_i h(p_i),$$

where the combination coefficients are independent of h. Moreover, we must have $\sum a_i = 1$ by considering $h \equiv 1$. □

Remark 1 This means that $M_B(h)$ is a weighted average of the values $h(p_0), h(p_1)$ and $h(p_2)$. Moreover, if one of the boundary points, for example p_2, is a nonjunction point, then by the fact that $h(p_2) = \frac{1}{2}(h(p_0) + h(p_1))$, we have

$$M_B(h) = a_0 h(p_0) + a_1 h(p_1) + \frac{1}{2} a_2 (h(p_0) + h(p_1)) = \widetilde{a}_0 h(p_0) + \widetilde{a}_1 h(p_1)$$

for $\widetilde{a}_0 = a_0 + \frac{1}{2}a_2$ and $\widetilde{a}_1 = a_1 + \frac{1}{2}a_2$. We also have $\widetilde{a}_0 + \widetilde{a}_1 = 1$. Hence in this case, we can also view $M_B(h)$ as a weighted average of the values of $h(p_0)$ and $h(p_1)$.

Remark 2 The proof of Proposition 5.1 shows that (a_0, a_1, a_2) depends only on the neighborhood type of C_w and the relative position of B associated to C_w, and does not depend on the particular choice of C_w. In other words, if we consider a cell C_w with a given neighborhood type, then for each set $B \in \mathcal{B}$ with the expression $B = B(c_0, c_1, c_2)$, the coefficients (a_0, a_1, a_2) of B depend only on (c_0, c_1, c_2).

The following is the main result in this section.

Theorem 5.1 *Given a point $x \in K \setminus V_0$, let C_w be a cell containing x, not intersecting V_0, and let D_w be the union of C_w and its neighboring cells of the same size. Then there exists a mean value neighborhood B of x satisfying $C_w \subset B \subset D_w$. Moreover, for each point $x \in K \setminus V_0$, there exists a system of mean value neighborhoods $B_k(x)$ with $\bigcap_k B_k(x) = \{x\}$.*

Proof We need to classify the distinct neighborhood types into three cases according to the number of nonjunction points in the set of boundary points of C_w.

Case 1 All boundary points of C_w are junction points.

This case is similar to what we have described in the $\mathcal{SG}$ setting. Let W denote the triangle in $\mathbb{R}^3$ with boundary points $P_0 = (1, 0, 0)$, $P_1 = (0, 1, 0)$ and $P_2 = (0, 0, 1)$ and π_W the plane containing W. Notice that from Proposition 5.1, $(a_0, a_1, a_2) \in \pi_W$ for each B. We use $\mathcal{T}$ to denote the map from $\mathcal{B}$ to π_W. From Remark 2 of Proposition 5.1, the map $\mathcal{T}$ is uniquely determined by the neighborhood type of C_w. Let $\mathcal{B}^*$ be a subfamily contained in $\mathcal{B}$ defined by

$$\begin{aligned}\mathcal{B}^* = &\{B(0, c_1, c_2) : 0 \le c_i \le 1\} \cup \{B(c_0, 0, c_2) : 0 \le c_i \le 1\}\\ &\cup \{B(c_0, c_1, 0) : 0 \le c_i \le 1\},\end{aligned}$$

i.e., those elements B in $\mathcal{B}$ which have the decomposition form $B = C_w \cup E_1 \cup E_2$ or $B = C_w \cup E_0 \cup E_2$, or $B = C_w \cup E_0 \cup E_1$. Then we have

Claim 1 *The map $\mathcal{T}$ from $\mathcal{B}$ to π_W fills out a region $\widetilde{W}$ which contains the triangle W. Moreover, $\mathcal{T}$ is one-to-one from $\mathcal{B}^*$ onto $\widetilde{W}$.*

Proof The proof is similar to the $\mathcal{SG}$ case. The only difference is the line segments $\overline{OQ_0}$ and $\overline{OP_2}$ described in the proof of Theorem 3.1 may become continuous curves $\widehat{OQ_0}$ and $\widehat{OP_2}$ in the general setting. □

Case 2 There is one nonjunction point (for example, p_2) among the boundary points of C_w.

