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Characterization of the Weak-Type Boundedness of the Hilbert Transform on Weighted Lorentz Spaces

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Abstract We characterize the weak-type boundedness of the Hilbert transform H on weighted Lorentz spaces $\Lambda_u^p(w)$, with $p > 0$, in terms of some geometric conditions on the weights u and w and the weak-type boundedness of the Hardy–Littlewood maximal operator on the same spaces. Our results recover simultaneously the theory of the boundedness of H on weighted Lebesgue spaces $L^p(u)$ and Muckenhoupt weights A_p, and the theory on classical Lorentz spaces $\Lambda^p(w)$ and Ariño-Muckenhoupt weights B_p.

Keywords Weighted Lorentz spaces · Hilbert transform · Muckenhoupt weights · B_p weights

Mathematics Subject Classification (2010) 26D10 · 42A50

1 Introduction and Motivation

In this paper, we characterize the weak-type boundedness of the Hilbert transform on weighted Lorentz spaces

$$H : \Lambda_u^p(w) \longrightarrow \Lambda_u^{p,\infty}(w), \tag{1.1}$$

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if $0 < p < \infty$, and H is the Hilbert transform defined by

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy,$$

whenever this limit exists almost everywhere. We recall (see [15, 16]) that, given u, a positive and locally integrable function (called weight) in $\mathbb{R}$ and given a weight w in $\mathbb{R}^+$, the Lorentz space $\Lambda_u^p(w)$ is defined as

$$\Lambda_u^p(w) = \left\{ f \in \mathcal{M}(\mathbb{R}) : \|f\|_{\Lambda_u^p(w)} = \left(\int_0^\infty \big(f_u^*(t)\big)^p w(t)\, dt \right)^{1/p} < \infty \right\},$$

where $\mathcal{M} = \mathcal{M}(\mathbb{R})$ is the set of Lebesgue measurable functions on $\mathbb{R}$, f_u^* is the decreasing rearrangement of f with respect to the weight u [5]

$$f_u^*(t) = \inf\big\{ y > 0 : u\big(\big\{x \in \mathbb{R} : \big|f(x)\big| > y\big\}\big) \le t \big\},$$

with $u(E) = \int_E u(x)\, dx$, and the weak-type Lorentz space is

$$\Lambda_u^{p,\infty}(w) = \left\{ f \in \mathcal{M} : \|f\|_{\Lambda_u^{p,\infty}(w)} = \sup_{t>0} f_u^*(t) W(t)^{1/p} < \infty \right\},$$

where $W(t) = \int_0^t w(s)\, ds$. In order to avoid trivial cases, we will assume that $u(x) > 0$, a.e. $x \in \mathbb{R}$.

The motivation for studying (1.1) comes naturally, as a unified theory, from the fact that weighted Lorentz spaces include, as particular examples, the weighted Lebesgue spaces $L^p(u)$ and the classical Lorentz spaces $\Lambda^p(w)$, and in both cases the boundedness of the Hilbert transform is already known [9, 12, 20]. They also include the case of the Lorentz spaces $L^{p,q}(u)$, where only some partial results were previously known [8].

(i) If $w = 1$, (1.1) is equivalent to the fact that

$$H : L^p(u) \to L^{p,\infty}(u)$$

is bounded, and this problem was solved by Hunt, Muckenhoupt, and Wheeden [12]. An alternative proof was provided in [9] by Coifman and Fefferman and the solution is the A_p class of weights, if $p > 1$ [17]:

$$\sup_I \left(\frac{1}{|I|} \int_I u(x)\, dx \right) \left(\frac{1}{|I|} \int_I u^{-1/(p-1)}(x)\, dx \right)^{p-1} < \infty,$$

where the supremum is considered over all intervals I of the real line.

This condition also characterizes the strong-type boundedness

$$H : L^p(u) \to L^p(u),$$

and if $p = 1$

$$H : L^1(u) \to L^{1,\infty}(u)$$

is bounded if and only if $u \in A_1$:

$$Mu(x) \leq Cu(x), \qquad \text{a.e. } x \in \mathbb{R},$$

with M being the Hardy–Littlewood maximal function:

$$Mf(x) = \sup_{x \in I} \frac{1}{|I|} \int_I \left|f(y)\right| dy,$$

where the supremum is taken over all intervals I containing $x \in \mathbb{R}$.

Recall [10] that a weight $u \in A_\infty$ if and only if there exist $C_u > 0$ and $\delta \in (0,1)$ such that, for every interval I and every measurable set $E \subset I$,

$$\frac{u(E)}{u(I)} \leq C_u \left(\frac{|E|}{|I|}\right)^\delta, \tag{1.2}$$

and it holds that

$$A_\infty = \bigcup_{p \geq 1} A_p.$$

(ii) On the other hand, if $u = 1$, the characterization of (1.1) is equivalent to the boundedness of

$$H : \Lambda^p(w) \longrightarrow \Lambda^{p,\infty}(w),$$

given by Sawyer [20]. A simplified description of the class of weights [19] that characterizes this property is $B_{p,\infty} \cap B_\infty^*$, where a weight $w \in B_\infty^*$ if

$$\int_0^r \frac{1}{t} \int_0^t w(s)\, ds\, dt \leq C \int_0^r w(s)\, ds, \tag{1.3}$$

for all $r > 0$, and $w \in B_{p,\infty}$ if the Hardy operator

$$Pf(t) = \frac{1}{t} \int_0^t f(s)\, ds$$

satisfies that

$$P : L^p_{\text{dec}}(w) \longrightarrow L^{p,\infty}(w)$$

is bounded, where

$$L^p_{\text{dec}}(w) = \left\{f \in L^p(w) \colon f \text{ is decreasing}\right\}.$$

These weights have been well studied (see [3, 6, 18]) and it is known that if $p \leq 1$ then, $w \in B_{p,\infty}$ if and only if W is p quasi-concave: for every $0 < r < t < \infty$

$$\frac{W(t)}{t^p} \leq C \frac{W(r)}{r^p},$$

and if $p > 1$, $B_{p,\infty} = B_p$, where $w \in B_p$ if

$$r^p \int_r^\infty \frac{w(t)}{t^p}\, dt \le C \int_0^r w(s)\, ds \tag{1.4}$$

for every $r > 0$. Moreover, for every $p > 0$,

$$M : \Lambda^p(w) \longrightarrow \Lambda^{p,\infty}(w),$$

if and only if $w \in B_{p,\infty}$.

If we consider the strong-type boundedness

$$H : \Lambda^p(w) \longrightarrow \Lambda^p(w),$$

this is equivalent to the condition $w \in B_p \cap B^*_\infty$.

In [1] we gave the following characterization of the weights w for which (1.1) holds under the assumption that $u \in A_1$:

$$H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w) \quad \Longleftrightarrow \quad w \in B_{p,\infty} \cap B^*_\infty, \quad p > 0.$$

We also proved that if $p > 1$ and $u \in A_1$, then

$$H : \Lambda^p_u(w) \to \Lambda^p_u(w) \quad \Longleftrightarrow \quad w \in B_p \cap B^*_\infty.$$

The main result of this paper solves the weak-type boundedness of H for a general weight u, as follows:

Theorem 1.1 *For every* $0 < p < \infty$,

$$H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$$

is bounded if and only if the following conditions hold:

(i) $u \in A_\infty$.
(ii) $w \in B^*_\infty$.
(iii) $M : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$ *is bounded.*

Remark 1.2 The necessity of the condition $u \in A_\infty$ in (i) was, for us, an unexpected result since in the case of the Hardy–Littlewood maximal operator it was proved in [6] that $u \in A_\infty$, or even the doubling property, was not necessary to have the corresponding weak-type boundedness; that is

$$M : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w) \quad \not\Rightarrow \quad u \in A_\infty.$$

Remark 1.3 It is worth mentioning that the characterization of the weak-type boundedness of the Hardy–Littlewood maximal operator in terms of the weights u and w was left open in [6], for $p \ge 1$. The case $p < 1$ is given by the following condition [6]:

for every finite family of disjoint intervals $\{I_j\}_{j=1}^J$, and every family of measurable sets $\{S_j\}_{j=1}^J$, with $S_j \subset I_j$, for every j, we have that

$$\frac{W(u(\bigcup_{j=1}^J I_j))}{W(u(\bigcup_{j=1}^J S_j))} \leq C \max_{1\leq j\leq J} \left(\frac{|I_j|}{|S_j|}\right)^p.$$

We list now several results that are important for our purposes [1, 6]:

Proposition 1.4 (a) *$\Lambda_u^p(w)$ and $\Lambda_u^{p,\infty}(w)$ are quasi-normed spaces if and only if w satisfies the Δ_2 condition; that is, for every $r > 0$,*

$$W(2r) \leq CW(r). \tag{1.5}$$

(b) *If $u \notin L^1(\mathbb{R})$, $w \notin L^1(\mathbb{R}^+)$ and $w \in \Delta_2$, then $\mathcal{C}_c^\infty(\mathbb{R})$ is dense in $\Lambda_u^p(w)$.*

Definition 1.5 The *associate space* of $\Lambda_u^{p,\infty}(w)$, denoted as $(\Lambda_u^{p,\infty}(w))'$, is defined as the set of all measurable functions g such that

$$\|g\|_{(\Lambda_u^{p,\infty}(w))'} := \sup_{f\in\Lambda_u^{p,\infty}(w)} \frac{|\int_{\mathbb{R}} f(x)g(x)u(x)\,dx|}{\|f\|_{\Lambda_u^{p,\infty}(w)}} < \infty.$$

In [6], these spaces were characterized as follows:

Proposition 1.6 [6] *If $0 < p < \infty$, then*

$$\left(\Lambda_u^{p,\infty}(w)\right)' = \Lambda_u^1\left(W^{-1/p}\right).$$

Proposition 1.7 [1] *Assume that the Hilbert transform H is well defined on $\Lambda_u^p(w)$ and that (1.1) holds. Then, we have the following conditions:*

(a) *$u \notin L^1(\mathbb{R})$ and $w \notin L^1(\mathbb{R}^+)$.*
(b) *There exists $C > 0$ such that, for every measurable set E and every interval I, such that $E \subset I$, we have that*

$$\frac{W(u(I))}{W(u(E))} \leq C\left(\frac{|I|}{|E|}\right)^p.$$

In particular, $W \circ u$ satisfies the doubling property; that is, there exists a constant $c > 0$ such that $W(u(2I)) \leq cW(u(I))$, for all intervals $I \subset \mathbb{R}$, where $2I$ denotes the interval with the same center as I and double the size length.