In this case, there is no neighboring cell intersecting C_w at the point p_2. Hence E_2 will always be empty. So $\mathcal{B} = \{B(c_0, c_1, 0) : 0 \le c_i \le 1\}$ for this case.

As shown in Remark 1 of Proposition 5.1, for any harmonic function h on K, $B \in \mathcal{B}$, $M_B(h)$ is a weighted average of $h(p_0)$ and $h(p_1)$, i.e.,

$$M_B(h) = a_0 h(p_0) + a_1 h(p_1)$$

with a_0, a_1 independent of h, satisfying $a_0 + a_1 = 1$. Let I denote the line segment in $\mathbb{R}^2$ with endpoints $P_0 = (1, 0)$, $P_1 = (0, 1)$ and ρ_I the line containing I. Notice that from Remark 1 of Proposition 5.1, $(a_0, a_1) \in \rho_I$ for each B. We still use $\mathcal{T}$ to denote the map from $\mathcal{B}$ to ρ_I. From Remark 2 of Proposition 5.1, the map $\mathcal{T}$ is uniquely determined by the neighborhood type of C_w. We may write $\mathcal{T}(B(c_0, c_1, 0)) = (a_0, a_1)$ for each set $B(c_0, c_1, 0)$. We will show the image of the map $\mathcal{T}$ covers the line segment I. Similar to Case 1, let $\mathcal{B}^*$ be a subfamily contained in $\mathcal{B}$ defined by

$$\mathcal{B}^* = \{B(c_0, 0, 0) : 0 \le c_0 \le 1\} \cup \{B(0, c_1, 0) : 0 \le c_1 \le 1\},$$

i.e., those elements B in $\mathcal{B}$ which have the decomposition form $B = C_w \cup E_0$ or $B = C_w \cup E_1$. Then we have

Claim 2 *The map $\mathcal{T}$ from $\mathcal{B}$ to ρ_I fills out the line segment I. Moreover, $\mathcal{T}$ is a one-to-one map on $\mathcal{B}^*$.*

Proof The proof is similar to Case 1. Denote by $O=(\frac{1}{2},\frac{1}{2})$ the midpoint of I. We only prove the map $\mathcal{T}$ from $\mathcal{B}$ to ρ_I fills out half of the line segment I. Then we will get the desired result by symmetry.

Let h be a harmonic function on K. We consider $\mathcal{T}(\{(B(c,0,0)) : 0\le c\le 1\})$. When $c=0$, $B(0,0,0)=C_w$ and $M_{C_w}(h)=\frac{1}{3}(h(p_0)+h(p_1)+h(p_2))$. Combining this with the fact that

$$h(p_2)=\frac{1}{2}(h(p_0)+h(p_1)),$$

we get

$$M_{C_w}(h)=\frac{1}{2}(h(p_0)+h(p_1)).$$

Hence $\mathcal{T}(B(0,0,0))$ is the midpoint O of I. When $c=1$, $B(1,0,0)=C_w\cup C_0$, and an easy calculation gives that $M_{C_w\cup C_0}=h(p_0)$. Hence $\mathcal{T}(B(1,0,0))$ is the endpoint P_0. So if we vary c continuously between 0 and 1, we can fill out the line segment joining O and P_0, which is half of I. □

Case 3 There are two nonjunction points (for example, p_1 and p_2) among the boundary points of C_w.

In this case, let h be any harmonic function on K. By the matching condition on both points p_1 and p_2, h must be constant on the whole cell C_w. Hence for every point $x\in C_w$, we could view C_w itself as the mean value neighborhood of x.

Hence the proof of Theorem 5.1 is completed by using a same argument as that of Theorem 3.2. □

We should mention here that the result can also be extended to some other p.c.f. fractals including the 3-dimensional Sierpinski gasket. However, it seems that some strong symmetric conditions of both the geometric and the harmonic structures should be required.

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