(c) *W is p quasi-concave. In particular, $w \in \Delta_2$.*
(d) *$w \in B_{p,\infty}$.*

As usual, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \leq CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

Taking into account Proposition 1.7, we shall assume from now on, and without loss of generality, that

$$w \in \Delta_2, \quad u \notin L^1(\mathbb{R}) \quad \text{and} \quad w \notin L^1(\mathbb{R}^+).$$

Also, we want to emphasize that, for a weight u in $\mathbb{R}$ we say that u satisfies the doubling property or $u \in \Delta_2$ if, for every interval I, $u(2I) \lesssim u(I)$, while in the case of a weight w in $\mathbb{R}^+$, the condition $w \in \Delta_2$ is given by (1.5).

Let us start by giving some important facts of each class of weights appearing in our results.

2 Several Classes of Weights

2.1 The B_∞^* Class

In this section we shall study weights satisfying (1.3) and we shall prove several properties that will be fundamental for our further results.

Lemma 2.1 *Let* $\varphi : (0,1] \to [0,1]$ *be an increasing submultiplicative function such that* $\varphi(\lambda) < 1$, *for some* $\lambda \in (0,1)$. *Then*,

$$\varphi(x) \lesssim \frac{1}{1 + \log(1/x)}.$$

Proof Since $0 < \lambda < 1$, given $x \in (0,1)$, there exists $k \in \mathbb{N} \cup \{0\}$ such that $x \in [\lambda^{k+1}, \lambda^k)$ and, using that $\varphi(\lambda) < 1$, it is clear that

$$\sup_{j \in \mathbb{N}} \varphi(\lambda)^j \big(1 + (j+1)\log(1/\lambda)\big) = C_\lambda < \infty.$$

Therefore,

$$\varphi(x) \le \varphi\big(\lambda^k\big) \le \varphi(\lambda)^k \lesssim \frac{1}{1 + (k+1)\log(1/\lambda)} \lesssim \frac{1}{1 + \log(1/x)},$$

as we wanted to see. □

Corollary 2.2 *If* $\varphi : (0,1] \to [0,1]$ *is an increasing submultiplicative function, the following conditions are equivalent*:

(1) *There exists* $\lambda \in (0,1)$ *such that* $\varphi(\lambda) < 1$.
(2) $\varphi(x) \lesssim (1 + \log(1/x))^{-1}$.
(3) *Given* $p > 0$, $\varphi(x) \lesssim (1 + \log(1/x))^{-p}$.
(4) $\lim_{x \to 0} \varphi(x) = 0$.

Proof Clearly (2), (3) and (4) imply (1) and, (2) and (3) imply (4). On the other hand, by Lemma 2.1, (1) implies (2). Hence, it only remains to prove that (1) implies (3). Suppose that $\varphi(\lambda) < 1$ and take $p > 0$. If $\psi = \varphi^{1/p}$, then ψ is also increasing, submultiplicative and $\psi(\lambda) < 1$, and by Lemma 2.1 we get (3). □

In what follows, the following function will play an important role,

$$\overline{W}(t) = \sup_{s>0} \frac{W(st)}{W(s)}.$$

Proposition 2.3 *The following statements are equivalent* (*see also* [2]):

(i) $w \in B_\infty^*$.
(ii) *There exists* $\lambda \in (0,1)$ *such that* $\overline{W}(\lambda) < 1$.
(iii) $\frac{W(t)}{W(s)} \lesssim (1+\log(s/t))^{-1}$, *for all* $0 < t \leq s$.
(iv) *Given* $p > 0$, $\frac{W(t)}{W(s)} \lesssim (1+\log(s/t))^{-p}$, *for all* $0 < t \leq s$.
(v) $\overline{W}(0^+) = 0$.
(vi) *For every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $W(t) \leq \varepsilon W(s)$, *provided* $t \leq \delta s$.

Proof Since $\overline{W}$ is submultiplicative we have, by Corollary 2.2 and letting $\varphi = \overline{W}_{|(0,1]}$, the equivalences between (ii), (iii), (iv) and (v). Also, note that if (vi) holds, then taking $\lambda = t/s$, we get $W(\lambda s) \leq \varepsilon W(s)$, for every $s \in [0,\infty)$ if $\lambda \leq \delta$, and hence we get (v). On the other hand, taking $t \leq \lambda s$, we get, by (v), that $W(t) \leq \varepsilon W(s)$ whenever $t \leq \delta s$.

Now, if (i) holds, for every $s \leq r$,

$$W(s) \log \frac{r}{s} \leq \int_s^r \frac{W(t)}{t} dt \lesssim W(r),$$

and since W is increasing we deduce that $W(s)(1 + \log \frac{r}{s}) \lesssim W(r)$, and (iii) holds. On the other hand if (iv) holds with $p = 2$, then

$$\int_0^r \frac{W(t)}{t} dt \lesssim W(r) \int_0^r \big(1 + \log(r/t)\big)^{-2} \frac{dt}{t} \lesssim W(r),$$

and hence (i) holds. □

Proposition 2.4 [2, 19] *Let* Q *be the conjugate Hardy operator defined by*

$$Qf(t) = \int_t^\infty f(s) \frac{ds}{s}.$$

Then, for every $0 < p < \infty$,

$$Q : L^p_{\mathrm{dec}}(w) \to L^{p,\infty}(w) \quad \Longleftrightarrow \quad w \in B_\infty^* \quad \Longleftrightarrow \quad Q : L^p_{\mathrm{dec}}(w) \to L^p(w).$$

Using now interpolation on the cone of decreasing functions [7], we obtain the following corollary:

Corollary 2.5 *Let* $0 < p < \infty$. *Then,*

$$w \in B_\infty^* \quad \Longleftrightarrow \quad Q : L^{p,\infty}_{\mathrm{dec}}(w) \to L^{p,\infty}(w).$$

2.2 The $B_{p,\infty}$ Class

As was mentioned in the introduction, if $p > 1$, $w \in B_{p,\infty}$ if and only if $w \in B_p$, and in this case the following result follows:

Proposition 2.6 *If* $1 < p < \infty$ *and* $w \in B_{p,\infty}$, *then*

$$\|\chi_E\|_{(\Lambda_u^{p,\infty}(w))'} \approx \frac{u(E)}{W^{1/p}(u(E))}.$$

Proof By Proposition 1.6, we obtain that

$$\|\chi_E\|_{(\Lambda_u^{p,\infty}(w))'} = \int_0^{u(E)} \frac{1}{W^{1/p}(t)}\, dt,$$

but, since $w \in B_p$, we have that [21],

$$\int_0^r \frac{1}{W^{1/p}(t)}\, dt \lesssim \frac{r}{W^{1/p}(r)},$$

and hence,

$$\frac{u(E)}{W^{1/p}(u(E))} \le \int_0^{u(E)} \frac{1}{W^{1/p}(t)}\, dt \lesssim \frac{u(E)}{W^{1/p}(u(E))},$$

as we wanted to see. □

2.3 $u \in A_\infty$ and $w \in B_\infty^*$

It is known that, if $u \in A_\infty$, then there exists $q > 1$ such that

$$\frac{u(I)}{u(E)} \lesssim \left(\frac{|I|}{|E|}\right)^q, \tag{2.1}$$

for every interval I and every set $E \subset I$ [14, p. 27].

Proposition 2.7 *We have that* $u \in A_\infty$ *and* $w \in B_\infty^*$ *if and only if the following condition holds*: *for every* $\varepsilon > 0$, *there exists* $0 < \eta < 1$ *such that*

$$W\big(u(S)\big) \le \varepsilon W\big(u(I)\big), \tag{2.2}$$

for every interval I *and every measurable set* $S \subseteq I$ *satisfying that* $|S| \le \eta |I|$.

Proof Let us first assume that $w \in B_\infty^*$ and $u \in A_\infty$. Then, by Proposition 2.3 we have that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $W(t) \le \varepsilon W(s)$, whenever $t \le \delta s$.

On the other hand, if $S \subset I$ is such that $|S| < \eta |I|$, for some $\eta > 0$,

$$\frac{u(S)}{u(I)} \le C_u \left(\frac{|S|}{|I|}\right)^r < C_u \eta^r,$$

where $r \in (0,1)$ and $C_u > 0$ are constants depending on the A_∞ condition. So, choosing $\eta \in (0,1)$ such that $C_u \eta^r < \delta$ we obtain the result.

Conversely, let us see first that $u \in A_\infty$. Let $\varepsilon = 1/2^{k-1}$, with $k \in \mathbb{N}$ and let $\varepsilon' < 1/c^k$, where $c > 1$ is the constant in the Δ_2 condition of w. Let $\delta = \delta(\varepsilon')$ be such that, by hypothesis, $|S| \le \delta |I|$ implies,

$$W\big(u(S)\big) \le \varepsilon' W\big(u(I)\big) < \frac{1}{c^k} W\big(u(I)\big).$$

If $\frac{u(I)}{u(S)} \le 2^{k-1}$ we get

$$W\big(u(S)\big) < \frac{1}{c^k} W\left(\frac{u(I)}{u(S)} u(S)\right) \le \frac{1}{c} W\big(u(S)\big),$$

which is a contradiction. Hence, necessarily $u(S) \le \frac{1}{2^{k-1}} u(I) = \varepsilon u(I)$. Thus, we have proved that,

$$\forall \varepsilon > 0, \exists \delta > 0; \quad |S| \le \delta |I| \quad \Longrightarrow \quad u(S) \le \varepsilon u(I),$$

and this implies that $u \in A_\infty$ [10].

Let us now prove that $w \in B_\infty^*$. By (2.2), we have that there exists $\lambda < 1$ such that $W(u(E))/W(u(I)) < 1/2$, provided $E \subset I$ and $|E| \le \lambda |I|$.

Now, since $u \in A_\infty$ we have by (2.1), that there exists $q > 1$ and $C_u > 0$ such that, for every $S \subset I$,

$$\frac{|S|}{|I|} \le C_u \left(\frac{u(S)}{u(I)}\right)^{1/q}, \tag{2.3}$$

and hence if we take δ such that $C_u \delta^{1/q} \le \lambda$, and $S \subset I$ such that $u(S)/u(I) \le \delta$, we obtain $W(u(S))/W(u(I)) < 1/2$.

Then, if $0 < t \le \delta s$ and we take an interval I such that $u(I) = s$ and $S \subset I$ satisfies $u(S) = t$, we obtain $W(t)/W(s) < 1/2$, and consequently $\overline{W}(\delta) < 1$. The result now follows from Proposition 2.3. □

3 Main Results

It is known (see [11, p. 256]) that if $f \in \mathcal{C}_c^\infty$, then

$$(Hf)^2 = f^2 + 2H(fHf), \tag{3.1}$$

and, using this equality, it was proved that, if $p > 1$,

$$H : L^p \to L^p \quad \Longrightarrow \quad H : L^{2p} \to L^{2p}.$$

Using the same sort of ideas we obtain the following result:

Theorem 3.1 *If* (1.1) *holds, for some* $0 < p < \infty$ *then, for every* $r > p$,

$$H : \Lambda_u^r(w) \longrightarrow \Lambda_u^r(w)$$

is bounded.

Proof By (3.1), we have that

$$\begin{aligned}\|Hf\|_{\Lambda_u^{2p,\infty}(w)} &= \|(Hf)^2\|^{1/2}_{\Lambda_u^{p,\infty}(w)} = \|f^2 + 2H(fHf)\|^{1/2}_{\Lambda_u^{p,\infty}(w)} \\ &\le C\big(\|f^2\|_{\Lambda_u^{p,\infty}(w)} + \|H(fHf)\|_{\Lambda_u^{p,\infty}(w)}\big)^{1/2} \\ &\le \big(C\|f\|^2_{\Lambda_u^{2p,\infty}(w)} + C_p\|fHf\|_{\Lambda_u^p(w)}\big)^{1/2}.\end{aligned}$$

Now, we have that

$$(fHf)_u^*(t) \le f_u^*(t/2)(Hf)_u^*(t/2)$$

and hence, since $w \in \Delta_2$, we obtain that

$$\begin{aligned}\|fHf\|_{\Lambda_u^p(w)} &\lesssim \left(\int_0^\infty \big(f_u^*(t)\big)^p \big((Hf)_u^*(t)\big)^p w(t)\,dt\right)^{1/p} \\ &= \left(\int_0^\infty \frac{(f_u^*(t))^p}{W^{1/2}(t)} \big(W^{1/2p}(t)(Hf)_u^*(t)\big)^p w(t)\,dt\right)^{1/p} \\ &\le \|Hf\|_{\Lambda_u^{2p,\infty}(w)}\|f\|_{\Lambda_u^{2p,p}(w)},\end{aligned}$$

where the $\Lambda_u^{q,p}(w)$ spaces are defined [6] by the condition

$$\|f\|_{\Lambda_u^{q,p}(w)} = \left(\int_0^\infty f^*(t)^p W^{\frac{p}{q}-1}(t)w(t)\,dt\right)^{1/p} < \infty.$$

Therefore, we have that

$$\|Hf\|^2_{\Lambda_u^{2p,\infty}(w)} \le C\|f\|^2_{\Lambda_u^{2p,\infty}(w)} + C_p\|f\|_{\Lambda_u^{2p,p}(w)}\|Hf\|_{\Lambda_u^{2p,\infty}(w)}$$

and, consequently,

$$\frac{\|Hf\|^2_{\Lambda_u^{2p,\infty}(w)}}{\|f\|^2_{\Lambda_u^{2p,p}(w)}} \le C\frac{\|f\|^2_{\Lambda_u^{2p,\infty}(w)}}{\|f\|^2_{\Lambda_u^{2p,p}(w)}} + C_p\frac{\|Hf\|_{\Lambda_u^{2p,\infty}(w)}}{\|f\|_{\Lambda_u^{2p,p}(w)}}.$$

Using that $\Lambda_u^{2p,p}(w) \hookrightarrow \Lambda_u^{2p,\infty}(w)$, we obtain that

$$\left(\frac{\|Hf\|_{\Lambda_u^{2p,\infty}(w)}}{\|f\|_{\Lambda_u^{2p,p}(w)}}\right)^2 \le C + C_p\frac{\|Hf\|_{\Lambda_u^{2p,\infty}(w)}}{\|f\|_{\Lambda_u^{2p,p}(w)}},$$

from which it follows that

$$\|Hf\|_{\Lambda_u^{2p,\infty}(w)} \lesssim \|f\|_{\Lambda_u^{2p,p}(w)}$$

and hence

$$H : \Lambda_u^{2p,p}(w) \longrightarrow \Lambda_u^{2p,\infty}(w)$$

is bounded. Finally, by interpolation (see [6, Theorem 2.6.5]), we obtain that, for every $p < r < 2p$,

$$H : \Lambda_u^r(w) \to \Lambda_u^r(w)$$

is bounded. The result now follows by iteration. □

Lemma 3.2 *Let* $0 < p < \infty$ *be fixed. If* (1.1) *holds, then*

$$\left\|H(uf)u^{-1}\right\|_{(\Lambda_u^p(w))'} \lesssim \|f\|_{(\Lambda_u^{p,\infty}(w))'}.$$

Proof The result follows easily from the definition of the associate spaces and the fact that

$$\int_{\mathbb{R}} (Hf)(x)g(x)\,dx = -\int_{\mathbb{R}} (Hg)(x)f(x)\,dx.$$ □

Lemma 3.3 *If* $p > 1$ *and* (1.1) *holds then, for every measurable set* E,

$$\sup_F \frac{\int_F |H(u\chi_E)(x)|\,dx}{W^{1/p}(u(F))} \lesssim \frac{u(E)}{W^{1/p}(u(E))},$$

where the supremum is taken over all measurable sets F.

Proof Using duality and Lemma 3.2, we can prove that (recall that $u(x) > 0$, a.e. $x \in \mathbb{R}$):

$$\begin{aligned}\int_F \left|H(u\chi_E)(x)\right| dx &= \int_F \left|H(u\chi_E)(x)u^{-1}(x)\right| u(x)\,dx \\ &\le \left\|H(u\chi_E)u^{-1}\right\|_{(\Lambda_u^p(w))'} \|\chi_F\|_{\Lambda_u^p(w)} \\ &\lesssim \|\chi_E\|_{(\Lambda_u^{p,\infty}(w))'} \|\chi_F\|_{\Lambda_u^p(w)},\end{aligned}$$

and the result follows by Proposition 2.6. □

As an immediate consequence, we obtain the following:

Corollary 3.4 *If* (1.1) *holds for some* $0 < p < \infty$, *then*

$$\sup_I \frac{1}{u(I)} \int_I \left|H(u\chi_I)(x)\right| dx < \infty, \tag{3.2}$$

where the supremum is taken over all intervals I.

Proof By Theorem 3.1, we can assume that $p > 1$ and therefore Lemma 3.3 holds. Taking $F = E = I$ in this lemma, we obtain the result. □

Theorem 3.5 *If H satisfies (1.1) for some $0 < p < \infty$, then $u \in A_\infty$.*

Proof It is known that if

$$Cf(\theta) = \text{p.v.} \int_0^1 \frac{f(x)}{\tan \pi(\theta - x)}\, dx$$

is the conjugate operator, then for an $f \in L^1(0,1)$ such that $Cf \in L^1(0,1)$, the non-tangential maximal operator $Nf \in L^1(0,1)$ [5]. Moreover, if $f \geq 0$, it is also known [5] that $Nf \approx Mf$ and, in fact,

$$\int_0^1 Mf(x)\, dx \lesssim \int_0^1 f(x)\, dx + \int_0^1 \big|Cf(x)\big|\, dx \lesssim \int_0^1 f(x)\, dx + \int_0^1 \big|Hf(x)\big|\, dx.$$

Now, if f is supported in an interval $I = (a,b)$, we can consider f_I defined on $(0,1)$ as $f_I(x) = f((b-a)x + a)$ and, by translation and dilation invariance of the operators M and H, we have that

$$\frac{1}{|I|} \int_I Mf(x)\, dx \lesssim \frac{1}{|I|} \int_I f(x)\, dx + \frac{1}{|I|} \int_I \big|Hf(x)\big|\, dx.$$

Consequently, if we take $f = u\chi_I$ and use (3.2) we obtain that, for every interval I,

$$\int_I M(u\chi_I)(x)\, dx \lesssim u(I),$$

and hence $u \in A_\infty$ [13, 22]. □

It was proved in [1] that if $u \in A_1$, the weak-type boundedness of H implies that $w \in B^*_\infty$. Now, an easy modification of that proof (we include the details for the sake of completeness) also shows that if $u \in A_\infty$, the same results holds.

Theorem 3.6 *If H satisfies (1.1) for some $0 < p < \infty$, then $w \in B^*_\infty$.*

Proof Let $0 < t \leq s < \infty$. Since $u \notin L^1(\mathbb{R})$, there exists $\nu \in (0,1]$ and $b > 0$ such that

$$t = \int_{-b\nu}^{b\nu} u(r)\, dr \leq \int_{-b}^{b} u(r)\, dr = s.$$

Now, simple computations of the Hilbert transform of the interval $(0,b)$ show [1] that, for every $b > 0$, and every $\nu \in (0,1]$,

$$\frac{W(\int_{-b\nu}^{b\nu} u(s)\,ds)}{W(\int_{-b}^{b} u(s)\,ds)} \lesssim \left(1+\log\frac{1}{\nu}\right)^{-p} \tag{3.3}$$

and hence

$$\frac{W(t)}{W(s)} \lesssim \left(1+\log\frac{1}{\nu}\right)^{-p}.$$

Let $S=(-b\nu, b\nu)$ and $I=(-b,b)$. Since $u\in A_\infty$, we obtain by (2.3), that there exists $q>1$ such that

$$\nu=\frac{|S|}{|I|}\lesssim\left(\frac{u(S)}{u(I)}\right)^{1/q}=\left(\frac{t}{s}\right)^{1/q}$$

and therefore

$$\frac{W(t)}{W(s)} \lesssim \left(1+\log\frac{s}{t}\right)^{-p}.$$

From here, it follows by Proposition 2.3 that $w\in B_\infty^*$. □

Our next goal is to prove that

$$H:\Lambda_u^p(w)\to\Lambda_u^{p,\infty}(w) \quad\Longrightarrow\quad M:\Lambda_u^p(w)\to\Lambda_u^{p,\infty}(w).$$

Let us start with some previous lemmas. We need to introduce the following notation: given a finite family of disjoint intervals $\{I_i\}_i$, we shall denote by $I_i^*=101I_i$. Then,

$$I_i^*=\bigcup_{j=-50}^{50} I_{i,j},$$

where $I_{i,j}$ is the interval with $|I_{i,j}|=|I_i|$,

$$\operatorname{dist}(I_{i,j}, I_i)=\big(|j|-1\big)|I_i|, \quad j\neq 0 \tag{3.4}$$

and such that $I_{i,j}$ is situated to the left of I_i, if $j<0$, and to the right, if $j>0$. Also, $I_{i,0}=I_i$.

If the family of intervals $\{I_i^*\}_i$ are pairwise disjoint, we say that $\{I_i\}_i$ is *well-separated*.

Lemma 3.7 *Let $u\in\Delta_2$. Then, given a well-separated finite family of intervals $\{I_i\}_i$, it holds that*

$$W^{1/p}\bigg(u\Big(\bigcup_i I_{i,j_i}\Big)\bigg)\approx W^{1/p}\bigg(u\Big(\bigcup_i I_i\Big)\bigg),$$

for any choice of $j_i\in[-50,50]$.

Proof Since w is also in Δ_2, we have that

$$W^{1/p}\Big(u\Big(\bigcup_i I_{i,j_i}\Big)\Big) \le W^{1/p}\Big(u\Big(\bigcup_i I_i^*\Big)\Big) = W^{1/p}\Big(\sum_i u(I_i^*)\Big)$$
$$\lesssim W^{1/p}\Big(\sum_i u(I_i)\Big) = W^{1/p}\Big(u\Big(\bigcup I_i\Big)\Big).$$

On the other hand, $I_i \subset I^*_{i,j_i}$ and hence

$$u\Big(\bigcup_i I_i\Big) = \sum_i u(I_i) \lesssim \sum_i u\big(I^*_{i,j_i}\big) \lesssim \sum_i u(I_{i,j_i}) = u\Big(\bigcup_i I_{i,j_i}\Big)$$

and therefore

$$W^{1/p}\Big(u\Big(\bigcup_i I_i\Big)\Big) \lesssim W^{1/p}\Big(u\Big(\bigcup_i I_{i,j_i}\Big)\Big),$$

and the result follows. □

Lemma 3.8 *Let f be a positive locally integrable function, $\lambda > 0$ and assume $\{I_i\}_{i=1}^m$ is a well separated family of intervals so that, for every i,*

$$\lambda \le \frac{\int_{I_i} f(y)\,dy}{|I_i|} \le 2\lambda.$$

Then, for every $1 \le i \le m$, there exists $j_i \in [-50, 50] \setminus \{0\}$ such that

$$\big|H(f\chi_{\bigcup_{i=1}^m I_i})(x)\big| \ge \frac{\lambda}{8}, \quad \textit{for every } x \in \bigcup_{i\in J} I_{i,j_i}.$$

Proof Given $1 \le i \le m$, let us define, for every $x \notin \bigcup_{i=1}^m I_i$,

$$A_i(x) = \sum_{j=1}^{i-1} \int_{I_j} \frac{f(y)}{x-y}\,dy, \qquad B_i(x) = \sum_{j=i+1}^{m} \int_{I_j} \frac{f(y)}{x-y}\,dy,$$

and

$$C_i(x) = A_i(x) + B_i(x).$$

If we write $g = f\chi_{\bigcup_{i=1}^m I_i}$, we have that

$$Hg(x) = C_i(x) + \int_{I_i} \frac{f(y)}{x-y}\,dy.$$

It also holds that if $I_i = (a_i, b_i)$, then A_i, B_i, and hence C_i, are decreasing functions in the interval (b_{i-1}, a_i).

Let us write $I_{i,-1} = (a_{i,-1}, b_{i,-1})$.

(a) If $C_i(a_{i,-1}) \leq \lambda/4$, then $C_i(x) \leq \lambda/4$, for every $x \in I_{i,-1}$ and since for these x,

$$\left|\int_{I_i} \frac{f(y)}{x-y}\,dy\right| = \int_{I_i} \frac{f(y)}{|x-y|}\,dy \geq \frac{\int_{I_i} f(y)\,dy}{2|I_i|} \geq \frac{\lambda}{2},$$

we obtain that, for every $x \in I_{i,-1}$

$$Hg(x) \leq \frac{\lambda}{4} - \frac{\lambda}{2} = -\frac{\lambda}{4}$$

and consequently $|Hg(x)| \geq \frac{\lambda}{4}$, for every $x \in I_{i,-1}$. Hence, in this case, we choose $j_i = -1$.

(b) If $C_i(a_{i,-1}) > \lambda/4$, then $C_i(x) \geq \lambda/4$, for every $x \in I_{i,j}$ with $j \in [-50, -2]$. Now, by (3.4), we have that if $x \in I_{i,j}$,

$$\left|\int_{I_i} \frac{f(y)}{x-y}\,dy\right| = \int_{I_i} \frac{f(y)}{|x-y|}\,dy \leq \frac{\int_{I_i} f(y)\,dy}{\operatorname{dist}(I_{i,j}, I_i)} \leq \frac{2\lambda}{|j|-1},$$

and thus, if we take $j = -17$, we obtain that, for every $x \in I_{i,-17}$

$$Hg(x) \geq \frac{\lambda}{4} - \frac{\lambda}{8} = \frac{\lambda}{8},$$

and consequently, in this case, with $j_i = -17$ the result follows. □

Theorem 3.9 *If* $p > 0$, *then*

$$H : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w) \quad \Longrightarrow \quad M : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w).$$

Proof Let us consider a positive locally integrable function f. Let $\lambda > 0$ and let us take a compact set K such that $K \subset \{x : Mf(x) > \lambda\}$. Then, for each $x \in K$, we can choose an interval I_x such that

$$\lambda < \frac{\int_{I_x} f(y)\,dy}{|I_x|} \leq 2\lambda.$$

Then, considering $K \subset \bigcup_{x\in K} I_x^*$, we can obtain, using a Vitali covering lemma, a well-separated finite family $\{I_i\}_{i=1}^m \subset \{I_x\}_x$, such that $K \subset \bigcup_i 3I_i^*$ and hence,

$$W^{1/p}\big(u(K)\big) \lesssim W^{1/p}\left(u\left(\bigcup_i 3I_i^*\right)\right) \lesssim W^{1/p}\left(u\left(\bigcup_i I_i\right)\right). \tag{3.5}$$

Now, by Lemma 3.8, we obtain that there exists j_i such that

$$\bigcup_{i=1}^m I_{i,j_i} \subset \left\{\left|H(f\chi_{\bigcup_{i=1}^m I_i})(x)\right| \geq \frac{\lambda}{8}\right\}.$$

Hence, by Lemma 3.7, we have that

$$\begin{aligned} W\left(u\left(\bigcup_i I_i\right)\right) &\approx W\left(u\left(\bigcup_{i=1}^m I_{i,j_i}\right)\right) \\ &\le W\left(u\left(\left\{\left|H(f\chi_{\bigcup_{i=1}^m I_i})(x)\right| \ge \frac{\lambda}{8}\right\}\right)\right) \\ &\lesssim \frac{1}{\lambda^p}\|f\|^p_{\Lambda^p_u(w)} \end{aligned}$$

and by (3.5), we obtain that

$$\lambda W^{1/p}\big(u(K)\big) \lesssim \|f\|_{\Lambda^p_u(w)}.$$

Finally, the result follows by taking the supremum on all compact sets $K \subset \{Mf > \lambda\}$. □

We finally present the proof of our main Theorem 1.1.

Proof of Theorem 1.1 If (1.1) holds, then we have, by Theorems 3.5 and 3.6, that $u \in A_\infty$ and $w \in B^*_\infty$. Also, by Theorem 3.9, the weak-type boundedness of M follows.

Conversely, it was proved in [4] that if $u \in A_\infty$,

$$\big(H^* f\big)^*_u(t) \lesssim \big(Q(Mf)^*_u\big)(t/4),$$

for all $t > 0$, provided the right hand side is finite, where

$$H^* f(x) = \frac{1}{\pi}\sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\,dy\right|$$

is the Hilbert maximal operator. Then, by Corollary 2.5 and the boundedness hypothesis on M, we have that

$$\begin{aligned} \big\|H^* f\big\|^p_{\Lambda^{p,\infty}_u(w)} &\lesssim \sup_{t>0} W(t)^{1/p} Q(Mf)^*_u(t/4) \\ &\lesssim \sup_{t>0} W(t)^{1/p}(Mf)^*_u(t) \lesssim \int_0^\infty f^*_u(t)^p w(t)\,dt, \end{aligned}$$

and therefore

$$H^* : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$$

is bounded. Now, since C^∞_c is dense in $\Lambda^p_u(w)$ and $Hf(x)$ is well defined at almost every point $x \in \mathbb{R}$, for every function $f \in C^\infty_c$, it follows by standard techniques that, for every $f \in \Lambda^p_u(w)$, $Hf(x)$ is well defined at almost every point $x \in \mathbb{R}$ and

$$H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$$

is bounded, from which the result follows. □

Observe that we have also proved the following result:

Theorem 3.10 *If* $0 < p < \infty$, *then*

$$H^* : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w)$$

is bounded if and only if conditions (i), (ii) *and* (iii) *of Theorem* 1.1 *hold.*

Taking into account Remark 1.3 and Proposition 2.7, we have the following characterization of (1.1), in terms of geometric conditions on the weights, in the case $0 < p < 1$.

Corollary 3.11 *If* $0 < p < 1$, (1.1) *holds if and only if* $u \in A_\infty$, $w \in B_\infty^*$ *and for every finite family of disjoint intervals* $\{I_j\}_{j=1}^J$, *and every family of measurable sets* $\{S_j\}_{j=1}^J$, *with* $S_j \subset I_j$, *for every* j, *we have that*

$$\frac{W(u(\bigcup_{j=1}^J I_j))}{W(u(\bigcup_{j=1}^J S_j))} \le C \max_{1\le j\le J} \left(\frac{|I_j|}{|S_j|}\right)^p, \tag{3.6}$$

or equivalently (3.6) *holds and, for every* $\varepsilon > 0$, *there exists* $0 < \eta < 1$ *such that*

$$W\big(u(S)\big) \le \varepsilon W\big(u(I)\big),$$

for every interval I *and every measurable set* $S \subseteq I$ *satisfying that* $|S| \le \eta |I|$.

As mentioned in Remark 1.3, the characterization of the weak-type boundedness of M in the case $p \ge 1$ was left open in [1] and it will be studied in a forthcoming paper.

3.1 Application to the $L^{p,q}(u)$ Spaces

In the case of the Lorentz spaces $L^{p,q}(u)$ we observe that $L^{p,q}(u) = \Lambda_u^q(w)$ and $L^{p,\infty}(u) = \Lambda_u^{q,\infty}(w)$, with $w(t) = t^{q/p-1}$ and since in this case $w \in B_\infty^*$ and the boundedness of

$$M : L^{p,q}(u) \to L^{p,\infty}(u)$$

is completely known (see [6, Theorem 3.6.1]), we have the following corollary, extending the result of [8, Theorem 5] in the case of the Hilbert transform.

Corollary 3.12 *For every* $p, q > 0$,

$$H : L^{p,q}(u) \longrightarrow L^{p,\infty}(u)$$

is bounded if and only if $p \ge 1$ *and*

(a) *if* $p > 1$ *and* $q > 1$: $u \in A_p$;

(b) *if* $p > 1$ *and* $q \leq 1$:

$$\frac{u(I)}{u(S)} \lesssim \left(\frac{|I|}{|S|}\right)^p$$

for every measurable set $S \subset I$;

(c) *if* $p = 1$, *then necessarily* $q \leq 1$ *and the condition is* $u \in A_1$.

Remark 3.13 We observe that Corollary 3.12, together with Theorem 3.9, gives us that, if $p > 1$, $q > 1$ and $u \in A_p$, then $M : L^{p,q}(u) \longrightarrow L^{p,\infty}(u)$, which was proved in [8].

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